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A Primitive Non-symmetric 3-Class Association Scheme on 36 Elements with $p_{11}^1 = 0$ Exists and is Unique

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In this paper we construct a primitive, non-symmetric 3-class association scheme with parameters $v = 36$, $v_1 = 7$, $p_{11}^1 = 0$ and $p_{11}^2 = 4$ and show that such a scheme is determined by its parameters.

1. Introduction

In [4] we gave a set of necessary conditions in order that a symmetric 2-scheme (we call an n-class association scheme briefly an n-scheme) is the symmetric closure of a non-symmetric 3-scheme. There we found that it is 'feasible' that a primitive symmetric 2-scheme $(\mathbf{X}, \bar{\mathbf{R}})$ of type $NL_2(6)$ (that is, a scheme with parameter $v = 36$, $\bar{v}_1 = 14$ and $\bar{p}_{11}^1 = 4$) can be 'split' into a non-symmetric 3-scheme with the parameters mentioned in the abstract. We set out to construct the latter scheme and from its construction its uniqueness follows.

We mention here that there exist two imprimitive non-symmetric 3-schemes $(\mathbf{X}, \mathbf{R})$ on 36 elements with $p_{11}^1 = 0$. Of the first one the graph of the first relation is the union of 12 directed 3-circuits. The graph of the first relation of the other scheme can be described as follows. Let $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ be three subsets of $\mathbf{X}$, each of cardinality 12 and forming a partition of $\mathbf{X}$. If $a \in \mathbf{A}$, $b \in \mathbf{B}$ and $c \in \mathbf{C}$, then $(a, b) \in R_1$, $(b, c) \in R_1$ and $(c, a) \in R_1$. Imprimitive non-symmetric 3-schemes are discussed in [5, 6].

For more details on non-symmetric 3-schemes we refer to [4, 5]. We shall use the notation of Delsarte as it was introduced for association schemes in [3].

2. Preliminaries

Definition 2.1. Let $\mathbf{X}$ be a set with v elements. Let $\mathbf{R} = \{R_0, R_1, \ldots, R_n\}$ be a family of $n + 1$ binary relations on $\mathbf{X}$. The pair $(\mathbf{X}, \mathbf{R})$ will be called an *association scheme with n classes* (also called an *n-scheme*) if the following conditions are satisfied:
(1) the family $\mathbf{R}$ is a partition of $\mathbf{X}^2$ and R_0 is the diagonal (equality) relation;
(2) for any $i \in \{0, 1, \ldots, n\}$ the inverse $R_i^{-1} = \{(y, x) \mid (x, y) \in R_i\}$ of the relation R_i belongs to $\mathbf{R}$ (the index of the relation R_i^{-1} is denoted by i_R);
(3) for $i, j, k \in \{0, 1, \ldots, n\}$ the so-called *intersection numbers*

$$p_{ij}^k = |\{z \in \mathbf{X} \mid (x, z) \in R_i, (z, y) \in R_j\}|$$

are independent of the choice of $(x, y) \in R_k$;
(4) for all $i, j, k \in \{0, 1, \ldots, n\}$ we have $p_{ij}^k = p_{ji}^k$.

For every i the number $p_{ii_R}^0$ is called the *valency* of R_i and is denoted by v_i. An

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association scheme $(\mathbf{X}, \mathbf{R})$ is called *symmetric* if all its relations are symmetric, i.e. $i = i_R$ for all i; otherwise it is called *non-symmetric.* We denote the symmetric closure of an n-scheme $(\mathbf{X}, \mathbf{R})$ by $(\mathbf{X}, \bar{\mathbf{R}})$ (here $\bar{\mathbf{R}} = \{R \cup R^{-1} \mid R \in \mathbf{R}\}$). The *adjacency matrix* of the relation R_i is denoted by D_i, while the $n+1$ *maximal common eigenspaces* of $(\mathbf{X}, \mathbf{R})$ are denoted by $\mathbf{V}_k$. L_i is the matrix with (k, j)-entry p^k_{ij}. I and J denote the identity matrix and the all-one matrix, respectively, of dimensions determined by the context.

From now on in this paper $(\mathbf{X}, \bar{\mathbf{R}})$ denotes a symmetric 2-scheme and its parameters are provided with a bar. $(\mathbf{X}, \mathbf{R})$ denotes a non-symmetric 3-scheme. We assume throughout this paper that $R_2 = R_1^{-1}$ and $\mathbf{V}_2^* = \mathbf{V}_1$ (following Delsarte, $\mathbf{V}_2^*$ denotes the space of all the complex conjugates of the vectors in $\mathbf{V}_2$).

In this paper we shall use the following shorthand notation for the parameters of $(\mathbf{X}, \mathbf{R})$: $u = v_1/v_3$, $\alpha = p^1_{11}$, $\beta = p^2_{11}$, $\gamma = p^1_{33}$, $\delta = p^1_{13}$, $\varepsilon = p^1_{23}$ and $\lambda = p^3_{33}$.

Using $L_iL_j = L_jL_i$, one proves that the matrices L_i of $(\mathbf{X}, \mathbf{R})$ have the following form: $L_0 = I$ and

$$L_1 = \begin{pmatrix} 0 & 0 & v_1 & 0 \\ 1 & \alpha & \alpha & \delta \\ 0 & \beta & \alpha & \varepsilon \\ 0 & u\varepsilon & u\delta & u\gamma \end{pmatrix}, \qquad L_2 = \begin{pmatrix} 0 & v_1 & 0 & 0 \\ 0 & \alpha & \beta & \varepsilon \\ 1 & \alpha & \alpha & \delta \\ 0 & u\delta & u\varepsilon & u\gamma \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & 0 & 0 & v_3 \\ 0 & \delta & \varepsilon & \gamma \\ 0 & \varepsilon & \delta & \gamma \\ 1 & u\gamma & u\gamma & \lambda \end{pmatrix}.$$

Let $(\mathbf{X}, \mathbf{R})$ be such that $(\mathbf{X}, \bar{\mathbf{R}})$ is its symmetric closure then we call $(\mathbf{X}, \mathbf{R})$ a *splitting* of $(\mathbf{X}, \bar{\mathbf{R}})$. We denote by s the index such that $\bar{R}_s = R_1 \cup R_2$ and by n the index such that $\bar{R}_n = R_3$.

In [4] we gave a set of necessary conditions in order that it is feasible that a symmetric 2-scheme can be split into a non-symmetric 3-scheme (the 'feasibility conditions'). We call a splitting *feasible* if the parameters of $(\mathbf{X}, \bar{\mathbf{R}})$ satisfy those conditions, and the splitting is called *realizable* if the splitting exists.

The feasibility conditions are such that the parameters of $(\mathbf{X}, \mathbf{R})$ can be calculated from its symmetric closure once the feasibility of the splitting has been established.

THEOREM 2.2. *Let $(\mathbf{X}, \bar{\mathbf{R}})$ be a symmetric 2-scheme the splitting of which is feasible. Then that splitting is realizable iff there are two matrices D_1 and D_2 of order v and with entries 0 and 1 such that*:
(1) $D_2^T = D_1$;
(2) $\bar{D}_s = D_1 + D_2$:
(3) $\bar{D}_s(D_1 - D_2) = (\alpha - \beta)(D_1 - D_2)$;
(4) $(D_1 - D_2)^2 = -v_s\bar{D}_0 - (\alpha - \beta)\bar{D}_s - 2u(\delta - \varepsilon)\bar{D}_n$.

PROOF. If the splitting is realizable then the conditions are obviously met.

So let D_1 and D_2 be two matrices satisfying the given conditions. By taking the transposes we find from the third condition $D_1D_2 = D_2D_1$. Since an entry of $D_1 - D_2$ is 0 iff the corresponding entry of $\bar{D}_s$ is 0, we derive from (3) (considering the diagonal entries) $JD_1 = JD_2$, while (2) implies $JD_1 + JD_2 = \bar{v}_sJ$ and so $JD_1 = D_1J = JD_2 = D_2J$. Therefore $D_iD_3 = D_3D_i$ for $i = 1, 2$, where $D_3 = J - I - D_1 - D_2$. Since $(D_1 + D_2)^2$ and $(D_1 - D_2)^2$ belong to the linear space $\mathbf{A}$, generated by I, D_1, D_2 and D_3 also $D_1D_2 \in \mathbf{A}$.

This implies that **A** is a commutative, normal algebra and therefore corresponds in this case to a non-symmetric 3-scheme; cf. [3, Theorem 2.1]. □

3. The Main Result

In this section we assume that $(\mathbf{X}, \mathbf{R})$ is a non-symmetric 3-scheme with parameters $v = 36$, $v_1 = 7$, $\alpha = 0$ and $\beta = 4$. From the elementary properties of an association scheme one derives $v_3 = 21$, $\gamma = 12$, $\delta = 6$, $\varepsilon = 3$, $\lambda = 12$, $u\gamma = 4$, $u\delta = 2$ and $u\varepsilon = 1$.

For any given $a \in \mathbf{X}$ we define $\mathbf{Out}(a) = \{d \in \mathbf{X} \mid (a, d) \in R_1\}$, $\mathbf{In}(a) = \{d \in \mathbf{X} \mid (a, d) \in R_2\}$ and $\mathbf{Not}(a) = \{d \in \mathbf{X} \mid (a, d) \in R_3\}$. Obviously, $|\mathbf{Out}(a)| = |\mathbf{In}(a)| = 7$ and $|\mathbf{Not}(a)| = 21$ for all $a \in \mathbf{X}$.

Throughout this section $x \in \mathbf{X}$, fixed. Then y with or without sub- or superscripts denotes an element of $\mathbf{Out}(x)$, z with or without sub- or superscripts denotes an element of $\mathbf{In}(x)$, and e and f with or without sub- or superscripts denote elements of $\mathbf{Not}(x)$.

$$(y, y'), (z, z') \in R_3 \quad \text{and} \quad (y, z) \in R_1 \cup R_3 \quad \text{all follow from} \quad \alpha = 0.$$

Lemma 3.1. $|\mathbf{In}(x) \cap \mathbf{Out}(y) \cap \mathbf{Out}(y')| = |\mathbf{Out}(x) \cap \mathbf{In}(z) \cap \mathbf{In}(z')| = 2.$

Proof. Since $(y, y') \in R_3$, $|\mathbf{Out}(y) \cap \mathbf{Out}(y')| = u\delta = 2$ and, from $u\varepsilon = 1$, $|\mathbf{Not}(x) \cap \mathbf{Out}(y) \cap \mathbf{Out}(y')| = 0$ follows. This implies the first assertion, while the second one follows analogously. □

Corollary 3.2. *The design* $(\mathbf{Out}(x), \mathbf{In}(x), R_3)$ *is isomorphic to the finite geometry* $PG(2, 2)$.

Proof. Since $\beta = 4$, $|\mathbf{In}(x) \cap \mathbf{Not}(y)| = |\mathbf{Out}(x) \cap \mathbf{Not}(z)| = 3$. Using this and Lemma 3.1, we obtain

$$|\mathbf{In}(x) \cap \mathbf{Not}(y) \cap \mathbf{Not}(y')| = |\mathbf{Out}(x) \cap \mathbf{Not}(z) \cap \mathbf{Not}(z')| = 1. \qquad \square$$

Henceforth we shall also consider $\mathbf{Out}(x)$ and $\mathbf{In}(x)$ as the sets of the points and the lines of $PG(2,2)$, respectively. So y and z are incident in $PG(2, 2)$ (that is, they form a *flag*) iff $(y, z) \in R_3$.

Lemma 3.3. $|\mathbf{Not}(x) \cap \mathbf{In}(y) \cap \mathbf{In}(y')| = |\mathbf{Not}(x) \cap \mathbf{Out}(z) \cap \mathbf{Out}(z')| = 1.$

Proof. Since $u\delta = 2$ and $|\mathbf{Not}(x)| = 21$, there corresponds to each of the 21 subsets $\{y, y'\}$ from $\mathbf{Out}(x)$ exactly one element $e \in \mathbf{Not}(x)$ such that $e \in \mathbf{In}(y) \cap \mathbf{In}(y')$. This effectively proves the lemma. □

The unique element $e \in \mathbf{Not}(x) \cap \mathbf{In}(y) \cap \mathbf{In}(y')$ will be denoted by $\mathbf{in}(y, y')$ and the unique element $f \in \mathbf{Not}(x) \cap \mathbf{Out}(z) \cap \mathbf{Out}(z')$ will be denoted by $\mathbf{out}(z, z')$.

The next lemma is an immediate consequence of $\alpha = 0$ and $u\varepsilon = 1$.

Lemma 3.4. *For any given* e, $|\mathbf{Out}(x) \cap \mathbf{In}(e)| = \mathbf{In}(x) \cap \mathbf{Out}(e)| = 1$ *holds and if* $y \in \mathbf{Out}(x) \cap \mathbf{In}(e)$ *and* $z \in \mathbf{In}(x) \cap \mathbf{Out}(e)$ *then* $(y, z) \in R_3$.

The unique elements in $\mathbf{Out}(x) \cap \mathbf{In}(e)$ and $\mathbf{In}(x) \cap \mathbf{Out}(e)$ will be denoted by y^e and z^e, respectively. Once $(y, z) \in R_3$ is given there is exactly one e such that $y = y^e$ and $z = z^e$. So the 21 elements of $\mathbf{Not}(x)$ can also be characterized as the flags of $PG(2, 2)$.

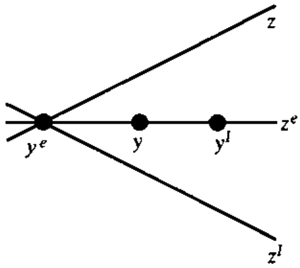

FIGURE 1. The points and lines in $PG(2,2)$ connected with $e \in \mathbf{Not}(x)$.

LEMMA 3.5. *If* $e = \mathbf{in}(y, y') = \mathbf{out}(z, z')$, *then* y, y' *and* y^e *are incident with* z^e *and* z, z' *and* z^e *are incident with* y^e.

PROOF. Since $(e, y) \in R_1$, $(e, y') \in R_1$ and $(e, z^e) \in R_1$, we find $(y, z^e) \in R_3$ and $(y', z^e) \in R_3$. But now it is easy to complete the proof of the lemma. □

Since y^e and z^e are incident, we find by Lemma 3.5 that y and y' are not incident with z or z'. So the situation is as in Figure 1. Plainly, each one of the three sets $\{y, y'\}$, $\{z, z'\}$ and $\{y^e, z^e\}$ determines e uniquely. Note that $(e, y_0) \in R_3$ if $y_0 \neq y, y', y^e$.

LEMMA 3.6. *The following hold*:
(1) $(e, f) \in R_1$ *if* $y^e \neq y^f$, $z^e \neq z^f$ *and* $(y^e, z^f) \in R_3$; *in that case* $(y^f, z^e) \in R_1$.
(2) $(e, f) \in R_3$ *in the case that either* $y^e = y^f$ *or* $z^e = z^f$ *or in the case that* $y^e \neq y^f$, $z^e \neq z^f$ *and both* $(y^e, z^f) \in R_1$ *and* $(y^f, z^e) \in R_1$.

PROOF. Let $e = \mathbf{in}(y_i, y_j) = \mathbf{out}(z_k, z_l)$ and $f = \mathbf{in}(y_m, y_n) = \mathbf{out}(z_p, z_q)$.
Since y^e is incident with z^f, $y^e \neq y^f$ and $z^e \neq z^f$ we find $y^e \in \{y_m, y_n\}$ and $z^f \in \{z_k, z_l\}$. But this implies that there are two directed paths $f \to z^f \to e$ and $f \to y^e \to e$ and so $(e, f) \in R_1$, necessarily.
If either $y^e = y^f$ or $z^e = z^f$ then one simply checks that $(e, f) \in R_3$ $(\alpha = 0)$. If neither y^e is incident with z^f nor is y^f incident with z^e we may assume that z^e and z^f intersect in $y_i = y_m$. But $(e, y_i) \in R_1$ and $(f, y_m) \in R_1$; hence $(e, f) \in R_3$. □

As $PG(2,2)$ is unique, the next theorem follows immediately from the results of this section.

THEOREM 3.7. *If* $(\mathbf{X}, \mathbf{R})$ *exists then it is uniquely determined by its parameters.*

In order to show that there exists a non-symmetric 3-scheme with the parameters $v = 36$, $v_1 = 7$, $\alpha = 0$ and $\beta = 4$, it is obvious what to do. Let $\mathbf{X}$ be the set consisting of the points and the lines of a $PG(2,2)$, of the flags of this geometry and of an element x not equal to the afore-mentioned entities. Then we define a set $\{R_0, R_1, R_2, R_3\}$ of four relations on $\mathbf{X}$ in accordance with the results that we found in the first part of this section.

Now form a square matrix D of order 36 as follows. For $a, b \in \mathbf{X}$ the entry $D_{a,b}$ of D is 0 if $(a, b) \in R_0 \cup R_3$, +1 if $(a, b) \in R_1$ and -1 if $(a, b) \in R_2$.

In order to be able to apply Theorem 2.2 we must first have a symmetric 2-scheme which acts as the symmetric closure for the scheme $(\mathbf{X}, \mathbf{R})$. To this end replace in D every -1 by $+1$ and call this matrix $\bar{D}_1$.

THEOREM 3.8. *Let I be the identity matrix of order* 36 *and let J be the all-one matrix of the same order. Then with $\bar{D}_0 = I$ and $\bar{D}_2 = J - \bar{D}_0 - \bar{D}_1$ the matrices $\bar{D}_0$, $\bar{D}_1$ and $\bar{D}_2$ are the adjacency matrices of a symmetric* 2-*scheme* $(\mathbf{X}, \bar{\mathbf{R}})$ *of the type* $NL_2(6)$.

PROOF. It obviously suffices to prove that $\bar{D}_1^2 = 14\bar{D}_0 + 4\bar{D}_1 + 6\bar{D}_2$. But this is not difficult to check, either by hand or by computer. We leave this to the reader. □

Seidel has shown in [7] that there are at least 105 non-isomorphic schemes of type $NL_2(6)$, one of which is the scheme found in the above theorem. Schemes of type $NL_2(6)$ are symmetric 2-schemes with $v = 36$, $\bar{v}_1 = 14$ and $\bar{p}_{11}^1 = 4$ and if we take the eigenvalue of $\bar{D}_1$ on $\bar{\mathbf{V}}_1$ as -4, then if the splitting of such a scheme exists this splitting must be according to case I, to use the terminology of [4].

THEOREM 3.9. *The following hold*:

(1) *A symmetric* 2-*scheme of the type* $NL_2(6)$ *can be split into a non-symmetric* 3-*scheme iff it is isomorphic to the scheme* $(\mathbf{X}, \bar{\mathbf{R}})$ *found in Theorem* 3.8.

(2) *Such a splitting* $(\mathbf{X}, \mathbf{R})$ *has the parameters* $v = 36$, $v_1 = 7\alpha = 0$ *and* $\beta = 4$.

(3) *There exists one, and up to isomorphisms, only one non-symmetric* 3-*scheme with these parameters.*

PROOF. First replace in D every -1 by 0 and call this matrix D_1. Then replace in $-D$ every -1 by 0 and call this matrix D_2.

Then apply Theorem 2.2 to $(\mathbf{X}, \bar{\mathbf{R}})$ in Theorem 3.8. For the D_1 and D_2 mentioned in Theorem 2.2 we can take the matrices found at the beginning of this proof and we have 'only' still to check the conditions (3) and (4) of Theoem 2.2.

The uniqueness of $(\mathbf{X}, \mathbf{R})$ follows from Theorem 3.7. □

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We are grateful for the following remarks made by the referee, putting the results of this paper in a wider context.

The automorphism group of $(\mathbf{X}, \mathbf{R})$ is in fact $G_2(2)'$ in its permutation representation of degree 36, and the automorphism group of $(\mathbf{X}, \bar{\mathbf{R}})$ is $G_2(2)$. The existence of $(\mathbf{X}, \mathbf{R})$ can easily be derived from the character table of this group, and from the decomposition of the permutation character in irreducibles; cf. [2].

Also, after Corollary 3.2 has been proved, uniqueness of $(\mathbf{X}, \bar{\mathbf{R}})$ immediately follows from the recent characterization of locally co-Heawood graphs in [1].

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