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# Defining relation for semi-invariants of three by three matrix triples

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ABSTRACT

The single defining relation of the algebra of $SL_3 \times SL_3$-invariants of triples of $3 \times 3$ matrices is explicitly computed. Connections to some other prominent algebras of invariants are pointed out.



## 1. Introduction

Denote by $M_n(K)$ the space of $n \times n$ matrices over an infinite field $K$. The direct product of two copies of the general linear group $G_n := GL_n(K) \times GL_n(K)$ acts linearly on $M_n(K)$: the group element $(g, h)$ maps $A \in M_n(K)$ to $gAh^{-1}$. Take the direct sum $M_n(K)^m := \underbrace{M_n(K) \oplus \cdots \oplus M_n(K)}_{m}$ of $m$ copies of this representation of $G_n$. The action of $G_n$ induces an action on the algebra of polynomial functions $K[M_n(K)^m]$ in the usual way. Let $R_{n,m}(K)$ be the subalgebra of the invariants of the subgroup $SL_n(K) \times SL_n(K)$ of $G_n$. It is called also the algebra of semi-invariants of $G_n$ on $M_n(K)^m$. The structure and minimal systems of generators of $R_{n,m}(K)$ are known in a few cases only. Over a field of characteristic 0 or $p > 2$ the algebra $R_{2,m}(K)$ is minimally generated by the determinants $\det(A_r)$, $r = 1, \ldots, m$, the mixed discriminants $M(A_{r_1}, A_{r_2})$, $1 \le r_1 < r_2 \le m$, and the discriminants $D(A_{r_1}, A_{r_2}, A_{r_3}, A_{r_4})$, $1 \le r_1 < r_2 < r_3 < r_4 \le m$ (see [11] and the last paragraph of Section 3 in [13], or [20, Theorem 11.47]). Here $M(A_1, A_2)$ is defined as the coefficient of $t_1t_2$ in

$$\det(t_1A_1 + t_2A_2) = t_1^2 \det(A_1) + t_1t_2M(A_1, A_2) + t_2^2 \det(A_2),$$

$$D(A_1, A_2, A_3, A_4) = \begin{vmatrix} a_{11}^{(1)} & a_{11}^{(2)} & a_{11}^{(3)} & a_{11}^{(4)} \\ a_{21}^{(1)} & a_{21}^{(2)} & a_{21}^{(3)} & a_{21}^{(4)} \\ a_{12}^{(1)} & a_{12}^{(2)} & a_{12}^{(3)} & a_{12}^{(4)} \\ a_{22}^{(1)} & a_{22}^{(2)} & a_{22}^{(3)} & a_{22}^{(4)} \end{vmatrix},$$

where $A_r = \left(a_{ij}^{(r)}\right)_{2\times 2}$, $r = 1, 2, 3, 4$. For $m \le 4$ the generators $\det(A_r)$ and $M(A_{r_1}, A_{r_2})$ are algebraically independent and for $m = 4$ the algebra $R_{2,4}(K)$ is a free module over the polynomial subalgebra generated by them with basis $1, D(A_1, A_2, A_3, A_4)$. It is pointed out in [13,20] that $R_{2,m}(K)$ can be interpreted as the ring of vector invariants of the special orthogonal group of

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degree 4. Therefore the relations among the generators can be deduced from classically known results, and even a Gröbner basis of the ideal of relations can be obtained from [13]. For $m = 2$ and any $n$ the algebra $R_{n,2}$ is generated by the algebraically independent coefficients of $\det(t_1A_1 + t_2A_2)$, [17], see also [19].

Apart from the cases $n = 2$ for any $m$ and $m = 2$ for any $n$, the only other case when a minimal system of generators of $R_{n,m}(K)$ is explicitly known is $n = m = 3$, and this algebra is the main object to study in the present paper. In the sequel we denote $R(K) := K[M_3(K)^3]^{SL_3(K)\times SL_3(K)}$, the algebra of $SL_3(K) \times SL_3(K)$-invariant polynomial functions on $M_3(K)^3$.

Define polynomial functions $f_{i,j,k}$ on $M_3(K)^3$ by the equality

$$\det(t_1A_1 + t_2A_2 + t_3A_3) = \sum_{i+j+k=3} t_1^i t_2^j t_3^k f_{i,j,k}(A_1, A_2, A_3)$$

for all $t_1, t_2, t_3 \in K$ and $A_1, A_2, A_3 \in M_3(K)$. Obviously the ten polynomials $f_{i,j,k}$ belong to $R(K)$. Furthermore, define $h$ as the coefficient of $t_1^2t_2^2t_3^2$ in

$$\det\begin{pmatrix} t_2A_2 & t_1A_1 \\ t_1A_1 & t_3A_3 \end{pmatrix}$$

and define $q$ as the coefficient of $t_1^2t_2t_3^2t_4t_5^2t_6$ in

$$\det\begin{pmatrix} 0 & t_1A_1 & t_2A_2 \\ t_4A_1 & 0 & t_3A_3 \\ t_5A_2 & t_6A_3 & 0 \end{pmatrix}.$$

Clearly $h$ and $q$ belong to $R(K)$. It is proved in [10] that $h$ and the ten polynomials $f_{i,j,k}$ (where $i + j + k = 3$) constitute a homogeneous system of parameters in $R(K)$. Denote by $P(K)$ the subalgebra generated by these eleven algebraically independent elements. In the case when the characteristic of the base field $K$ is zero, using a result of Teranishi [23] it was established in [10] that $R(K)$ is a free $P(K)$-module generated by 1 and $q$:

$$R(K) = P(K) \oplus P(K)q. \tag{1}$$

A similar description of $R(K)$ is stated by Mukai without proof in [20, Proposition 11.49]. It follows from (1) that $q$ satisfies a monic quadratic relation with coefficients from $P(K)$. In the present paper we find the explicit form of this relation.

A crucial role in our considerations is played by the following right action of the general linear group $GL_3(K)$ on $M_3(K)^3$: For $g = (g_{ij})_{3\times3} \in GL_3(K)$ and $(A_1, A_2, A_3) \in M_3(K)^3$ we have

$$(A_1, A_2, A_3) \cdot g := \left(\sum_{i=1}^{3} g_{i1}A_i, \sum_{i=1}^{2} g_{i2}A_i, \sum_{i=1}^{3} g_{i3}A_i\right).$$

This induces a left action of $GL_3(K)$ on the coordinate ring of $M_3(K)^3$: for a polynomial function $f$ on $M_3(K)^3$ and $g \in GL_3(K)$, the function $g \cdot f$ maps $(A_1, A_2, A_3) \in M_3(K)^3$ to $f((A_1, A_2, A_3) \cdot g)$. Since this action of $GL_3(K)$ commutes with the action of $SL_3(K) \times SL_3(K)$ introduced above, $R(K)$ is a $GL_3(K)$-submodule of the coordinate ring of $M_3(K)^3$.

First in Section 2 we treat the case when $K$ is the field $\mathbb{Q}$ of rational numbers. By the theory of polynomial representations of $GL_3(\mathbb{Q})$ one can read off from (1) that $h$ and $q$ can be replaced by $H$ and $Q$ that are *highest weight vectors* with respect to $GL_3(\mathbb{Q})$. In fact $H$ and $Q$ are invariants with respect to the subgroup $SL_3(\mathbb{Q})$ of $GL_3(\mathbb{Q})$ and they are uniquely determined up to non-zero scalar multiples. The relation among the new generators $Q, H, f_{i,j,k}$ takes place in the subalgebra of $SL_3(\mathbb{Q})$-invariants in $R(\mathbb{Q})$. This is a "small" subalgebra of $R(\mathbb{Q})$, and a "large" part of it can be identified with the algebra of $SL_3(\mathbb{C})$-invariants of ternary cubic forms, whose explicit generators $S$ and $T$ are known from a famous classical computation of Aronhold [3]. It is an easy matter to find $H$ and $Q$ explicitly, and then most of the computational difficulty in finding the relation among $Q, H, f_{i,j,k}$ is already contained in Aronhold's computation, so one gets easily the desired relation (cf. Theorem 1).

Rewriting the relation found in Section 2 in terms of our original generators $q$, $h$, $f_{i,j,k}$, we obtain a relation $A(q, h, f_1, \ldots, f_{10}) = 0$ with integer coefficients. This yields a uniform description for $R(K)$ in terms of a minimal generating system and the corresponding defining relations, valid over any infinite base field $K$ and also for $K = \mathbb{Z}$, the ring of integers, see Theorem 3.

The results in Theorems 1 and 3 can be applied to recover in a transparent way known results in three other topics of independent interest. In Remark 2 we mention the connection to the explicit determination of the Jacobian of a cubic curve, and to the description of $SL_3(\mathbb{C}) \times SL_3(\mathbb{C}) \times SL_3(\mathbb{C})$-invariants of tensors in $\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$. Furthermore, in Section 4 we deduce from Theorem 3 the explicit combinatorial description of the ring of conjugation invariants of pairs of $3 \times 3$ matrices. In particular, we recover the complicated relation due to Nakamoto [21] as a simple consequence of our results on $R(K)$. In summary, the complicated relation mentioned above comes from the simple relation in Theorem 1 by specialization and change of variables.

Let us note finally that $R$ is an instance of a semi-invariant algebra of a quiver, and Theorems 1 and 3 give information on the homogeneous coordinate ring of the moduli space of semistable (3, 3)-dimensional representations (cf. [18]) of the generalized Kronecker quiver with three arrows.

## 2. Characteristic zero

Throughout this section we assume that $K = \mathbb{Q}$, the field of rational numbers. (Everything would hold for any characteristic zero base field.) To simplify notation set $R := R(\mathbb{Q})$, $P := P(\mathbb{Q})$. The homogeneous components of $R$ are polynomial $GL_3(\mathbb{Q})$-modules. Recall that given a representation of $GL_3(\mathbb{Q})$ on some vector space and $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}^3$ we say that a non-zero vector $v$ is a weight vector of weight $\alpha$ if $\mathrm{diag}(z_1, z_2, z_3) \cdot v = z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} v$ for all diagonal elements $\mathrm{diag}(z_1, z_2, z_3) \in GL_3(\mathbb{Q})$. A polynomial $GL_3(\mathbb{Q})$-module is completely reducible, and the isomorphism classes of irreducible polynomial $GL_3(\mathbb{Q})$-modules are labeled by partitions $\lambda$ with at most three non-zero parts, i.e., $\lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{N}_0^3$ is a triple of non-negative integers with $\lambda_1 \geq \lambda_2 \geq \lambda_3$. Write $V_\lambda$ for the irreducible polynomial $GL_3(\mathbb{Q})$-module corresponding to $\lambda$. Given a polynomial representation of $GL_3(\mathbb{Q})$, a weight vector is called a *highest weight vector* if it is fixed by all unipotent upper triangular elements in $GL_3(\mathbb{Q})$. Then its weight is necessarily a partition $\lambda$, and it generates a $GL_3(\mathbb{Q})$-submodule isomorphic to $V_\lambda$.

The $GL_3(\mathbb{Q})$-module structure of $R$ is encoded in its 3-variable Hilbert series

$$H(R; t_1, t_2, t_3) := \sum_{\alpha \in \mathbb{N}_0^3} \dim_{\mathbb{Q}}(R_\alpha) t_1^{\alpha_1} t_2^{\alpha_2} t_3^{\alpha_3} \in \mathbb{Z}[[t_1, t_2, t_3]],$$

where $R_\alpha$ denotes the $\alpha$ weight subspace of $R$. From (1) we know that

$$H(R; t_1, t_2, t_3) = \frac{1 + t_1^3 t_2^3 t_3^3}{(1 - t_1^2 t_2^2 t_3^2) \prod_{i+j+k=3} (1 - t_1^i t_2^j t_3^k)}.$$

This shows that up to degree $\leq 6$, the homogeneous components of $R$ coincide with those of $P$. Hence $P$ is a $GL_3(\mathbb{Q})$-submodule in $R$. Denote by $P_0$ the subalgebra of $P$ generated by the $f_{i,j,k}$. For $g = (g_{ij})_{3\times 3} \in GL_3(\mathbb{Q})$ we have

$$\begin{aligned}\sum_{i+j+k=3} t_1^i t_2^j t_3^k f_{i,j,k}((A_1, A_2, A_3) \cdot g) &= \det\left(\sum_{j=1}^3 \left(t_j \sum_{i=1}^3 g_{ij} A_i\right)\right) = \det\left(\sum_{i=1}^3 \left(\sum_{j=1}^3 g_{ij} t_j\right) A_i\right) \\ &= \sum_{l+m+n=3} f_{l,m,n}(A_1, A_2, A_3) \left(\sum_{r=1}^3 g_{1r} t_r\right)^l \left(\sum_{r=1}^3 g_{2r} t_r\right)^m \left(\sum_{r=1}^3 g_{3r} t_r\right)^n,\end{aligned}$$

hence

$$\sum_{i+j+k=3} (g \cdot f_{i,j,k}) t_1^i t_2^j t_3^k = \sum_{l+m+n=3} f_{l,m,n} \left(\sum_{r=1}^3 g_{1r} t_r\right)^l \left(\sum_{r=1}^3 g_{2r} t_r\right)^m \left(\sum_{r=1}^3 g_{3r} t_r\right)^n. \quad (2)$$

So the $f_{i,j,k}$ span a $GL_3(\mathbb{Q})$-submodule, hence $P_0$ is also a $GL_3(\mathbb{Q})$-submodule, and we see from the Hilbert series that the degree 6 homogeneous component of $P_0$ has a $GL_3(\mathbb{Q})$-module direct complement in the degree 6 homogeneous component of $P$ isomorphic to $V_{(2,2,2)}$. Taking the multidegree into account we conclude that there exist unique scalars $\beta_i$, $i = 1, 2, 3, 4$, such that $H := h + \beta_1 f_{2,1,0} f_{0,1,2} + \beta_2 f_{2,0,1} f_{0,2,1} + \beta_3 f_{1,2,0} f_{1,0,2} + \beta_4 f_{1,1,1}^2$ is a highest weight vector (and hence spans the submodule $V_{(2,2,2)}$ mentioned above). To find the values $\beta_i$ note that

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

generate a Zariski dense subgroup in the subgroup of unipotent upper triangular matrices in $GL_3(\mathbb{Q})$. Therefore the condition that $H$ is a highest weight vector is equivalent to the condition that the above two elements of $GL_3(\mathbb{Q})$ fix $H$. This gives a system of linear equations for the $\beta_i$, that can be easily solved (we used CoCoA [8]), and we get that

$$H = h - \frac{1}{3} f_{2,1,0} f_{0,1,2} - \frac{1}{3} f_{2,0,1} f_{0,2,1} + \frac{2}{3} f_{1,2,0} f_{1,0,2} + \frac{1}{12} f_{1,1,1}^2. \quad (3)$$

Denote by $P_+$ the sum of the positive degree homogeneous components of $P$. Then by the above considerations we know that $P_+ R$ is a $GL_3(\mathbb{Q})$-submodule in $R$, and the Hilbert series of $R$ shows that $P_+ R$ has a $GL_3(\mathbb{Q})$-module direct complement isomorphic to $V_{(0)} \oplus V_{(3,3,3)}$ (where $V_{(0)}$ is the trivial $GL_3(\mathbb{Q})$-module). Consequently, there is a unique weight vector $v$ in $P$ of weight $(3, 3, 3)$ (i.e., a multihomogeneous element of multidegree $(3, 3, 3)$) such that $Q := q + v$ is a highest weight vector (and hence spans the submodule $V_{(3,3,3)}$ mentioned above). Solving a small system of linear equations as in the case of $H$ we obtain

$$\begin{aligned}Q &= q - \frac{1}{2} h f_{1,1,1} + \frac{3}{2} f_{3,0,0} f_{0,3,0} f_{0,0,3} - \frac{1}{2} f_{3,0,0} f_{0,2,1} f_{0,1,2} - \frac{1}{2} f_{0,3,0} f_{2,0,1} f_{1,0,2} - \frac{1}{2} f_{0,0,3} f_{2,1,0} f_{1,2,0} \\ &\quad - \frac{1}{2} f_{1,1,1} f_{1,2,0} f_{1,0,2} + \frac{1}{2} f_{2,1,0} f_{1,0,2} f_{0,2,1} + \frac{1}{2} f_{1,2,0} f_{2,0,1} f_{0,1,2}. \end{aligned} \quad (4)$$

It follows from (3), (4) and (1) that $Q$, $H$ and the $f_{i,j,k}$ constitute a minimal generating system of $R$, and that $Q$ satisfies a monic quadratic polynomial with coefficients in $P$. Note that $Q, H \in R^{SL_3(\mathbb{Q})}$. Next we introduce two other distinguished elements $\widetilde{S}$, $\widetilde{T}$ in $R^{SL_3(\mathbb{Q})}$. Eq. (2) shows that the algebraically independent invariants $f_{i,j,k}$ span a $GL_3(\mathbb{Q})$-submodule in $R$ isomorphic to the dual of the space of ternary cubic forms, hence $P_0^{SL_3(\mathbb{Q})}$ is isomorphic to the algebra of $SL_3(\mathbb{Q})$-invariants of ternary cubic forms. The latter was determined in [3], and is generated by two algebraically independent elements $S$ and $T$. Here $S$ and $T$ are homogeneous polynomials of degree four and six in the coefficients of the general cubic ternary form

$$aX^3 + bY^3 + cZ^3 + 3a_2X^2Y + 3a_3X^2Z + 3b_1XY^2 + 3b_3Y^2Z + 3c_1XZ^2 + 3c_2YZ^2 + 6mXYZ.$$

The expressions $S$ and $T$ can be found in [22], in [9, page 160], or in [2]. Now substitute in $S$ and $T$ the coefficients of the general ternary form by the $f_{i,j,k}$ to get elements $\widetilde{S}$ and $\widetilde{T}$ in $P_0$. The exact substitution is given by the following table:

| $a$ | $a_2$ | $a_3$ | $b$ | $b_1$ | $b_3$ | $c$ | $c_1$ | $c_2$ | $m$ |
|---|---|---|---|---|---|---|---|---|---|
| $f_{3,0,0}$ | $\frac{1}{3}f_{2,1,0}$ | $\frac{1}{3}f_{2,0,1}$ | $f_{0,3,0}$ | $\frac{1}{3}f_{1,2,0}$ | $\frac{1}{3}f_{0,2,1}$ | $f_{0,0,3}$ | $\frac{1}{3}f_{1,0,2}$ | $\frac{1}{3}f_{0,1,2}$ | $\frac{1}{6}f_{1,1,1}$ |

By (2) this substitution induces a $GL_3(\mathbb{Q})$-module isomorphism from the dual of the space of ternary cubic forms to the subspace of $R$ spanned by the $f_{i,j,k}$. Hence $\widetilde{S}$, $\widetilde{T}$ are $SL_3(\mathbb{Q})$-invariants in $P_0$, and by Aronhold [3] we have $P_0^{SL_3(\mathbb{Q})} = \mathbb{Q}[\widetilde{S}, \widetilde{T}]$. Moreover, since $H$ is $SL_3(\mathbb{Q})$-invariant, we have $P^{SL_3(\mathbb{Q})} = (P_0[H])^{SL_3(\mathbb{Q})} = P_0^{SL_3(\mathbb{Q})}[H] = \mathbb{Q}[\widetilde{S}, \widetilde{T}, H]$, a three-variable polynomial algebra. Since $Q$ is also $SL_3(\mathbb{Q})$-invariant, we conclude from $R = P \oplus P \cdot Q$ that

$$R^{SL_3(\mathbb{Q})} = P^{SL_3(\mathbb{Q})} \oplus Q \cdot P^{SL_3(\mathbb{Q})} = \mathbb{Q}[\widetilde{S}, \widetilde{T}, H] \oplus Q \cdot \mathbb{Q}[\widetilde{S}, \widetilde{T}, H].$$

Taking the degrees into account it follows that $Q^2 = \alpha H^3 + \beta H\widetilde{S} + \gamma\widetilde{T}$ for some unique scalars $\alpha, \beta, \gamma \in \mathbb{Q}$. The scalars can be easily found by substituting special matrix triples into the above equality: on skew-symmetric triples all the $f_{i,j,k}$ vanish, hence $\widetilde{T}$ and $\widetilde{S}$ vanish. On the other hand, the value of $H$ on the triple

$$\left(\begin{pmatrix} 0 & -x_1 & -y_1 \\ x_1 & 0 & -z_1 \\ y_1 & z_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -x_2 & -y_2 \\ x_2 & 0 & -z_2 \\ y_2 & z_2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -x_3 & -y_3 \\ x_3 & 0 & -z_3 \\ y_3 & z_3 & 0 \end{pmatrix}\right)$$

is $\det^2\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$, whereas the value of $Q$ on this triple is $\det^3\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$. This shows that $\alpha = 1$. Note that

$$\det\begin{pmatrix} t_1 & t_3 & 0 \\ 0 & at_1 + t_2 & -bt_1 + t_3 \\ bt_1 + t_3 & 0 & -at_1 + t_2 \end{pmatrix} = t_3^3 + t_2^2 t_1 - b^2 t_1^2 t_3 - a^2 t_1^3$$

(the Weierstrass canonical form of a plane cubic in homogeneous coordinates $(t_1 : t_2 : t_3)$ on $\mathbb{P}^2$). The values of the invariants $\widetilde{S}, \widetilde{T}, H, Q$ on the corresponding matrix triple

$$\left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & -b \\ b & 0 & -a \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}\right)$$

are $-b^2/27$, $-4a^2/27$, $-b$, $-a$. It follows that $\beta = 27$ and $\gamma = -27/4$. Hence we proved the following:

**Theorem 1.** *We have the equality*

$$Q^2 = H^3 + 27H\widetilde{S} - \frac{27}{4}\widetilde{T} \tag{5}$$

*where $H$ and $Q$ are given explicitly in (3) and (4), and they are characterized (up to non-zero scalar multiples) in $R$ as the unique degree 6 and degree 9 $SL_3(\mathbb{Q})$-invariants in $R$.*

**Remark 2.** (i) The relation (5) essentially coincides with the relation

$$J^2 = 4\Theta^3 + 108S\Theta H^4 - 27TH^6$$

among the basic covariants of plane cubics (cf. [22]). It was observed by Weil [26] that this gives the equation of the Jacobian of a plane cubic, see [2] for a proof. As it is pointed out in the book of Mukai [20, page 430], an alternative approach to this result can be based on the study of $R$ and its defining relation (5). We mention that the results on the equation of the Jacobian of a plane cubic are extended to arbitrary characteristic (including 2 and 3) in [4]. Our results in Section 3 have relevance for this.

(ii) The algebra $R^{SL_3(\mathbb{Q})}$ can be identified with the algebra of $SL_3(\mathbb{Q}) \times SL_3(\mathbb{Q}) \times SL_3(\mathbb{Q})$-invariants on $\mathbb{Q}^3 \otimes \mathbb{Q}^3 \otimes \mathbb{Q}^3$. The arguments above show that this is a three-variable polynomial algebra generated by $\widetilde{S}$, $H$, and $Q$. This result is well known, see [7,24,25,6]. Our results provide an alternative proof, and an alternative interpretation of the basic invariants.

## 3. Relation over the integers

To shorten the expressions, set

$$f_1 := f_{3,0,0}, \quad f_2 := f_{2,1,0}, \quad f_3 := f_{2,0,1}, \quad f_4 := f_{1,2,0}, \quad f_5 := f_{1,1,1},$$
$$f_6 := f_{1,0,2}, \quad f_7 := f_{0,3,0}, \quad f_8 := f_{0,2,1}, \quad f_9 := f_{0,1,2}, \quad f_{10} := f_{0,0,3}.$$

It turns out that $Q^2 - H^3 - 27H\widetilde{S} + \frac{27}{4}\widetilde{T} = A(q, h, f_1, \dots, f_{10})$, where $A$ is a 12-variable polynomial with integer coefficients, given explicitly as follows:

$$\begin{aligned} A(q,h,f_1,\dots,f_{10}) &= q^2 - qhf_5 + 3qf_1f_7f_{10} - qf_1f_8f_9 - qf_2f_4f_{10} + qf_2f_6f_8 + qf_3f_4f_9 - qf_3f_6f_7 - qf_4f_5f_6 \\ &\quad - h^3 + h^2f_2f_9 + h^2f_3f_8 - 2h^2f_4f_6 \\ &\quad + 3hf_1f_4f_8f_{10} - hf_1f_4f_9^2 - 6hf_1f_5f_7f_{10} + hf_1f_5f_8f_9 + 3hf_1f_6f_7f_9 - hf_1f_6f_8^2 \\ &\quad - hf_2^2f_8f_{10} + 3hf_2f_3f_7f_{10} - hf_2f_3f_8f_9 + hf_2f_4f_5f_{10} + hf_2f_4f_6f_9 - hf_2f_6^2f_7 \\ &\quad - hf_3^2f_7f_9 - hf_3f_4^2f_{10} + hf_3f_4f_6f_8 + hf_3f_5f_6f_7 - hf_4^2f_6^2 + 9f_1^2f_7^2f_{10}^2 \\ &\quad - 6f_1^2f_7f_8f_9f_{10} + f_1^2f_7f_9^3 + f_1^2f_8^3f_{10} - 6f_1f_2f_4f_7f_{10}^2 + f_1f_2f_4f_8f_9f_{10} \\ &\quad + 3f_1f_2f_5f_7f_9f_{10} - f_1f_2f_5f_8^2f_{10} + 3f_1f_2f_6f_7f_8f_{10} - 2f_1f_2f_6f_7f_9^2 \\ &\quad + 3f_1f_3f_4f_7f_9f_{10} - 2f_1f_3f_4f_8^2f_{10} + 3f_1f_3f_5f_7f_8f_{10} - f_1f_3f_5f_7f_9^2 - 6f_1f_3f_6f_7^2f_{10} \\ &\quad + f_1f_3f_6f_7f_8f_9 + f_1f_4^3f_{10}^2 - f_1f_4^2f_5f_9f_{10} + f_1f_4^2f_6f_8f_{10} + f_1f_4f_5^2f_8f_{10} \\ &\quad - 3f_1f_4f_5f_6f_7f_{10} + f_1f_4f_6^2f_7f_9 - f_1f_5^3f_7f_{10} + f_1f_5^2f_6f_7f_9 - f_1f_5f_6^2f_7f_8 + f_1f_6^3f_7^2 \\ &\quad + f_2^3f_7f_{10}^2 - 2f_2^2f_3f_7f_9f_{10} + f_2^2f_3f_8^2f_{10} - f_2^2f_4f_6f_8f_{10} - f_2^2f_5f_6f_7f_{10} + f_2^2f_6^2f_7f_9 \\ &\quad - 2f_2f_3^2f_7f_8f_{10} + f_2f_3^2f_7f_9^2 - f_2f_3f_4f_5f_8f_{10} + 4f_2f_3f_4f_6f_7f_{10} + f_2f_3f_5^2f_7f_{10} \\ &\quad - f_2f_3f_5f_6f_7f_9 + f_2f_4^2f_5f_6f_{10} - f_2f_4f_6^3f_7 + f_3^3f_7^2f_{10} + f_3^2f_4^2f_8f_{10} - f_3^2f_4f_5f_7f_{10} \\ &\quad - f_3^2f_4f_6f_7f_9 - f_3f_4^3f_6f_{10} + f_3f_4f_5f_6^2f_7. \end{aligned}$$

For $i, j, k \in \{1, 2, 3\}$ denote by $x_{ij}^{(r)}$ the coordinate function on $M_3(\mathbb{Q})^3$ mapping the matrix triple $(A_1, A_2, A_3)$ to the $(i, j)$-entry of $A_r$. Then $R(\mathbb{Q})$ contains the subring

$$R(\mathbb{Z}) := R(\mathbb{Q}) \cap \mathbb{Z}[x_{ij}^{(r)} \mid i, j, r = 1, 2, 3].$$

**Theorem 3.** *Let $K$ be an infinite field or the ring of integers. Then $R(K)$ is minimally generated as a $K$-algebra by the twelve elements $q, h, f_j$, $j = 1, \dots, 10$, satisfying the single algebraic relation $A(q, h, f_1, \dots, f_{10}) = 0$ (where $A$ is given explicitly above). Moreover, $R(K)$ is a free module with basis $1, q$ over its $K$-subalgebra generated by the eleven algebraically independent elements $h, f_1, \dots, f_{10}$.*

**Proof.** We know already from [10] and Section 2 that the statement holds when $K$ is a field of characteristic zero. We also know already that for any $K$, the given twelve elements satisfy the relation $A(q, h, f_1, \dots, f_{10}) = 0$.

Suppose next that $K$ is an infinite field of positive characteristic. We claim that 1 and $q$ generate a free $P(K)$-submodule in $R(K)$. Indeed, otherwise $q$ belongs to the field of fractions of $P(K)$. By the above relation $q$ is integral over $P(K)$. Since $P(K)$ is a unique factorization domain, it follows that $q$ belongs to $P(K)$. Taking the grading of $R$ into account, we conclude that $q = hc + d$, where $c$ is a linear combination of the $f_i$, and $d$ is a cubic polynomial in the $f_i$. Now substitute into this equality a triple $(A_1, A_2, A_3)$, where the $A_i$ constitute a basis of the space of $3 \times 3$ skew-symmetric matrices. All the $f_i$ vanish on this triple, hence $(hc + d)(A_1, A_2, A_3) = 0$, whereas $q$ does not vanish on this triple as we pointed out in Section 2. So $P(K) \oplus P(K)q \subseteq R(K)$. It follows from the theory of modules with good filtration (cf. [16, page 399]) that the Hilbert series of $R(K)$ coincides with the Hilbert series of $R(\mathbb{Q})$. We know already that the latter coincides with the Hilbert series of $P(K) \oplus P(K)q$, hence we have the equality $R(K) = P(K) \oplus P(K)q$. This shows both the statement on the generators and the relation.

Finally we turn to $R(\mathbb{Z})$. Denote by $P(\mathbb{Z})$ the $\mathbb{Z}$-subalgebra of $R(\mathbb{Z})$ generated by the eleven elements $h, f_1, \dots, f_{10}$. From the case $K = \mathbb{Q}$ we know that $P(\mathbb{Z})$ is a polynomial ring, and $R(\mathbb{Z})$ contains the free $P(\mathbb{Z})$-submodule $M := P(\mathbb{Z}) \oplus P(\mathbb{Z})q$. Take any $f \in R(\mathbb{Z})$. It follows from the case $K = \mathbb{Q}$ that some positive integer multiple $mf$ of $f$ belongs to $M$, so $mf = c + dq$, where $c, d \in P(\mathbb{Z})$. We may assume that $m$ is minimal. If $m \neq 1$, then let $p$ be a prime divisor of $m$, and let $L$ be an infinite field of characteristic $p$. Reduction mod $p$ of coefficients gives a ring homomorphism $\pi : Z := \mathbb{Z}[x_{ij}^{(r)} \mid i, j, r = 1, 2, 3] \to L[M_3(L)^3]$, and this restricts to a ring homomorphism $\pi : R(\mathbb{Z}) \to R(L)$ and $\pi : P(\mathbb{Z}) \to P(L)$. Since $\pi(mf) = 0$, we get that $\pi(c) + \pi(d)q = 0$ holds in $R(L)$. From the case $K = L$ of our theorem we know that $1, q$ are independent over $P(L)$, hence $\pi(c) = 0$ and $\pi(d) = 0$, i.e. $c, d \in P(\mathbb{Z}) \cap pZ$. Clearly $P(\mathbb{Z}) \cap pZ = pP(\mathbb{Z})$, since the eleven generators of $P(\mathbb{Z})$ are mapped under $\pi$ to algebraically independent elements of $R(L)$. Consequently, $(m/p)f \in M$, contradicting the minimality of $m$. Thus we have proved the equality $R(\mathbb{Z}) = P(\mathbb{Z}) \oplus P(\mathbb{Z})q$. This implies both the statement on the generators of $R(\mathbb{Z})$ and the statement on the relation. □

**Remark 4.** The fact that a minimal $\mathbb{Z}$-algebra generating system of $R(\mathbb{Z})$ stays a minimal $K$-algebra generating system of $R(K)$ when exchanging the base ring to any infinite field $K$ is accidental, and the analogous property does not hold in general in similar situations. For example, denote by $R_{n,m}(K)$ the ring of $SL_n(K) \times SL_n(K)$-invariants of $m$-tuples of $n \times n$ matrices. It is proved in [15] that the method we used to construct generators in the special case $n = m = 3$ (i.e. polarization of the determinant of block matrices) yields in general an (infinite) generating system of $R_{n,m}(K)$ for any infinite base field $K$, hence also for $K = \mathbb{Z}$. (The latter claim follows in the same way as it is explained by Donkin [16] in a related situation.) However, Proposition 5 below and the results of [14] imply that if $m$ is sufficiently large, then a minimal $\mathbb{Z}$-algebra generating system of $R_{n,m}(\mathbb{Z})$ becomes redundant over fields $K$ whose characteristic is zero or greater than $n$.

## 4. Conjugation invariants of pairs of 3 × 3 matrices

The general linear group $GL_3(K)$ acts on $M_3(K)^2 = M_3(K) \oplus M_3(K)$ by simultaneous conjugation: for $g \in GL_3(K)$ and $A, B \in M_3(K)$ we set $g \cdot (A, B) = (gAg^{-1}, gBg^{-1})$. For any infinite field $K$ denote by $U(K) := K[M_3(K)^2]^{GL_3(K)}$ the corresponding algebra of invariants. Similarly to Section 3, consider

$$U(\mathbb{Z}) := U(\mathbb{Q}) \cap \mathbb{Z}[x_{ij}^{(r)} \mid i, j = 1, 2, 3;\ r = 1, 2]$$

where $x_{ij}^{(r)}$ is the coordinate function assigning to the pair $(A_1, A_2) \in M_3(\mathbb{Q})^2$ the $(i, j)$-entry of $A_r$. A minimal system of generators of $U(K)$ was given by Teranishi [23] when $\mathrm{char}(K) = 0$; Nakamoto [21] extended the result for any infinite base field $K$ or $K = \mathbb{Z}$, and determined the single defining relation among the generators. An exact description of $U(K)$ can also be obtained from Theorem 3, using the following statement (proved in [10, Proposition 4.1]):

**Proposition 5.** *The specialization $x_{ij}^{(3)} \mapsto \delta_j^i$ (where $\delta_j^i = 1$ if $i = j$ and $\delta_j^i = 0$ otherwise) induces a surjection $\varphi : R(K) \to U(K)$.*

**Corollary 6.** *Let $K$ be an infinite field or the ring of integers. Then $U(K)$ is minimally generated as a $K$-algebra by the eleven elements $\varphi(q), \varphi(h), \varphi(f_j), j = 1, \ldots, 9$, satisfying the single algebraic relation $A(\varphi(q), \varphi(h), \varphi(f_1), \ldots, \varphi(f_9), 1) = 0$ (where $A$ is given explicitly in Section 3). Moreover, $R(K)$ is a free module with basis $1, \varphi(q)$ over its $K$-subalgebra generated by the ten algebraically independent elements $\varphi(h), \varphi(f_1), \ldots, \varphi(f_9)$.*

**Proof.** First we express the $\varphi$-images of the generators of $R(K)$ in terms of the usual generators of $U(K)$. Define the functions $t, s, d$ on $M_3(K)$ by the equality

$$\det(zI + A) = z^3 + t(A)z^2 + s(A)z + d(A),$$

where $I$ is the $3 \times 3$ identity matrix and $z \in K$ arbitrary. One has the equality

$$s(AB) = t(A^2B^2) + t(AB)t(A)t(B) - t(A^2B)t(B) - t(AB^2)t(A) - s(A)s(B)$$

for $A, B \in M_3(K)$ (see [12, Lemma 2] for a generalization). Furthermore, we have

$$\begin{aligned}
&\varphi(f_{3,0,0})(A, B) = d(A), \qquad \varphi(f_{0,3,0})(A, B) = d(B), \qquad \varphi(f_{0,0,3})(A, B) = 1,\\
&\varphi(f_{2,0,1})(A, B) = s(A), \qquad \varphi(f_{0,2,1})(A, B) = s(B),\\
&\varphi(f_{1,0,2}(A, B) = t(A), \qquad \varphi(f_{0,1,2})(A, B) = t(B),\\
&\varphi(f_{1,1,1})(A, B) = t(A)t(B) - t(AB),\\
&\varphi(f_{2,1,0})(A, B) = t(A^2B) - t(AB)t(A) + s(A)t(B),\\
&\varphi(f_{1,2,0})(A, B) = t(AB^2) - t(AB)t(B) + t(A)s(B).
\end{aligned}$$

Applying Amitsur's formula [1] one gets

$$\begin{aligned}
&\varphi(h)(A, B) = -t(A^2B^2) + t(A^2B)t(B) - t^2(A)s(B) + 2s(A)s(B),\\
&\varphi(q)(A, B) = t(B^2A^2BA) - s(A)s(B)t(AB) - t(A^2B)t(AB)t(B) - t(AB^2)t(AB)t(A) + t^2(AB)t(A)t(B).
\end{aligned}$$

Therefore by Theorem 3 and Proposition 5, the eleven elements

$$t(A), s(A), d(A), t(B), s(B), d(B), t(AB), t(A^2B), t(AB^2), t(A^2B^2), t(B^2A^2BA) \tag{6}$$

generate $U(K)$. Moreover, $t(B^2A^2BA)$ satisfies a monic quadratic relation over the subalgebra $W(K)$ of $U(K)$ generated by the first ten elements. Since by general principles on group actions, the transcendence degree of $U(K)$ is ten, the first ten generators are algebraically independent. Moreover, $t(B^2A^2BA)$ does not vanish on the pair $(E_{21} - E_{32}, E_{12} + E_{23})$ (where $E_{ij}$ is the matrix unit whose only non-zero entry is a 1 in the $(i, j)$-position), whereas all the first nine generators vanish on this pair. Since the tenth generator has degree 4 and the eleventh generator has degree 6, it follows that $t(B^2A^2BA)$ is not contained in $W(K)$. Hence by the integral closedness of $W(K)$ and by the quadratic relation we conclude that $U(K) = W(K) \oplus t(B^2A^2BA)W(K)$. The statements in our corollary obviously follow. □

**Corollary 7.** *When $K$ is an infinite field with $\mathrm{char}(K) \neq 2$ or 3, then the algebra $U(K)$ is minimally generated by $\varphi(Q)$, $\varphi(H)$, $\varphi(f_1), \ldots, \varphi(f_9)$, and these generators satisfy the single algebraic relation*

$$\varphi(Q)^2 = \varphi(H)^3 + 27\varphi(H)\varphi(\widetilde{S}) - \frac{27}{4}\varphi(\widetilde{T}).$$

**Remark 8.** (i) Expressing the left-hand side of $A(\varphi(q), \varphi(h), \varphi(f_1), \ldots, \varphi(f_9), 1) = 0$ in terms of the generators (6) we obtain a transparent derivation of the relation found originally by hard computational labor by Nakamoto [21]. We include this relation in the Appendix.

(ii) The form of the relation in Corollary 7 is rather simple (or better to say that the complication is built into the nineteenth century expressions for $S$ and $T$ due to [3]): indeed, the quartic or sextic generators appear only in three terms, and the remaining 9 generators appear only in two prominent classically known (though complicated) expressions.

(iii) We note that working over a characteristic zero base field, another minimal generating system of $U(K)$ is found in [5], such that the relation between them takes a simpler form than the relation in [21]. This relation is of a different nature than the one in Corollary 7.

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**Appendix**

Setting

$$t_1 := t(A), \quad s_1 := s(A), \quad d_1 := d(A), \quad t_2 := t(B), \quad s_2 := s(B), \quad d_2 := d(B)$$
$$r := t(B^2A^2BA), \quad k := t(A^2B^2), \quad w_1 := t(A^2B), \quad w_2 := t(AB^2), \quad z := t(AB)$$

we have

$$\begin{aligned}
0 = {} & r^2 - rkz + rkt_1t_2 - rw_1w_2 - rw_1t_1t_2^2 - rw_2t_1^2t_2 + rzt_1^2t_2^2 + 3rd_1d_2 - rd_1s_2t_2 - rd_2s_1t_1 - rs_1s_2t_1t_2 \\
& + k^3 - 2k^2w_1t_2 - 2k^2w_2t_1 + k^2zt_1t_2 - 5k^2s_1s_2 + k^2s_1t_2^2 + k^2s_2t_1^2 \\
& + kw_1^2s_2 + kw_1^2t_2^2 + kw_1w_2z + 2kw_1w_2t_1t_2 - kw_1zs_2t_1 - kw_1zt_1t_2^2 - 3kw_1d_2s_1 \\
& + kw_1d_2t_1^2 + 9kw_1s_1s_2t_2 - 2kw_1s_1t_2^3 - 2kw_1s_2t_1^2t_2 + kw_2^2s_1 + kw_2^2t_1^2 - kw_2zs_1t_2 \\
& - kw_2zt_1^2t_2 - 3kw_2d_1s_2 + kw_2d_1t_2^2 + 9kw_2s_1s_2t_1 - 2kw_2s_1t_1t_2^2 - 2kw_2s_2t_1^3 + kz^2s_1s_2 \\
& - 6kzd_1d_2 + 4kzd_1s_2t_2 - kzd_1t_2^3 + 4kzd_2s_1t_1 - kzd_2t_1^3 - 8kzs_1s_2t_1t_2 + 2kzs_1t_1t_2^3 \\
& + 2kzs_2t_1^3t_2 + 3kd_1d_2t_1t_2 - 2kd_1s_2^2t_1 - 2kd_2s_1^2t_2 + 8ks_1^2s_2^2 - 2ks_1^2s_2t_2^2 - 2ks_1s_2^2t_1^2 \\
& + w_1^3d_2 - w_1^3s_2t_2 - w_1^2w_2s_2t_1 - 2w_1^2zd_2t_1 + 2w_1^2zs_2t_1t_2 + 4w_1^2d_2s_1t_2 - w_1^2d_2t_1^2t_2 \\
& - w_1^2s_1s_2^2 - 4w_1^2s_1s_2t_2^2 + w_1^2s_1t_2^4 + w_1^2s_2t_1^2t_2^2 - w_1w_2^2s_1t_2 + w_1w_2zs_1t_2^2 + w_1w_2zs_2t_1^2 \\
& - 6w_1w_2d_1d_2 + 4w_1w_2d_1s_2t_2 - w_1w_2d_1t_2^3 + 4w_1w_2d_2s_1t_1 - w_1w_2d_2t_1^3 - 8w_1w_2s_1s_2t_1t_2 \\
& + 2w_1w_2s_1t_1t_2^3 + 2w_1w_2s_2t_1^3t_2 + w_1z^2d_2s_1 + w_1z^2d_2t_1^2 - w_1z^2s_1s_2t_2 - w_1z^2s_2t_1^2t_2 \\
& + 6w_1zd_1d_2t_2 + w_1zd_1s_2^2 - 4w_1zd_1s_2t_2^2 + w_1zd_1t_2^4 - 8w_1zd_2s_1t_1t_2 + 2w_1zd_2t_1^3t_2 \\
& + w_1zs_1s_2^2t_1 + 8w_1zs_1s_2t_1t_2^2 - 2w_1zs_1t_1t_2^4 - 2w_1zs_2t_1^3t_2^2 - 3w_1d_1d_2s_2t_1 - 2w_1d_1d_2t_1t_2^2 \\
& + 2w_1d_1s_2^2t_1t_2 + 4w_1d_2s_1^2s_2 + 2w_1d_2s_1^2t_2^2 - w_1d_2s_1s_2t_1^2 - 8w_1s_1^2s_2^2t_2 + 2w_1s_1^2s_2t_2^3 \\
& + 2w_1s_1s_2^2t_1^2t_2 + w_2^3d_1 - w_2^3s_1t_1 - 2w_2^2zd_1t_2 + 2w_2^2zs_1t_1t_2 + 4w_2^2d_1s_2t_1 - w_2^2d_1t_1t_2^2 \\
& - w_2^2s_1^2s_2 - 4w_2^2s_1s_2t_1^2 + w_2^2s_1t_1^2t_2^2 + w_2^2s_2t_1^4 + w_2z^2d_1s_2 + w_2z^2d_1t_2^2 - w_2z^2s_1s_2t_1 \\
& - w_2z^2s_1t_1t_2^2 + 6w_2zd_1d_2t_1 - 8w_2zd_1s_2t_1t_2 + 2w_2zd_1t_1t_2^3 + w_2zd_2s_1^2 - 4w_2zd_2s_1t_1^2 \\
& + w_2zd_2t_1^4 + w_2zs_1^2s_2t_2 + 8w_2zs_1s_2t_1^2t_2 - 2w_2zs_1t_1^2t_2^3 - 2w_2zs_2t_1^4t_2 - 3w_2d_1d_2s_1t_2 \\
& - 2w_2d_1d_2t_1^2t_2 + 4w_2d_1s_1s_2^2 - w_2d_1s_1s_2t_2^2 + 2w_2d_1s_2^2t_1^2 + 2w_2d_2s_1^2t_1t_2 - 8w_2s_1^2s_2^2t_1 \\
& + 2w_2s_1^2s_2t_1t_2^2 + 2w_2s_1s_2^2t_1^3 + z^3d_1d_2 - z^3d_1s_2t_2 - z^3d_2s_1t_1 + z^3s_1s_2t_1t_2 - 5z^2d_1d_2t_1t_2 \\
& + 4z^2d_1s_2t_1t_2^2 - z^2d_1t_1t_2^4 + 4z^2d_2s_1t_1^2t_2 - z^2d_2t_1^4t_2 - z^2s_1^2s_2^2 - 4z^2s_1s_2t_1^2t_2^2 + z^2s_1t_1^2t_2^4 \\
& + z^2s_2t_1^4t_2^2 + 6zd_1d_2s_1s_2 + zd_1d_2s_1t_2^2 + zd_1d_2s_2t_1^2 + 2zd_1d_2t_1^2t_2^2 - 4zd_1s_1s_2^2t_2 \\
& + zd_1s_1s_2t_2^3 - 2zd_1s_2^2t_1^2t_2 - 4zd_2s_1^2s_2t_1 - 2zd_2s_1^2t_1t_2^2 + zd_2s_1s_2t_1^3 + 8zs_1^2s_2^2t_1t_2 \\
& - 2zs_1^2s_2t_1t_2^3 - 2zs_1s_2^2t_1^3t_2 + 9d_1^2d_2^2 - 6d_1^2d_2s_2t_2 + d_1^2d_2t_2^3 + d_1^2s_2^3 - 6d_1d_2^2s_1t_1 + d_1d_2^2t_1^3 \\
& - 2d_1d_2s_1s_2t_1t_2 + 2d_1s_1s_2^3t_1 + d_2^2s_1^3 + 2d_2s_1^3s_2t_2 - 4s_1^3s_2^3 + s_1^3s_2^2t_2^2 + s_1^2s_2^3t_1^2.
\end{aligned}$$

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