Linear Transformations Which Map the Classes of ω-Matrices and τ-Matrices into or onto Themselves

Daniel Hershkowitz
Department of Mathematics
University of Wisconsin, Madison
Madison, Wisconsin 53706

and

Volker Mehrmann*
Fakultät für Mathematik
Universität Bielefeld
Postfach 8640
D-4800 Bielefeld 1, F.R.G.

Submitted by Hans Schneider

ABSTRACT

A characterization is given for linear transformations on $n \times n$ matrices which map the classes of ω- and τ-matrices into themselves, under certain nonsingularity assumptions on the mapping. These results are also used in obtaining the characterization of those linear transformations which map the above classes onto themselves.

1. INTRODUCTION

The question of characterizing the linear transformations which preserve certain properties of square matrices has been studied in several recent papers. In particular, it has been of interest to characterize those linear transformations which map certain classes of matrices into or onto themselves. The "into" case is, in general, harder than the "onto" case, and it is usually solved under some additional hypothesis, such as nonsingularity of the transformation or a somewhat weaker condition.

The discussion here is devoted to the classes of ω- and τ-matrices, introduced by Engel and Schneider [3]. These classes, defined by eigenvalue

*The work of this author was carried out while a visitor at the University of Wisconsin, Madison.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/86/$3.50

monotonicity, contain the important classes of Hermitian matrices, totally nonnegative matrices, Z- and M-matrices, and triangular matrices with real diagonal elements, and hence studying their properties has been the theme of many papers, e.g. [5] and the references there. There are some interesting open problems concerning these matrices, especially the localization of their spectra; see [3] and [7].

In this paper we characterize the linear transformations which map the class $\omega_{\langle n\rangle}$ $[\tau_{\langle n\rangle}]$ of all $n\times n$ ω-matrices [τ-matrices] into or onto itself. Our results and the methods used in our proofs seem to help in understanding the properties of these matrices better, and they may provide useful tools in solving the other open problems.

The main results are introduced in Section 3 and proved in Section 4. In the same spirit as in [1] and [4], we show that a linear transformation which preserves $\omega_{\langle n\rangle}$ or $\tau_{\langle n\rangle}$ is a composition of obvious types of mappings, namely transposition, diagonal similarity, permutation similarity, multiplication by a positive scalar, and addition of a scalar matrix.

Among other results we show that a linear transformation L satisfies $L(\omega_{\langle n\rangle})=\omega_{\langle n\rangle}$ if and only if $L(\omega_{\langle n\rangle})\subseteq\omega_{\langle n\rangle}$ and L is nonsingular, and that $L(\tau_{\langle n\rangle})\subseteq\tau_{\langle n\rangle}$ implies $L(\omega_{\langle n\rangle})\subseteq\omega_{\langle n\rangle}$. The theorems for $\omega_{\langle n\rangle}$ and $\tau_{\langle n\rangle}$ are very similar. However, our assertions hold also when we restrict ourselves to real τ-matrices, but not for real ω-matrices.

As in [4], our "into" results are proven under the assumption that the restriction of the transformation to the set of all matrices with zero diagonal elements is nonsingular. In part of our proof we follow the lines of the proofs given in [4], and some of our first propositions are similar to those in [4].

We remark that the problem of characterizing transformations which are not necessarily linear and preserve properties of matrices is also of interest. A result in this direction concerning $\omega_{\langle n\rangle}$ and $\tau_{\langle n\rangle}$ is found in [6].

2. DEFINITIONS AND NOTATION

NOTATION 2.1. We denote:

$|\alpha|$ = the cardinality of a set α.
$\mathbb{R}$ = the field of real numbers.
$\mathbb{C}$ = the field of complex numbers.

NOTATION 2.2. For a field F and a positive integer n we denote:

$\langle n\rangle$ = the set $\{1,2,\dots,n\}$.
$F^{n,n}$ = the set of all $n\times n$ matrices over F.
E_{ij} = the matrix in $F^{n,n}$ all of whose entries are zero except for the one in the (i,j) position, whose value is 1.

NOTATION 2.3. For $A \in F^{n,n}$ and $\alpha \subseteq \langle n \rangle$ we denote:

$A[\alpha]$ = the principal submatrix of A whose rows and columns are indexed by α.
$\sigma(A)$ = the spectrum of A.

DEFINITION 2.4. For $A \in \mathbb{C}^{n,n}$ we define the number $l(A)$ by

$$l(A) = \begin{cases} \min\{\sigma(A) \cap \mathbb{R}\}, & \sigma(A) \cap \mathbb{R} \neq \varnothing, \\ \infty, & \sigma(A) \cap \mathbb{R} = \varnothing. \end{cases}$$

For $A \in \mathbb{C}^{3,3}$ we also define the number $h(A)$ by

$$h(A) = \min\{l(A[\{1,2\}]), l(A[\{1,3\}]), l(A[\{2,3\}])\}.$$

DEFINITION 2.5. A matrix $A \in \mathbb{C}^{n,n}$ is said to be an *ω-matrix* if

$$l(A[\alpha]) < \infty \qquad \text{for all} \quad \alpha \subseteq \langle n \rangle, \quad \alpha \neq \varnothing,$$

and if

$$l(A[\alpha]) \leqslant l(A[\beta]) \qquad \text{for all} \quad \alpha, \beta \subseteq \langle n \rangle, \quad \phi \neq \beta \subseteq \alpha.$$

An ω-matrix is said to be a *τ-matrix* if further $l(A) \geqslant 0$. We denote by $\omega_{\langle n \rangle}$ $[\tau_{\langle n \rangle}]$ the set of all $n \times n$ ω-matrices [τ-matrices], and by $\omega^k_{\langle n \rangle}$ $[\tau^k_{\langle n \rangle}]$ the set of all $n \times n$ complex matrices whose principal submatrices of order less than or equal to k are all ω-matrices [τ-matrices].

DEFINITION 2.6. A matrix $A \in \mathbb{C}^{n,n}$ is said to be a *P^0-matrix* if

$$\det A[\alpha] \geqslant 0 \qquad \text{for all} \quad \alpha \subseteq \langle n \rangle, \quad \alpha \neq \varnothing.$$

The set of all $n \times n$ P^0-matrices is denoted by $P^0_{\langle n \rangle}$.

NOTATION 2.7. For a linear transformation L on $F^{n,n}$ and for $i, j \in \langle n \rangle$ we denote:

$N(L)$ = the null space (kernel) of L.
S_{ij} = the set $\{E_{lm} : l, m \in \langle n \rangle, \; L(E_{ij})_{lm} \neq 0\}$.
S_{ij}^T = the set $\{E_{ml} : E_{lm} \in S_{ij}\}$.

$\tilde{L}$ = the $n^2 \times n^2$ matrix which represents L in the basis $\{E_{11}, \dots, E_{nn}, E_{12}, E_{21}, E_{13}, \dots, E_{n,n-1}\}$ of $F^{n,n}$. We partition $\tilde{L}$ by

$$\tilde{L} = \begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix},$$

where $\tilde{L}_{11}$ is an $n \times n$ matrix.

DEFINITION 2.8. A (directed) *graph* $G = (V, E)$ is a pair of finite sets with $E \subseteq V \times V$. An element of V is called a *vertex* of G, and an element of E is called an *arc* of G. An arc of the type (i, i) where $i \in V$ is called a *loop*.

DEFINITION 2.9. Let G be a graph. A cycle in G of length k is a set of k arcs of G

$$\{(i_1, i_2), \dots, (i_{k-1}, i_k), (i_k, i_1)\},$$

where $i_1, \dots, i_k$ are distinct.

DEFINITION 2.10. Let A be an $n \times n$ matrix. A *cyclic product* of length k of A is a product $a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{k-1} i_k} a_{i_k i_1}$, where $\{(i_1, i_2), \dots, (i_{k-1}, i_k), (i_k, i_1)\}$ is a cycle in the graph whose vertex set is $\langle n \rangle$ and whose arc set is $\langle n \rangle \times \langle n \rangle$.

3. RESULTS

Our first two theorems characterize the linear transformations which map $\omega_{\langle n \rangle}$ *into* itself, under some nonsingularity assumption.

THEOREM 3.1. *Let L be a linear transformation on $\mathbb{C}^{n,n}$, $n \geqslant 3$, satisfying*

$$N(L) \cap \{A \in \mathbb{C}^{n,n} : a_{ii} = 0, 1 \leqslant i \leqslant n\} = \{0\}.$$

Then $L(\omega_{\langle n \rangle}) \subseteq \omega_{\langle n \rangle}$ if and only if L is a composition of one or more of the *following types of transformations*:

(i) $A \to \alpha DAD^{-1}$, *in which* α *is a positive scalar and* D *is nonsingular diagonal matrix* (*a combination of multiplication by a positive scalar and diagonal similarity*);

(ii) $A \to A^T$ (*transposition*);

(iii) $A \to PAP^T$, *in which* P *is a permutation matrix* (*permutation similarity*); *and*

(iv) $A \to A + \alpha I$, *in which* α *is a real linear combination of the diagonal entries of* A.

Moreover, the same is true if $\omega_{\langle n \rangle}$ *is replaced by* $\omega^k_{\langle n \rangle}$, $k \geqslant 3$.

THEOREM 3.2. *Let* L *be a linear transformation on* $\mathbb{C}^{2,2}$ *satisfying*

$$N(L) \cap \{ A \in \mathbb{C}^{2,2} : a_{11} = a_{22} = 0 \} = \{0\}.$$

Then $L(\omega_{\langle 2 \rangle}) \subseteq \omega_{\langle 2 \rangle}$ *if and only if* L *is a composition of one or more transformation of types* (i) *and* (ii) *as listed in Theorem* 3.1, *and*

(v) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} k_1 a_{11} + k_2 a_{22} & a_{12} \\ a_{21} & t_1 a_{11} + t_2 a_{22} \end{bmatrix}$, *in which* k_1, k_2, t_1, *and* t_2 *are real.*

The following theorem characterizes the linear transformations which map $\omega_{\langle n \rangle}$ *onto* itself.

THEOREM 3.3. *Let* L *be a linear transformation on* $\mathbb{C}^{n,n}$. *Then* $L(\omega_{\langle n \rangle}) = \omega_{\langle n \rangle}$ *if and only if* $L(\omega_{\langle n \rangle}) \subseteq \omega_{\langle n \rangle}$ *and* L *is nonsingular.*

In the following theorems we discuss linear transformations which map the class $\tau_{\langle n \rangle}$ into or onto itself. Our results are given over the field F, which can be either $\mathbb{R}$ or $\mathbb{C}$. In fact, since the reality of the images under L is not used in the proofs, our theorems hold also for transformations L satisfying $L(\tau_{\langle n \rangle} \cap \mathbb{R}^{n,n}) \subseteq \tau_{\langle n \rangle}$.

THEOREM 3.4. *Let* L *be a linear transformation on* $F^{n,n}$, $n \geqslant 3$, *satisfying*

$$N(L) \cap \{ A \in F^{n,n} : a_{ii} = 0, 1 \leqslant i \leqslant n \} = \{0\}.$$

Then $L(\tau_{\langle n \rangle} \cap F^{n,n}) \subseteq \tau_{\langle n \rangle} \cap F^{n,n}$ *if and only if* L *is a composition of one or* *more transformations of types (i), (ii), and (iii) as listed in Theorem* 3.1, *and*

(vi) $A \to A + \alpha I$, *in which α is a nonnegative linear combination of the diagonal entries of A.*

Moreover, the same is true if $\tau_{\langle n \rangle}$ is replaced by $\tau^k_{\langle n \rangle}$, $k \geqslant 3$.

THEOREM 3.5. *Let L be a linear transformation on $F^{2,2}$ satisfying*

$$N(L) \cap \left\{ A \in F^{2,2} : a_{11} = a_{22} = 0 \right\} = \{0\}.$$

Then $L(\tau_{\langle 2 \rangle} \cap F^{2,2}) \subseteq \tau_{\langle 2 \rangle} \cap F^{2,2}$ if and only if L is a composition of one or more transformations of types (i) *and* (ii) *as listed in Theorem* 3.1, *and*

(vii) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} k_1 a_{11} + k_2 a_{22} & a_{12} \\ a_{21} & t_1 a_{11} + t_2 a_{22} \end{bmatrix}$, *in which* $k_1, k_2, t_1, t_2 \geqslant 0$ *satisfy either*

$$k_1 t_2 + k_2 t_1 \geqslant 1$$

or

$$1 - 2(k_1 t_2 + k_2 t_1) + (k_1 t_2 - k_2 t_1)^2 \leqslant 0.$$

THEOREM 3.6. *Let L be a linear transformation on $F^{n,n}$, $n \geqslant 3$. Then $L(\tau_{\langle n \rangle} \cap F^{n,n}) = \tau_{\langle n \rangle} \cap F^{n,n}$ if and only if L is a composition of one or more transformations of types* (i), (ii) *and* (iii), *as listed in Theorem* 3.1.

THEOREM 3.7. *Let L be a linear transformation on $F^{2,2}$. Then $L(\tau_{\langle 2 \rangle} \cap F^{2,2}) = \tau_{\langle 2 \rangle} \cap F^{2,2}$ if and only if L is a composition of one or more transformations of types* (i) *and* (ii) *as listed in Theorem* 3.1, *and*

(viii) $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \to \begin{bmatrix} k_1 a_{11} + k_2 a_{22} & a_{12} \\ a_{21} & t_1 a_{11} + t_2 a_{22} \end{bmatrix}$, *in which either*

$$k_1 = t_2 = 0, \qquad k_2, t_1 > 0, \qquad k_2 t_1 = 1$$

or

$$k_2 = t_1 = 0, \qquad k_1, t_2 > 0, \qquad k_1 t_2 = 1.$$

4. PROOFS

Proof of Theorems 3.1 and 3.4. The proofs of these two theorems are almost identical, and hence they are united. For convenience our proof is split into several propositions.

It is easy to verify that each of the given transformations in Theorem 3.1 [3.4] maps the class $\omega_{\langle n \rangle}$ $[\tau_{\langle n \rangle} \cap F^{n,n}]$ into itself. Thus it suffices to prove the reverse direction, namely that every transformation which maps $\omega_{\langle n \rangle}$ $[\tau_{\langle n \rangle} \cap F^{n,n}]$ into itself is a composition of transformations of the types specified in Theorem 3.1 [3.4].

In our proof we use the following four lemmas, which describe properties of ω- and τ-matrices.

LEMMA 4.1 (Lemma 3.3 in [3]). *Every ω-matrix has real principal minors.*

LEMMA 4.2 (Theorem 3.6 in [3]). *We have*

$$\tau_{\langle n \rangle} \subseteq P^{0}_{\langle n \rangle}.$$

The following lemma is essentially known. A proof is provided for the sake of completeness.

LEMMA 4.3. *Let $A \in \mathbb{C}^{2,2}$. Then*

$$A \in \omega_{\langle 2 \rangle}$$

if and only if a_{11} and a_{22} are real and $a_{12}a_{21} \geqslant 0$.

Proof. Let λ be the following eigenvalue of A:

$$\lambda = \tfrac{1}{2}\left[a_{11} + a_{22} - \sqrt{(a_{11} - a_{22})^2 + 4a_{12}a_{21}}\right].$$

If $A \in \omega_{\langle 2 \rangle}$, then a_{11} and a_{22} are real and $l(A) = \lambda$. Hence, since $l(A) \leqslant \min\{a_{11}, a_{22}\}$, it follows that $a_{12}a_{21} \geqslant 0$.

Conversely, if a_{11} and a_{22} are real and $a_{12}a_{21} \geqslant 0$, then λ is real and furthermore

$$\lambda \leqslant \min\{a_{11}, a_{22}\}. \qquad \blacksquare$$

LEMMA 4.4. *Let $A \in \mathbb{C}^{3,3}$. Then*

$$A \in \omega_{\langle 3 \rangle}$$

if and only if a_{11}, a_{22}, and a_{33} are real,

$$a_{ij}a_{ji} \geqslant 0, \qquad i, j \in \langle 3 \rangle, \quad i \neq j,$$

and

$$\det[A - h(A)I] \leqslant 0.$$

Proof. Our claim follows from Lemma 4.3 and Proposition 2 in [5]. ■

As mentioned in the introduction and in the previous section, Theorem 3.4 holds also when we restrict ourselves to real matrices. The same does not hold for Theorem 3.1, which is valid only when complex matrices are considered [see transformation (ix) after the proof]. In order to prove all cases we assume from now on (until the end of the proof of Theorem 3.1 [3.4]) that

$$L(\tau_{\langle n \rangle}) \subseteq \omega_{\langle n \rangle}, \tag{4.5}$$

or

$$L(\tau_{\langle n \rangle} \cap \mathbb{R}^{n,n}) \subseteq \tau_{\langle n \rangle} \cap \mathbb{R}^{n,n}. \tag{4.6}$$

Observe that the assumption (4.5) is weaker than the assumption

$$L(\omega_{\langle n \rangle}) \subseteq \omega_{\langle n \rangle}.$$

In fact, we shall have to distinguish between (4.5) and (4.6) only in Propositions 4.7 and 4.9, Lemma 4.17, and Corollary 4.31.

PROPOSITION 4.7. *We have*

$$\tilde{L}_{12} = 0.$$

Proof. Let (4.5) hold, and assume that for some $i, j, k \in \langle n \rangle$, $i \neq j$, we have

$$L(E_{ij}) = aE_{kk} + \cdots, \qquad a \neq 0. \tag{4.8}$$

Since $cE_{ij} \in \tau_{\langle n \rangle}$ for all $c \in \mathbb{C}$, it follows that $B(c) = L(cE_{ij}) \subseteq \omega_{\langle n \rangle}$. But for an appropriate value of c the matrix $B(c)$ has a nonreal diagonal element, in contradiction to Lemma 4.1. Therefore the assumption (4.8) is false and our claim follows.

If (4.6) holds, then since for $i, j \in \langle n \rangle$, $i \neq j$, we have $\pm E_{ij} \in \tau_{\langle n \rangle} \cap \mathbb{R}^{n,n}$, it follows that $\pm L(E_{ij}) \in \tau_{\langle n \rangle}$. By Lemma 4.2, the diagonal entries of $L(E_{ij})$ are both nonnegative and nonpositive, and therefore zero. ∎

Proposition 4.9. *We have*

$$S_{ij} \cap S_{ij}^T = \varnothing, \qquad i, j \in \langle n \rangle, \quad i \neq j.$$

Proof. Let $i, j \in \langle n \rangle$, $i \neq j$, and assume that

$$E_{lm} \in S_{ij} \cap S_{ij}^T \qquad \text{for some} \quad l, m \in \langle n \rangle. \tag{4.10}$$

Then

$$L(E_{ij}) = aE_{lm} + bE_{ml} + \cdots,$$

where $ab \neq 0$. By Proposition 4.7 we have $l \neq m$, and by (4.5) or (4.6) and Lemma 4.3 we have $ab > 0$.

If (4.5) holds, then since $cE_{ij} \in \tau_{\langle n \rangle}$ for all $c \in \mathbb{C}$, it follows that $B(c) = L(cE_{ij}) \subseteq \omega_{\langle n \rangle}$. But, in view of Proposition 4.7, for an appropriate value of c the $\{l, m\}$ principal minor of $B(c)$ is complex, in contradiction to Lemma 4.1.

If (4.6) holds, then since $E_{ij} \in \tau_{\langle n \rangle} \cap \mathbb{R}$, it follows that $L(E_{ij}) \in \tau_{\langle n \rangle}$. Observe that by Proposition 4.7 we have

$$\det L(E_{ij})[\{l, m\}] = -ab < 0,$$

in contradiction to Lemma 4.2.

In any case our assumption (4.10) is false. ∎

Proposition 4.11. *We have*

$$S_{ij} \cap S_{lm}^T = \varnothing, \qquad i, j, l, m \in \langle n \rangle, \quad i \neq j, \quad l \neq m, \quad \{i, j\} \neq \{l, m\}.$$

Proof. Let $i, j, l, m \in \langle n \rangle$, $i \neq j$, $l \neq m$, $\{i, j\} \neq \{l, m\}$, and assume that

$$E_{pq} \in S_{ij} \cap S_{lm}^T \quad \text{for some} \quad p, q \in \langle n \rangle. \tag{4.12}$$

Then,

$$L(E_{ij}) = aE_{pq} + \cdots$$

and

$$L(E_{lm}) = bE_{qp} + \cdots,$$

where $ab \neq 0$. Since $E_{ij} \pm E_{lm} \in \tau_{\langle n \rangle} \cap \mathbb{R}^{n,n}$, it follows from (4.5) or (4.6) that $B = L(E_{ij} + E_{lm}) \in \omega_{\langle n \rangle}$ and $C = L(E_{ij} - E_{lm}) \in \omega_{\langle n \rangle}$. By Proposition 4.9 we have

$$b_{pq} = c_{pq} = a, \qquad b_{qp} = -c_{qp} = b.$$

It now follows from Lemma 4.3 that the product ab is positive as well as negative, which is impossible. Therefore, the assumption (4.12) is false. ■

PROPOSITION 4.13. *If*

$$N(L) \cap \{A \in \mathbb{C}^{n,n} : a_{ii} = 0, 1 \leqslant i \leqslant n\} = \{0\}, \tag{4.14}$$

then

$$|S_{ij}| = 1 \text{ and } S_{ij} = S_{ji}^T \quad \text{for all} \quad i, j \in \langle n \rangle, \quad i \neq j.$$

Furthermore, $\tilde{L}_{22}$ is a generalized monomial matrix, i.e. a product of a permutation matrix and a nonsingular diagonal matrix.

Proof. Since the restriction of L to the subspace $\{A \in \mathbb{C}^{n,n} : a_{ii} = 0, 1 \leqslant i \leqslant n\}$ is nonsingular, we have

$$\left| \bigcup_{\substack{l \neq m \\ (l,m) \neq (i,j)}} S_{lm} \right| \geqslant \dim \operatorname{span}(\{L(E_{lm}) : l, m \in \langle n \rangle, l \neq m, (l,m) \neq (i,j)\})$$

$$= n^2 - n - 1.$$

Our claims now follow from Propositions 4.9 and 4.11. ■

We remark that for the real case in Theorem 3.4 we replace $\mathbb{C}$ by $\mathbb{R}$ in (4.14) and in the proof of Proposition 4.13.

From now on we assume that L satisfies (4.14) too.

PROPOSITION 4.15. *We have*

$$\tilde{L}_{21} = 0.$$

Proof. Let $i \in \langle n \rangle$ and assume that

(4.16)

$$L(E_{ii}) = aE_{lm} + bE_{ml} + \cdots, \quad a \neq 0, \qquad \text{for some} \quad l, m \in \langle n \rangle, \quad l \neq m.$$

By Proposition 4.13 there exist $p, q \in \langle n \rangle$, $p \neq q$, such that $L(E_{pq}) = cE_{ml}$ with $c \neq 0$. The matrix

$$A = E_{ii} - \frac{\bar{a} + b}{c} E_{pq}$$

is in $\tau_{\langle n \rangle}$, so by (4.5) or (4.6), $B = L(A) \in \omega_{\langle n \rangle}$. Observe that $b_{lm} = a$ and $b_{ml} = -\bar{a}$. Hence, $b_{lm} b_{ml} < 0$, in contradiction to Lemma 4.3. We thus conclude that the assumption (4.16) is false, proving that $\tilde{L}_{21} = 0$. ∎

Let $\mathcal{G}$ be the set of all directed graphs whose vertex set is $\{1, \ldots, n\}$ and whose arc set consists of three arcs none of which is a loop. Denote by $E(G)$ the arc set of a directed graph G. In view of Proposition 4.13 we define a one-to-one function f_L from $\mathcal{G}$ into itself by

$$E(f_L(G)) = \left\{(i, j) : S_{kl} = \{E_{ij}\}, (k, l) \in E(G)\right\}, \qquad G \in \mathcal{G}.$$

LEMMA 4.17. *Let $G \in \mathcal{G}$. If G has no cycle then $f_L(G)$ has no cycle.*

Proof. Let $E(G) = \{(i, j), (k, l), (s, t)\}$. If G has no cycle, then the matrix

$$A = aE_{ij} + bE_{kl} + cE_{st}$$

is in $\tau_{\langle n \rangle}$ for every choice of a, b, and c, and hence $L(A) \in \tau_{\langle n \rangle}$. Assume that

$f_L(G)$ has a cycle. If (4.5) holds, then since for appropriate complex values of a, b, and c the matrix $L(A)$ has a complex principal minor, we have a contradiction to Lemma 4.1. If (4.6) holds, then since for appropriate real values of a, b, and c the matrix $L(A)$ has a negative principal minor, we have a contradiction to Lemma 4.2. ∎

LEMMA 4.18. *Let $G \in \mathcal{G}$. If G has a cycle of length k $(k = 2,3)$, then $f_L(G)$ has a cycle of length k.*

Proof. Since $\mathcal{G}$ is a finite set and f_L is a one-to-one function, it follows from Lemma 4.17, using counting arguments, that if G has a cycle then $f_L(G)$ has a cycle. Our claim now follows observing that by Proposition 4.13, $f_L(G)$ has a cycle of length 2 if and only if G does. ∎

PROPOSITION 4.19. *If $S_{12} = \{E_{i_1 j_1}\}$ and $S_{13} = \{E_{i_2 j_2}\}$, then either $i_1 = i_2$ or $j_1 = j_2$.*

Proof. Let $S_{23} = \{E_{i_3 j_3}\}$, and let $G \in \mathcal{G}$ be such that

$$E(G) = \{(1,2),(2,3),(3,1)\}.$$

Since by Proposition 4.13 $S_{31} = \{E_{j_2 i_2}\}$, we have

$$E(f_L(G)) = \{(i_1, j_1),(i_3, j_3),(j_2, i_2)\}.$$

By Lemma 4.18, $f_L(G)$ has a cycle of length 3, and hence either $i_1 = i_2$ or $j_1 = j_2$. ∎

If follows from Propositions 4.13 and 4.19 that by applying an appropriate permutation similarity, as well as a transposition if needed, we may assume that for all $i \in \langle n \rangle$, L maps the off-diagonal elements of the ith row onto the off-diagonal elements of the ith row, and the same holds for columns. Thus, from now on we may assume that

$$L(E_{ij}) = b_{ij} E_{ij}, \quad b_{ij} \neq 0, \qquad i, j \in \langle n \rangle, \quad i \neq j. \tag{4.20}$$

We define the $n \times n$ matrix C by

$$c_{ij} = \begin{Bmatrix} b_{ij}, & i \neq j \\ 0, & i = j \end{Bmatrix}, \qquad i, j \in \langle n \rangle.$$

LEMMA 4.21. *All cyclic products of length* 2 *and* 3 *of the matrix* C *are positive.*

Proof. The positivity of the cyclic products of length 2 follows easily using Lemma 4.2. To prove the positivity of cyclic products of length 3 we consider, without loss of generality, the cycle $\{(1,2),(2,3),(3,1)\}$. Let A be the direct sum of the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

and the identity matrix of order $n-3$. By Proposition 4.15 and by (4.20), the matrix $L(A)$ is a direct sum of a matrix

$$M = \begin{bmatrix} \alpha_1 & c_{12} & 0 \\ 0 & \alpha_2 & c_{23} \\ -c_{31} & 0 & \alpha_3 \end{bmatrix}$$

and a diagonal matrix. Since $A \in \tau_{\langle n \rangle} \cap \mathbb{R}^{n,n}$, it follows by (4.5) or (4.6) that $L(A) \in \omega_{\langle n \rangle}$, and hence the matrix M is in $\omega_{\langle 3 \rangle}$. Observe that

$$h = h(M) = \min\{\alpha_1, \alpha_2, \alpha_3\},$$

and thus

$$\det(M - hI) = -c_{12}c_{23}c_{31}.$$

By Lemma 4.4, since $c_{ij} \neq 0$ for $i \neq j$, it follows that $c_{12}c_{23}c_{31} > 0$. ∎

COROLLARY 4.22. *After an appropriate diagonal similarity we have*

$$c_{ij} > 0, \qquad i, j \in \langle n \rangle, \quad i \neq j. \tag{4.23}$$

Proof. After an appropriate diagonal similarity we may obtain

$$c_{1j} > 0, \qquad j = 2, \ldots, n. \tag{4.24}$$

By Lemma 4.21 we have

$$c_{i1} > 0, \qquad i = 2, \ldots, n. \tag{4.25}$$

Let $p,q \in \langle n \rangle$, $p,q \neq 1$, $p \neq q$. By Lemma 4.21 we have

$$c_{1p}c_{pq}c_{q1} > 0,$$

and by (4.24) and (4.25) we have

$$c_{pq} > 0. \qquad \blacksquare$$

In view of Corollary 4.22 we may assume from now on that (4.23) holds.

PROPOSITION 4.26. *The image under L of the identity matrix is a real scalar matrix.*

Proof. Let $D = L(I)$. By Proposition 4.15 the matrix D is diagonal. Since $D \in \omega_{\langle n \rangle}$, it follows that D is a real matrix. Without loss of generality (applying an appropriate permutation similarity) we may assume that

$$d_{11} \leqslant d_{22} \leqslant \cdots \leqslant d_{nn}.$$

There are three possibilities for the relations between d_{11}, d_{22}, and d_{33}:

$$d_{11} = d_{22} < d_{33}, \tag{4.27}$$

$$d_{11} < d_{22} \leqslant d_{33}, \tag{4.28}$$

and

$$d_{11} = d_{22} = d_{33}. \tag{4.29}$$

If (4.27) holds, then let $A \in \mathbb{C}^{n,n}$ be defined by

$$a_{ij} = \begin{cases} t, & i = j = 1,\dots,n, \\ 1, & (i,j) \in \{(1,3),(2,1),(2,3),(3,1),(3,2)\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $t \geqslant 1$. It is easy to verify that $A \in \tau_{\langle n \rangle} \cap \mathbb{R}^{n,n}$ and hence $L(A) \in \omega_{\langle n \rangle}$. Therefore, the matrix $M = L(A) - td_{11}I$ is in $\omega_{\langle n \rangle}$. Observe that since $t(d_{33} - d_{11}) > 0$, we have

$$h = h(M[\{1,2,3\}]) < 0.$$

Furthermore, for t sufficiently large we have

$$|h| < \min\left\{\frac{c_{13}c_{32}c_{21}}{c_{13}c_{31}}, \frac{c_{13}c_{32}c_{21}}{c_{23}c_{32}}\right\}. \tag{4.30}$$

It now follows from (4.30) that

$$\det(M - hI)[\{1,2,3\}] > 0,$$

in contradiction to Lemma 4.4. Hence, the possibility (4.27) is false.

If (4.28) holds, then let $A \in \mathbb{C}^{n,n}$ be defined by

$$a_{ij} = \begin{cases} t, & i = j = 1,\dots,n, \\ 1, & (i,j) \in \{(1,2),(1,3),(2,1),(2,3),(3,2)\}, \\ 0 & \text{otherwise,} \end{cases}$$

where $t \geqslant 1$. Here too $A \in \tau_{\langle n \rangle} \cap \mathbb{R}^{n,n}$ and hence $L(A) \in \omega_{\langle n \rangle}$. Therefore, the matrix $M = L(A) - td_{11}I$ is in $\omega_{\langle n \rangle}$. Since $t(d_{22} - d_{11}) > 0$ and $t(d_{33} - d_{11}) > 0$, we have $h = h(M[\{1,2,3\}]) = l(M[\{1,2\}]) < 0$. As before, for t sufficiently large we have

$$|h| < \frac{c_{13}c_{32}c_{21}}{c_{23}c_{32}}$$

and thus

$$\det(M - hI)[\{1,2,3\}] > 0,$$

in contradiction to Lemma 4.4. Therefore, the possibility (4.28) is false too, and thus (4.29) holds. Similarly we show that

$$d_{ii} = d_{i+1,i+1} = d_{i+2,i+2}, \qquad i \in \langle n-2 \rangle,$$

proving that D is a scalar matrix. ∎

Corollary 4.31. *If* (4.5) *holds then*

$$L(\omega_{\langle n \rangle}) \subseteq \omega_{\langle n \rangle}.$$

If (4.6) *holds then*

$$L\left(\omega_{\langle n\rangle} \cap \mathbb{R}^{n,n}\right) \subseteq \omega_{\langle n\rangle} \cap \mathbb{R}^{n,n}.$$

Proof. Let $A \in \omega_{\langle n\rangle}[\omega_{\langle n\rangle} \cap \mathbb{R}^{n,n}]$. Then $A + dI \in \tau_{\langle n\rangle}[\tau_{\langle n\rangle} \cap \mathbb{R}^{n,n}]$ for $d \geqslant -l(A)$. By (4.5) [(4.6)] we have

$$L(A+dI) = L(A) + dL(I) \in \omega_{\langle n\rangle}\left[\tau_{\langle n\rangle} \cap \mathbb{R}^{n,n}\right].$$

Since by Proposition 4.26 $L(I)$ is a real scalar matrix, it follows that $L(A) \in \omega_{\langle n\rangle}[\omega_{\langle n\rangle} \cap \mathbb{R}^{n,n}]$. ∎

PROPOSITION 4.32. *After applying appropriate diagonal similarity and multiplication by a positive scalar we have*

$$c_{ij} = 1, \qquad i, j \in \langle n\rangle, \quad i \neq j.$$

Proof. Let

$$g = \max\{c_{ij}c_{ji} : i, j \in \langle n\rangle, i \neq j\}. \tag{4.33}$$

By applying an appropriate permutation similarity we may assume, without loss of generality, that

$$g = c_{12}c_{21}. \tag{4.34}$$

We prove our assertion by induction on n. For $n = 1$ there is nothing to prove. For $n = 2$ the claim follows immediately from (4.23). Assume that our proposition holds for $n < m$, and let $n = m$. Since the restriction of L to matrices with zero nth row and column induces a transformation on $\mathbb{C}^{m-1,m-1}$ $[\mathbb{R}^{m-1,m-1}]$ which preserves $\omega_{\langle m-1\rangle}$ $[\tau_{\langle m-1\rangle} \cap \mathbb{R}^{m-1,m-1}]$, it follows by the inductive assumption that we may assume that

$$c_{ij} = 1, \qquad i, j \in \langle m-1\rangle, \quad i \neq j. \tag{4.35}$$

Let $p, q \in \langle m-1\rangle$, $p \neq q$, and define the $n \times n$ matrix A by

$$a_{ij} = \begin{cases} 0, & i = j = 1, \dots, n, \\ 1, & (i,j) \in \{(p,q),(q,p),(p,n),(q,n),(n,q)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $A \in \omega_{\langle n \rangle} \cap \mathbb{R}^{n,n}$ and hence by Corollary 4.31

$$B = L(A) \in \omega_{\langle n \rangle}. \tag{4.36}$$

The matrix $M = B[\{p, q, n\}]$ is

$$\begin{bmatrix} 0 & 1 & c_{pn} \\ 1 & 0 & c_{qn} \\ 0 & c_{nq} & 0 \end{bmatrix}$$

up to permutation similarity. In view of (4.33), (4.34), and (4.35) we have $h = h(M) = l(M[\{1,2\}]) = -1$. Thus,

$$\det(M - hI) = c_{nq}(c_{pn} - c_{qn}),$$

and by (4.36) and Lemma 4.4 we have

$$c_{nq}(c_{pn} - c_{qn}) \leqslant 0.$$

Since by (4.23) $c_{nq} > 0$, we derive that

$$c_{qn} \geqslant c_{pn}. \tag{4.37}$$

Changing the roles of p and q, we similarly show that

$$c_{pn} \geqslant c_{qn},$$

and by (4.37) we obtain

$$c_{pn} = c_{qn}.$$

In a similar way one can prove that

$$c_{np} = c_{nq}.$$

Thus, after applying an appropriate diagonal similarity (where we change only the nth row and column) we have

$$c_{1n} = c_{2n} = \cdots = c_{n-1,n} = c_{n1} = c_{n2} = \cdots = c_{n,n-1} = a > 0. \tag{4.38}$$

Now let $A \in \mathbb{R}^{n,n}$ be defined by

$$a_{ij} = \begin{cases} 1, & (i,j) \in \{(1,2),(1,n),(2,n),(n,1),(n,2)\} \\ 0 & \text{otherwise.} \end{cases}$$

Observe that $A \in \omega_{\langle n \rangle} \cap \mathbb{R}^{n,n}$ and hence by Corollary 4.31 we have

$$B = L(A) \in \omega_{\langle n \rangle}.$$

The matrix $M = B[\{1,2,n\}]$ is

$$\begin{bmatrix} 0 & 1 & a \\ 0 & 0 & a \\ a & a & 0 \end{bmatrix}.$$

We have

$$h = h(M) = -a,$$

and so

$$\det(M - hI) = a(a - a^2). \tag{4.39}$$

By Lemma 4.4 it follows from (4.39) that

$$a \geqslant 1,$$

and by (4.33), (4.34), (4.35), and (4.38) we obtain

$$a = 1. \qquad \blacksquare$$

PROPOSITION 4.40. *We have*

$$(L(E_{ii}))_{jj} = (L(E_{ii}))_{kk}$$

for all distinct $i, j, k \in \langle n \rangle$.

Proof. Let i, j, and k be distinct elements of $\langle n \rangle$, and let

$$a_1 = (L(E_{ii}))_{ii}, \qquad a_2 = (L(E_{ii}))_{jj}, \qquad a_3 = (L(E_{ii}))_{kk}.$$

Let $A \in \mathbb{R}^{n,n}$ be defined by

$$a_{pq} = \begin{cases} 1, & (p,q) \in \{(i,i),(i,j),(i,k),(j,k),(k,i),(k,j)\}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that $A \in \omega_{\langle n \rangle} \cap \mathbb{R}^{n,n}$, and hence it follows from Corollary 4.31 that

$$B = L(A) \in \omega_{\langle n \rangle}.$$

By Proposition 4.32, the matrix $M = B[\{i, j, k\}]$ is

$$\begin{bmatrix} a_1 & 1 & 1 \\ 0 & a_2 & 1 \\ 1 & 1 & a_3 \end{bmatrix}$$

up to permutation similarity. Let t be a real number such that $h(M - tI) = 0$, and let $G = M - tI$. Since $h(G) = 0$, it follows that $g_{11}, g_{22}, g_{33} > 0$ and that

$$\det G[\{1,3\}] = 0 \tag{4.41}$$

and/or

$$\det G[\{2,3\}] = 0. \tag{4.42}$$

Assume that

$$g_{33} > g_{22}. \tag{4.43}$$

Since $\det G[\{2,3\}] \geqslant 0$, it follows from (4.43) that

$$g_{33} > 1. \tag{4.44}$$

If (4.41) holds, then it follows from (4.44) that $g_{11} < 1$, but then

$$\det G = 1 - g_{11} > 0,$$

in contradiction to Lemma 4.4.

If (4.42) holds, then it follows from (4.44) that $g_{22} < 1$ and

$$\det G = 1 - g_{22} > 0,$$

which is again a contradiction to Lemma 4.4. Thus our assumption (4.43) is false and hence

$$g_{33} \leqslant g_{22}.$$

Similarly (changing the roles of j and k) we show that

$$g_{22} \leqslant g_{33},$$

and therefore

$$g_{22} = g_{33}.$$

Since

$$g_{ii} = a_i - t, \qquad i = 1,2,3,$$

our claim follows. ∎

LEMMA 4.45. *The matrix*

$$A = \begin{bmatrix} a & b & b \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \qquad b > 0, \quad a \text{ real},$$

is in $\omega_{\langle 3 \rangle}$ *if and only if* $a \geqslant b - 1$.

Proof. Let $h = h(A)$. By Lemma 4.4, $A \in \omega_{\langle 3 \rangle}$ if and only if

$$\det(A - hI) \leqslant 0. \tag{4.46}$$

Observe that for $a < b - 1$ we have

$$\frac{\sqrt{a^2 + 4b} - a}{2} > 1. \tag{4.47}$$

We now have

$$h = \begin{cases} l(A[\{2,3\}]) = -1, & a \geqslant b - 1, \\ l(A[\{1,3\}]) = \dfrac{a - \sqrt{a^2 + 4b}}{2} < -1, & a < b - 1, \end{cases}$$

and hence

$$\text{(4.48)} \qquad \det(A-hI)=\begin{cases}0, & a \geqslant b-1,\\ b-\dfrac{\sqrt{a^2+4b}+a}{2}, & a<b-1.\end{cases}$$

But since for $a<b-1$ it follows from (4.47) that

$$\frac{\sqrt{a^2+4b}+a}{2}=b\Bigg/\frac{\sqrt{a^2+4b}-a}{2}<b,$$

it now follows from (4.48) that (4.46) holds if and only if $a \geqslant b-1$. ■

In view of Proposition 4.40 we have

$$\text{(4.49)} \qquad L(E_{ii})=b_iE_{ii}+d_iI, \qquad i\in\langle n\rangle.$$

Observe that if (4.6) holds, then $d_i \geqslant 0$.

In order to complete the proof of Theorem 3.1 [3.4] we have to show that $b_i=1$, $i\in\langle n\rangle$.

Let i, j, and k be distinct elements of $\langle n\rangle$, and let $A\in\mathbb{R}^{n,n}$ be defined by

$$a_{pq}=\begin{cases}2, & (p,q)\in\{(i,j),(i,k)\},\\ 1, & (p,q)\in\{(i,i),(j,k),(k,i),(k,j)\},\\ 0 & \text{otherwise.}\end{cases}$$

Observe that $A\in\omega_{\langle n\rangle}$ and hence by Corollary 4.31 we have

$$\text{(4.50)} \qquad B=L(A)\in\omega_{\langle n\rangle}.$$

By (4.49) and Proposition 4.32, the matrix $B[\{i,j,k\}]$ is

$$\begin{bmatrix} b_i & 2 & 2\\ 0 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}+d_iI$$

up to permutation similarity. By Lemma 4.45 it follows from (4.50) that

$$\text{(4.51)} \qquad b_i \geqslant 1.$$

Consider now the matrix $A \in \mathbb{R}^{n,n}$ defined by

$$a_{pq} = \begin{cases} \frac{1}{2}, & (p,q) \in \{(i,j),(i,k)\}, \\ -\frac{1}{2}, & p = q = i, \\ 1, & (p,q) \in \{(j,k),(k,i),(k,j)\}, \\ 0 & \text{otherwise.} \end{cases}$$

Here too $A \in \omega_{\langle n \rangle}$ and hence $B = L(A) \in \omega_{\langle n \rangle}$. The matrix $M = B[\{i, j, k\}]$ is now

$$\begin{bmatrix} -\frac{1}{2}b_i & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} - \tfrac{1}{2}d_i I$$

up to permutation similarity. Since $M \in \omega_{\langle 3 \rangle}$, it follows from Lemma 4.45 that

$$-\tfrac{1}{2}b_i \geqslant -\tfrac{1}{2},$$

or

$$b_i \leqslant 1.$$

Together with (4.51) we have

$$b_i = 1.$$

The proof of Theorem 3.1 [3.4] is now completed. ■

Since, in all the propositions and lemmas, only principal submatrices of order no more than 3 were needed, it follows that Theorem 3.1 [3.4] holds also for the classes $\tau^k_{\langle n \rangle}$ $[\omega^k_{\langle n \rangle}]$, $k \geqslant 3$. However, the "only if" part of Theorem 3.1 [3.4] does not hold for the class $\tau^2_{\langle n \rangle}$ $[\omega^2_{\langle n \rangle}]$, as demonstrated by the transformation

$$L: A \to M \circ A,$$

where M is an $n \times n$ matrix all of whose entries are positive and further

$$m_{ii} \geqslant 1, \qquad i \in \langle n \rangle,$$

and

$$m_{ij}m_{ji}=1, \qquad i,j\in\langle n\rangle, \quad i\neq j.$$

Here $M\circ A$ denotes the Hadamard product of M and A, i.e. the matrix B whose entries are $b_{ij}=m_{ij}a_{ij}$, $i,j\in\langle n\rangle$. Using Lemmas 4.2 and 4.3 it is easy to verify that L preserves $\tau^2_{\langle n\rangle}$ $[\omega^2_{\langle n\rangle}]$, although L is not a composition of transformations of the types specified in Theorem 3.1 [3.4].

The "only if" part of Theorem 3.1 does not hold if we restrict ourselves to real ω-matrices. Observe that transformations of the type

(ix) $A\to A+\alpha I$, in which α is a real linear combination of the (not necessarily diagonal) entries of A,

map $\omega_{\langle n\rangle}\cap\mathbb{R}^{n,n}$ into itself. We pose the following open problem.

QUESTION 4.52. Let L be a linear transformation on $\mathbb{R}^{n,n}$, $n\geqslant 3$, satisfying

$$N(L)\cap\{A\in\mathbb{R}^{n,n}: a_{ii}=0, 1\leqslant i\leqslant n\}=\{0\}.$$

Is it true that $L(\omega_{\langle n\rangle}\cap\mathbb{R}^{n,n})\subseteq\omega_{\langle n\rangle}\cap\mathbb{R}^{n,n}$ if and only if L is a composition of transformations of types (i), (ii), (iii), and (ix)?

We remark that in the case $n=2$ there is at least one more type of transformation,

(x) $\begin{bmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\to\begin{bmatrix} a_{11} & k_1a_{12}+k_2a_{21}\\ t_1a_{21}+t_2a_{21} & a_{22}\end{bmatrix}$, in which k_1, k_2, t_1, and t_2 are nonnegative. It is easy to verify, using Lemma 4.3, that such a transformation maps the class $\omega_{\langle n\rangle}\cap\mathbb{R}^{2,2}$ into itself.

We remark that Theorems 3.1, 3.2, 3.4, and 3.5 do not hold in general when the nonsingularity assumption (4.14) (where $\mathbb{C}$ is replaced by $\mathbb{R}$ if needed) is omitted, as demonstrated by Example 5 in [4].

Before proving Theorems 3.2 and 3.5, we note that Lemmas 4.1, 4.2, and 4.3 and Propositions 4.7, 4.9, 4.11, 4.13, and 4.15 hold also in the case $n=2$.

Proof of Theorem 3.2. By Lemma 4.3, transformations of types (i), (ii), and (v) map $\omega_{\langle 2\rangle}$ into itself. Conversely, if L satisfies (4.5) and (4.14) ($n=2$), then it follows from Proposition 4.13 that after applying a transposition if needed we have

$$L(E_{12})=aE_{12}, \qquad L(E_{21})=bE_{21}.$$

Furthermore, it easily follows from Lemma 4.3 that $ab > 0$. Hence, by applying an appropriate diagonal similarity and multiplying by a positive scalar we obtain

$$L(E_{12}) = E_{12}, \qquad L(E_{21}) = E_{21}. \tag{4.53}$$

It now follows that L is of type (v). The reality of k_1, k_2, t_1, and t_2 follows from Lemma 4.1. ∎

Proof of Theorem 3.5. Clearly, transformations of types (i) and (ii) map the class $\tau_{\langle 2\rangle} \cap F^{2,2}$ into itself. Let L be a transformation of type (vii), and let $A \in \tau_{\langle 2\rangle} \cap F^{2,2}$. By Lemma 4.3 we have

$$a_{12}a_{21} \geqslant 0. \tag{4.54}$$

Since, by Lemma 4.2, $A \in P^0_{\langle 2\rangle}$, it follows from Proposition 11 in [4] (and the remark after the proof there concerning the fact that the arguments used in the proof are equivalences) that $L(A) \in P^0_{\langle 2\rangle}$. Observe that L does not affect a_{12} and a_{21}, and therefore it follows from (4.54) and Lemmas 4.2 and 4.3 that $L(A) \in \tau_{\langle 2\rangle} \cap F^{2,2}$.

Conversely, assume that

$$L\left(\tau_{\langle 2\rangle} \cap F^{2,2}\right) \subseteq \tau_{\langle 2\rangle} \cap F^{2,2}, \tag{4.55}$$

and that

$$N(L) \cap \left\{A \in F^{2,2} : a_{11} = a_{22} = 0\right\} = \{0\}.$$

As in the proof of Theorem 3.2, we may assume that (4.53) holds. We now prove that

$$L\left(P^0_{\langle 2\rangle} \cap F^{2,2}\right) \subseteq P^0_{\langle 2\rangle} \cap F^{2,2}. \tag{4.56}$$

Let $A \in P^0_{\langle 2\rangle} \cap F^{2,2}$. If $a_{12}a_{21} \geqslant 0$, then it follows from Lemmas 4.2 and 4.3 that

$$A \in \tau_{\langle 2\rangle} \cap F^{2,2},$$

and hence by (4.55) we have

$$L(A) \in \tau_{\langle 2\rangle} \cap F^{2,2} \subseteq P^0_{\langle 2\rangle} \cap F^{2,2}.$$

If $a_{12}a_{21} < 0$, then, since by (4.55) and Proposition 4.15 the matrix

$$L\left(\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix}\right)$$

is a nonnegative diagonal matrix, it follows that $L(A) \in P^0_{\langle 2\rangle} \cap F^{2,2}$. Therefore, (4.56) holds and by Proposition 11 of [4] the transformation L is of type (vii). ∎

In order to prove our "onto" theorems we need the following immediate lemma.

LEMMA 4.57. *If the range of L contains the class* $\tau_{\langle n\rangle} \cap \mathbb{R}^{n,n}$, *then L is nonsingular.*

Proof. Our claim follows observing that $\tau_{\langle n\rangle} \cap \mathbb{R}^{n,n}$ contains the basis $\{E_{11}, \dots, E_{nn}, E_{12}, E_{21}, E_{13}, \dots, E_{n,n-1}\}$ of $F^{n,n}$. ∎

Proof of Theorem 3.3. If $L(\omega_{\langle n\rangle}) = \omega_{\langle n\rangle}$, then clearly $L(\omega_{\langle n\rangle}) \subseteq \omega_{\langle n\rangle}$, and L is nonsingular by Lemma 4.57. Conversely, assume that

$$L(\omega_{\langle n\rangle}) \subseteq \omega_{\langle n\rangle}, \tag{4.58}$$

and that L is nonsingular. Then L satisfies the nonsingularity conditions of Theorems 3.1 and 3.2. We distinguish between two cases:

(1) $n = 2$. By Theorem 3.2, $L(\omega_{\langle 2\rangle}) \subseteq \omega_{\langle 2\rangle}$ if and only if $\tilde{L}_{11}$ is real, $\tilde{L}_{12} = 0$, $\tilde{L}_{21} = 0$ and $\tilde{L}_{22}$ is a nonnegative generalized monomial. Observe that in that case $\tilde{L}^{-1}$ has the same form. Thus, it follows from (4.58) that $L^{-1}(\omega_{\langle 2\rangle}) \subseteq \omega_{\langle 2\rangle}$ and hence $L(\omega_{\langle 2\rangle}) = \omega_{\langle 2\rangle}$.

(2) $n \geqslant 3$. In view of Theorem 3.1, since clearly transformations of types (i), (ii), and (iii) map the class $\omega_{\langle n\rangle}$ onto itself, it is enough to show that a nonsingular transformation of type (iv) maps the class $\omega_{\langle n\rangle}$ onto itself. Observe that L is such a transformation if and only if

$$\tilde{L}_{12} = 0, \quad \tilde{L}_{21} = 0, \quad \tilde{L}_{22} = I, \text{ and } \tilde{L}_{11} = I + ev^T,$$

where e is an $n \times 1$ matrix all of whose entries are 1, v is an $n \times 1$ matrix, and $\tilde{L}_{11}$ is nonsingular. Observe that in that case $v^Te \neq -1$ (otherwise $\tilde{L}_{11}$ would have zero row sums) and that

$$\tilde{L}_{11}^{-1} = I + ez^T,$$

where

$$z = \frac{-1}{1+v^Te}v.$$

Therefore, $\tilde{L}^{-1}$ has the same form as $\tilde{L}$. Thus, it follows from (4.58) that $L^{-1}(\omega_{\langle n \rangle}) \subseteq \omega_{\langle n \rangle}$ and hence $L(\omega_{\langle n \rangle}) = \omega_{\langle n \rangle}$. ■

Proof of Theorem 3.6. Since clearly transformations of types (i), (ii), or (iii) map the class $\tau_{\langle n \rangle} \cap F^{n,n}$ onto itself, it follows that if L is a composition of such transformations, then

$$L\left(\tau_{\langle n \rangle} \cap F^{n,n}\right) = \tau_{\langle n \rangle} \cap F^{n,n}. \tag{4.59}$$

Conversely, we assume that (4.59) holds. By Lemma 4.57 L is nonsingular, and by Theorem 3.4 $\tilde{L}$ is the direct sum of $\tilde{L}_{11}$ and $\tilde{L}_{22}$, where $\tilde{L}_{11}$ is a nonnegative matrix whose diagonal entries are greater than or equal to 1. Furthermore, we have $L^{-1}(\tau_{\langle n \rangle} \cap F^{n,n}) \subseteq \tau_{\langle n \rangle} \cap F^{n,n}$, and hence $\tilde{L}_{11}^{-1}$ too is a nonnegative matrix with diagonal entries greater than or equal to 1. As is well known (e.g. [2, p. 84]), a nonnegative matrix which has a nonnegative inverse is a generalized monomial. Thus, the matrix $\tilde{L}_{11}$ is a generalized monomial. Since both $\tilde{L}_{11}$ and $\tilde{L}_{11}^{-1}$ have diagonal entries greater than or equal to 1, it follows that $\tilde{L}_{11} = I$. Therefore, L is a composition of transformations of types (i), (ii), and (iii) only. ■

Proof of Theorem 3.7. Assume that

$$L\left(\tau_{\langle 2 \rangle} \cap F^{2,2}\right) = \tau_{\langle 2 \rangle} \cap F^{2,2}. \tag{4.60}$$

By Lemma 4.57 L is nonsingular, and by Theorem 3.5 L is a composition of transformations of types (i), (ii), and (vii). Since clearly transformations of types (i) or (ii) map the class $\tau_{\langle 2 \rangle} \cap F^{2,2}$ onto itself, we may assume that L is of type (vii). Observe that L is such a transformation if and only if $\tilde{L}_{12} = 0$, $\tilde{L}_{21} = 0$, $\tilde{L}_{22} = I$, and

$$\tilde{L}_{11} = \begin{bmatrix} k_1 & t_1 \\ k_2 & t_2 \end{bmatrix}$$

is a nonnegative matrix, where either

$$k_1 t_2 + k_2 t_1 \geqslant 1 \tag{4.61}$$

or

$$1-2(k_1t_2+k_2t_1)+(k_1t_2-k_2t_1)^2 \leqslant 0. \tag{4.62}$$

By (4.60) we have

$$L^{-1}\left(\tau_{\langle 2\rangle} \cap F^{2,2}\right) \subseteq \tau_{\langle 2\rangle} \cap F^{2,2},$$

and hence the matrix $\tilde{L}^{-1}$ has the same form as $\tilde{L}$. Since $\tilde{L}_{11}$ and $\tilde{L}^{1}_{11}$ are nonnegative, it follows that $\tilde{L}_{11}$ is a generalized monomial and so either

$$k_2=t_1=0, \qquad k_1, t_2>0, \tag{4.63}$$

or

$$k_1=t_2=0, \qquad k_2, t_1>0. \tag{4.64}$$

If (4.63) holds, then it follows from (4.61) and (4.62) that $k_1t_2 \geqslant 1$. Similarly, considering $\tilde{L}^{-1}$, we deduce that $1/k_1t_2 \geqslant 1$, and therefore $k_1t_2=1$. Using the same arguments, we show that if (4.64) holds, then we have $k_2t_1=1$. Therefore, the transformation L is of type (viii).

Conversely, it is enough to show that a transformation L of type (viii) maps the class $\tau_{\langle 2\rangle} \cap F^{2,2}$ onto itself. Observe that L is a nonsingular transformation of type (vii) and thus, by Theorem 3.5,

$$L\left(\tau_{\langle 2\rangle} \cap F^{2,2}\right) \subseteq \tau_{\langle 2\rangle} \cap F^{2,2}. \tag{4.65}$$

Also, the transformation L^{-1} is of type (viii) and hence

$$L^{-1}\left(\tau_{\langle 2\rangle} \cap F^{2,2}\right) \subseteq \tau_{\langle 2\rangle} \cap F^{2,2}. \tag{4.66}$$

The inclusions (4.65) and (4.66) imply (4.60). ■

REFERENCES

1 A. Berman, D. Hershkowitz, and C. R. Johnson, Linear operators which preserve certain positivity classes of matrices, *Linear Algebra Appl.* 68:9–29 (1985).

2 A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York, 1979.

3 G. M. Engel and H. Schneider, The Hadamard-Fischer inequality for a class of matrices defined by eigenvalue monotonicity, *Linear and Multilinear Algebra* 4:155–176 (1976).
4 D. Hershkowitz and C. R. Johnson, Linear transformations that map the P-matrices into themselves, *Linear Algebra Appl.*, 74:23–38 (1986).
5 D. Hershkowitz, V. Mehrmann, and H. Schneider, Eigenvalue interlacing for certain classes of matrices with real principal minors, to appear.
6 R. S. Varga, A note on an open question on ω- and τ-matrices, *Linear Algebra Appl.* 18:45–52 (1977).
7 R. S. Varga, Recent results in linear algebra and their applications (in Russian), in *Numerical Methods in Linear Algebra (Proceedings of the Third Seminar on Methods of Numerical Applied Mathematics, Novosibirsk, 1978)*, Akad. Nauk SSSR Sibirsk. Otdel., Vyčisl. Centr, Novosibirsk, 1978, 5–15.

Received 4 March 1985; revised 21 June 1985