The Strong Ekeland Variational Principle,
the Strong Drop Theorem and Applications

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INTRODUCTION

This article is motivated by the papers of J.-P. Penot [17] and Daneš [4], where it is proved that the Drop theorem [3] and the Ekeland variational principle [7, 8] are equivalent. We refer the reader to the references of the Penot paper [17], where the areas of applications of the Ekeland variational principle are shown: nonlinear functional analysis, convex analysis, generalized differential calculus, optimization theory, sensitivity, fixed-point theory, and global analysis. In the same paper some areas of applications of a geometrical result of Daneš [3] known as the Drop theorem are mentioned. The generalized Drop theorem of Daneš [4] states: if $B$ is a closed, convex, and bounded subset of a Banach space $(E, \| \cdot \|)$, $A$ is a closed and nonempty subset of $E$, $d(A, B) := \inf\{\|x - y\| : x \in A, y \in B\} > 0$, $s \in A$, then there exists a point $a \in A \cap \text{co}\{\{s \cup B\} \cap A = \{a\}$. Here we prove the generalized Drop theorem in a different way and we show that the above-mentioned point $a$ can be chosen from $A$ in such a way, that $x_n \to a$ whenever $\{x_n\}_{n \geq 1} \subset \text{co}\{\{a\} \cup B\}$ and $d(x_n, A) \to 0$.

Also, here we strengthen in a similar manner the Ekeland variational principle, a lemma of Phelps [18] and the Flower Petal theorem of Penot [17], showing that some additional stability properies are valid. In Section 2 it is shown that all these strengthened assertions are equivalent (and are equivalent to the original versions).

In Sections 3 and 4, using the strengthened forms of the Ekeland variational principle and the Drop theorem, we present new results and new proofs of known results about generic properties of convex and non-convex minimization problems.

We note that for topological spaces there is an assertion analogous to the Ekeland variational principle due to Čoban and Kenderov [2], and also, that there is another direction of investigations connected with the
Drop theorem and with the geometry of Banach spaces (see Rolewicz [21], Kutzarova [14], and Montesinos [16]).

Some of the results presented here are announced in [10].

1. The Statements

Let $E$ be a real Banach space with norm $\| \cdot \|$, with $B(x; r)$ (resp. $B[x; r]$) denoted as the open (resp. the closed) ball with center $x$ and radius $r$. For $x, y \in E$ denote $[x, y] = \{tx + (1 - t)y : t \in [0, 1]\}$; $bdA$ for $A \subset E$ is the boundary of $A$. The drop associated with a point $a \in E$ and a convex subset $B$ of $E$ is the convex hull of $\{a\} \cup B$. Denote $d(a, B) = \inf \{\|a - b\| : a \in A, b \in B\}$.

The Hausdorff distance between two subsets $A$ and $B$ of $E$ is defined as follows: $d(A, B) = \max \{\sup_{a \in A} \inf_{b \in B} \|a - b\|, \sup_{b \in B} \inf_{a \in A} \|a - b\|\}$.

Denote by $N$ the set of all positive integers.

**Theorem 1.1** (Generalized Drop theorem, Danes [4]). Let $A$ be a non-empty and closed subset of $E$, $B$ be a closed, bounded, and convex subset of $E$ such that $d(A, B) > 0$, $s \in A$. Then there exists a point $a \in A \cap D(s, B)$ such that $D(a, B) \cap A = \{a\}$.

**Proof.** If $B$ is empty, then there is nothing to prove. Let $d = d(A, B)$. First we will prove

$$\forall \varepsilon > 0, \forall A' \subset A, A' \neq \emptyset, \exists a \in A' \quad \text{such that} \quad d(A(a, B) \cap A') \leq \varepsilon. \quad (1)$$

For $\delta \in (0, \min \{dM/2, 1\}]$, where $M = \diam B + d(A', B) + 1$, there exists $a \in A'$ such that $d(a, B) < d(A', B) + \delta$. Let $z \in D(a, B) \cap A'$. Then $z = ta + (1 - t)b$ for some $t \in [0, 1]$ and $b \in B$. Since the distance function to a convex set is convex, we have: $d(A', B) \leq d(z, B) \leq td(a, B) < td(A', B) + \delta$, whence $1 - t < \delta/d(A', B) \leq \delta/d$ and $\|z - a\| = (1 - t)\|a - b\| < \delta M/d < \varepsilon/2$, therefore $diam(D(a, B) \cap A') \leq \varepsilon$ and (1) is proved.

By (1) we can construct sequences $\{A_i\}_{i \geq 1}$, $\{a_i\}_{i \geq 1}$ with the properties:

$$diam A_i < 1/i, \quad (2)$$
$$a_i \in A_{i-1}, \quad (3)$$
$$A_i = D_i \cap A_{i-1}, \quad \text{where} \quad D_i = D(a_i, B), A_0 = A \cap D(s, A). \quad (4)$$

By the Cantor theorem $\bigcap_{i=1}^{\infty} A_i = \{a\}$. For $D := D(a, B)$ we will prove

$$\bigcap_{i=1}^{\infty} D_i = D. \quad (5)$$
Since $a \in D$ we have $D \subset D_i$, for every $i \in N$ and therefore $D \subset \bigcap_{i=1}^{\infty} D_i$. Let $x \in \bigcap_{i=1}^{\infty} D_i$. Then $x = t_i a_i + (1-t_i) b_i$, where $t_i \in [0,1]$, $b_i \in B$. Take a convergent subsequence \( \{t_{n_k}\} \), $t_{n_k} \to t$. Consider the cases:

(a) $t = 1$. Then $\|x - a_{n_k}\| = (1-t_{n_k})\|a_{n_k} - b_{n_k}\| \leq (1-t_{n_k}) \text{diam } D_1 \to 0$.

(b) $t < 1$. Then there exists $\nu$ such that $1-t_{n_k} > 0$ for every $k > \nu$ and we can write $b_{n_k} = x/(1-t_{n_k}) - t_{n_k} a_{n_k}/(1-t_{n_k}) \to x/(1-t) - t a/(1-t) = b \in B$ (because $B$ is closed). Hence $x = ta + (1-t) b$, $x \in D$.

Since $D_i \subset D_{i-1} \subset D(s, A)$ for $i \geq 2$, by (4) we have

$$A_i = D_i \cap A_{i-1} = D_i \cap D_{i-1} \cap A_{i-2} = D_i \cap A \cap D(s, A) = D_i \cap A$$

and by (5) we have

$$D \cap A = \left( \bigcap_{i=1}^{\infty} D_i \right) \cap A = \bigcap_{i=1}^{\infty} (D_i \cap A) = \bigcap_{i=1}^{\infty} A_i = \{a\}.$$  

Now we will see, that the point $a$ from Theorem 1.1 can be chosen from $A$ in such a way, that every minimizing sequence \( \{x_n\}_{n \geq 1} \subset D(a, B), \) $d(x_n, A) \to 0$, must converge to $a$.

**Theorem 1.2 (Strong Drop theorem).** Let $B$ be a closed, convex, and bounded subset of $E$, $A$ be nonempty and closed subset of $E$ and $d := d(A, B) > 0$, $B_\varepsilon := \{x \in E: d(x, B) \leq \varepsilon\}$. Then for every $\varepsilon \in (0, d)$, for every $z \in E$ such that $D(z, B_\varepsilon) \cap A \neq \emptyset$ there exists a point $a \in A \cap D(z, B_\varepsilon)$ such that $A \cap D(a, B_\varepsilon) = \{a\}$ and $x_n \to a$ whenever \( \{x_n\}_{n \geq 1} \subset D(a, B)$ and $d(x_n, A) \to 0$.

**Proof.** By the generalized Drop theorem (Theorem 1.1) applied for $A \cup \{z\}$ and $B_\varepsilon$ there exists a point $a \in D(z, B_\varepsilon) \cap A$ such that $A \cap D(a, B_\varepsilon) = \{a\}$. Let \( \{x_n\}_{n \geq 1} \subset D(a, B)$, $x_n \neq a$ for every $n \in N$ and $d(x_n, A) \to 0$. Then $x_n = t_n a + (1-t_n) b_n$ for some $t_n \in [0,1]$, $b_n \in B$. There exist $y_n \in A$ such that $\|x_n - y_n\| \to 0$. Let $z_n \in [x_n, y_n] \cap bdD(a, B_\varepsilon)$ and $c_n := z_n/(1-t_n) - t_n a/(1-t_n)$. If $c_n \notin int B_\varepsilon$ for some $m \in N$, then $z_m \in int D(a, B_\varepsilon)$, a contradiction. Therefore $d(c_n, B) \geq \varepsilon$ for every $n \in N$ and we have

$$\|x_n - z_n\| = (1-t_n)\|b_n - c_n\| \geq (1-t_n) d(c_n, B) \geq (1-t_n) \varepsilon,$$

whence $1-t_n \leq \|x_n - z_n\|/\varepsilon \to 0$. Hence

$$\|x_n - a\| = (1-t_n)\|b_n - a\| \leq (1-t_n) \text{diam } D(a, B) \to 0.$$
THEOREM 1.3 (Phelps [18, Lemma 1.2]). Suppose that $B$ is a bounded, closed, convex, and nonempty subset of $E$ with $0 \not\in B$ and that $A$ is a closed subset of $E$. If for some $z \in E \ A \cap (K+z)$ (where $K = \mathbb{R}^+ B$) is bounded and nonempty, then there exists a point $a \in A \cap (K+z)$ such that $A \cap (K+a) = \{a\}$.

Here we present the strengthened variants of the Phelps lemma, of the Ekeland variational principle and of the Penot Flower Petal theorem [17] and we will prove in Section 3 that they are equivalent.

THEOREM 1.4 (strong Phelps lemma). Suppose that $B$ is closed, convex, bounded and nonempty subset of $E$ with $0 \not\in B$ and that $A$ is a closed non-empty subset of $E$, $B_e := \{x \in E: d(x, B) \leq \varepsilon\}$. If $\varepsilon \in (0, d(0, B))$ and $A \cap (K_e + z)$ is nonempty and bounded for some $z \in E$, where $K_e = \mathbb{R}^+ B_e$, then there exists a point $a \in A \cap (K_e + z)$ such that $A \cap (K+a) = \{a\}$ and $x_n \to a$ whenever $\{x_n\}_{n \geq 1} \subset K + a$ and $d(x_n, A) \to 0$ ($K = \mathbb{R}^+ B$).

THEOREM 1.5 (amended strong Ekeland's variational principle). Let $f: M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function on a complete metric space $(M, d)$. If $f$ is bounded below on $M$ and not improper, then for any $\delta > 0$, $\gamma > 0$, and any $x_0 \in M$ there exists a point $a \in M$ such that:

1. $f(a) < f(x) + \gamma d(a, x)$ for every $x \in M$, $x \neq a$,
2. $f(a) < f(x_0) - \gamma d(a, x_0) + \delta$,
3. $x_n \to a$ whenever $\{x_n\}_{n \geq 1} \subset M$ and $f(x_n) + \gamma d(a, x_n) \to f(a)$.

THEOREM 1.6 (strong Ekeland's variational principle). Let $f: M \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous, bounded below on $M$ and not improper function on a complete metric space $(M, d)$, $\varepsilon > 0$ and $x_0 \in M$ be such that $f(x_0) < \inf f(M) + \varepsilon$. Then for any $\delta_1 > 0$, $\delta_2 > 0$, $\gamma > 0$ with $\gamma \delta_1 \geq \varepsilon$ there exists a point $a \in M$ such that:

1. $f(a) < f(x) + \gamma d(a, x)$ for every $x \in M$, $x \neq a$,
2. $d(a, x_0) < \delta_1 + \delta_2$,
3. $x_n \to a$ whenever $\{x_n\}_{n \geq 1} \subset M$ and $f\{x_n\} + \gamma d(a, x_n) \to f(a)$.

It is easy to see that Theorem 1.6 follows from Theorem 1.5. Indeed, by (2) of Theorem 1.5 for $\delta = \delta_2 \gamma$ we have

$$d(a, x_0) < \frac{f(x_0) - f(a)}{\gamma} + \delta_2 \leq \varepsilon \gamma + \delta_2 \leq \delta_1 + \delta_2.$$

In [17] Penot introduced a notion “petal,” namely the petal $P_\gamma(a, b)$ associated with $\gamma > 0$ and points $a, b$ of a metric space $(M, d)$ is the set $P_\gamma(a, b) = \{x \in M: \gamma d(a, x) + d(x, b) \leq d(a, b)\}$ and proved that his
"Flower Petal Theorem" is equivalent to the Ekeland variational principle and to the Drop theorem.

Analogously, for a function \( f: m \to \mathbb{R} \) on a metric space \( (M, d) \), the set \( P_{\gamma, \delta}(a, f) := \{ x \in M : \gamma d(a, x) + f(x) \leq f(a) + \delta \} \) will be called a petal associated with \( \delta \geq 0, \gamma > 0, a \in M \) and \( f \). If \( f(\cdot) = d(\cdot, b) \) for \( b \in M \), then \( P_{\gamma, 0}(a, f) \) is the petal \( P_{\gamma}(a, b) \) in the Penot definition.

**Theorem 1.7 (strong Flower Petal theorem).** Let \( A \) be a closed subset of a complete metric space \( (M, d) \), \( f: M \to \mathbb{R} \) be a Lipschitz function bounded below on \( A \) and let \( \delta > 0, \gamma > 0, x_0 \in A \). Then there exists a point \( a \in A \cap P_{\gamma, \delta}(x_0, f) \) such that

1. \( P_{\gamma, \delta}(a, f) \cap A = \{ a \} \) and
2. \( x_n \to a \) whenever \( \{ x_n \}_{n \geq 1} \subset P_{\gamma, \delta}(a, f) \) and \( d(x_n, A) \to 0 \).

2. The Implications

In this section we will prove the following implications: The generalized Drop theorem \( \Rightarrow \) the Phelps lemma \( \Rightarrow \) the strong Phelps lemma \( \Rightarrow \) the amended strong Ekeland's variational principle \( \Rightarrow \) the strong Flower Petal theorem \( \Rightarrow \) the Petal theorem (assertion (1) of Theorem 1.7) \( \Rightarrow \) the generalized Drop theorem.

Let \( K(x, B) \) be the cone generated by \( x \notin B \) and the set \( B \) of a normed space \( (E, \| \cdot \|) \): \( K(x, B) := \{ z \in E : z = x + t(b - x), t \geq 0, b \in B \} \).

**Proposition 2.1.** The generalized Drop theorem (Theorem 1.1) implies the Phelps lemma (Theorem 1.3).

**Proof.** Let us adopt the notations of the Phelps lemma. Since \( A \cap (K+z) \) is bounded, there exists \( r > 0 \) such that \( A \cap (K+z) \subset B(z; r) \). Let \( t_0 = 2r/d(0, B) \) and \( C = z + t_0B \). Then \( d(z, C) = t_0d(0, B) = 2r \) and \( d(C, A \cap (K+z)) \geq r \). Thus we can apply Theorem 1.1 for \( C \) and \( A \cap (K+z) \) (it is easy to see that \( K \) is closed). There exists a point \( a \in A \cap (K+z) \) such that:

\[
A \cap (K+z) \cap D(a, C) = \{ a \}. \tag{1}
\]

We will prove that

\[
a + K \subset K(a, C). \tag{2}
\]

We have: \( a - z = t_1b_1 \) for some \( t_1 \geq 0, b_1 \in B \) and \( t_1d(0, B) \leq t_1 \| b_1 \| = \| a - z \| < r \), whence \( t_1 < r/d(0, B) < 2r/d(0, B) = t_0 \). Let \( x \in a + K \) and
Then \( x = a + t_2 b_2 \) for some \( t_2 \geq 0 \), \( b_2 \in B \), and \( b_3 := t_1 b_1 / t_0 + t_2 b_2 / t_0 \in B \). For \( s := t_2 / t_3 \) we have: \( x = a + t_2 b_2 = a + s t_3 b_2 = a + s (t_0 b_3 - t_1 b_1) = a + s (t_0 b_3 + z - a) \), therefore \( x \in K(a, C) \). Since \( B(z; r) \cap C = \emptyset \), we have

\[
B(z; r) \cap K(a, C) = B(z; r) \cap D(a, C)
\]

and by (2)

\[
(K + a) \cap B(z; r) \subset K(a, C) \cap B(z; r) = B(z; r) \cap D(a, C).
\]

Since \( K \) is a cone and \( a \in K + z \), we have \( K + a \subset K + z \) and by (1),

\[
(K + a) \cap A = (K + a) \cap (K + z) \cap A \cap B(z; r) \subset B(z; r) \cap D(a, C) \cap (K + z) \cap A = \{a\}. \]

**Proposition 2.2.** The Phelps lemma (Theorem 1.3) implies the strong Phelps lemma (Theorem 1.4).

**Proof.** The proof is analogous to the proof that the generalized Drop theorem (Theorem 1.1) implies the strong Drop theorem (Theorem 1.2). It is easy to see that \( B \) and \( K \) are closed. By the Phelps lemma (Theorem 1.3) there exists a point \( a \in (K + z) \cap A \) such that

\[
A \cap (K + z) \cap (K + a) = \{a\}.
\]

Since \( K \) is a cone and \( a \in (K + z) \), we have \( K + a \subset K + z \), therefore \( A \cap (K + a) = \{a\} \).

Let \( \{x_n\}_{n \geq 1} \subset K + a \), \( x_n \neq a \) and \( d(x_n, A) \rightarrow 0 \). Then \( x_n = a + t_n b_n \) for some \( t_n > 0 \), \( b_n \in B \). There exist \( y_n \in A \) such that \( \|x_n - y_n\| \rightarrow 0 \). Let \( z_n \in [x_n, y_n] \cap bd(K + a) \) and \( c_n := (z_n - a) / t_n \). If \( c_m \in \text{int} B \) for some \( m \), then \( z_m \in \text{int}(K + a) \), a contradiction. Therefore \( d(c_n, B) \geq \varepsilon \) for every positive integer \( n \) and we have

\[
\|x_n - z_n\| = t_n \|b_n - c_n\| \geq t_n \varepsilon, \text{ whence } t_n \leq \|x_n - z_n\| / \varepsilon \rightarrow 0.
\]

Hence

\[
\|x_n - a\| = t_n \|b_n\| \leq t_n \text{ diam } \{B \cup \{0\}\} \rightarrow 0. \]

Let \( E_1 := E \times \mathbb{R} \) be furnished with the "max" norm:

\[
\|(x_1, t_1) - (x_2, t_2)\| = \max \{\|x_1 - x_2\|, |t_1 - t_2|\}.
\]
Denote $B_r = \{(x, 0) \in E_1 : \|x\| \leq r\}$,

$$B_{r, \delta} = \{(x, t) \in E_1 : d_1((x, t), B_r) \leq \delta\},$$

where

$$d_1((x_1, t_1), (x_2, t_2)) = \|(x_1, t_1) - (x_2, t_2)\|.$$ 

**Lemma 2.3.** For every $r, s, \varepsilon > 0$ there exists $\delta > 0$ such that

$$K((0, s), B_{r, \delta}) \subset K((0, s), B_{r+\varepsilon}).$$

**Proof.** Let $0 < \delta < \varepsilon s/(r + \varepsilon + s)$, $(x, t) \in K((0, s), B_{r, \delta})$. Then $(r + \delta)/(s - \delta) < r + \varepsilon$, $(x, t) = (0, s) + \lambda_1((x_1, t_1) - (0, s))$ for some $\lambda_1 \geq 0$ and $(x_1, t_1) \in B_{r, \delta}$. Let $\lambda_2 = (s - t_1)/s, x_2 = x_1/\lambda_2$. Then

$$\|x_2\| = \|x_1\|/\lambda_2 \leq s(r + \delta)/(s - t_1) < r + \varepsilon$$

and

$$(x, t) = (0, s) + \lambda_1((x_1, t_1) - (0, s)) = (0, s) + \lambda_1((\lambda_2 x_2, s - s\lambda_2) - (0, s))$$

$$= (0, s) + \lambda_1((\lambda_2 x_2, 0) - (0, s\lambda_2)) = (0, s) + \lambda_1 \lambda_2((x_2, 0) - (0, s)),$$

which shows that $(x, t) \in K((0, s), B_{r+\varepsilon})$.

**Proposition 2.4.** The strong Phelps lemma (Theorem 1.4) implies the amended strong Ekeland variational principal (Theorem 1.5).

**Proof.** Let us adopt the notations of Theorem 1.5. If $f(x_0) = \inf f(M)$, then $a := x_0$ fulfills (1), (2), and (3) of Theorem 1.5. Assume $f(x_0) > \inf f(M)$. We follow the Penot idea from [17]: replacing $d$ by $d' := \min\{s, d\}$, where $s = 1/\gamma(f(x_0) - \inf fM))$, we may suppose that $d$ is bounded. Indeed, it is easy to check that if (1), (2), and (3) are fulfilled for $d'$, then they are fulfilled for $d$, too.

$(M, d)$ can be isometrically embedded into some Banach space $E$, for instance, into the Banach space of the bounded functions on $M$ with the supremum norm via the mapping $x \mapsto d_x$, where $d_x(y) = d(x, y)$, for $x, y \in M$.

Thus we may suppose, that $M$ is a bounded closed subset of a Banach space $(E, \|\cdot\|)$. Since $f$ is lower semicontinuous and $M$ is closed, $\text{epi} f$ is a closed subset of $E_1 := E \times \mathbb{R}$ with the "max" norm (see before Lemma 2.3). We may suppose that $0 < \delta < \gamma \text{diam } M$. Let $\varepsilon > 0$, $d = \varepsilon/\gamma$, $0 < \delta_1 < \delta d/(\gamma \text{diam } M - \delta)$,

$$B_1 = \{(x, -\delta) \in E_1 : \|x\| \leq d + \delta_1\}, \quad K_1 = K(0, B_1) \quad (= \mathbb{R}^+ B_1),$$

$$B_2 = \{(x, -\delta) \in E_1 : \|x\| \leq d\}, \quad K_2 = K(0, B_2) \quad (= \mathbb{R}^+ B_2).$$
By Lemma 2.3 there exists $\delta_2 > 0$ such that

$$K_3 := \mathbb{R}^+ \times \{ (x, t) \in E_1 : d_1((x, t), B_2) \leq \delta_2 \} = K_1.$$ 

We will prove that there exists $t_0 \in \mathbb{R}$ such that $((x_0, t_0) + K_2) \cap \text{epi} f \neq \emptyset$. If $f(x_0) < +\infty$, take $t_0 = f(x_0)$. Since $f$ is not improper, there exists a point $x_1 \in M$ with $f(x_1) < +\infty$. Take $\lambda_1 \geq \|x_1 - x_0\|/d$, $z_1 = (x_1 - x_0)/\lambda_1$, $t_0 = f(x_1) + \lambda_1 \epsilon$. Then $(x_1, f(x_1)) = (x_0, t_0) + \lambda_1(z_1, -\epsilon)$, $\|z_1\| \leq d$ which shows that $(x_1, f(x_1)) \in (x_0, t_0) + K_2$.

Further, $((x_0, t_0) + K_1) \cap \text{epi} f$ is a bounded subset of $E_1$. Indeed if $(x_2, t_2) \in ((x_0, t_0) + K_1) \cap \text{epi} f$, then $(x_2, t_2) = (x_0, t_0) + \lambda_2(z_2, -\epsilon)$, where $t_2 \geq f(x_2)$, $\|z_2\| \leq d + \delta_1$, $\lambda_2 \geq 0$ and we have

$$\inf f(M) < f(x_2) \leq t_2 = t_0 - \lambda_2 \epsilon \leq t_0,$$

$$\|x_2\| \leq \|x_0\| + \lambda_2 \|z_2\| \leq \|x_0\| + (t_0 - t_2)(d + \delta_1)/\epsilon$$

$$\leq \|x_0\| + (t_0 - \inf f(M))(d + \delta_1)/\epsilon = : t_3,$$

whence $\|x_2\| \leq \max\{t_0, t_3\}$. Thus we can apply the strong Phelps lemma: there exists a point $(a, \alpha) \in ((x_0, t_0) + K_3) \cap \text{epi} f = ((x_0, t_0) + K_1) \cap \text{epi} f$ such that

(i) $((a, \alpha) + K_3) \cap \text{epi} f = \{(a, \alpha)\}$

(ii) $w_n \to (a, \alpha)$ whenever $\{w_n\}_{n \geq 1} \subset (a, \alpha) + K_2$ and $d_1(w_n, \text{epi} f) \to 0$.

Obviously $\alpha = f(a)$. Thus $(a, f(a)) = (x_0, t_0) + p(y, -\epsilon)$, where $p \geq 0$, $\|y\| \leq d + \delta_1$. If $y \neq 0$ we have $p = \|a - x_0\|/\|y\| \geq \|a - x_0\|/(d + \delta_1)$ and

$$f(a) = t_0 - \epsilon \leq t_0 - \|a - x_0\| \epsilon/(d + \delta_1) \leq f(x_0) - \epsilon \|a - x_0\|/(d + \delta_1)$$

$$\leq f(x_0) - \gamma \|a - x_0\| + \delta_1 \gamma \text{diam } M/(d + \delta_1) < f(x_0) - \gamma \|a - x_0\| + \delta,$$

which is the condition (2) of Theorem 1.5. If $y = 0$, then $a = x_0$ and (2) is fulfilled.

Let $a \neq x \in M$, $\lambda = \|a - x\|/d$, $z = (x - a)/\lambda$, $t = f(a) - \lambda \epsilon$. Then $\|z\| = d$, $(x, t) = (a, f(a)) + \lambda(z, -\epsilon)$, which shows that $(x, t) \in (a, f(a)) + K_2$, whence by (i) $(x, t) \notin \text{epi} f$, therefore $f(x) > t = f(a) - \lambda \epsilon = f(a) - \gamma \|a - x\|$, which is condition (1) of Theorem 1.5.

Let $\{x_n\}_{n \geq 1} \subset M$, $x_n \neq a$. $f(x_n) + \gamma \|a - x_n\| \to f(a)$. Denote $\lambda_n = \|a - x_n\|/d$, $z_n = (x_n - a)/\lambda_n$, $t_n = f(a) - \lambda_n \epsilon$. Then $\|z_n\| = d$, $(x_n, t_n) = (a, f(a)) + \lambda_n(z_n, -\epsilon)$, which shows that $(x_n, t_n) \in (a, f(a)) + K_2$ and we have

$$f(x_n) - t_n = f(x_n) - f(a) + \epsilon \|a - x_n\|/d \to 0,$$
which means \( d_i((x_n, t_n), \text{epi} f) \to 0 \). Now, (ii) shows that \((x_n, t_n) \to (a, f(a))\), which completes the proof. □

**Proposition 2.5.** The amended strong Ekeland variational principle implies the strong Flower Petal theorem.

**Proof.** Let us adopt the notations of Theorem 1.7. By Theorem 1.5 there exists a point \( a \in A \) such that:

1. \( f(a) < f(x) + \gamma d(a, x) \) for every \( x \in A, x \neq a \),
2. \( f(a) < f(x_0) - \gamma d(a, x_0) + \delta \), and
3. \( x_n \to a \) whenever \( \{x_n\}_{n \geq 1} \subseteq A \) and \( f(x_n) + \gamma d(a, x_n) \to f(a) \).

Condition (1) shows that for each \( x \in A \setminus \{a\} \) \( x \notin P_{\gamma,0}(a,f) \), (2) shows that \( a \in P_{\gamma,\delta}(x_0,f) \).

Let \( \{x_n\}_{n \geq 1} \subseteq P_{\gamma,0}(a,f) \), \( d(x_n, A) \to 0 \). There exist \( a_n \in A \) with \( d(x_n, a_n) \to 0 \) and, using (1), we can write

\[
0 \leq \gamma d(a_n, a) + f(a_n) - f(a) \leq \gamma d(a, x_n) + \gamma d(x_n, a_n) + f(a_n) - f(x_n) + f(x_n) - f(a) \\
\leq \gamma d(x_n, a_n) + Ld(a_n, x_n) \to 0,
\]

where \( L \) is the Lipschitz constant for \( f \). Now (3) shows that \( a_n \to a \), whence \( x_n \to a \). □

Let now \( M = E \) be a Banach space, \( A \subseteq M \), \( f \) be a convex function, \( B \) be a closed, convex, and bounded subset, \( \varepsilon > 0 \), \( k_1 := \sup f(B) < \inf f(A) =: k_2 \), \( 0 < \gamma < (k_2 - k_1)/(d + \varepsilon + \text{diam } B) \), \( d := d(A, B) > 0 \), \( a \in A \), \( d(a, B) < d + \varepsilon \). We will show that

\[
D(a, B) \subseteq P_{\gamma,0}(a,f).
\]

Let \( z \in D(a, B) \). Then \( z = ta + (1 - t)b \) for some \( t \in [0, 1] \), \( b \in B \) and we have

\[
\gamma d(a, z) + f(z) \leq \gamma (1 - t)\|a - b\| + tf(a) + (1 - t)f(b) \\
= (1 - t)(\gamma \|a - b\| + f(b)) + tf(a) \\
\leq (1 - t)[\gamma (d(a, B) + \text{diam } B) + f(b)] + tf(a) \\
< (1 - t)[\gamma (d + \varepsilon + \text{diam } B) + k_1] + tf(a) \\
< (1 - t)k_2 + tf(a) \leq (1 - t)f(a) + tf(a) = f(a),
\]

therefore \( z \in P_{\gamma,0}(a,f) \).

If we take \( f(\cdot) = d(\cdot, B) \), then \( f \) is a convex Lipschitz function and it is clear how to prove the following.
PROPOSITION 2.6. The Flower Petal theorem (Assertion (1) of Theorem 1.7) implies the generalized Drop theorem.

3. APPLICATIONS OF THE STRONG EKELAND VARIATIONAL PRINCIPLE

Let $(X, d)$ be a complete metric space and $B(X)$ stand for the set of all lower semicontinuous (l.s.c.) bounded from below real-valued functions on $X$. $B(X)$ is a complete metric space under the distance

$$
\rho(f_1, f_2) = \sup_{x \in X} |f_1(x) - f_2(x)|/(1 + |f_1(x) - f_2(x)|), \quad f_1, f_2 \in B(X).
$$

Denote by $\mathcal{Y}(X)$ the set of all nonempty closed subsets of $X$. If the metric $d$ is not bounded, we may replace it by the complete and bounded metric $d'(x, y) = \min\{d(x, y), 1\}$. It is not difficult to see that the metrics $d$ and $d'$ are equivalent and that the Hausdorff metrics $h$ and $h'$ on $\mathcal{Y}(X)$ generated, respectively, by $d$ and $d'$ are equivalent. Thus we may suppose that $d$ is a bounded metric. It is well known that $(\mathcal{Y}(X), h)$ is a complete metric space (see [31, p. 417]).

Let us recall the well-known Tyhonov and Hadamard well posedness (the last is taken wrt the Hausdorff distance $h$ on $\mathcal{Y}(X)$ and the metric $\rho$ on $B(X)$). Let $(A, f) \in \mathcal{Y}(X) \times B(X)$ and $\arg\min_A f$ stands for the solution set (possible empty) of the minimization problem $(A, f)$: find $x_0 \in A$ such that $f(x_0) = \inf f(A)$.

A minimization problem $(A_0, f_0)$ is called well posed in sense of Tyhonov (resp. Hadamard) if it has unique solution $x_0 \in A$ and every minimizing sequence $\{x_n\}_{n \geq 1} \subset A$, i.e., $f(x_n) \to f(x_0)$ (resp. $\{x_n\}_{n \geq 1} \subset X$, $x_n \in \arg\min_{A_n} f_n$ and $(A_0, f_0) \to (A_0, f_0)$) converges to $x_0$. It is easy to see that the minimization problem $(A_0, f_0)$ is well posed in the sense of Hadamard if and only if the multivalued mapping $\mathcal{Y}(X) \times B(X) \ni (A, f) \mapsto \arg\min_A f$ is single-valued and upper semicontinuous at $(A_0, f_0)$.

Recall that the multivalued mapping $F: \tilde{X} \to Y$ is said to be upper semicontinuous at $x \in \tilde{X}$, where $\tilde{X}$ and $Y$ are topological spaces, if for every open set $\mathcal{V} \supset F(x)$ there exists an open set $U \ni x$ such that $F(z) \subset \mathcal{V}$ for every $z \in U$.

Let $L_{A,f}(\varepsilon) = \{x \in X: f(x) \leq \inf f(A) + \varepsilon \text{ and } d(x, A) \leq \varepsilon\}$, for $\varepsilon > 0$. Following Revalski [19] we will say that the minimization problem $(A, f)$ is called well posed if $\inf_{\varepsilon > 0} \text{diam } L_{A,f}(\varepsilon) = 0$. It is easy to see that the minimization problem $(A, f)$ is well posed in the sense of Revalski if and only if $\arg\min_A f = \{x_0\}$ and $x_n \to x_0$ whenever $\{x_n\}_{n \geq 1} \subset X$, $f(x_n) \to f(x_0)$ and $d(x_n, A) \to 0$.

In [19] it is proved that for the assertions:

(a) the problem $(A, f)$ is well posed in the sense of Hadamard,
(b) the problem \((A,f)\) is well posed in the sense of Revalski,

(c) the problem \((A,f)\) is well posed in the sense if Tyhonov are valid

the implications: (a) \(\Rightarrow\) (b) \(\Rightarrow\) (c). If in addition \(f\) is continuous, then

(b) \(\Rightarrow\) (a).

Here we shall see how with the help of the strong Ekeland variational principle one can prove the following.

**Theorem 3.1 (Revalski \([19]\)).** The set \(\mathcal{R} \) of these elements \((A,f)\) of

\[ \Gamma := \mathcal{V}(X) \times C_b(X) \]

for which the corresponding minimization problem \((A,f)\) is well posed in the sense of Hadamard is a dense \(G_\delta\) subset of \(\Gamma\). Here \(C_b(X) = \{f \in B(X) : f \text{ is continuous}\} \).

Before giving the proof, we need the following lemma.

Define

\[ d'((A_1,f_1),(A_2,f_2)) = \max(h(A_1,A_2),\rho(f_1,f_2)) \text{ for } A_1,A_2 \subseteq X \]

\[ f_1,f_2 \in B(X). \]

**Lemma 3.2.** The function \(\varphi: (\Gamma, d') \ni (A,f) \to \inf f(A)\) is upper semicontinuous (u.s.c.).

**Proof.** Let \(\varepsilon > 0\) and \((A_0,f_0) \in \Gamma\) be fixed. There exists \(x_0 \in A_0\) such that

\[ f_0(x_0) < \inf f_0(A_0) + \frac{\varepsilon}{3}. \]

Since \(f\) is continuous, there exists \(\delta \in (0,\varepsilon/(3 + \varepsilon))\) such that

\[ |f_0(x) - f_0(x_0)| < \frac{\varepsilon}{3} \]

whenever \(x \in X\) and \(d(x,x_0) < \delta\). Let \((A,f) \in \Gamma\) and \(d'((A,f),(A_0,f_0)) < \delta\). Then there exists \(x \in A\) such that \(d(x,x_0) < \delta\) and

\[ |f(x) - f_0(x)|/(1 + |f(x) - f_0(x)|) < \varepsilon, \]

whence

\[ |f(x) - f_0(x)| < \delta/(1 - \delta) < \frac{\varepsilon}{3}. \]

Thus we have \(\varphi(A,f) = \inf f(A) \leqslant f(x) < f_0(x) + \frac{\varepsilon}{3} < f_0(A_0) + 2\varepsilon/3 < \inf f_0(A_0) + \varepsilon = \varphi(A_0,f_0) + \varepsilon. \]

**Proof of Theorem 3.1.** (a) **Denseness.** Let \((A,f) \in \Gamma, \varepsilon > 0, 0 < \gamma < \varepsilon/\text{diam } X\). By the strong Ekeland variational principle (Theorem 1.5 or 1.6) there exists a point \(a \in B[A;\varepsilon]: = \{x \in X : d(x,A) \leqslant \varepsilon\}\) such that:

(i) \(f(a) < f(x) + \gamma d(a,x)\) whenever \(a \neq x \in B[A;\varepsilon]\) and

(ii) \(x_n \to a\) whenever \(\{x_n\}_{n \geq 1} \subseteq B[A;\varepsilon]\) and \(f(x_n) + \gamma d(a,x_n) \to f(a)\).

For \(\rho(f,f_\varepsilon) := f(\cdot) + \gamma d(\cdot,a)\) we have \(f_\varepsilon \in C_b(X)\) and

\[ \rho(f,f_\varepsilon) = \sup_{x \in X} |f(x) - f_\varepsilon(x)|/(1 + |f(x) - f_\varepsilon(x)|) \]

\[ = \sup_{x \in X} \gamma d(a,x)/(1 + \gamma d(a,x)) \leqslant \gamma \text{ diam } X < \varepsilon. \]

For \(A_\varepsilon := A \cup \{a\}\) we have \(h(A,A_\varepsilon) \leqslant \varepsilon\) and \(f_\varepsilon(a) = \inf f_\varepsilon(A_\varepsilon)\).
Let \( \{ x_n \}_{n \geq 1} \subset X, \ d(x_n, A_\varepsilon) \to 0, \ f_\varepsilon(x_n) \to f_\varepsilon(a). \) If \( d(a, A) < \varepsilon, \) then for some \( \varepsilon_1 > 0 \) \( x_n \in B[A; \varepsilon] \) for every \( n > \varepsilon_1 \) and by (ii) \( x_n \to a. \) Assume that \( d(a, A) = \varepsilon \) and \( x_n \not\to a. \) Then there exist \( \delta > 0 \) and a subsequence \( \{ x_{n_k} \} \) such that \( \| x_{n_k} - a \| > \delta \) for every integer \( k. \) From \( d(x_{n_k}, A_\varepsilon) \to 0 \) it follows that for some \( \delta_2 \) we have \( x_{n_k} \in B[A; \varepsilon] \) for \( k > \delta_2. \) By (ii) we obtain \( x_{n_k} \not\to a, \) a contradiction. Therefore the minimization problem \( (A_\varepsilon, f_\varepsilon) \) is well posed in the sense of Revalski and, as it was mentioned above, in the sense of Hadamard.

(b) \( G_\delta. \) We will prove that for every integer \( n \) the set \( \Gamma_n = \{ (A, f) \in \Gamma: \inf_{\varepsilon > 0} \text{diam } L_{A, f}(\varepsilon) < 1/n \} \) is open. Let \( (A_0, f_0) \in \Gamma_n. \) Then there exists \( \varepsilon > 0 \) such that \( \text{diam } L_{A_0, f_0}(\varepsilon) < 1/n. \) By Lemma 3.2 there exists \( 0 < \delta < \min \{ \varepsilon/(3 + \varepsilon), \ 2\varepsilon/3, 1 \} \) such that \( \inf f(A) - \inf f_0(A_0) < \varepsilon/3 \) whenever \( d'((A, f), (A_0, f_0)) < \delta, \ (A, f) \in \Gamma. \) Let \( (A, f) \in \Gamma, \ d'((A, f), (A_0, f_0)) < \delta, \ x \in L_{A, f}(\varepsilon/3). \) Then \( \rho(f, f_0) < \delta \) implies \( |f(x) - f_0(x)| < \delta/(1 - \delta) < \varepsilon/3 \) and we have

\[
\frac{\varepsilon}{3} + f(x) < \inf f(A) + 2\varepsilon/3 < \inf f_0(A_0) + \varepsilon
\]

and

\[
d(x, A_0) \leq d(x, A) + h(A, A_0) \leq 3\varepsilon/3 + \delta < \varepsilon + 2\varepsilon/3 - \varepsilon,
\]

which shows that \( x \in L_{A_0, f_0}(\varepsilon). \) Hence \( L_{A, f}(\varepsilon/3) \subset L_{A_0, f_0}(\varepsilon) \) and \( (A, f) \in \Gamma_n. \) Obviously \( \Gamma_0 = \bigcap_{n=1}^{\infty} \Gamma_n. \) ■

Analogous theorem for a class topological spaces containing all metrizable compacta was proved by Kenderov [12].

Let us now consider the class of convex minimization problems. Let \( X \) be a nonempty closed convex subset of a real Banach space \((E, \| \cdot \|)).\) Denote by \( \text{CONV}_{\varepsilon}(X) \) those \( f \in C_b(X) \) which are convex. \( \text{CONV}_{\varepsilon}(X) \) is a complete metric space under the distance \( \rho. \) Let \( K(X) = \{ A \subset X: A \neq \emptyset, \ A \text{ is closed and convex} \}. \) When \( X \) is bounded, then \( (K(X), h) \) is a complete metric space too.

In the same way, as Theorem 3.1, we can prove the following.

**Theorem 3.3 (Revalski [19]).** Let \( X \) be bounded. Then the set \( S_0 \) of those elements \( (A, f) \in K(X) \times \text{CONV}_{\varepsilon}(X) \) for which the corresponding minimization problem \( (A, f) \) is well posed in the sense of Hadamard is a dense \( G_\delta \) subset of \( K(X) \times \text{CONV}_{\varepsilon}(X). \)

**Proof.** Let us adopt the notations of the proof of Theorem 3.1. The new part of the proof is of the choice of the set \( A_\varepsilon \) (in the proof of Theorem 3.1): here we take \( A_\varepsilon \) to be \( \text{co}\{ A \cup \{ a \} \}. \)

Let \( \{ x_n \}_{n \geq 1} \subset X, \ d(x_n, A) \to 0, \ f_\varepsilon(x_n) \to f_\varepsilon(a). \) Assume that \( d(a, A) = \varepsilon \) and \( x_n \not\to a. \) Then there exists \( \delta \in (0, 2 \text{diam } A) \) and a subsequence \( \{ x_{n_k} \} \)
such that \( \|x_n - a\| > \delta \) for every \( k = 1, 2, \ldots \). Let \( z \in A \setminus B(a; \delta/2) \). Then \( z = ta + (1 - t)b \) for some \( t \in [0, 1] \), \( b \in A \) and we have \( \|z - a\| = (1 - t)\|a - b\| > \delta/2 \), whence \( 1 - t > \delta/(2\|a - b\|) > \delta/(2\text{diam } A_z) \), \( t < 1 - \delta/(2\text{diam } A_z) \), and \( d(z, A) \leq td(a, A) < \varepsilon - \varepsilon\delta/(2 \text{diam } A_z) \), because of the convexity of the distance function \( d(\cdot, A) \). There exist \( \{y_k\}_{k \geq 1} \subset A \) such that \( \|x_n - y_k\| \to 0 \). Then for some \( v \) we have \( y_k \notin B(a; \delta/2) \) and \( \|x_n - y_k\| < \varepsilon\delta/(2 \text{diam } A_z) \) for \( k > v \). Thus we obtain (for \( k > v \)):
\[
d(x_n, A) \leq \|x_n - y_k\| + d(y_k, A) < \varepsilon\delta/(2 \text{diam } A_z) + \varepsilon - \varepsilon\delta/(2 \text{diam } A_z) = \varepsilon.
\]

Hence by (ii) (from the proof of Theorem 3.1) it follows that \( x_n \to a \), a contradiction. 1

We note that in [19] it is proved, that the sets \( S_0 \) and \( \Gamma_0 \) (in Theorem 3.1) contain dense \( G_\delta \) subsets. Similar results are obtained in [6] and [15].

Using the strong Ekeland variational principle, in an analogous way one can prove another theorem of Revalski [19, Theorem 3].

4. Applications of the Strong Drop Theorem

Denote by \( \mathcal{V} \) the set of all closed, convex, bounded, and nonempty subsets of \( E \) furnished with the Hausdorff metric \( h \). It is well known that \( (\mathcal{V}, h) \) is a complete metric space (see [13], p.417).

**Lemma 4.1.** Let \( X \subset E \) be a nonempty bounded subset, \( c \in \mathbb{R} \), \( f: E \to \mathbb{R} \) be a continuous convex function, \( A(c) = \{x \in E : f(x) \leq c\} \) and let there exists \( z \in E \) with \( f(z) < c \). Then for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( d(x, A(c)) < \varepsilon \) whenever \( x \in X \) and \( f(x) < c + \delta \).

**Proof.** Assume the contrary. Then there exists \( \varepsilon_0 > 0 \), \( x_n \in X \), such that \( f(x_n) < c + 1/n \) and \( d(x_n, A(c)) \geq \varepsilon_0 \) for every positive integer \( n \). Let \( t > 0 \), \( 0 < 1 - t < \varepsilon_0/h(z, X) \), \( 0 < \delta_1 < (1 - t)(c - f(z))/t \), \( m > 1/\delta_1 \), \( y = tx_m + (1 - t)z \). Then we can write
\[
f(y) \leq tf(x_m) + (1 - t)f(z) < t(c + 1/m) + (1 - t)c - t\delta_1 < c,
\]
whence \( y \in A(c) \). But
\[
\varepsilon_0 \leq d(x_n, A(c)) \leq \|x_n - y\| = (1 - t)\|x_m - z\| \leq (1 - t)h(z, X) < \varepsilon_0,
\]
a contradiction. 1

**Theorem 4.2.** Let \( f: E \to \mathbb{R} \) be a convex continuous function bounded from below on the bounded sets and satisfying the condition: either
(1) \( \inf f(E) = -\infty \) or

(2) there exists \( z_0 \in E \) such that \( f(z_0) = \inf f(E) \) and \( x_n \to z_0 \) whenever \( \{x_n\}_{n \geq 1} \subset E \) and \( f(x_n) \to f(z_0) \).

Let \( \mathcal{V}_0 \) be the set of those elements \( X \) of \( \mathcal{Y} \) for which the corresponding minimization problem \( (X, f) \) is well posed in the sense of Hadamard. Then \( \mathcal{V}_0 \) is a dense \( G_\delta \) subset of \( \mathcal{Y} \).

**Proof.** We will work with the well posedness in the sense of Revalski, because it is equivalent to the well posedness in the sense of Hadamard, as it was mentioned above (see [19]).

(a) **Denseness.** Let \( \epsilon > 0 \), \( X \in \mathcal{V} \), \( m = \inf f(B[X; \epsilon/2]) \), \( A = \{x \in E: f(x) \leq m\} \), \( B[X; \epsilon/2] := \{x \in E: d(x, X) \leq \epsilon/2\} \). If the case (2) (in the definition of \( f \)) holds and if \( z_0 \in B[X; \epsilon] \), then there is nothing to prove.

Assume that either the case (1) is fulfilled, or \( z_0 \notin B[X; \epsilon] \). Denote

\[
x_0 = \begin{cases} 
  z_0 & \text{if case (2) holds,} \\
  \hat{x} & \text{with } f(\hat{x}) < m, \text{ otherwise.}
\end{cases}
\]

Then \( f(x_0) < m \). By Lemma 4.1, \( d(B[X; \epsilon/2], A) = 0 \). Hence \( d(X, A) \leq \epsilon/2 \).

If \( d(X, A) < \epsilon/2 \), then there exist \( z \in A, \delta > 0 \) with \( B(z; \delta) \subset B[X; \epsilon/2] \). Let \( t \in (0, 1), 1 - t < \delta/h(x_0, B[X; \epsilon/2]), y = tz + (1 - t)x_0 \). Then

\[
\|y - z\| = (1 - t)\|z - x_0\| < (1 - t)h(x_0, B[X; \epsilon/2]) < \delta,
\]

therefore \( y \in B[X; \epsilon/2] \) and

\[
m \leq f(y) \leq tf(z) + (1 - t)f(x_0) < tm + (1 - t)m = m,
\]
a contradiction. Therefore \( d(X, A) = \epsilon/2 \) and, since \( f \) is continuous, \( A \) is closed. Thus we can apply the strong Drop theorem: there exists \( a \in B(X; \epsilon) \cap A \) such that

(i) \( D(a, X) \cap A = \{a\} \) and

(ii) \( x_n \to a \) whenever \( \{x_n\}_{n \geq 1} \subset D(a, X) \) and \( d(x_n, A) \to 0 \).

By (i) and by the continuity of \( f \) we have \( m = f(a) = \inf f(D(a, X)) \).

Let \( \{y_n\}_{n \geq 1} \subset E, d(y_n, D(a, X)) \to 0, f(y_n) \to f(a) \). Then by Lemma 4.1 \( d(y_n, A) \to 0 \). There exists \( x_n \in D(a, X) \) such that \( \|x_n - y_n\| \to 0 \). So \( d(x_n, A) \leq \|x_n - y_n\| + d(y_n, A) \to 0 \), and by (ii) we obtain \( x_n \to a \), whence \( y_n \to a \). The set \( X_1 := D(a, X) \) is closed and thus we obtain that the minimization problem \( (X_1, f) \) is well posed in the sense of Revalski and, as it was mentioned above, in the sense of Hadamard. For \( x \in X_1 \) we have \( x = \lambda a + (1 - \lambda) b \), where \( \lambda \in [0, 1], b \in X \) and since \( d(\cdot, B) \) is a convex function (because \( B \) is convex), \( d(x, X) \leq \lambda d(a, X) \leq d(a, X) < \epsilon \). Therefore \( h(X, X_1) \leq \epsilon \) and the denseness is proved.

The "\( G_\delta \)" part of the proof is the same as in the proof of Theorem 3.1.
It is easy to see that if $f : E \to \mathbb{R}$ is continuous and sublinear (i.e., $f(x + y) \leq f(x) + f(y)$) then the well posedness in the sense of Tyhonov and Rvervalski (therefore and in the sense of Hadamard) coincide. Indeed, let the minimization problem $(X, f)$ be well posed in the sense of Tyhonov and let $\{x_n\}_{n \geq 1} \subset E$, $d(x_n, X) \to 0$, $f(x_n) \to \inf f(X) = f(x_0)$ for some $x_0 \in E$. Then there exists $y_n \in E$ with $\|x_n - y_n\| \to 0$ and we have

$$f(x_0) \leq f(y_n) = f(x_n) + \left[ f(y_n) - f(x_n) \right] \leq f(x_n) + f(y_n - x_n) \to f(x_0),$$

whence $y_n \to x_0$ and $x_n \to x_0$.

Now, if we take $f(\cdot) = (1 - y)\|\cdot - y\|$ for some $y \in E$, we obtain the following.

**COROLLARY 4.3** (De Blasi and Myjak [5]). Let $y \in E$ be fixed and $\mathcal{V}_0$ be the set of those elements $X$ of $\mathcal{V}_0$ for which the metric projection for $y$ is single-valued, $P_X(y) := \{x \in X : \|x - y\| = \inf_{z \in X} \|z - y\|\} = \{y\}$, and every minimizing sequence $\{x_n\}_{n \geq 1} \subset X$, $\|x_n - y\| \to \|x_0 - y\|$, converges to $x_0$. Then $\mathcal{V}_0$ is a dense $G_\delta$ subset of $\mathcal{V}$.

In addition we obtain that the set $\mathcal{V}_0$ from Corollary 4.3 is precisely the set $\{X \in \mathcal{V} : P_X(y) = \text{single-valued}\}$.

Let us recall that $l \in E^*$ (resp. $l \in E$) is said to be strongly (resp. $w^*$-strongly) exposing functional for $X \subset E$ (resp. $X \subset E^*$) if there exists $x_0 \in X$ for which $l(x_0) = \sup l(X)$ and $x_n \to x_0$, whenever $\{x_n\}_{n \geq 1} \subset X$ and $l(x_n) \to \sup l(X)$. The point $x_0$ in the above definition is said to be a strongly (resp. $w^*$-strongly) exposed point of $X$.

**COROLLARY 4.4** [9]. Let $l \in E^*$, $l \neq 0$, $\mathcal{V}_1$ be the set of those elements $X$ of $\mathcal{V}$ for which $l$ is strongly exposing for $X$, $\mathcal{V}_2$ be the set of those $X \in \mathcal{V}$ for which the multivalued mapping $X \mapsto \{x \in X : l(x) = \sup l(X)\}$ is single-valued and upper semicontinuous at $X$. Then $\mathcal{V}_1 = \mathcal{V}_2$ and $\mathcal{V}_1$ is a dense $G_\delta$ subset of $\mathcal{V}$.

The set $X \subset E$ is said to be dentable, if it has slices of arbitrarily small diameter. A slice is the set $S(X, l, \alpha) = \{x \in X : l(x) > \sup l(X) - \alpha\}$, where $l \in E^*$, $\alpha > 0$. Obviously if $X$ has a strongly exposed point, then $X$ is dentable.

**COROLLARY 4.5** [9]. Almost all (in sense of the Baire category) closed, convex, bounded, and nonempty subsets of a Banach space are dentable.

Let $\mathcal{V}^*$ be the set of all convex, $w^*$-compact and nonempty subset of $E^*$ (the dual space of $E$), furnished with the Hausdorff metric $h$. We shall see that $(\mathcal{V}^*, h)$ is a complete metric space.

Let $L$ be the set of all sublinear, positive-homogeneous and continuous functionals on $E$: $l \in L$ if and only if $l(x+y) \leq l(x) + l(y)$, $l(tx) = tl(x)$, for
$t \geq 0$, $l$ is continuous. For $l \in L$ define $K(l) = \{ x^* \in E^*: \langle x, x^* \rangle \leq l(x), \forall x \in E \}$. Obviously $K(l)$ is a $w^*$-compact convex set. By the Hörmander theorem [11], (see also [20, p. 17]) $K(l) \neq \emptyset$ and $l(\cdot) = \sigma_K(\cdot)$, where $\sigma_K$ is the support function for $K \in Y^*$: $\sigma_K(x) = \sup_{x^* \in K} \langle x, x^* \rangle$ (which easily follows in our situation also by the separation theorem in $E \times \mathbb{R}$ for epi $l$ and $(x, (l(x) - \varepsilon), \varepsilon > 0)$. The separation theorem shows that $\forall x^* \in K = \{ x^* \in E^*: \langle x, x^* \rangle \leq \sigma_K(x), \forall x \in E \}$. Therefore the mapping $I: L \ni l \mapsto K(l)$ is a bijection between $L$ and $Y^*$. Also it is known and easy to prove that

$$h(K_1, K_2) = \sup_{x \in S} |\sigma_{K_1}(x) - \sigma_{K_1}(x)|, \quad \text{where} \quad K_1, K_2 \in Y^*, S = \{ x \in E: \| x \| = 1 \}.$$ 

Thus $I$ is an isometric isomorphism (Minkowski's duality) between $(L, \tau)$ and $(Y^*, h)$, where

$$\tau(l_1, l_2) := \sup_{x \in S} |l_1(x) - l_2(x)|.$$ 

Analogically $(Y^*, h)$ is isometrically isomorphic to $(L^*, \tau^*)$, where $L^*$ is the space of all continuous, $w^*$-lower semicontinuous, sublinear, and positively homogenous functionals on $E^*$ furnished with the metric $\tau^*$,

$$\tau^*(l^*_1, l^*_2) := \sup_{x^* \in S^*} |l^*_1(x^*) - l^*_2(x^*)|, \quad S^* = \{ x^* \in E^*: \| x^* \| = 1 \}, \quad l^*_1, l^*_2 \in L^*.$$ 

From this duality it follows that $(L^*, \tau^*)$ is a complete metric space. It is easy to see that $(L, \tau)$ is a complete metric space, therefore $(Y^*, h)$ is a complete metric space.

**Theorem 4.6** [9]. Let $W_0 \subset Y \times E^*$ be the set of those elements $(X, l)$ for which $l$ is a strongly exposing functional for $X$. Then $W_0$ contains a dense $G_\delta$ subset of $Y \times E^*$.

**Proof.** The denseness follows from Corollary 4.4. The part "$G_\delta$" is analogous to the part "$G_\delta$" in Theorem 3.1.

As an immediate consequence of Theorem 4.6, of its dual analogy assertion and of a theorem of Kuratowski and Ulam [13, p. 255], we obtain

**Theorem 4.7** [9]. Let $E^*$ (resp. $E$) be separable. Then there exists a dense $G_\delta$ subset $Y^*_0 \subset Y^*$ (resp. $Y^* \subset Y^*$) such that for every $X \in Y^*_0$ (resp. for every $X^* \in Y^*_0$) the set $L(X)$ of the strongly (resp. the set $L(X^*)$ of the $w^*$-strongly) exposing functionals contains a dense $G_\delta$ subset of $E^*$ (resp.
of $E$). In particular every $X \in \mathcal{V}_0$ (resp. every $X^* \in \mathcal{V}_0^*$) is the closed (resp. $w^*$-closed) convex hull of its strongly (resp. $w^*$-strongly) exposed points.

Actually $L(X)$ and $L(X^*)$ are $G_\delta$ sets; it follows from the almost obvious assertion that for every bounded subset $X \subset E$ (resp. $X^* \subset E^*$) the set of strongly (resp. $w^*$-strongly) exposing functionals form a $G_\delta$ subset of $E^*$ (resp. of $E$) (perhaps empty). Thus Theorem 4.7 follows also from Corollary 4.4, from its dual analogy assertion and from this assertion.

Using the well-known duality between Fréchet differentiability and strong exposedness, that $x \in X \in \mathcal{V}$ is strongly exposed by $x^*$ (resp. $x^* \in X \in \mathcal{V}^*$ is $w^*$-strongly exposed by $x \in E$) if and only if $\sigma_{x^*}(\cdot)$ is Fréchet differentiable at $x$ (resp at $x^*$) (see for instance [1, p. 159]) we obtain

**Theorem 4.8.** Let $E$ (resp. $E^*$) be separable. Then there exists a dense $G_\delta$ subset $L_0 \subset L$ (resp. $L_0^* \subset L^*$) such that every $l \in L_0$ (resp. every $l^* \in L_0^*$) is Fréchet differentiable on a dense $G_\delta$ subset of $E$ (resp. of $E^*$).

For a comparison with Asplund and weak* Asplund spaces see [1].

In the end we will present a generic result about the metric projection in an arbitrary Banach space $(E, \| \cdot \|)$.

Let $P$ be the set of all equivalent to $\| \cdot \|$ norms in $E$, furnished with the metric $\tau: \tau(p_1, p_2):= \sup_{x \in S} | p_1(x) - p_2(x) |$, $p_1, p_2 \in P$.

First we will prove the following simple

**Proposition 4.9.** Let $p_0 \in P$ be fixed. Then there exist $C_1 > 0$, $C_2 > 0$, $C'_2 > 0$ such that for $0 < \delta < 1/C'_2$ it is fulfilled:

$$C_1 p(x) \leq \| x \| \leq C_1 p(x)$$

whenever $x \in E$, $p \in P$ and $\tau(p, p_0) < \delta$.

**Proof.** Since $p_0$ and $\| \cdot \|$ are equivalent norms, there exist $C'_1 > 0$, $C'_2 > 0$ such that $C'_1 p_0(x) \leq \| x \| \leq C'_2 p_0(x)$ for every $x \in E$. Let $0 < \delta < 1/C'_2$, $C_1 := C'_1/(1 + \delta C'_1)$, $C_2 := C'_2/(1 - \delta C'_2)$, $p \in P$, $\tau(p, p_0) < \delta$. Then

$$\| x \| \leq C'_2 p_0(x) = C'_2 \| x \| p_0(x/\| x \|)$$

$$< C'_2 \| x \| [ p(x/\| x \|) + \delta ] = C'_2 p(x) + \delta \| x \|,$$

whence $\| x \| (1 - \delta C'_2) < C'_2 p(x)$, $\| x \| < C_2 p(x)$. Analogically $\| x \| > C_1 p(x)$.

From this proposition it follows that $P$ is an open subset of the complete metric space $\bar{P}$ of all continuous seminorms on $(E, \| \cdot \|)$, under the distance $\tau$. Therefore $P$ is a Baire space.

Further we adopt the following notations: for $p \in P$, $B(x; r; p)$, $B(x; r; p)$,
$S[x; r; p]$ will denote respectively the closed, the open ball, and the sphere with center $x$ and radius $r$ with respect to the norm $p$:

$p(x, M)$ will denote the distance from $x$ to the subset $M \subset E$ with respect to the norm $p$: $p(x, M) := \inf_{y \in M} p(x - y)$;

$P_{M, p}(x) := \{ y \in M : p(x - y) = \inf_{z \in M} p(x - z) \}$ — the metric projection of $x$ over $M$ with respect to the norm $p$.

**Theorem 4.10.** Let $M$ be a closed nonempty subset of $E$, $\Gamma$ be the set of those elements $(x, p)$ of $E \times P$ for which the metric projection $P_{M, p}$ is single-valued at $x$, $P_{M, p}(x) = \{ y_x \}$ and every minimizing sequence $(x, c_n, a_n, c \in M)$ (that is $p(x - x_n) \to p(x - y_x)$) converges to $y_x$. Then $\Gamma$ is a dense $G_{\delta}$ subset of $E \times P$, as $E \times P$ is furnished with the metric $\rho$:

$$\rho((x_1, p_1), (x_2, p_2)) = \max \{ \| x_1 - x_2 \|, \tau(p_1, p_2) \},$$

for $p_i \in P$, $x_i \in E$, $i = 1, 2$.

**Proof.** (a) **Denseness.** Let $(x_0, p_0) \in E \times P$, $x_0 \notin M$, $\varepsilon > 0$, $r_0 = p_0(x_0, M)$, $0 < \delta < \min \{ \varepsilon/C_2, 1/C_2, 2\varepsilon C_1 r_0/(2 + C_1 \varepsilon), 2r_0/3 \}$, $r = r_0 - \delta/2$, where $C_2$, $C_1$, and $C_0$ are the constants for $p_0$ from Proposition 4.9. By the strong Drop theorem there exists $a \in B(x_0; r_0 + \delta/2, p_0) \cap M$ such that:

1. $D(a, B[x_0; r, p_0]) \cap M = \{ a \}$ and
2. $x_n \to a$ whenever $\{ x_n \}_{n \geq 1} \subset D(a, B[x_0; r, p_0])$ and $p_0(x_n, M) \to 0$.

Let $z_0 \in [x_0, a]$, $p_0(x_0 - z_0) = \delta$, $a' = 2z_0 - a$. Then $B[z_0; r - \delta, p_0] \subset B[x_0; r, p_0]$ and $a' \in B[x_0; r, p_0]$. Denote $A := c\{ \{ a \} \cup \{ a' \} \cup B[z_0; r - \delta, p_0] \}$. Then $A \subset D(a, B[x_0, r, p_0])$, therefore by (1) and (2) we have

3. $A \cap M = \{ a \}$ and
4. $x_n \to a$ whenever $\{ x_n \}_{n \geq 1} \subset A$ and $p_0(x_n, M) \to 0$.

Denote $a_1 = (a - z_0)/(r - \delta)$, $a_2 = (a' - z_0)/(r - \delta)$, $B_0 = \{ x \in E : p_0(x) \leq 1 \}$, $B_1 = (A - z_0)/(r - \delta)$. Then $B_1 = c\{ \{ a_1 \} \cup \{ a_2 \} \cup B_0 \}$ and $p_0(a_1) = p_0(a - z_0)/(r - \delta) = (p_0(a - x_0) - p_0(x_0 - z_0))/(r - \delta) < r/(r - \delta)$. Define $p_1(x) := \inf \{ t > 0 : x \in tB_1 \}$, $b(x) = \{ tx : t \geq 0 \} \cap bdB_1$. We can write

$$p_1(x) = p_0(x)/p_0(b(x))$$

(because $b(x) = t(x)x$ for some $t(x) \geq 0$, whence $p_0(b(x)) = t(x)p_0(x)$ and $1 = p_1(b(x)) = t(x)p_1(x)$). But $b(x) = (\lambda x a_1 + \mu x a_2 + v x z)$, where $\lambda x$, $\mu x$, $v x \in [0, 1]$, $\lambda x + \mu x + v x = 1$, $z \in B_0$, whence

$$1 \leq p_0(b(x)) \leq \lambda x p_0(a_1) + \mu x p_0(a_2) + v x p_0(z) < p_0(a_1).$$
Thus we obtain

\[
\tau(p_1, p_0) = \sup_{x \in S} |p_1(x) - p_0(x)| = \sup_{x \in S} |p_0(x) - p_0(b(x)) - p_0(x)|
\]

\[
= \sup_{x \in S} p_0(x)(1 - 1/p_0(b(x))) \leq \sup_{x \in S} \|x\|(1 - 1/p_0(a_1))/C_1
\]

\[
= (1 - 1/p_0(a_1))/C_1 < (1 - (r - \delta)/r)/C_1
\]

\[
- \delta/(C_1 r) = \delta/(C_1(r_0 - \delta/2)) < \varepsilon,
\]

and

\[
\|z_0 - x_0\| \leq C_2 p_0(z_0 - x_0) = C_2 \delta < \varepsilon.
\]

By the construction \(B[z_0; p_1(z_0 - a); p_1] = A\) and \(p_1(z_0, M) = p_1(z_0 - a) = r - \delta =: r_1\).

Let \(\{z_n\}_{n \geq 1} \subset M\), \(p_1(z_n - z_0) \rightarrow p_1(z_0 - a)\), and let \(y_n = [z_0, z_n] \cap S[z_0; r_1; p_1]\). Then \(p_1(y_n - z_n) \rightarrow 0\). Since \(p_1\) and \(p_0\) are equivalent (because \(B_1/p_0(a_1) \subset B_0 \subset B_1\), \(p_0(y_n - z_n) \rightarrow 0\), which means \(p_0(y_n, M) \rightarrow 0\). By (4) we obtain \(y_n \rightarrow a\), and \(p_1(z_n - a) \leq p_1(z_n - y_n) + p_1(y_n - a) \rightarrow 0\).

(b) \(G_\delta\). First we will prove that

\[
\text{(5) } \|p(x) - p_0(x)\| \leq \|x\| \tau(p, p_0) \text{ and}
\]

\[
\text{(6) } p(x, M) \leq p_0(x_0, M) + \delta + \delta/C_1 + \delta C_2(p_0(x_0, M) + \delta/C_1 + \delta) = f(\delta), \text{ whenever } x_0 \in E, \|x - x_0\| < \delta < 1/C_2, \tau(p, p_0) < \delta, p, p_0 \in P, \text{ where}
\]

\(C_1, C_2\) and \(C_2\) are the constants for \(p_0\) from Proposition 4.9.

Write

\[
\|p(x) - p_0(x)\| = \|x\| \|p(x) - p_0(x) - \tau(p, p_0)\| \leq \|x\| \tau(p, p_0),
\]

which is (5).

For (6) let \(\|x - x_0\| < \delta < 1/C_2\), \(\tau(p, p_0) < \delta, p \in P\). Choose \(y \in M\) with \(p_0(x - y) < p_0(x, M) + \delta\). Then, using (5) and Proposition 4.9, we have:

\[
p(x, M) \leq p(x - y) \leq p_0(x - y) + \|x - y\| \delta \leq p_0(x, M) + \delta + \delta C_2 p_0(x - y)
\]

\[
< p_0(x_0, M) + p_0(x_0 - x) + \delta + \delta C_2(p_0(x_0 - x) + \delta)
\]

\[
\leq p_0(x_0, M) + \delta/C_1 + \delta + \delta C_2(p_0(x_0, M) + p_0(x_0 - x) + \delta)
\]

\[
< p_0(x_0, M) + \delta/C_1 + \delta + \delta C_2(p_0(x_0, M) + \delta/C_1 + \delta)
\]

and (6) is proved.

Let \(L(x, p, \varepsilon) = \{z \in M: p(x - z) \leq p(x, M) + \varepsilon\} \subset P, x \in E, \varepsilon > 0\) and let \(\Gamma_n = \{(x, p) \in E \times P: \inf_{s \geq 0} \text{diam } L(x, p, \varepsilon) < 1/n\}\). We will prove that \(\Gamma_n\) is open for every \(n \in N\). Let \(n \in N\) and \((x_0, p_0) \in \Gamma_n\). There exists \(\varepsilon > 0\) such that \(\text{diam } L(x_0, p_0, \varepsilon) < 1/n\). Denote \(g(\delta) = f(\delta) - p_0(x_0, M)\) and since
there exist $\delta \in (0, 1/C_2)$ such that $g(\delta) + f(\delta) \delta C_2 + \delta \varepsilon C_3/2 + \delta/C_1 < \varepsilon/2$. We will check that

(7) $B(x; p(x, M) + \varepsilon/2; p) \subset B(x_0; p_0(x_0, M) + \varepsilon; p_0)$ whenever $p \in P$, $\tau(p, p_0) < \delta$, and $\|x - x_0\| < \delta$.

For such $p$ and $x$, let $z \in B(x; p(M) + \varepsilon/2; p)$. Then, using (5), (6), and Proposition 4.9 we have

$$p_0(z - x_0) \leq p_0(z - x) + p_0(x - x_0) \leq p(z - x) + \delta \|z - x\| + \delta/C_1$$

$$< p(x, M) + \varepsilon/2 + \delta C_2 p(z - x) + \delta/C_1$$

$$< p(x, M) + \varepsilon/2 + \delta C_2 (p(x, M) + \varepsilon/2) + \delta/C_1$$

$$= p(x, M)(1 + \delta C_2) + \delta C_2 \varepsilon/2 + \delta/C_1 + \varepsilon/2$$

$$\leq f(\delta)(1 + \delta C_2) + \delta C_2 \varepsilon/2 + \delta/C_1 + \varepsilon/2$$

$$= p_0(x_0, M) + g(\delta) + f(\delta) \delta C_2 + \delta C_2 \varepsilon/2 + \delta/C_1 + \varepsilon/2$$

$$< p_0(x_0, M) + \varepsilon$$

and (7) is proved.

But $L(x, p, \gamma) = B(x; p(x, M) + \gamma; p) \cap M$ for $\gamma > 0$; therefore $L(x, p, \varepsilon/2) \subset L(x_0, p_0, \varepsilon)$ which means that $(x, p) \in \Gamma_n$. Thus $\Gamma_n$ is open. Obviously $\Gamma = \cap_{n=1}^{\infty} \Gamma_n$.

Using a theorem of Kuratowski and Ulam [12, p. 255], we obtain

**THEOREM 4.11.** Let $E$ be separable, $M$ be a closed nonempty subset of $E$. Then there exists a dense $G_\delta$ subset $P_0$ of $P$ such that for every $p \in P_0$ there exists a dense $G_\delta$ subset $E_0$ of $E$ (depending on $p$) such that for every $x \in E_0$, the metric projection $P_{M, p}$ is single-valued at $x$, $P_{M, p}(x) = \{y_x\}$ and every minimizing sequence $\{x_n\}_{n \geq 1} \subset M$, $p(x_n - x) \to p(x - y_x)$ converges to $y_x$ (and, therefore, $P_{M, p}$ is upper semicontinuous at $x$).

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