Edge guards in rectilinear polygons

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Abstract

In this paper we consider the problem of placing edge guards to supervise a rectilinear art gallery. We show that no gallery with \( n \) vertices requires more than \( \lfloor (3n + 4)/16 \rfloor \) guards. Since this number of edge guards is necessary for some galleries the bound is tight. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

A rectilinear polygon is a polygon whose edges are parallel to a pair of orthogonal axes. In this paper the axes are assumed to be the horizontal axis (x-axis) and the vertical axis (y-axis). The edges of the rectilinear polygon thus alternate between vertical and horizontal. Our problem is to guard the interior of a rectilinear polygon. Therefore we consider only polygons that are simple; the term rectilinear polygon in this paper will imply a simple rectilinear polygon. We are concerned with choosing a subset of edges such that every point in the polygon can be seen from the edges in the set. Each edge in the set is an edge guard. A point \( p \) is visible to an edge guard \( g \) if and only if for some point \( x \in g \) the line segment \( xp \) does not intersect the exterior of the polygon.

The problem of determining how many edge guards are always sufficient and sometimes necessary to see every point in a rectilinear polygon is a variation of the original art gallery problem which asked "What is the minimum number of guards needed to guard an art gallery with \( n \) walls?". This problem and its many variations are the focus of a book by O'Rourke [9] and survey papers by Shermer [11] and Urrutia [13]. Some of the variations involve mobile guards. A mobile guard moves back and forth between two points never straying outside the polygon, that is a mobile guard is a line segment, which never intersects the exterior of the polygon. If the two points are consecutive vertices of the polygon then the guard is an edge guard. If the two points are nonconsecutive vertices, then the guard is a diagonal guard and if there are no restrictions on the locations of the points, then the guard is a line guard. Several results exist for diagonal guards and line guards, in particular Aggarwal [1] showed that \( \lfloor (3n + 4)/16 \rfloor \) line guards are necessary and sufficient to guard a rectilinear polygon with \( n \) vertices, and Györi et al. [5] showed that \( \lfloor (3n + 4(h + 1))/16 \rfloor \) line guards are necessary and sufficient for rectilinear polygons with \( h \)
holes. However, not much is known about edge guards. Theorems exist only for monotone polygons [3] and spiral polygons [2,14].

In this paper, we show that \( [(3n + 4)/16] \) edge guards are always sufficient to guard any rectilinear polygon.

That \( [(3n + 4)/16] \) edge guards are sometimes necessary, follows from the fact that \( [(3n + 4)/16] \) line guards are sometimes necessary, as shown by Aggarwal. The end points of a line guard are any two points in \( P \), restricting our choice of endpoints to consecutive vertices will not result in fewer guards being necessary.

2. Proof of sufficiency

**Theorem 1.** \( [(3n + 4)/16] \) edge guards are always sufficient to guard any rectilinear polygon with \( n \) vertices.

**Proof.** We begin by partitioning the rectilinear polygon \( P \) into convex quadrilaterals, using diagonals between vertices. That such a partitioning always exists was established by Kahn et al. [6] in 1980. Additional proofs of this result and methods for constructing such partitionings have been given by Győri [4], Lubiw [7], O’Rourke [8], Sack and Toussaint [10] and Urrutia [13]. Here we will use Lubiw’s quadrilateralization method.

Lubiw’s method starts at a specified edge \( ab \) and finds two other vertices \( c \) and \( d \), in such a way that the four vertices together form a convex quadrilateral. This quadrilateral is removed leaving one or more polygonal pieces, each consisting of original edges of \( P \) and one diagonal. Only the diagonal can be a slanted edge, all other edges alternate between vertical and horizontal. The step of removing a quadrilateral is repeated for each polygonal piece, with the diagonal as the specified edge \( ab \), until all remaining pieces are quadrilaterals. For a detailed description of the process, the reader is referred to Lubiw’s paper. Here we merely mention the following properties of the resulting quadrilateralization:

1. For a polygon with \( n \) vertices the number of quadrilaterals is \( (n - 2)/2 \).
2. The nose of a diagonal \( d \) (by which we mean the right triangle with \( d \) as its hypotenuse) is empty, that is it contains no part of \( P \)’s boundary.
3. A quadrilateral will have one, two, three or four sides that are diagonals. The remaining sides are edges of the polygon. We use the term *quadrilateral of degree x* to denote a quadrilateral with \( x \) diagonal sides.
4. A dual of the quadrilateralization is a tree with one node for each quadrilateral. Two nodes are connected if the corresponding quadrilaterals share a diagonal. Thus the nodes may have degree 1, 2, 3 or 4. At least two nodes have degree 1, since every tree of two or more nodes must have at least two leaves.
5. The number of possible configurations for a quadrilateral of a given degree is restricted as follows.
   - A quadrilateral of degree 1 has two possible configurations (see Fig. 1(a)).
   - A quadrilateral of degree 2 has six possible configurations (see Fig. 1(b)).
   - A quadrilateral of degree 3 has four possible configurations (see Fig. 1(c)).
   - A quadrilateral of degree 4 has one possible configuration (see Fig. 1(d)).

For convenience the degree 3 quadrilaterals will henceforth be referred to as being of type I, II, III or IV as indicated in the figure.
Fig. 1. Possible configurations for different degree quadrilaterals. (a) Degree 1 quadrilaterals. (b) Degree 2 quadrilaterals. (c) Degree 3 quadrilaterals. (d) Degree 4 quadrilaterals.

6. Not all configurations of quadrilaterals can lie next to each other. To determine which configurations can be adjacent we look at the shared diagonal. The charge on a diagonal $d$ for a given quadrilateral $Q$ is determined as follows: If three of the edges adjacent to $d$ and the quadrilateral $Q$ lie in the same
Fig. 2. The subtree contains four nodes. (a) Possible configurations of subtree. (b) Configurations where a single guard does not see all four quadrilaterals in the tree. The shaded quadrilaterals are covered, but quadrilateral L remains as a dangling leaf.

half-plane determined by $d$, then the charge on $d$ is $+$. If two of the edges adjacent to $d$ lie in the same half-plane as $Q$, then the charge is 0, and finally if only one edge adjacent to $d$ lies in the half-plane with $Q$, then the charge is $-$ (see Fig. 1). It follows from this definition that for two adjacent quadrilaterals the charge on the shared diagonal must be $+$ in one quadrilateral and $-$ in the other, or 0 in both quadrilaterals. Since all diagonals in a quadrilateral of degree 4 have $-$ charges two degree 4 quadrilaterals cannot be adjacent, nor can they exist with only degree 3 quadrilaterals in between.

7. In the dual tree two nodes with degree 4 cannot be adjacent nor can they be connected by a path consisting of degree 3 nodes only.

The last property follows immediately from property 6. For proofs of properties 5 and 6 see O'Rourke [9].

The remainder of the proof is constructive. We work simultaneously with the quadrilateralized polygon and the dual tree and place guards in such a way that no more than three guards are used for every eight quadrilaterals. We proceed as follows.

1. In the tree, pick as root a node with degree 1 (at least two such nodes exist).
2. Find the leaves on the deepest level of the tree.
3. Choose the smallest possible subtree with at least three nodes that contains at least one of these leaves. Such a subtree will contain three, four, five, six or seven nodes. No subtree will contain more than seven nodes since its root has at most three children (the degree is at most four and one edge leads to the parent) and each child in turn has at most one child (or it would be the root of a subtree with at least three children).
4. Place guards in the partial polygon corresponding to the subtree.
5. Remove the covered quadrilaterals from the polygon and remove the corresponding nodes from the tree.

6. Repeat steps 2–5 until all quadrilaterals are covered.

Occasionally, a single quadrilateral in the group corresponding to a subtree is left unguarded. The corresponding node is always a child of the root of the subtree and is not removed (see Fig. 2(b)). It remains in the tree to be included as a leaf of another subtree. We call this leaf a dangling leaf and indicate it by a dashed edge in the tree.

We next show how to place guards in the partial polygon corresponding to a subtree in such a way that no more than three guards are used for every eight quadrilaterals. For any quadrilaterals leftover at the end we place one guard for every whole or partial group of three quadrilaterals. We have five cases, one for each possible size of a subtree. We first consider the cases where no dangling leaves are present and then discuss the situations where such a presence make a difference.

**Case 1. The subtree contains three nodes**

One guard is used to cover three quadrilaterals.

There are two possible configurations of the subtree. A simple path consisting of the root, one child and one grandchild or a root with two children. We look at the central of the three nodes. In the first configuration this node has degree 2. Originally the degree could have been higher, if we take into account that other subtrees may already have been removed from the tree. Similarly the degree of the central node in the second configuration is 3, but could have started out as 4. Thus the quadrilateral corresponding to the central node has 2, 3 or 4 sides that are interior diagonals in the first configuration, and 3 or 4 diagonals in the second configuration. If the central quadrilateral has 2 or 3 sides that are diagonals, then either the two surrounding quadrilaterals share a vertex or there exists an edge which connects them (see Figs. 1(b) and (c)). A guard on an edge incident to the shared vertex or on the connecting edge sees all three quadrilaterals. If the central quadrilateral has 4 diagonals, then either the two surrounding quadrilaterals share a vertex or an edge incident to a vertex of one quadrilateral lies opposite the other quadrilateral with no obstructions between, since the noses must be empty (see Fig. 1(d)). A guard on an edge incident to the shared vertex or on the specified edge sees all three quadrilaterals.

**Case 2. The subtree contains four nodes**

One guard is used to cover at least three of the quadrilaterals.

There are three possible configurations of the subtree, two of which are symmetric (see Fig. 2(a)). We look at the root of the subtree. The degree of this node is 3 or 4. If it is 3, then it could have started out as 4. Thus the corresponding quadrilateral has three or four sides that are interior diagonals. If the subtree has two leaves, we will choose a guard as in Case 1 to cover the quadrilaterals corresponding to the root and the two nodes of the longer branch in the tree. This guard may cover all the quadrilaterals, but there are configurations where the fourth quadrilateral remains uncovered (see Fig. 2(b)). If so it is left as a dangling leaf. If the subtree has three leaves, then in the corresponding group of quadrilaterals, there exists an edge which is incident to a vertex shared by two leaves and the root quadrilateral, and which lies opposite the third leaf with no intervening obstructions (see Fig. 1(d)). A guard on this edge sees all four quadrilaterals.
Case 3. The subtree contains five nodes

There are four possible configurations of the subtree (see Fig. 3(a)). The root node can have degree 3 or 4.

The five quadrilaterals corresponding to nodes in the subtree can in most cases be covered by a single edge guard, but in the worst case two guards are necessary (see Fig. 3(b)). We never need more than two guards since we can view the subtree as two overlapping subtrees with three nodes each and assign guards as in Case 1. Let $v$ denote the root of the subtree and let $u$ be the parent of $v$. In all cases where two guards are necessary, these same two guards also cover one or more additional quadrilaterals in the remaining polygon. To see this we use the following argument.

- Look at the quadrilateral $V$ corresponding to node $v$. Either $V$ has one side which is an edge of the polygon and three sides which are diagonals or $V$ has four diagonal sides. Assume that guards cannot be placed in such a way that the quadrilateral $U$ corresponding to node $u$ is covered.

Let the two overlapping subtrees be $S_1$ and $S_2$ and let $p, q, r, s$ be the vertices of $V$ with $p, q$ being the diagonal shared by $U, q, r$ the diagonal leading to the rest of $S_1, r, s$ the diagonal leading to the rest of $S_2$ and $s, p$ being an edge of the polygon. Vertex $q$ cannot be shared by the three quadrilaterals in $S_1$ since if that was the case an edge incident to $q$ could be chosen as guard for $S_1$ and $U$ would be covered. Vertex $q$ cannot be the endpoint of an edge which connects the leaf in $S_1$ to $V$ for the same reason. Thus $r$ must be a vertex shared by the quadrilaterals in $S_1$. Similarly vertex $s$ cannot be shared...
by the quadrilaterals in $S_2$, since in that case edge $s, p$ could be chosen as guard and quadrilateral $U$
would be covered. Thus either $r$ is shared by the quadrilaterals in subtree $S_2$ or an edge incident to $r$
connects the leaf in $S_2$ to the $V$. In either case a single guard on this edge sees all five quadrilaterals.
If instead $p, q$ is the edge leading to the rest of $S_1$ and $q, r$ is the diagonal shared with $U$, then for the
same reasons as above, vertex $p$ must be shared by the quadrilaterals in $S_1$ and vertex $s$ must be shared
by the quadrilaterals in $S_2$. A single guard on edge $s, p$ covers all five quadrilaterals. The situation
when all sides are diagonals is analogous.

Using a single guard to cover five quadrilaterals is within our bound. If instead we used two guards then
the quadrilateral corresponding to $u$ is also covered. We look at $u$. If $u$ has degree 1 (after the removal
of $v$), then we simply include $u$ as the new root of $v$’s subtree. The corresponding quadrilateral is already
covered. The two guards cover six quadrilaterals. This is within our bound. If $u$ has current degree 2,
then $v$ has a sibling. If this sibling is a leaf, then we again include $u$ as the new root of $v$’s subtree. The
quadrilateral corresponding to the sibling may or may not be covered by the two assigned guards. If it is,
then two guards cover seven quadrilaterals, and if it is not, then we leave the sibling as a dangling leaf
to be included in another subtree. If $v$’s sibling is not a leaf, then $u$ is the root of a subtree containing
exactly three vertices. If there were more, another subtree would have been formed and removed below $u$.
After assigning one guard to cover the quadrilaterals corresponding to $u$’s subtree, we have used three
guards to cover eight quadrilaterals. This is within our bound. Finally, if $u$ has current degree 3, then
$v$ has two siblings and $u$ is the root of a subtree with three, four or five nodes. If there are three or
four nodes in $u$’s subtree, then we assign a guard as in Case 1 or 2, respectively. We have used three
guards to cover eight or nine quadrilaterals depending on whether a leaf is left dangling or not. This is
within our bound. If $u$’s subtree has five nodes, then we again use two guards for the five quadrilaterals.
In all we have used four guards to cover ten quadrilaterals. We have moved the problem of being one
quadrilateral short one step closer to the root. The parent of $u$ cannot have degree 3 (after $u$’s subtree
has been removed), since two degree 4 nodes cannot be adjacent. Therefore, in the next step, we will
encounter a node with current degree 1 or with current degree 2. The corresponding quadrilateral will be
included and the deficit made up or we assign an additional guard to cover a subtree with three nodes. In
the latter case after two such steps, we have used two guards for six quadrilaterals in addition to the four
guards for the ten quadrilaterals, or a total of six guards for sixteen quadrilaterals. Our deficit has been
made up.

Case 4. The subtree contains six nodes

We use two guards to cover six quadrilaterals. The root $v$ of the subtree must have degree 4 (see
Fig. 4(a)). Let the diagonals of the degree 4 quadrilateral be $A, B, C$ and $D$ in counter clockwise order,
with diagonal $A$ leading to the remaining part of the polygon (see Fig. 4(b)). The other diagonals lead to
groups of two quadrilaterals, or for one diagonal to a single leaf quadrilateral $L$. If $L$ lies on the other
side of $B$, then a guard on one of the vertical edges incident to an endpoint of $C$ and a guard on one of
the horizontal edges incident to an endpoint of $D$ see all six quadrilaterals. If $L$ lies on the other side
of $C$, then a guard on one of the horizontal edges incident to the endpoints of $D$ and a guard on one of
the horizontal edges incident to the endpoints of $B$ see all six quadrilaterals. If $L$ lies on the other side
of $D$, then a guard on the vertical edge incident to the vertex shared by $C$ and $D$ and the horizontal edge
incident to an endpoint of $B$ see all six quadrilaterals.
Fig. 4. The subtree contains six nodes. (a) Possible configurations of subtree. (b) Two guards cover all quadrilaterals.

Fig. 5. The subtree contains seven nodes. (a) Single possible configuration of subtree. (b) Three guards are necessary to cover all quadrilaterals in the tree.
Case 5. The subtree contains seven nodes

There is only one possible configuration of this subtree (see Fig. 5(a)). The root node must have
degree 4. In the worst case (see Fig. 5(b)) three guards are necessary to cover the seven quadrilaterals.
Note that the three guards also cover one or more quadrilaterals in the remaining polygon. Let \( v \) denote
the root of the subtree and let \( u \) be the parent of \( v \). We look at \( u \). If \( u \) has degree 1 (after the removal of
\( v \)'s subtree), then we simply include \( u \) as the new root of \( v \)'s subtree. The corresponding quadrilateral
is already covered and the three guards cover eight quadrilaterals. This is within our bound. If \( u \) has
current degree 2, then \( v \) has a sibling. If this sibling is a leaf, then we again include \( u \) as the new root
of \( v \)'s subtree. The quadrilateral corresponding to the sibling may or may not be covered by the three
assigned guards. If it is, then three guards cover nine quadrilaterals, and if it is not, then we leave the
sibling as a dangling leaf to be included in the next subtree. If \( v \)'s sibling is not a leaf, then \( u \) is the
root of a subtree containing exactly three vertices. If there were more, another subtree would have been
formed and removed below \( u \). After assigning one guard to cover the quadrilaterals corresponding to \( u \)'s
subtree, we have used 4 guards to cover 10 quadrilaterals. We have also moved the problem of being one
quadrilateral short one step closer to the root. Note that \( u \) cannot have degree 3 (after the removal of \( v \)),
since two degree 4 nodes cannot be adjacent and cannot be connected by a path consisting of nodes with
degree 3. Therefore if we continue the process working our way up in the tree, we will either be able to
include an additional quadrilateral to make up our deficit, or we will keep removing subtrees containing
three nodes assigning one additional guard each time. After three such steps we have used three guards
for nine quadrilaterals in addition to the three guards for the seven quadrilaterals, or a total of six guards
for sixteen quadrilaterals. Our deficit has been made up.

In the above cases we assumed that no dangling leaves are present. We next discuss how to resolve the
situations where the presence of a dangling leaf makes a difference.

In general, when a dangling leaf is left, there are two edges in the remaining polygon (excluding the
edges of the leaf quadrilateral itself) from which the quadrilateral is visible. These edges are the edge
incident to the vertex shared by the quadrilateral and the rest of the polygon and an edge on the other
side of the diagonal that connected the leaf to the previous subtree root with no obstructions in between
(see leftmost example in Fig. 2(b)). A dangling leaf of this kind can be treated as a regular leaf, which
is completely attached along diagonal \( d \) (see leftmost example in Fig. 1(a)). In each of the five cases
above such a regular leaf is always guarded by a guard on one of the edges incident to \( d \)'s endpoints.
For the dangling leaf this corresponds to placing a guard on one of the two edges from which the leaf is
visible. Thus dangling leaves of this type present no problems and guard assignment is handled by the
cases above.

The other type of dangling leaf is one where the leaf quadrilateral is visible only from a single edge of
the remaining polygon (excluding the edges of the leaf itself). This situation can occur if the root node \( v \)
of the removed subtree corresponds to a degree 3 quadrilateral of type I or II (see central and rightmost
example in Fig. 2(b)). We look at both possibilities. If \( v \)'s quadrilateral is of type I, then the diagonal
leading to the root has a charge of \(-\). The next quadrilateral on way to the root must be a degree 2
quadrilateral or a degree 3 quadrilateral of type III or IV, since the charge on the same diagonal in this
quadrilateral must be \(+\). Similarly, if \( v \)'s quadrilateral is of type II, then the diagonal leading to the root
has a charge of 0. In the neighboring quadrilateral the charge on the diagonal must also be 0 and the
quadrilateral is a degree 2 quadrilateral or a degree 3 quadrilateral of type I or II. Note that a dangling
leaf of this type is never produced when the root of the removed subtree is a quadrilateral of degree 4.
Thus we never have the situation that the quadrilaterals in the removed tree were used as part of a group of eight or sixteen quadrilaterals. Instead this type of dangling leaf is produced in two cases, namely Case 2, after one guard has been used to cover three quadrilaterals, and in Case 3 after two guards have been used to cover six quadrilaterals.
Let $u$ be the parent of the dangling leaf. We look at the five cases under the assumption that dangling leaves are present.

**Case 1a. The subtree has three nodes**

If $u$ has degree 2, then a guard on the edge incident to the vertex shared by the leaf and the remaining polygon sees all three quadrilaterals. If $u$ has degree 3 (see Fig. 6), then whether the corresponding quadrilateral is of type I, II, III or IV, a guard on one of the leaf edges or the edge incident to the shared vertex sees all three quadrilaterals in the subtree.

**Case 2a. The subtree has four nodes**

We use one guard for the dangling leaf and one guard for the remaining subtree with three nodes. Including the one guard assigned when the dangling leaf was originally left, we have used a total of three guards to cover seven nodes. The problematic dangling leaf is no longer present and we proceed as in regular Case 5 above.

**Case 3a. The subtree has five nodes**

We view the subtree as two trees overlapping at the root and assign two guards as in Case 1a. We then proceed as in regular Case 3 above.
Case 4a. The subtree has six nodes

The root of the subtree must have degree 4 (see Fig. 7(a)). Two dangling leaves may be present. We assign two guards to cover the two longer branches of the tree. These guards will also see the sixth quadrilateral except in one case (see Fig. 7(b)). In this last case we leave the sixth quadrilateral as a dangling leaf. Note that in this case the dangling leaf is none of the problematic kind. We have used two guards for five quadrilaterals and including one guard used when one of the dangling leaves were produced, we have used a total of three guards for eight quadrilaterals. This is within our bound.

Case 5a. The subtree has seven nodes

In the worst case we must use three guards for seven quadrilaterals. We never need more since we can view the tree as three trees overlapping at the root and assign one guard per tree as in Case 1a above. We then proceed as in regular Case 5.

The assignment of edge guards as described above ensures that no more than three guards are used for every group of eight quadrilaterals and that for any remaining quadrilaterals at most one guard is used for every group of three quadrilaterals. With one additional guard for any partial group of one or two quadrilaterals left over at the end, this gives an upper bound of $3\lfloor q/8 \rfloor + \lceil (q \mod 8)/3 \rceil$ edge guards for a polygon with $q$ quadrilaterals. Equivalently, with $q = (n - 2)/2$ we have an upper bound of $\lfloor (3n + 4)/16 \rfloor$ edge guards.

Given the necessity of $\lfloor (3n + 4)/16 \rfloor$ edge guards and the theorem above we can state the following theorem.

**Theorem 2.** $\lfloor (3n + 4)/16 \rfloor$ edge guards are always sufficient and sometimes necessary to guard a rectilinear polygon with $n$ vertices.

3. Concluding remarks

We have shown that $\lfloor (3n + 4)/16 \rfloor$ is a tight bound on the number of edge guards required to guard a rectilinear polygon. This solves the question for rectilinear polygons without holes. The problem for rectilinear polygons with holes remains open as does the problem of finding a tight bound for the number of edge guards required in any arbitrarily shaped simple polygon. In the latter case an upper bound of $\lfloor n/3 \rfloor$ is easily achieved by simply taking the $\lfloor n/3 \rfloor$ vertex guards known to cover the polygon and assigning edges incident to these vertices as edge guards. A slight improvement of this bound is provided by Shermer [11] who shows that $\lceil 3n/10 \rceil$ edge guards are sufficient except for certain values of $n$. Toussaint conjectures that with the qualification that certain polygons with a small number of vertices be excluded, $\lfloor n/4 \rfloor$ is a tight bound on the number of required edge guards.

References


