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Existence of positive solutions for a class of $p\&q$ elliptic problems with critical growth on $\mathbb{R}^N$

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ABSTRACT

This paper is concerned with the existence of positive solutions to the class of $p\&q$ elliptic problems with critical growth type

$$-\operatorname{div}\big(a\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u\big)+b\big(|u|^p\big)|u|^{p-2}u=\lambda f(u)+|u|^{\gamma^*-2}u,$$
$$u(z)>0,\quad \forall x\in\mathbb{R}^N,$$

where λ is a positive parameter, $a:\mathbb{R}\to\mathbb{R}$ is a function of C^1 class and $b, f:\mathbb{R}\to\mathbb{R}$ are continuous functions.

1. Introduction

The purpose of this article is to investigate the existence of positive solutions for the following class of quasilinear problem

$$\begin{cases} -\operatorname{div}\big(a\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u\big)+b\big(|u|^p\big)|u|^{p-2}u=\lambda f(u)+|u|^{\gamma^*-2}u & \text{in } \mathbb{R}^N,\\ u\in X,\quad 1<p<N,\\ u(z)>0,\quad \forall z\in\mathbb{R}^N, \end{cases} \tag{P_λ}$$

where γ^* and X will be stated later.

Let us introduce the set $\mathcal{W}$ as being the collection of all functions $k:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the following properties:

There exist constants $a_0, b_0>0$, $a_1, b_1\geqslant 0$, $q>p$ such that

$$a_0+H(b_1)a_1t^{\frac{q-p}{p}}\leqslant k(t)\leqslant b_0+b_1t^{\frac{q-p}{p}}\quad \text{for all } t\geqslant 0, \tag{k_1}$$

where $H(s)=1$ if $s>0$ and $H(s)=0$ if $s=0$.

There exist constants α and θ such that $\gamma<\theta<\gamma^*$ and

$$K(t)\geqslant \frac{1}{\alpha}k(t)t\quad \text{with } 1<\frac{q}{p}\leqslant\alpha<\frac{\theta}{p}, \tag{k_2}$$

for all $t\geqslant 0$, where $K(t)=\int_0^t k(s)\,ds$ and where $\gamma=(1-H(b_1))p+H(b_1)q$ and $\gamma^*=(1-H(b_1))p^*+H(b_1)q^*$.

The function

$$t\to k\big(t^p\big)t^{p-2}\ \text{is increasing.} \tag{k_3}$$

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The hypotheses on functions a, b and f are the following: $a, b \in \mathcal{W}$ and the nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and since we are looking for positive solutions, we suppose that

$$f(t) = 0 \quad \text{for all } t < 0. \tag{f_1}$$

Moreover, we assume the following growth conditions at the origin and at infinity:

$$\lim_{|t|\to 0} \frac{|f(t)|}{|t|^{p-1}} = 0 \tag{f_2}$$

and there exists $s \in (\gamma, \gamma^*)$ verifying

$$\lim_{|t|\to\infty} \frac{|f(t)|}{|t|^{s-1}} = 0. \tag{f_3}$$

In this article, we use the classical Palais–Smale condition. Related with this condition, we suppose that f verifies the well-known Ambrosetti–Rabinowitz superlinear condition,

$$0 < \theta F(t) = \int_0^t f(\xi)\, d\xi \leqslant t f(t) \quad \text{for all } t > 0, \tag{f_4}$$

where θ appeared in (a_2).

Our main result is

Theorem 1.1. *Assume that $a \in C^1(\mathbb{R}^+, \mathbb{R}^+) \cap \mathcal{W}$, $b \in C(\mathbb{R}^+, \mathbb{R}^+) \cap \mathcal{W}$ and that the conditions (f_1)–(f_4) hold. Then, there exists $\lambda^* > 0$, such that problem (P_λ) has a ground-state positive solution in $C^{1,\alpha}(\mathbb{R}^N)$, with $0 < \alpha < 1$, for all $\lambda \geqslant \lambda^*$.*

Now, we will give some examples of functions a and b in order to illustrate the degree of generality of the kind of operators studied here.

Example 1.1. Considering $a(t) = b(t) = 1$, we have that $a, b \in \mathcal{W}$ with $a_0 = b_0 = 1$ and $b_1 = 0$ and $a_1 > 0$. Hence, Theorem 1.1 is valid for the problem

$$-\Delta_p u + |u|^{p-2}u = \lambda f(u) + |u|^{p^*-2}u \quad \text{in } \mathbb{R}^N.$$

Note that, in this case, Theorem 1.1 is the main result in [1] with $\lambda = 1$.

Example 1.2. Considering $a(t) = b(t) = 1 + t^{\frac{q-p}{p}}$, we have that $a, b \in \mathcal{W}$ with $a_0 = b_0 = a_1 = b_1 = 1$. Hence, Theorem 1.1 is valid for the problem

$$-\Delta_p u - \Delta_q u + |u|^{p-2}u + |u|^{q-2}u = \lambda f(u) + |u|^{q^*-2}u \quad \text{in } \mathbb{R}^N.$$

Example 1.3. Considering $a(t) = 1 + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ and $b(t) = 1$, we have that $a \in \mathcal{W}$ with $a_0 = 1$, $b_0 = 2$ and $b_1 = 0$, $a_1 > 0$ and $b \in \mathcal{W}$ with $a_0 = b_0 = 1$ and $b_1 = 0$ and $a_1 > 0$. Hence, Theorem 1.1 is valid for the problem

$$-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u + \frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}\right) + |u|^{p-2}u = \lambda f(u) + |u|^{p^*-2}u \quad \text{in } \mathbb{R}^N.$$

Example 1.4. Considering $a(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$ and $b(t) = 1 + t^{\frac{q-p}{p}}$, we have that $a \in \mathcal{W}$ with $a_0 = 1$, $b_0 = 2$ and $b_1 = a_1 = 1$ and $b \in \mathcal{W}$ with $a_0 = a_1 = b_0 = b_1 = 1$. Hence, Theorem 1.1 is valid for the problem

$$-\Delta_p u - \Delta_q u - \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}\right) + |u|^{p-2}u + |u|^{q-2}u = \lambda f(u) + |u|^{q^*-2}u \quad \text{in } \mathbb{R}^N,$$

or still more complex problems, for example:

Example 1.5. Considering $a(t) = b(t) = 1 + t^{\frac{q-p}{p}} + \frac{1}{(1+t)^{\frac{p-2}{p}}}$, we have that $a, b \in \mathcal{W}$ with $a_0 = 1$, $b_0 = 2$ and $b_1 = a_1 = 1$

$$-\Delta_p u - \Delta_q u - \operatorname{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{(1+|\nabla u|^p)^{\frac{p-2}{p}}}\right) + \left(|u|^{p-2}u + |u|^{q-2}u + \frac{|u|^{p-2}u}{(1+|u|^p)^{\frac{p-2}{p}}}\right) = \lambda f(u) + |u|^{q^*-2}u \quad \text{in } \mathbb{R}^N.$$

Other combinations can be made with the functions presented in the examples above, generating very interesting elliptic problems from the mathematical point of view.

The quasilinear equation of the type $p\&q$-Laplacian has received special attention in the last years, see for example the articles [6–8,10,11,13,14,17,20] and the references therein.

The existence and multiplicity of solutions of quasilinear problem

$$\begin{cases} -\operatorname{div}\big(a\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u\big) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^N$, were studied by J.M. B do O (see [7,8]). In [7] the author showed a result of multiplicity using a $\mathbb{Z}_2$ version of the Mountain Pass Theorem [18], f being a function with subcritical exponential growth. In [8] the author used an argument of minimization, f being a function with subcritical growth.

The subcritical problem

$$\begin{cases} -\Delta_p u - \Delta_q u + v(x)|u|^{p-2}u + w(x)|u|^{q-2}u = \lambda f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

was studied by L. Cherfils and V. Il'yasov in [6]. In this article, the authors showed a result of existence and nonexistence using a variational principle.

In [17], Medeiros and Perera showed the existence of two solutions for the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = \lambda|u|^{p-2}u - \mu|u|^{q-2}u + f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The first solution was obtained via Mountain Pass Theorem and the second solution was obtained via cohomological linking theorem.

In [14], the problem with critical growth on bounded domain of $\mathbb{R}^N$ was treated by Li and Guo. The authors showed a result of multiplicity of solutions for the problem

$$\begin{cases} -\Delta_p u - \Delta_q u = |u|^{p^*-2}u + \mu|u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The case on $\mathbb{R}^N$ was studied in [10] by He and Li. More precisely, the authors studied the subcritical problem

$$\begin{cases} -\Delta_p u - \Delta_q u + m|u|^{p-2}u + n|u|^{q-2}u = f(x,u) & \text{in } \mathbb{R}^N, \\ u \in W^{1,p}\big(\mathbb{R}^N\big) \cap W^{1,q}\big(\mathbb{R}^N\big), \end{cases}$$

using the Mountain Pass Theorem and the concentration–compactness principle of Lions [15].

Other solutions' existence's results of $p\&q$-problems can be seen in [13,20]. For a result of regularity, see [11].

Moreover, that class of equations comes, for example, from a general reaction–diffusion system:

$$u_t = \operatorname{div}\big[D(u)\nabla u\big] + c(x,u), \tag{1.1}$$

where $D(u) = (|\nabla u|^{p-2} + |\nabla u|^{q-2})$. This system has a wide range of applications in physics and related sciences, such as biophysics, plasma physics and chemical reaction design. In such applications, the function u describes a concentration, the first term on the right-hand side of (1.1) corresponds to the diffusion with a diffusion coefficient $D(u)$; whereas the second one is the reaction and relates to source and loss processes. Typically, in chemical and biological applications, the reaction term $c(x,u)$ is a polynomial of u with variable coefficients (see [11,13,20]).

Our theorem extends or complements the articles above, because we consider a more general class of operators, f has a critical growth and problem (P_λ) is on $\mathbb{R}^N$.

2. Variational framework

We say that $u \in X$ with $u > 0$ on $\mathbb{R}^N$ is a weak solution of the problem (P_λ) if it verifies

$$\int_{\mathbb{R}^N} a\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u\nabla\phi\,dx + \int_{\mathbb{R}^N} b\big(|u|^p\big)|u|^{p-2}u\phi\,dx - \lambda\int_{\mathbb{R}^N} f(u)\phi\,dx - \int_{\mathbb{R}^N} |u|^{\gamma^*-2}u\phi\,dx = 0$$

for all $\phi \in X$, where X denotes the Sobolev space $W^{1,p}(\mathbb{R}^N) \cap W^{1,\gamma}(\mathbb{R}^N)$ endowed with the norm

$$\|u\| = \|u\|_{1,p} + H(b_1)\|u\|_{1,q},$$

where

$$\|u\|_{1,m}^m = \int_{\mathbb{R}^N} |\nabla u|^m\,dx + \int_{\mathbb{R}^N} |u|^m\,dx.$$

We will look for solutions of (P_λ) by finding critical points of the C^1-functional $I: X \to \mathbb{R}$ given by

$$I(u) = \frac{1}{p}\int_{\mathbb{R}^N} A\big(|\nabla u|^p\big)\,dx + \frac{1}{p}\int_{\mathbb{R}^N} B\big(|u|^p\big)\,dx - \lambda\int_{\mathbb{R}^N} F(u)\,dx - \frac{1}{\gamma^*}\int_{\mathbb{R}^N} u_+^{\gamma^*}\,dx.$$

Note that

$$I'(u)\phi = \frac{1}{p}\int_{\mathbb{R}^N} a\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u\nabla\phi\,dx + \frac{1}{p}\int_{\mathbb{R}^N} b\big(|u|^p\big)|u|^{p-2}u\phi\,dx - \lambda\int_{\mathbb{R}^N} f(u)\phi\,dx - \int_{\mathbb{R}^N} u_+^{\gamma^*-1}\phi\,dx,$$

for all $\phi \in X$.

In order to use critical point theory we firstly derive the results related to the Palais–Smale compactness condition.

We say that a sequence (u_n) is a Palais–Smale sequence for the functional I if

$$I(u_n) \to c_*$$

and

$$\big\|I'(u_n)\big\| \to 0 \quad \text{in } (X)',$$

where $c_* = \inf_{\eta\in\Gamma}\max_{t\in[0,1]} I(\eta(t)) > 0$ and $\Gamma := \{\eta \in C([0,1], X):\ \eta(0) = 0,\ I(\eta(1)) < 0\}$.

If every Palais–Smale sequence of I has a strong convergent subsequence, then one says that I satisfies the Palais–Smale condition ((PS) for short).

Firstly one proves that functional I has the geometry of Mountain Pass Theorem.

Lemma 2.1. *For each $\lambda > 0$, the functional I satisfies the following conditions*:

(i) *There exist $r, \rho > 0$ such that*:

$$I(u) \geqslant \rho \quad \text{with } \|u\| = r.$$

(ii) *There exists $e \in B_r^c(0)$ with $I(e) < 0$.*

Proof. (i) By (f_2) and (f_3), we get

$$I(u) \geqslant \frac{1}{p}\int_{\mathbb{R}^N} A\big(|\nabla u|^p\big)\,dx + \frac{1}{p}\int_{\mathbb{R}^N} B\big(|u|^p\big)\,dx - \lambda\frac{\epsilon}{p}\int_{\mathbb{R}^N} |u|^p\,dx - \lambda\frac{C_\epsilon}{s}\int_{\mathbb{R}^N} |u|^s\,dx - \frac{1}{\gamma^*}\int_{\mathbb{R}^N} |u|^{\gamma^*}\,dx.$$

Now, by (k_1) we derive

$$\begin{aligned} I(u) \geqslant{}& \frac{a_0}{p}\int_{\mathbb{R}^N} |\nabla u|^p\,dx + H(b_1)\frac{a_1}{q}\int_{\mathbb{R}^N} |\nabla u|^q\,dx + \frac{a_0}{p}\int_{\mathbb{R}^N} |u|^p\,dx + H(b_1)\frac{a_1}{q}\int_{\mathbb{R}^N} |u|^q\,dx - \lambda\frac{\epsilon}{p}\int_{\mathbb{R}^N} |u|^p\,dx \\ &- \lambda\frac{C_\epsilon}{s}\int_{\mathbb{R}^N} |u|^s\,dx - \frac{1}{\gamma^*}\int_{\mathbb{R}^N} |u|^{\gamma^*}\,dx. \end{aligned}$$

So

$$I(u) \geqslant C_1\big(\|u\|_{1,p}^p + H(b_1)\|u\|_{1,q}^q\big) - \lambda\frac{C_\epsilon}{s}\int_{\mathbb{R}^N} |u|^s\,dx - \frac{1}{\gamma^*}\int_{\mathbb{R}^N} |u|^{\gamma^*}\,dx.$$

Choosing $0 < r = \|u\| < 1$, we get $\|u\|_{1,p}^{(q-p)} < 1$ and, hence,

$$I(u) \geqslant C_1\big(\|u\|_{1,p}^q + H(b_1)\|u\|_{1,q}^q\big) - \lambda\frac{C_\epsilon}{s}\int_{\mathbb{R}^N} |u|^s\,dx - \frac{1}{\gamma^*}\int_{\mathbb{R}^N} |u|^{\gamma^*}\,dx.$$

Hence,

$$I(u) \geqslant C_2\|u\|^q - \lambda\frac{C_\epsilon}{s}\int_{\mathbb{R}^N} |u|^s\,dx - \frac{1}{\gamma^*}\int_{\mathbb{R}^N} |u|^{\gamma^*}\,dx.$$

By the Sobolev embedding we get

$$I(u) \geqslant C_2\|u\|^q - \lambda C_3\|u\|^s - C_4\|u\|^{\gamma^*}$$

and, since that $q < s < \gamma^*$, the lemma is proved.

(ii) From (f_4), there exist $C_4, C_5 > 0$ such that

$$F(t) \geqslant C_4 t^\theta - C_5, \quad \forall t > 1.$$

Thus, fixing $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\phi > 0$ in $\mathbb{R}^N$, we get

$$I(t\phi) \leqslant \frac{1}{p}\int_{\mathbb{R}^N} A\big(t^p|\nabla\phi|^p\big)\,dx + \frac{1}{p}\int_{\mathbb{R}^N} B\big(t^p|\phi|^p\big)\,dx - C_4 t^\theta \int_{\mathbb{R}^N}\phi^\theta\,dx + C_5|\operatorname{supp}\phi| - \frac{t^{\gamma^*}}{\gamma^*}\int_{\mathbb{R}^N}|u|^{\gamma^*}\,dx.$$

From (k_1) we derive

$$\begin{aligned} I(t\phi) \leqslant{}& \frac{b_0 t^p}{p}\int_{\mathbb{R}^N}|\nabla\phi|^p\,dx + \frac{b_1 t^q}{q}\int_{\mathbb{R}^N}|\nabla\phi|^q\,dx + \frac{b_0 t^p}{p}\int_{\mathbb{R}^N}|\phi|^p\,dx + \frac{b_1 t^q}{q}H(b_1)\int_{\mathbb{R}^N}|\phi|^q\,dx \\ &- C_4 t^\theta\int_{\mathbb{R}^N}\phi^\theta\,dx + C_5|\operatorname{supp}\phi| - \frac{t^{\gamma^*}}{\gamma^*}\int_{\mathbb{R}^N}|u|^{\gamma^*}\,dx. \end{aligned}$$

Since $\gamma^* > \theta > \gamma$, there exists $\bar{t} > 1$ such that $e = \bar{t}\phi$ satisfies $I(e) < 0$ and $\|e\| > \rho$. □

We devote the rest of this section to show that c_* is attained by a positive function. We start by defining the best constant of the Sobolev embedding $W^{1,\gamma}(\mathbb{R}^N) \hookrightarrow L^{\gamma^*}(\mathbb{R}^N)$ as

$$S := \inf\left\{\int_{\mathbb{R}^N}|\nabla u|^\gamma\,dx\colon\ u \in X,\ \int_{\mathbb{R}^N}|u|^{\gamma^*}\,dx = 1\right\}.$$

As in [5] and arguing as in [2], we are able to compare the minimax level c_* with a suitable number which involves the constant S.

Lemma 2.2. *If the conditions (k_1)–(k_3) and (f_1)–(f_4) hold, then there exists $\lambda_* > 0$ such that $c_* \in (0, (\frac{1}{\theta} - \frac{1}{\gamma^*})(a_0 S)^{N/\gamma})$ for all $\lambda \geqslant \lambda^*$.*

Proof. Considering $\phi \in C_0^\infty(\mathbb{R}^N)$ with $\phi > 0$, there exists $t_\lambda > 0$ verifying $I(t_\lambda\phi) = \max_{t\geqslant 0} I(t\phi)$ and $t_\lambda\phi \in \mathcal{N}$, that is,

$$\int_{\mathbb{R}^N} a\big(|t_\lambda\nabla\phi|^p\big)|t_\lambda\nabla\phi|^p\,dx + \int_{\mathbb{R}^N} b\big(|t_\lambda\phi|^p\big)|t_\lambda\phi|^p\,dx = \lambda\int_{\mathbb{R}^N} f(t_\lambda\phi)t_\lambda\phi\,dx + t_\lambda^{\gamma^*}\int_{\mathbb{R}^N}\phi^{\gamma^*}\,dx.$$

Using (k_1), we have

$$b_0\int_{\mathbb{R}^N}|t_\lambda\nabla\phi|^p\,dx + b_1\int_{\mathbb{R}^N}|t_\lambda\nabla\phi|^q\,dx + b_0\int_{\mathbb{R}^N}|t_\lambda\phi|^p\,dx + b_1\int_{\mathbb{R}^N}|t_\lambda\phi|^q\,dx \geqslant \lambda\int_{\mathbb{R}^N} f(t_\lambda\phi)t_\lambda\phi\,dx + t_\lambda^{\gamma^*}\int_{\mathbb{R}^N}\phi^{\gamma^*}\,dx. \tag{2.1}$$

Since $b_1 \geqslant 0$, then using (2.1) we get

$$t_\lambda^p b_0\int_{\mathbb{R}^N}|\nabla\phi|^p\,dx + b_1 t_\lambda^q\int_{\mathbb{R}^N}|\nabla\phi|^q\,dx + b_0 t_\lambda^p\int_{\mathbb{R}^N}|\phi|^p\,dx + b_1 t_\lambda^q\int_{\mathbb{R}^N}|\phi|^q\,dx \geqslant t_\lambda^{\gamma^*}\int_{\mathbb{R}^N}\phi^{\gamma^*}\,dx$$

which implies that (t_λ) is bounded. Thus, there exists a sequence $(\lambda_n) \subset \mathbb{R}$ such that

$$t_{\lambda_n} \to t_0 \geqslant 0 \quad \text{when } \lambda_n \to +\infty.$$

Note that if $t_0 > 0$ then there exists $K > 0$ such that

$$K \geqslant b_0\int_{\mathbb{R}^N}|t_{\lambda_n}\nabla\phi|^p\,dx + b_1\int_{\mathbb{R}^N}|t_{\lambda_n}\nabla\phi|^q\,dx + b_0\int_{\mathbb{R}^N}|t_{\lambda_n}\phi|^p\,dx + b_1\int_{\mathbb{R}^N}|t_{\lambda_n}\phi|^q\,dx$$

and

$$\lambda_n\int_{\mathbb{R}^N} f(t_{\lambda_n}\phi)t_{\lambda_n}\phi\,dx + t_{\lambda_n}^{\gamma^*}\int_{\mathbb{R}^N}\phi^{\gamma^*}\,dx \to +\infty$$

which is an absurd. Hence we conclude that $t_0 = 0$. Thus, if we define $\eta_*(t) = te$ for $t \in [0,1]$, it follows that $\eta_* \in \Gamma$ and thus

$$0 < c_* \leqslant \max_{t\in[0,1]} I\big(\eta_*(t)\big) = I(t_\lambda \phi) \leqslant b_0 t_\lambda^p \int_{\mathbb{R}^N} |\nabla\phi|^p\,dx + t_\lambda^q b_1 \int_{\mathbb{R}^N} |\nabla\phi|^q\,dx + b_0 t_\lambda^p \int_{\mathbb{R}^N} |\phi|^p\,dx + b_1 t_\lambda^q \int_{\mathbb{R}^N} |\phi|^q\,dx.$$

This way, if λ is large enough we derive

$$b_0 t_\lambda^p \int_{\mathbb{R}^N} |\nabla\phi|^p\,dx + t_\lambda^q b_1 \int_{\mathbb{R}^N} |\nabla\phi|^q\,dx + b_0 t_\lambda^p \int_{\mathbb{R}^N} |\phi|^p\,dx + b_1 t_\lambda^q \int_{\mathbb{R}^N} |\phi|^q\,dx < \left(\frac{1}{\theta} - \frac{1}{\gamma^*}\right)(a_0 S)^{N/\gamma},$$

which leads to

$$0 < c_* < \left(\frac{1}{\theta} - \frac{1}{\gamma^*}\right)(a_0 S)^{N/\gamma}. \qquad \square$$

Remark 2.3. Note that, from the lemma above, if $\lambda \to \infty$, then $c_* \to 0$.

Lemma 2.4. *Let (u_n) be a sequence in X such that $I(u_n) \to c_*$ and $I'(u_n) \to 0$ as $n \to \infty$. Then*

i) *$u_n \rightharpoonup u$ in X.*
ii) *There exists $\lambda^* > 0$ such that $I'(u) = 0$ for all $\lambda \geqslant \lambda^*$.*
iii) *$u_n \geqslant 0$ for $n \in \mathbb{N}$.*

Proof. i) We shall prove that (u_n) is bounded in X. Indeed, from (f_3) we get

$$C\big(1 + \|u_n\|\big) \geqslant I(u_n) - \frac{1}{\theta} I'(u_n)u_n.$$

So

$$C\big(1 + \|u_n\|\big) \geqslant \frac{1}{p}\int_{\mathbb{R}^N} A\big(|\nabla u_n|^p\big)\,dx - \frac{1}{\theta}\int_{\mathbb{R}^N} a\big(|\nabla u_n|^p\big)|\nabla u_n|^p\,dx + \frac{1}{p}\int_{\mathbb{R}^N} B\big(|u_n|^p\big)\,dx - \frac{1}{\theta}\int_{\mathbb{R}^N} b\big(|u_n|^p\big)|u_n|^p\,dx.$$

By (k_2) we derive

$$C\big(1 + \|u_n\|\big) \geqslant \left(\frac{1}{p\alpha} - \frac{1}{\theta}\right)\left[\int_{\mathbb{R}^N} a\big(|\nabla u_n|^p\big)|\nabla u_n|^p\,dx + \int_{\mathbb{R}^N} b\big(|u_n|^p\big)|u_n|^p\right]dx.$$

From (k_1) we have

$$C\big(1 + \|u_n\|\big) \geqslant \left(\frac{1}{p\alpha} - \frac{1}{\theta}\right)a_0 \int_{\mathbb{R}^N} \big[|\nabla u_n|^p + |u_n|^p\big]\,dx + \left(\frac{1}{p\alpha} - \frac{1}{\theta}\right)H(b_1)a_1 \int_{\mathbb{R}^N} \big[|\nabla u_n|^q + |u_n|^q\big]\,dx.$$

Hence,

$$C\big(1 + \|u_n\|\big) \geqslant C_1\|u_n\|_{1,p}^p + C_2 H(b_1)\|u_n\|_{1,q}^q. \tag{2.2}$$

Thus, if $b_1 = 0$, then (u_n) is bounded in X. If $b_1 > 0$, suppose, for contradiction, that, up to a subsequence, $\|u_n\| \to +\infty$. We consider several cases:

a) $\|u_n\|_{1,p} \to +\infty$ and $\|u_n\|_{1,q} \to +\infty$;
b) $\|u_n\|_{1,p} \to +\infty$ and $\|u_n\|_{1,q}$ is bounded;
c) $\|u_n\|_{1,p}$ is bounded and $\|u_n\|_{1,q} \to +\infty$.

In the first case, for n sufficiently large, $\|u_n\|_{1,q}^{q-p} \geqslant 1$ and $\|u_n\|_{1,q}^q \geqslant \|u_n\|_{1,q}^p$. Thus, recalling (2.2),

$$C\big(1 + \|u_n\|\big) \geqslant C_1\|u_n\|_{1,p}^p + C_2\|u_n\|_{1,q}^p \geqslant C_3\big(\|u_n\|_{1,p} + \|u_n\|_{1,q}\big)^p = C_3\|u_n\|^p,$$

which is an absurd.

In case b), by (f_4), we have

$$C\big(1+\|u_n\|_{1,p}+\|u_n\|_{1,q}\big)=C\big(1+\|u_n\|\big)\geqslant\left(\frac{1}{p}-\frac{1}{\theta}\right)\|u_n\|_{1,p}^{p}.$$

Thus, we derive

$$C\left(\frac{1}{\|u_n\|_{1,p}^{p}}+\frac{1}{\|u_n\|_{1,p}^{p-1}}+\frac{\|u_n\|_{1,q}}{\|u_n\|_{1,p}^{p}}\right)\geqslant\left(\frac{1}{p}-\frac{1}{\theta}\right).$$

Since $p-1>0$, passing to the limit as $n\to\infty$, we obtain $0<(\frac{1}{p}-\frac{1}{\theta})\leqslant 0$, which is an absurd.

The last case is similar to the case b).

Thus $u_n\rightharpoonup u$ in X.

ii) Since, up to a subsequence, $u_n\to u$ in $L^m_{loc}(\mathbb{R}^N)$ for $1\leqslant m<\gamma^*$,

$$\int_{\mathbb{R}^N}|u_n|^{\gamma}\phi\,dx\to\int_{\mathbb{R}^N}|u|^{\gamma}\phi\,dx$$

and

$$\int_{\mathbb{R}^N}|u_n|^{s}\phi\,dx\to\int_{\mathbb{R}^N}|u|^{s}u\phi\,dx.$$

Hence, from generalized Lebesgue's Theorem

$$\int_{\mathbb{R}^N}b\big(|u_n|^p\big)|u_n|^p\phi\,dx\to\int_{\mathbb{R}^N}b\big(|u|^p\big)|u|^p\phi\,dx \tag{2.3}$$

and

$$\int_{\mathbb{R}^N}f(u_n)u_n\phi\,dx\to\int_{\mathbb{R}^N}f(u)u\phi\,dx, \tag{2.4}$$

for all $\phi\in X$. Moreover, $u_n(x)\to u(x)$ a.e. in $\mathbb{R}^N$ and recalling a result due to Brezis and Lieb [4] (see also [9, Lemma 4.6])

$$\int_{\mathbb{R}^N}|u_n|^{\gamma-2}u\phi\,dx\to\int_{\mathbb{R}^N}|u|^{\gamma-2}u\phi\,dx,$$

$$\int_{\mathbb{R}^N}|u_n|^{s-2}u\phi\,dx\to\int_{\mathbb{R}^N}|u|^{s-2}u\phi\,dx$$

and

$$\int_{\mathbb{R}^N}|u_n|^{\gamma^*-2}u\phi\,dx\to\int_{\mathbb{R}^N}|u|^{\gamma^*-2}u\phi\,dx. \tag{2.5}$$

Hence, from generalized Lebesgue's Theorem

$$\int_{\mathbb{R}^N}b\big(|u_n|^p\big)|u_n|^{p-2}u\phi\,dx\to\int_{\mathbb{R}^N}b\big(|u|^p\big)|u|^{p-2}u\phi\,dx \tag{2.6}$$

and

$$\int_{\mathbb{R}^N}f(u_n)u\phi\,dx\to\int_{\mathbb{R}^N}f(u)u\phi\,dx, \tag{2.7}$$

for all $\phi\in X$.

We claim that

$$\int_{\mathbb{R}^N}|u_n|^{\gamma^*}\phi\,dx\to\int_{\mathbb{R}^N}|u|^{\gamma^*}\phi\,dx. \tag{2.8}$$

In order to prove the claim we note that, taking a subsequence, we may suppose that

$$|\nabla u_n|^{\gamma} \rightharpoonup |\nabla u|^{\gamma} + \mu \quad \text{and} \quad |u_n|^{\gamma^*} \rightharpoonup |u|^{\gamma^*} + \nu \quad \big(\text{weak}^*\text{-sense of measures}\big).$$

Using the concentration–compactness principle due to Lions (cf. [16, Lemma 1.2]), we obtain an at most countable index set Λ, sequences $(x_i) \subset \mathbb{R}^N$, $(\mu_i), (\nu_i) \subset (0, \infty)$, such that

$$\nu = \sum_{i \in \Lambda} \nu_i \delta_{x_i}, \qquad \mu \geqslant \sum_{i \in \Lambda} \mu_i \delta_{x_i} \quad \text{and} \quad S\nu_i^{\gamma/\gamma^*} \leqslant \mu_i, \tag{2.9}$$

for all $i \in \Lambda$, where δ_{x_i} is the Dirac mass at $x_i \in \mathbb{R}^N$.

Now, for every $\varrho > 0$, we set $\psi_\varrho(x) := \psi((x - x_i)/\varrho)$ where $\psi \in C_0^\infty(\mathbb{R}^N, [0, 1])$ is such that $\psi \equiv 1$ on $B_1(0)$, $\psi \equiv 0$ on $\mathbb{R}^N \setminus B_2(0)$ and $|\nabla\psi|_\infty \leqslant 2$. Since $(\psi_\varrho u_n)$ is bounded, $I'(u_n)(\psi_\varrho u_n) \to 0$, that is,

$$\int_{\mathbb{R}^N} a\big(|\nabla u_n|^p\big)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\psi_\varrho \, dx + \int_{\mathbb{R}^N} b\big(|u_n|^p\big)|u_n|^{p-2}u_n\psi_\varrho \, dx$$
$$= -\int_{\mathbb{R}^N} \psi_\varrho a\big(|\nabla u_n|^p\big)|\nabla u_n|^p \, dx - \int_{\mathbb{R}^N} \psi_\varrho b\big(|u_n|^p\big)|u_n|^p \, dx + \lambda \int_{\mathbb{R}^N} f(x, u_n)\psi_\varrho u_n \, dx + \int_{\mathbb{R}^N} \psi_\varrho |u_n|^{\gamma^*} \, dx + o_n(1).$$

Since by (k_1) $a(t), b(t) \geqslant a_0 >$ and arguing as in [3], we can prove that

$$\lim_{\varrho \to 0}\left[\lim_{n\to\infty} \int_{\mathbb{R}^N} a\big(|\nabla u_n|^p\big)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\psi_\varrho \, dx\right] = 0$$

and

$$\lim_{\varrho \to 0}\left[\lim_{n\to\infty} \int_{\mathbb{R}^N} b\big(|u_n|^p\big)|u_n|^{p-2}u_n \cdot \psi_\varrho \, dx\right] = 0.$$

Moreover, since $u_n \to u$ in $L^m_{loc}(\mathbb{R}^N)$ for all $1 \leqslant m < \gamma^*$ and ψ_ϱ has compact support, we can let $n \to \infty$ in the above expression to obtain

$$\int_{\mathbb{R}^N} \psi_\varrho \, d\nu \geqslant \int_{\mathbb{R}^N} a_0\psi_\varrho \, d\mu.$$

Letting $\varrho \to 0$ we conclude that $\nu_i \geqslant a_0\mu_i$. It follows from (2.9) that $\nu_i \geqslant (a_0 S)^{N/\gamma}$. Thus, we derive

$$\nu_i \geqslant \left(\frac{1}{\theta} - \frac{1}{\gamma^*}\right)(a_0 S)^{N/\gamma}. \tag{2.10}$$

Now we shall prove that the above expression cannot occur, and therefore the set Λ is empty. Indeed, arguing by contradiction, let us suppose that $\nu_i \geqslant (\frac{1}{\theta} - \frac{1}{\gamma^*})(a_0 S)^{N/\gamma}$ for some $i \in \Lambda$. Thus,

$$c_* = I(u_n) - \frac{1}{\theta} I'(u_n)u_n + o_n(1).$$

From (f_4), (k_1) and (k_2) we have

$$c_* \geqslant \left(\frac{1}{\theta} - \frac{1}{\gamma^*}\right)\int_{\mathbb{R}^N} |u_n|^{\gamma^*} \, dx + o_n(1) \geqslant \left(\frac{1}{\theta} - \frac{1}{\gamma^*}\right)\int_{B_\varrho(x_i)} \psi_\varrho |u_n|^{\gamma^*} \, dx + o_n(1).$$

Letting $n \to \infty$, we get

$$c_* \geqslant \left(\frac{1}{\theta} - \frac{1}{\gamma^*}\right)\sum_{i\in\Lambda} \psi_\varrho(x_i)\nu_i = \left(\frac{1}{\theta} - \frac{1}{\gamma^*}\right)\sum_{i\in\Lambda} \nu_i \geqslant \left(\frac{1}{\theta} - \frac{1}{\gamma^*}\right)(a_0 S)^{N/\gamma},$$

which does not make sense for all $\lambda > \lambda^*$. Hence Λ is empty and it follows that

$$\int_{\mathbb{R}^N} |u_n|^{\gamma^*} \, dx \to \int_{\mathbb{R}^N} |u|^{\gamma^*} \, dx.$$

The next step is to prove that

$$\int_{\mathbb{R}^N} a\big(|\nabla u_n|^p\big)|\nabla u_n|^{p-2}\nabla u_n \nabla\phi\, dx = \int_{\mathbb{R}^N} a\big(|\nabla u|^p\big)|\nabla u|^{p-2}\nabla u \nabla\phi\, dx + o_n(1), \tag{2.11}$$

for all $\phi \in X$.

To this end, we will prove the inequality below:

$$C|x-y|^p \leqslant \big\langle a\big(|x|^p\big)|x|^{p-2}x - a\big(|y|^p\big)|y|^{p-2}y, x-y\big\rangle,$$

for all $x, y \in \mathbb{R}^N$. Indeed, firstly note that

$$\big\langle a\big(|x|^p\big)|x|^{p-2}x - a\big(|y|^p\big)|y|^{p-2}y, x-y\big\rangle = \sum_{j=1}^{N}\big(a\big(|x|^p\big)|x|^{p-2}x_j - a\big(|y|^p\big)|y|^{p-2}y_j\big)(x_j - y_j)$$

and for all $z, \xi \in \mathbb{R}^N$ we get

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\big(a\big(|z|^p\big)|z|^{p-2}z_j\big)\xi_i\xi_j = (p-2)|z|^{p-4}\sum_{i,j=1}^{N} a\big(|z|^p\big)z_i z_j \xi_i \xi_j + \sum_{i,j=1}^{N} a\big(|z|^p\big)|z|^{p-2}\delta_{i,j}\xi_i\xi_j + p\sum_{i,j=1}^{N} a'\big(|z|^p\big)|z|^{2p-4}z_i z_j \xi_i\xi_j.$$

Hence

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\big(a\big(|z|^p\big)|z|^{p-2}z_j\big)\xi_i\xi_j = (p-2)|z|^{p-4}a\big(|z|^p\big)\sum_{i,j=1}^{N} z_i z_j \xi_i \xi_j + \sum_{i,j=1}^{N} a\big(|z|^p\big)|z|^{p-2}|\xi|^2 + pa'\big(|z|^p\big)|z|^{2p-4}\sum_{i,j=1}^{N} z_i z_j \xi_i\xi_j.$$

Since

$$\sum_{i,j=1}^{N} z_i z_j \xi_i \xi_j = \left(\sum_{j=1}^{N} z_j \xi_j\right)^2,$$

we have

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\big(a\big(|z|^p\big)|z|^{p-2}z_j\big)\xi_i\xi_j = \left(\sum_{j=1}^{N} z_j \xi_j\right)^2 |z|^{p-4}\big[(p-2)a\big(|z|^p\big) + pa'\big(|z|^p\big)|z|^p\big] + a\big(|z|^p\big)|z|^{p-2}|\xi|^2.$$

By (k_3), we derive

$$\sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\big(a\big(|z|^p\big)|z|^{p-2}z_j\big)\xi_i\xi_j \geqslant a\big(|z|^p\big)|z|^{p-2}|\xi|^2. \tag{2.12}$$

Moreover, if $|y| \geqslant |x|$, we have $\frac{1}{2}|x-y| \leqslant |y|$ and for $t \in [0, \frac{1}{4}]$ we get

$$\big|y + t(x-y)\big| \geqslant |y| - t|x-y| \geqslant \frac{1}{4}|x-y|.$$

Making $z = x-y$ and $\xi = x-y$, from direct calculations we get

$$\sum_{j=1}^{N}\big(a\big(|x|^p\big)|x|^{p-2}x_j - a\big(|y|^p\big)|y|^{p-2}y_j\big)(x_j - y_j) = \int_0^1 \sum_{i,j=1}^{N} \frac{\partial}{\partial z_i}\big(a\big(|z|^p\big)|z|^{p-2}z_j\big)\xi_i\xi_j.$$

Using (2.12) we derive

$$\big\langle a\big(|x|^p\big)|x|^{p-2}x - a\big(|y|^p\big)|y|^{p-2}y, x-y\big\rangle \geqslant a\big(\big|y+t(x-y)\big|^p\big)\big|y+t(x-y)\big|^{p-2}|x-y|^2.$$

By (k_1) we conclude

$$\left\langle a\left(|x|^p\right)|x|^{p-2}x - a\left(|y|^p\right)|y|^{p-2}y, x-y\right\rangle \geqslant \frac{a_0}{4}|x-y|^{p-2}|x-y|^2 = \frac{a_0}{4}|x-y|^p.$$

Now, considering

$$P_N = \left\langle a\left(|\nabla u_n|^p\right)|\nabla u_n|^{p-2}\nabla u_n - a\left(|\nabla u|^p\right)|\nabla u|^{p-2}\nabla u, \nabla u_n - \nabla u\right\rangle$$

and $\psi \in C_0^\infty(\mathbb{R}^N)$ such that $\psi \equiv 1$ in $B_1(0)$ and $\psi \equiv 0$ in $\mathbb{R}^N \setminus B_2(0)$, we have

$$0 \leqslant \frac{a_0}{4} \int_{B_1(0)} |\nabla u_n - \nabla u|^p \, dx \leqslant \int_{B_1(0)} P_N \, dx \leqslant \int_{\mathbb{R}^N} P_N \psi \, dx.$$

Hence

$$0 \leqslant \frac{a_0}{4} \int_{B_1(0)} |\nabla u_n - \nabla u|^p \, dx \leqslant \int_{\mathbb{R}^N} a\left(|\nabla u_n|^p\right)|\nabla u_n|^p \psi \, dx - \int_{\mathbb{R}^N} a\left(|\nabla u_n|^p\right)|\nabla u_n|^{p-2}\nabla u_n \nabla u \psi \, dx + o_n(1).$$

Using (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8) we get

$$0 \leqslant \frac{a_0}{4} \int_{B_1(0)} |\nabla u_n - \nabla u|^p \, dx \leqslant I'(u_n)(u_n\psi) - I'(u_n)(u\psi) = o_n(1).$$

Thus

$$\frac{\partial u_n}{\partial x_i} \to \frac{\partial u}{\partial x_i} \quad \text{in } L^p\left(B_1(0)\right)$$

and, up to a subsequence,

$$\frac{\partial u_n}{\partial x_i}(x) \to \frac{\partial u}{\partial x_i}(x) \quad \text{a.e. in } \mathbb{R}^N.$$

Using also a result due to Brezis and Lieb [4] (see also [9, Lemma 4.6]), we conclude that (2.11) holds. Hence, $I'(u)\psi = 0$ for all $\psi \in X$ and for $\lambda \geqslant \lambda^*$.

iii) In view of (f_1) and (a_1), we have

$$\begin{aligned} o_n(1) = I'(u_n)u_n^- &= -\int_{\mathbb{R}^N} a\left(\left|\nabla u_n^-\right|^p\right)\left|\nabla u_n^-\right|^p dx - \int_{\mathbb{R}^N} b\left(\left|u_n^-\right|^p\right)\left|u_n^-\right|^p dx \\ &\leqslant -a_0 \int_{\mathbb{R}^N} \left|\nabla u_n^-\right|^p dx - b_1 \int_{\mathbb{R}^N} \left|\nabla u_n^-\right|^q dx - a_0 \int_{\mathbb{R}^N} \left|u_n^-\right|^p dx - H(b_1) \int_{\mathbb{R}^N} \left|u_n^-\right|^q dx = -\left\|u_n^-\right\|_{1,p}^p - H(b_1)\left\|u_n^-\right\|_{1,q}^q. \end{aligned}$$

Hence, $\|u_n^-\|_{1,p} = \|u_n^-\|_{1,q} = o_n(1)$ which implies $\|u_n^-\| = o_n(1)$. Thus, we can easily compute

$$I(u_n) = I\left(u_n^+\right) + o_n(1)$$

and

$$I'(u_n) = I'\left(u_n^+\right) + o_n(1).$$

Thus, we will assume hereafter that (u_n) is nonnegative. □

The next proposition is a version of Lions' results [15] to problem with $p\&q$-Laplacian.

Proposition 2.5. *Let $(u_n) \subset X$ be a $(PS)_{c_*}$ sequence for I with $u_n \rightharpoonup 0$ in X. Then we have either*:

a) *$u_n \to 0$ in X or*
b) *there exist a sequence $(y_n) \in \mathbb{R}^N$ and constants $R, \beta > 0$ such that*

$$\liminf_{n\to+\infty} \int_{B_R(y_n)} |u_n|^\gamma \, dx \geqslant \beta > 0.$$

Proof. Suppose that b) does not occur. Thus, $\liminf_{n\to+\infty}\sup_{y\in\mathbb{R}^N}\int_{B_R(y)}|u_n|^\gamma\,dx=0$. Using Lemma 8.4 in [15], we get $u_n\to 0$ in $L^m(\mathbb{R}^N)$, for all $m\in(\gamma,\gamma^*)$. This fact implies that

$$\int_{\mathbb{R}^N} f(u_n)u_n\,dx\to 0.$$

It follows that

$$\int_{\mathbb{R}^N} a\big(|\nabla u_n|^p\big)|\nabla u_n|^p\,dx+\int_{\mathbb{R}^N} b\big(|u_n|^p\big)|u_n|^p\,dx=\int_{\mathbb{R}^N}|u_n|^{\gamma^*}\,dx+o_n(1).$$

Since, up to a subsequence,

$$\int_{\mathbb{R}^N} a\big(|\nabla u_n|^p\big)|\nabla u_n|^p\,dx+\int_{\mathbb{R}^N} b\big(|u_n|^p\big)|u_n|^p\,dx\to L_\lambda$$

and

$$\int_{\mathbb{R}^N}|u_n|^{\gamma^*}\,dx\to L_\lambda.$$

If $L_\lambda\to 0$ as $\lambda\to\infty$, then

$$C\big(\|u_n\|_{1,p}^p+H(b_1)\|u_n\|_{1,q}^q\big)\leqslant\int_{\mathbb{R}^N} a\big(|\nabla u_n|^p\big)\nabla u_n|^p\,dx+\int_{\mathbb{R}^N} b\big(|u_n|^p\big)|u_n|^p\,dx=o_n(1)$$

and therefore

$$\|u_n\|\to 0.$$

If there exists $M>0$, independent of λ, such that $L_\lambda\geqslant M$, then

$$o_n(1)+c_*=\frac{1}{p}\int_{\mathbb{R}^N} A\big(|\nabla u_n|^p\big)\,dx+\frac{1}{p}\int_{\mathbb{R}^N} B\big(|u_n|^p\big)\,dx-\frac{1}{\gamma^*}\int_{\mathbb{R}^N}|u_n|^{\gamma^*}\,dx.$$

By (k_2) we get

$$o_n(1)+c_*\geqslant\left(\frac{1}{p\alpha}-\frac{1}{\gamma^*}\right)L_\lambda\geqslant\left(\frac{1}{p\alpha}-\frac{1}{\gamma^*}\right)M>0,$$

which is an absurd from Remark 2.3. Hence

$$\|u_n\|\to 0.\qquad\square$$

2.1. Proof of Theorem 1.1

By Lemma 2.4, there exists $u\in X$ such that $I'(u)=0$ and $u\geqslant 0$. Suppose that $u\not\equiv 0$. Adapting arguments from [12], we conclude that $u\in L^\infty(\mathbb{R}^N)\cap C^{1,\alpha}(\mathbb{R}^N)$ for some $0<\alpha<1$, and therefore it follows from Harnack's inequality [19] that $u(x)>0$ for all $x\in\mathbb{R}^N$. If $u\equiv 0$, then u_n no converges strongly to zero, because for the contrary case, we get $c_*=0$. Thus, from Proposition 2.5, there is a sequence $(y_n)\in\mathbb{R}^N$ and $R,\alpha>0$ such that

$$\liminf_{n\to+\infty}\int_{B_R(y_n)}|u_n|^\gamma\,dx>\beta. \tag{2.13}$$

Now, letting $\widetilde{u}_n(x)=u_n(x+y_n)$, using the invariance of $\mathbb{R}^N$ for translation, by a routine calculus we obtain $\|\widetilde{u}_n\|=\|u_n\|$, $I(\widetilde{u}_n)=I(u_n)$ and $I'(\widetilde{u}_n)=o_n(1)$. Then, there exists $\widetilde{u}$ such that $\widetilde{u}_n\rightharpoonup\widetilde{u}$ weakly in X and as before it follows that $I'(\widetilde{u})=0$. Now, by (2.13), taking a subsequence and R bigger we conclude that $\widetilde{u}$ is nontrivial and the proposition is proved.

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