Spectra and essential spectral radii of composition operators on weighted Banach spaces of analytic functions

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Abstract

We determine the spectra of composition operators acting on weighted Banach spaces \(H^\infty_v\) of analytic functions on the unit disc defined for a radial weight \(v\), when the symbol of the operator has a fixed point in the open unit disc. We also investigate in this case the growth rate of the Koenigs eigenfunction and its relation with the essential spectral radius of the composition operator.

Keywords: Weighted Bergman spaces of infinite order; Composition operators; Spectrum; Essential spectral radius; Koenigs eigenfunction

1. Introduction

The purpose of this paper is to determine the spectrum of a composition operator which is continuous on a weighted Banach space of analytic functions on the open unit disc \(\mathbb{D}\) of type \(H^\infty\), and to investigate how the essential spectral radius of the operator determines the growth of the Koenigs eigenfunction for the symbol. We denote by \(H(\mathbb{D})\) the space of holomorphic functions on \(\mathbb{D}\). As usual, \(H^\infty\) is the space of bounded analytic functions on \(\mathbb{D}\) endowed with the norm \(\|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)|\). Given an analytic self map \(\varphi\) on \(\mathbb{D}\), the composition operator on \(H(\mathbb{D})\) is defined by \(C_\varphi(f) = f \circ \varphi\). Clearly, for each positive \(n\), \(C^n_\varphi = C_{\varphi^n}\), where \(\varphi^n\) is the \(n\)th iterate of \(\varphi\). We refer the reader to the books of Cowen and MacCluer [7] and Shapiro [19] for a deep study of composition operators on classical spaces of holomorphic functions on the disc.

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A weight \( v : \mathbb{D} \to \mathbb{R} \) is a radial bounded continuous strictly positive function on the unit disc \( \mathbb{D} \) of the complex plane. We consider the weighted Bergman spaces of infinite order

\[
H_v^\infty = \left\{ f \in H(\mathbb{D}): \| f \|_v := \sup_{z \in \mathbb{D}} v(z) |f(z)| < \infty \right\}
\]

and

\[
H_v^0 = \left\{ f \in H_v^\infty: \lim_{|z| \to 1} v(z) |f(z)| = 0 \right\},
\]

endowed with the norm \( \| \cdot \|_v \). Notice that the norm topology of \( H_v^\infty \) is finer than the compact-open topology induced on the space. For more details about spaces of this type we refer to [2,3,14] and the references therein.

In this paper we determine the spectrum of the composition operator \( C_\varphi \) on both \( H_v^\infty \) and \( H_v^0 \) for more general weights \( v \) than the standard weights \( v_p(z) = (1 - |z|^2)^p \), \( p > 0 \), when \( \varphi \) has an attractive fixed point in \( \mathbb{D} \); thus we extend the results obtained by Aron and Lindström in [1]. This is presented in Theorem 3.5 and Corollary 3.6. The proof of Theorem 3.5 has its origins in the work of H. Kamowitz [13] who described the spectrum of \( C_\varphi \) on \( H^p \) spaces for several types of symbols; particularly, in the case of a function \( \varphi \) analytic on the closed disc, not inner and having an interior fixed point. Later C. Cowen and B. MacCluer [8] made a substantial improvement by, among other things, turning to \( C_\varphi \). Those results are the grounds on which L. Zheng [21] based her corresponding theorem in the \( H^\infty \) case. The proofs of the above mentioned results follow a similar pattern, which led to the proof of Theorem 8 in [1], that we follow closely to show Theorem 3.5. Other recent investigations on the spectrum of non-compact composition operators acting on Banach spaces of analytic functions are [15,17,18].

For general radial weights \( v \), which are typical in the sense defined below, and holomorphic self maps \( \varphi \) having an attractive fixed point \( w \) in \( \mathbb{D} \) we also study how the essential spectral radius of \( C_\varphi \) on both \( H_v^\infty \) and \( H_v^0 \) determines whether the Koenigs eigenfunction \( \sigma \) of \( C_\varphi \) belongs to \( H_v^\infty \) and \( H_v^0 \) respectively. Or in other words, whether \( \varphi'(w)^p \) belongs to the point spectrum of \( C_\varphi \). Every holomorphic self map \( \varphi \) having non-zero derivative at its Denjoy–Wolf point \( w \in \mathbb{D} \) has a unique Koenigs eigenfunction \( \sigma \in H(\mathbb{D}) \) determined by \( \sigma \circ \varphi = \varphi'(w)\sigma \), \( \sigma'(w) = 1 \). We refer the reader to chapters 5 and 6 of Shapiro’s book [19] and to the survey [20]. P. Bourdon [5] proved that the Koenigs eigenfunction \( \sigma \in H_v^0 \) if and only if \( |\varphi'(0)| > r_{e,H_v^0}(C_\varphi) \), in case \( \varphi \) has an attractive fixed point, say 0, in \( \mathbb{D} \); see [5, Theorem 4.4]. He also proved that if \( \sigma \in H_v^\infty \), then \( |\varphi'(0)| \geq r_{e,H_v^\infty}(C_\varphi) \), and that the converse does not hold; see [5, Section 5]. Bourdon established his results in terms of what he calls the “essential angular derivative \( \alpha(\varphi) \) of \( \varphi \),” which satisfies \( r_{e,H_v^\infty}(C_\varphi) = 1/\alpha(\varphi)^p \); see more details in Section 4. Our results and examples in this Section extend part of Bourdon results, and show that some of his results do not hold for arbitrary radial weights on the unit disc.

2. Preliminaries

A radial weight \( v \) is called typical if it is non-increasing with respect to \( |z| \) and satisfies \( \lim_{|z| \to 1} v(z) = 0 \). The associated weight \( \tilde{v} \) is defined by

\[
\tilde{v}(z) = \left( \sup \left\{ |f(z)|: f \in H_v^\infty, \| f \|_v \leq 1 \right\} \right)^{-1}.
\]

If \( v \) is a radial weight, then also \( \tilde{v} \) is a radial weight and it is non-increasing. If we take \( \tilde{v} \) instead of \( v \) and \( v \) is typical, then \( \tilde{v} \) is also typical and both the spaces \( H_v^\infty \) and \( H_v^0 \) do not change when we replace \( v \) by \( \tilde{v} \). Moreover, \( v \leq \tilde{v} \). We say that \( v \) is an essential weight if there is a constant \( C \) such that \( v(z) \leq \tilde{v}(z) \leq Cv(z) \) for all \( z \in \mathbb{D} \). For the standard weights \( v_p(z) = (1 - |z|^2)^p \), \( p > 0 \), we have that \( \tilde{v}_p = v_p \). Further, given \( z \in \mathbb{D} \), the element \( \delta_z \in (H_v^\infty)^* \) defined by \( \delta_z(f) = f(z) \) satisfies \( \| \delta_z \|_v = 1/\tilde{v}(z) \), and for each \( z \in \mathbb{D} \) there is \( f_z \in H_v^\infty, \| f_z \|_v \leq 1 \), such that \( |f_z(z)| = 1/\tilde{v}(z) \). More information about the associated weight \( \tilde{v} \) can be found in [2,3]. For a typical weight \( v \) the polynomials are a dense subset of \( H_v^0 \). Note that for non-typical weights it may happen that \( H_v^0 \) contains only the zero function. Two weights \( v \) and \( w \) are equivalent if there are positive constants \( c, C \) such that \( cv \leq w \leq Cva \) on \( \mathbb{D} \). A real function \( f \) defined on \([0, 1]\) is called almost decreasing if there is \( C > 0 \) such that \( s < t \) implies \( f(t) \leq Cf(s) \).

For \( a \in \mathbb{D} \), let \( \varphi_a(z) = (a - z)/(1 - \bar{a}z) \), so that \( \varphi_a \) is an automorphism of \( \mathbb{D} \) that exchanges the points 0 and \( a \). If \( v \) is a typical weight that satisfies the Lusky condition [14]

\[
\inf_n \frac{\tilde{v}(1 - 2^{-n})}{\tilde{v}(1 - 2^{-n})} > 0,
\]

(*
them Theorem 2.3 in [3] ensures that all operators $C_\psi$ are bounded on both $H^0_v$ and $H^\infty_v$. Several conditions equivalent to (**) can be seen in [9]. If condition (**) is satisfied, then $C_{\psi_a}$ is an invertible bounded operator on both $H^0_v$ and $H^\infty_v$ for every $a \in \mathbb{D}$.

In Section 4 we investigate the growth of Koenigs eigenfunction for an analytic self map $\psi$ on the unit disc satisfying $\psi(0) = 0, 0 < |\psi'(0)| < 1$. Our results yield immediately consequences for analytic self maps with a Denjoy–Wolff point $w \in \mathbb{D}$ such that $\psi'(w) \neq 0$, at least if the weight $v$ satisfies Lusky’s condition (**). Indeed, if $w \neq 0$, define $\psi = \psi_w \circ \psi \circ \psi_w$. The Koenigs eigenfunctions $\sigma_\psi$ for $\psi$ and $\sigma$ for $\psi$ are related by the simple formula $\sigma = (|w|^2 - 1)\sigma_\psi \circ \psi_w$. Here the factor $(|w|^2 - 1)$ yields the normalization of the derivative $\sigma'(w) = 1$; see [5, p. 571]. Now

$$v(z)|\sigma(z)| = \frac{v(z)}{\overline{v} (\psi_w(z))} (|w|^2 - 1) \overline{v} (\psi_w(z)) |\sigma_\psi (\psi_w(z))|$$

is bounded or tends to 0 as $|z|$ tends to 1 if and only if $v(z)|\sigma_\psi (z)|$ does, since $\frac{v(z)}{\overline{v} (\psi_w(z))}$ is bounded above and bounded away from 0 by [3]. An analogous consideration holds for $\sigma^n$ and $\sigma^n_\psi$. On the other hand, since the composition operators $C_\psi$ and $C_\psi$ are similar, the spectrum, and the essential spectrum radius, defined below, of both coincide.

The essential spectrum $\sigma_{e, X}(T)$ of a bounded operator $T$ on the Banach space $X$ is the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda I$ is not Fredholm. It is known that $\sigma_{e, X}(T) = \sigma_{e, X^*}(T^*)$ [11]. The essential spectral radius of $T$ on $X$ is given by

$$r_{e, X}(T) = \sup \{|\lambda| : \lambda \in \sigma_{e, X}(T)\}.$$ 

Another way of expressing the essential spectral radius is

$$r_{e, X}(T) = \lim_{n} \|T^n\|^{\frac{1}{n}}_{e, X},$$

where $\|T\|_{e, X}$ denotes the essential norm of $T$, i.e., the distance from the compact operators on $X$.

Let $\lambda \in \sigma_{X}(T)$ be such that $|\lambda| > r_{e, X}(T)$. Then $\lambda$ lies in the unbounded component of $\mathbb{C} \setminus \sigma_{e, X}(T)$. Now the Fredholm theory gives that $\lambda$ is an isolated point of $\sigma_{X}(T)$ which also is an eigenvalue of finite multiplicity. Let us state this well-known result as a lemma; see [11].

**Lemma 2.1.** Let $T : X \to X$ be a bounded operator. If $\lambda \in \sigma_{X}(T)$ is such that $|\lambda| > r_{e, X}(T)$, then $\lambda$ is an isolated eigenvalue of finite multiplicity.

If the weight $v$ is typical, then the following formula of the essential norm of $C_\psi$ on $H^\infty_v$ in terms of the weight has been obtained in [16] or [6], in [6] with the extra assumption that $H^0_v$ is isomorphic to $c_0$; see also [4],

$$\|C_\psi\|_{e, H^\infty_v} = \lim_{r \to 1} \sup_{|\psi(z)| > r} \frac{v(z)}{\overline{v} (\psi(z))}.$$

If $C_\psi$ is also bounded on $H^0_v$, then $\|C_\psi\|_{e, H^0_v} = \lim_{r \to 1} \sup_{|\psi(z)| > r} \frac{v(z)}{\overline{v} (\psi(z))}$.

By Theorem 2.1 in [3], for any typical weight $v$, $C_\psi : H^\infty_v \to H^\infty_v$ is bounded if and only if $C_\psi$ is bounded on $H^0_v$. Actually, in this case, $C_\psi : H^\infty_v \to H^\infty_v$ is the bitranspose map of $C_\psi : H^0_v \to H^0_v$. Therefore,

$$r_{e, H^0_v}(C_\psi) = r_{e, H^\infty_v}(C_\psi) \quad \text{and also} \quad \sigma_{H^0_v}(C_\psi) = \sigma_{H^\infty_v}(C_\psi)$$

for every typical weight $v$ such that $C_\psi$ is bounded on $H^0_v$ or $H^\infty_v$. In particular, we have

$$r_{e, H^0_v}(C_\psi) = r_{e, H^\infty_v}(C_\psi) = \left( \lim_{n} \sup_{|z| \to 1} \frac{v(z)}{\overline{v} (\psi_n(z))} \right)^{1/n}.$$  

If $\varphi(0) = 0$ and $0 \neq |\varphi'(0)| < 1$, then Koenigs Theorem (see 6.1 in [19]) states that the sequence of functions $\sigma_k(z) := \frac{\varphi_k(z)}{\varphi(0)}$ converges uniformly on compact subsets of $\mathbb{D}$ to a non-constant function $\sigma$, which is called Koenigs function, that satisfies $\sigma \circ \varphi = \varphi'(0)\sigma$. More generally, if $f$ and $\lambda$ solve $f \circ \varphi = \lambda f$, then there is a positive integer $n$ such that $\lambda = \varphi'(0)^n$ and $f$ is a constant multiple of $\sigma^n$. Note also that $|\varphi'(0)| < 1$ when $\varphi(0) = 0$ and $\varphi$ is not an automorphism.
3. The spectrum of \( C_\phi \). Main result

In this section we assume that \( \varphi(0) = 0, 0 < |\varphi'(0)| < 1 \). The assumption \( \varphi(0) = 0 \) implies that \( C_\phi \) is bounded both on \( H^0_v \) and \( H^\infty_v \).

**Lemma 3.1.** Let \( v \) be a typical weight and assume that \( \varphi(0) = 0 \) and \( 0 < |\varphi'(0)| < 1 \). Both \( \sigma_{H^0_v}(C_\phi) \) and \( \sigma_{H^\infty_v}(C_\phi) \) contain \( \varphi'(0)^n \) for all non-negative integers \( n \).

**Proof.** We use an argument of [12] to show that \( \varphi \) means that \( \sigma_{H^0_v}(C_\phi) \) and \( \sigma_{H^\infty_v}(C_\phi) \) contain \( \varphi'(0)^n \) for all non-negative integers \( n \).

For a positive integer \( m \) and an arbitrary weight \( v \), let \( H^\infty_{v,m} \) denote the closed subspace of \( H^\infty_v \) given by

\[
H^\infty_{v,m} := \{ f \in H^\infty_v : f \text{ has a zero of at least order } m \text{ at } 0 \}.
\]

We denote by \( \| \cdot \|_{m,v} \) the induced norm on \( H^\infty_{v,m} \). The following result is taken from [1].

**Lemma 3.2.** Let \( v \) be a weight and \( w \in \mathbb{D} \) such that \( |w| \geq 1/2 \). Then

\[
\| \delta_w \|_{v,m} \| \delta_w \|_v \leq 2^m \| \delta_w \|_{v,m}.
\]

**Proof.** Since \( H^\infty_{v,m} \subset H^\infty_v \), it follows that \( \| \delta_w \|_{v,m} \leq \| \delta_w \|_v \), so that the inequality on the left holds for all \( w \in \mathbb{D} \). Fix \( w \in \mathbb{D} \) with \( |w| \geq 1/2 \). For \( f \) in the unit ball of \( H^\infty_v \), the function \( g(z) := z^m f(z) \) belongs to the unit ball of \( H^\infty_{v,m} \) and we have

\[
|f(w)| \leq 2^m |\delta_w(g)| \leq 2^m \| \delta_w \|_{v,m}.
\]

Then taking the supremum over all \( f \) in the unit ball of \( H^\infty_v \), we have \( \| \delta_w \|_v \leq 2^m \| \delta_w \|_{v,m} \).

Now we want to estimate the norm of the evaluation map acting on the subspaces \( H^\infty_{v,m} \) of \( H^\infty_v \) for a general weight \( v \). For the standard weight \( v_p \) this result has been obtained in [1] with a different, longer proof.

**Proposition 3.3.** Let \( v \) be a weight and \( m \in \mathbb{N} \). Then there is a constant \( M_m > 0 \) such that

\[
|f(w)| \leq M_m \frac{1}{v(w)} \| f \|_v \| w \|_m^m
\]

for all \( f \in H^\infty_{v,m} \) and \( w \in \mathbb{D} \).

**Proof.** It is easy to see that \( H^\infty_{v,m} = z^m H^\infty_v \). Consequently, we can apply the closed graph theorem to get that the map \( f \mapsto f/z^m \) is well defined, linear and continuous from \( H^\infty_{v,m} \) into \( H^\infty_v \). Therefore, there is \( M_m > 0 \) such that \( \| f/z^m \|_v \leq M_m \| f \|_{m,v} \) for each \( f \in H^\infty_{v,m} \). If \( w \in \mathbb{D} \) and \( f \in H^\infty_{v,m} \), we have

\[
|f(w)| = |w|^m \left| \frac{f(w)}{z^m} \right| \leq |w|^m \| \delta_w \|_v \left| \frac{f(z)}{z^m} \right| \leq M_m |w|^m \| f \|_{m,v} \frac{1}{v(w)}.
\]

Recall that \( (z_k) \) is an iteration sequence for \( \varphi \) if \( \varphi(z_k) = z_{k+1} \) for all \( k \). We need the following crucial lemma due to Cowen and MacCluer [7].

**Lemma 3.4.** If \( \varphi \) is not an automorphism and \( \varphi(0) = 0 \), then given \( 0 < r < 1 \), there exists \( 1 \leq M < \infty \) such that if \( (z_k)_k \) is an iteration sequence with \( |z_k| \geq r \) for some non-negative integer \( n \) and \( (w_k)_k \) are arbitrary numbers, then there exists \( f \in H^\infty_v \) with \( f(z_k) = w_k, -K \leq k \leq n \) and \( \| f \|_\infty \leq M \sup \{|w_k| : -K \leq k \leq n\} \). Further there exists \( b < 1 \) such that for any iteration sequence \( (z_k) \) we have \( |z_{k+1}|/|z_k| \leq b \) whenever \( |z_k| \leq 1/2 \).
Theorem 3.5. Let \( v \) be a typical weight. Suppose \( \varphi \), not an automorphism, has fixed point \( 0 \in \mathbb{D} \). Then

\[
\sigma_{H_v^\infty}(C_\varphi) = \{ \lambda \in \mathbb{C} : |\lambda| \leq r_{e, H_v^\infty}(C_\varphi) \} \cup \{ \varphi'(0)^n \}_{n=0}^\infty.
\]

Proof. By Lemmas 2.1 and 3.1 it remains to show that

\[
\{ \lambda \in \mathbb{C} : |\lambda| \leq r_{e, H_v^\infty}(C_\varphi) \} \subset \sigma_{H_v^\infty}(C_\varphi).
\]

If \( r_{e, H_v^\infty}(C_\varphi) = 0 \), we are done since \( 0 \in \sigma_{H_v^\infty}(C_\varphi) \) when \( \varphi \) is not an automorphism. So we assume that \( \rho := r_{e, H_v^\infty}(C_\varphi) > 0 \). Since \( \varphi(0) = 0 \), we have \( \varphi(z) = z \psi(z) \), with \( \psi \in H^\infty \). Hence \( H_{v,m}^\infty \) is an invariant subspace under \( C_\varphi \). By Lemma 7.17 in [7], which is also valid for Banach spaces, we get that \( \sigma_{H_{v,m}^\infty}(C_\varphi) \subset \sigma_{H_v^\infty}(C_\varphi) \). Thus it is enough to show that every \( \lambda \) with \( 0 < |\lambda| < \rho \) belongs to \( \sigma_{H_{v,m}^\infty}(C_\varphi) \) for some \( m \) to be found. Let \( C_m \) denote the restriction of \( C_\varphi \) to the invariant closed subspace \( H_{v,m}^\infty \). We find below \( m \) such that \( (C_m - \lambda I)^n \) is bounded from below, which completes the proof.

Next we find a lower bound for \( \| C_m \|_{v,m} \). Define the linear functional \( L_\zeta \) on \( H_{v,m}^\infty \) by

\[
L_\zeta(f) = \sum_{k=-K}^{\infty} \lambda^{-k} f(z_k).
\]

To verify that \( L_\zeta \) is bounded use Proposition 3.3 on each summand and then apply inequalities (3.1) and (3.2) to terms with \( k \geq n + 1 \).

Next we find a lower bound for \( |L_\zeta|_{v,m} \). To do this, select \( f_{z_0} \in H_v^\infty \) with \( \| f_{z_0} \|_v \leq 1 \), so that \( |f_{z_0}(z_0)| = 1/\hat{v}(z_0) \).

By Lemma 3.4, there is \( f_1 \in H^\infty \) with \( \| f_1 \|_\infty \leq M \), satisfying \( |f_1(z_0)| = 1 \), \( z_0^m f_1(z_0) f_{z_0}(z_0) > 0 \) and \( f_1(z_k) = 0 \) for \(-K \leq k \leq n \), \( k \neq 0 \). Now, the function \( g(z) := z^m f_1(z) f_{z_0}(z) \) belongs to \( H_{v,m}^\infty \) and \( \| g \|_v \leq M \). Further,

\[
L_\zeta(g) = z_0^m f_1(z_0) f_{z_0}(z_0) + \sum_{k=n+1}^{\infty} \lambda^{-k} z_0^m f_1(z_k) f_{z_0}(z_k).
\]

If, in addition, we choose \( m \) so that \( M \frac{1}{|\lambda|^m} \frac{1}{\mathbb{N}(z_0)} b_m^m < \frac{1}{\hat{v}(z_0)} \), and use again inequalities (3.1) and (3.2), we obtain

\[
\left| \sum_{k=n+1}^{\infty} \lambda^{-k} g(z_k) \right| \leq \frac{|z_0|^m}{2\hat{v}(z_0)} \leq \frac{|z_0|^m}{2\hat{v}(z_0)}, \quad \text{hence} \quad |L_\zeta(g)| \geq \frac{|z_0|^m}{2\hat{v}(z_0)}.
\]

Therefore, using Proposition 3.3, we obtain the desired lower bound:

\[
\| L_\zeta \|_{v,m} \geq \frac{|z_0|^m}{2MM_m^m} \geq \frac{1}{2MM_m^m} \| f_{z_0} \|_{v,m}.
\]

The final step is to estimate \( \| (C_m^* - \lambda I) L_\zeta \|_{v,m} \) for a suitable iteration sequence \( \zeta \). First observe that \( (C_m^* - \lambda I) L_\zeta = -\lambda K^1 \delta_{z,-K} \).

Pick \( \mu \) so that \( |\lambda| < \mu < \rho \). Since \( \rho \) is the essential spectral radius, there is \( n_0 \) so that for every \( l \geq n_0 \),

\[
\| C_{\varphi}^l \|_{e, H_v^\infty} > \mu^l.
\]

Hence by (2.2) for each \( l \geq n_0 \) we can find a \( w \in \mathbb{D} \) so that

\[
\frac{v(w)}{\hat{v}(\varphi_l(w))} \geq \mu^l > 0, \quad \text{and} \quad |\varphi_l(w)| \geq 1/2.
\]
Thus, we apply Lemma 3.2 to get
\[
\frac{\|\delta_{\psi}(w)\|_{v,m}}{\|\delta_{w}\|_{v,m}} \geq \frac{1}{2^m} \frac{\|\delta_{\psi}(w)\|_{v}}{\|\delta_{w}\|_{v}} = \frac{1}{2^m} \frac{\tilde{v}(w)}{\tilde{\psi}(w)} \geq \frac{\mu}{2^m}.
\]
For every \( K \geq n_0 \) define the iteration sequence \((z_k)_{k=-K}^{\infty}\) by \( z_{-K} = w \) and \( z_{k+1} = \psi(z_k) \) for \( k \geq -K \). Then \( |z_0| = |\psi_K(w)| \geq 1/2 \), and
\[
\frac{\|C_m - \lambda I\|_{L_\psi} \|_{v,m}}{\|\delta_{\psi}(w)\|_{v,m}} \leq \frac{2MM_m}{\|\delta_{\psi}(w)\|_{v,m}} |\lambda|^{K+1} \|\delta_{w}\|_{v,m} \leq |\lambda|^{MM_m} 2^{m+1} \left( \frac{|\lambda|}{\mu} \right)^K.
\]
Choosing \( K \geq n_0 \) big enough, it follows that \( C_m - \lambda I \) is not bounded from below. \( \square \)

On account of (2.1) we conclude:

**Corollary 3.6.** Let \( v \) be a typical weight. Suppose that \( \psi \), not an automorphism, has fixed point \( 0 \in \mathbb{D} \). Then
\[
\sigma_{H_0^v}(C_\psi) = \{ \lambda \in \mathbb{C} : |\lambda| \leq r_{e,H_0^v}(C_\psi) \} \cup \{ \psi'(0)^n \}_{n=0}^{\infty}.
\]

Theorem 3.5 is valid for arbitrary typical weights, but it is less general than \([1, \text{Theorem 8}]\) in that the composition operators are unweighted. However, E. Wolf (Paderborn) has observed that our results are also valid for weighted composition operators.

4. **Remarks and examples about the Koenigs function and the essential spectral radius of \( C_\psi \)**

In this section we investigate when \( \psi'(0)^n, n \in \mathbb{N} \), belongs to the point spectrum of \( C_\psi \) on \( H_v^{\infty} \) and \( H_0^v \), respectively.

**Theorem 4.1.** Let \( n \in \mathbb{N} \) and \( v \) be a typical weight. Suppose \( \psi(0) = 0 \) and \( 0 < |\psi'(0)| < 1 \). If \( |\psi'(0)|^n > r_{e,H_0^v}(C_\psi) = r_{e,H_0^v}(C_\psi) \), then the Koenigs eigenfunction \( \sigma^n \) belongs to \( H_0^v \) with eigenvalue \( \psi'(0)^n \).

**Proof.** Since \( \psi'(0)^n \in \sigma_{H_0^v}(C_\psi) \) and \( |\psi'(0)|^n > r_{e,H_0^v}(C_\psi) \), it follows from Lemma 2.1 that \( \psi'(0)^n \) is an eigenvalue of finite multiplicity. Further, by the work of Koenigs, only constant multiples of \( \sigma^n \) can be corresponding eigenfunctions, and consequently \( \sigma^n \in H_0^v \). \( \square \)

We now find conditions to obtain a partial converse of Theorem 4.1. The connection between our presentation and Bourdon’s work in [5] is pointed out by the relationship \( r_{e,H_0^v}(C_\psi) = 1/\alpha(\psi)^p \), where \( \alpha(\psi) \) is the essential angular derivative defined in [5]. Such identity follows from (2.2) and the comments in [5, p. 566].

**Theorem 4.2.** Let \( n \in \mathbb{N} \) and \( v \) be a typical weight which is essential. Suppose that \( \psi(0) = 0, 0 < |\psi'(0)| < 1 \) and let \( 0 < q \leq p \). Assume the following two conditions:

(a) There is \( D > 0 \) such that \( v_p \leq Dv \) on \( \mathbb{D} \),
(b) The function \( v/v_q \) is almost decreasing with respect to \( |z| \).

If the eigenfunction \( \sigma^n \) for \( C_\psi \) belongs to \( H_0^v \), then \( |\psi'(0)|^n \geq r_{e,H_0^v}(C_\psi)^{p/q} \).

**Proof.** First of all, by (a), \( H_0^v \subset H_0^v \), hence \( \sigma^n \in H_0^v \). By Bourdon’s results in [5], it follows that \( |\psi'(0)|^n = r_{e,H_0^v}(C_\psi) \).

On the other hand, by our general assumptions on \( \psi, |\psi_n(z)| \leq |z| \) for all \( z \in \mathbb{D} \) and \( n \in \mathbb{N} \). We can apply (b) to get \( C > 0 \) with
\[
\frac{v(z)}{v(\psi_n(z))} \leq C \frac{v_q(\psi_n(z))}{v_q(\psi_n(z))} \quad \text{for all} \quad z \in \mathbb{D} \quad \text{and} \quad n \in \mathbb{N}.
\]
Since
\[ r_{e,H_v^\infty}(C_\varphi) = \lim_n \left( \limsup_{|z| \to 1} \frac{v(z)}{\tilde{v}(\varphi_n(z))} \right)^{1/n} \]
and \( \tilde{v} \) is essential, we get \( r_{e,H_v^\infty}(C_\varphi) \leq r_{e,H_q^\infty}(C_\varphi) \). It is easy to check that \( r_{e,H_v^\infty}(C_\varphi)^q = r_{e,H_q^\infty}(C_\varphi)^q \). Then
\[ |\varphi'(0)|^q \geq r_{e,H_v^\infty}(C_\varphi) = r_{e,H_q^\infty}(C_\varphi)^q. \]

**Corollary 4.3.** Suppose that \( \varphi(0) = 0 \), \( 0 < |\varphi'(0)| < 1 \). Let \( v \) be a typical weight which is essential and such that, for some \( q > 0 \), \( v(z)/v_q(z) \) is almost decreasing with respect to \( |z| \), and, further, there is \( \varepsilon_0 > 0 \) such that for each \( 0 < \varepsilon < \varepsilon_0 \) there is \( C(\varepsilon) > 0 \) with \( (1 - |z|^2)^{v_q(z)/v(z)} \leq C(\varepsilon) \) for every \( z \in \mathbb{D} \). If for some \( n \in \mathbb{N} \) the eigenfunction \( \sigma_n \) for \( C_\varphi \) belongs to \( H_v^\infty \), then \( |\varphi'(0)|^n \geq r_{e,H_v^0}(C_\varphi) = r_{e,H_q^0}(C_\varphi) \).

**Proof.** Fix \( 0 < \varepsilon < \varepsilon_0 \). We show that \( v \) satisfies the assumptions (a) and (b) of Theorem 4.2 for \( q < p = q + \varepsilon \). Indeed, (b) is trivial, as \( v/v_q \) is assumed to be almost decreasing with respect to \( |z| \). On the other hand
\[ \frac{v_p(z)}{v(z)} = (1 - |z|^2)^{v_q(z)/v(z)} \leq C(\varepsilon). \]

We apply Theorem 4.2 to conclude \( |\varphi'(0)|^n \geq r_{e,H_v^\infty}(C_\varphi)(q+\varepsilon)/q. \) Since this holds for each \( 0 < \varepsilon < \varepsilon_0 \), the conclusion follows. \( \Box \)

Observe that the second assumption in Corollary 4.3 holds if for each \( 0 < \varepsilon < \varepsilon_0 \), \( \lim_{r \to 1} \frac{v(r)}{v_q(r)(1-r^2)^\alpha} > 0. \)

The weights \( v(z) = (1 - |z|^2)^p(1 - \log(1 - |z|^2))^{-\alpha}, \alpha > 0 \) and \( 0 < p < \infty \), are essential by a theorem of Seip (see Proposition 2 in [9]), and satisfy the assumptions of Corollary 4.3, as a calculation shows. In fact \( (1 - \log(1 - |z|^2))^{-\alpha} \) is even decreasing. The following example yields that for these weights we cannot obtain a characterization like Bourdon’s Theorem 4.4 in [5] “\( \sigma \in H_q^0 \) if and only if \( |\varphi'(0)| > r_{e,H_q^0}(C_\varphi) \)” in particular, the converse of Theorem 4.1 does not hold for general weights \( v \).

**Example 1.** The function \( \sigma(z) = \frac{z}{1-z} \) is the Koenigs function for \( \varphi(z) = \frac{z}{1-z} \). Clearly \( \sigma \in H_q^0 \) for the weight \( v(z) = (1 - |z|^2)(1 - \log(1 - |z|^2))^{-1} \) and \( \varphi'(0) = 1/2 \). It follows from Corollary 4.3 that \( r_{e,H_q^0}(C_\varphi) \leq \frac{1}{2} \). Notice that
\[ \varphi_n(z) = \frac{z}{2^n - (2^n - 1)z} \]
and
\[ \lim_{r \to 1} \frac{1 - r^2}{1 - |\varphi_n(r)|^2} = \frac{2^n}{2^n - (2^n - 1)z}. \]

Then using
\[ \frac{v(z)}{v(\varphi_n(z))} = \frac{1 - |z|^2}{1 - |\varphi_n(z)|^2} \left( 1 - \frac{\log(1 - |\varphi_n(z)|^2)}{\log(1 - |z|^2)} \right), \]
we conclude that
\[ \lim_n (\limsup_{|z| \to 1} \frac{v(z)}{v(\varphi_n(z))})^{1/n} = \frac{1}{2}. \]

Hence \( r_{e,H_q^0}(C_\varphi) = 1/2 = \varphi'(0) \). This shows that “greater or equal” cannot be replaced by “strictly greater” in Corollary 4.3.

**Example 2.** By Theorem 4.1 and Corollary 4.3 we have
\[ |\varphi'(0)| > r_{e,H_q^0}(C_\varphi) \Rightarrow \sigma \in H_q^0 \Rightarrow |\varphi'(0)| \geq r_{e,H_q^0}(C_\varphi). \]

Example 1 shows that the first arrow cannot be reversed. We exhibit an example of a Koenigs eigenfunction \( \sigma \in H_v^\infty \setminus H_q^0 \) for a self map \( \varphi \) such that \( |\varphi'(0)| = r_{e,H_q^0}(C_\varphi) = r_{e,H_q^\infty}(C_\varphi) \), thus showing that the second arrow cannot be reversed either. To see this, take the weight \( v(z) = (1 - |z|)(1 - \log(1 - |z|))^{-1} \), which is equivalent to the one in Example 1, and \( \sigma(z) := (1/(1-z)) \log(1/(1-z)) \), as in the first example on p. 578 of [5]. Then \( \sigma \) is the Koenigs eigenfunction of \( \varphi = \sigma^{-1} \circ \sigma/2, \) so \( \varphi(0) = 0 \) and \( \varphi'(0) = 1/2. \) For the weight \( v_1(z) = 1 - |z|, \) it is shown in [5] that
$|\psi'(0)| = r_{e,H^0_1}(C_\psi)$. Clearly $\sigma \in H^\infty \setminus H^0_1$, and we can apply Corollary 4.3 to get $|\psi'(0)| = r_{e,H^0_1}(C_\psi) \geq r_{e,H^0_1}(C_\psi)$. Thus the second implication cannot be reversed. To see the equality of the two radii, observe that

$$\limsup_{|z| \to 1} \frac{v(z)}{v(\varphi_n(z))} = \limsup_{|z| \to 1} \frac{v_1(z)}{v_1(\varphi_n(z))} \left( \frac{1 - \log(1 - |\varphi_n(z)|)}{1 - \log(1 - |z|)} \right).$$

Now, for each $n$ there is a sequence $(z^n_k)_k$ in $D$ with $|z^n_k| \to 1$ as $k \to \infty$, such that the sequence $((1 - |z^n_k|)/(1 - |\varphi_n(z^n_k)|))_k$ converges. This implies that

$$\lim_k \left( \frac{1 - \log(1 - |\varphi_n(z^n_k)|)}{1 - \log(1 - |z^n_k|)} \right) = 1,$$

and we see that the inequality $r_{e,H^0_1}(C_\psi) \leq r_{e,H^0_1}(C_\psi)$ also holds.

We conclude with a remark related to the type of questions of this section when $C_\psi$ acts on the algebra $H^\infty$. As in Lemma 3.1, one has that $\psi'(0) \in \sigma_{H^\infty}(C_\psi)$. Further, $\sigma \in H^\infty$ if $r_{e,H^\infty}(C_\psi) = 0$. This follows as in Theorem 4.1, using Lemma 2.1 and Koenigs Theorem. The converse has been proved by T. Gamelin [10].

**Theorem 4.4.** Let $\psi(0) = 0$ and $0 < |\psi'(0)| < 1$. Then $\sigma \in H^\infty$ if and only if $r_{e,H^\infty}(C_\psi) = 0$.

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**References**


