CLOSED STRUCTURES ON REFLECTIVE SUBCATEGORIES OF THE CATEGORY OF TOPOLOGICAL SPACES

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Let \( \mathcal{A} \) be a reflective subcategory of the category of all topological spaces which contains a discrete doubleton. We prove that \( \mathcal{A} \) admits a symmetric monoidal closed structure if and only if it is closed under the formation of function spaces endowed with the topology of pointwise convergence. Moreover, if a symmetric monoidal closed structure on \( \mathcal{A} \) exists it is (up to isomorphism) unique.

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Introduction

We shall use the term closed category (see [16]) instead of the term symmetric monoidal closed category used in [6]. It is known (see [3]) that any epireflective subcategory of the category \( \mathcal{T} \) of all topological spaces and continuous maps admits exactly one structure of closed category. Brandenburg and Hušek proved in [2] that any reflective subcategory of the category \( \mathcal{T} \) containing a discrete doubleton is not cartesian closed. The result presented in this paper completes these two results. We shall prove that if \( \mathcal{A} \) is a reflective subcategory of the category \( \mathcal{T} \) containing a discrete doubleton and \((\square, H)\) a closed structure on \( \mathcal{A} \), then for any \( X, Y \in \mathcal{A} \), \( H(X, Y) \) is the space of all continuous maps from \( X \) to \( Y \) endowed with the topology of pointwise convergence, i.e., there exists at most one closed structure on \( \mathcal{A} \). Moreover, the property of being closed under the formation of function spaces with the topology of pointwise convergence is sufficient for the existence of a closed structure on \( \mathcal{A} \). As an application we shall show that, for example, many known epireflective (i.e., productive and closed-hereditary) subcategories of the category of all Hausdorff spaces do not admit a closed structure so that they are not closed under the formation of function spaces with the topology of pointwise convergence.

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1. Preliminaries and notations

All undefined terminology is that of [10]. We shall always use the following notations: \( \mathcal{A}(X, Y) \) denotes the set of all \( \mathcal{A} \)-morphisms \( X \to Y \). If \( A, B, C \) are sets and \( f: A \times B \to C \) is a map, then \( f^* \) is the map \( A \to C^B \) given by \( f^*(a)(b) = f(a, b) \) for all \( a \in A, b \in B \). If \( g: A \to C^B \) is a map, then \( g_* \) is the map \( A \times B \to C \) given by \( g_*(a, b) = g(a)(b) \) for all \( a \in A, b \in B \). The term clopen means closed and open (simultaneously). \( UX \) denotes the underlying set of \( X \).

Recall (see [16]) that a triple \( (\mathcal{A}, \square, H) \) is said to be a closed category provided that \( (\mathcal{A}, \square) \) is a symmetric monoidal category (see [16, p. 180]), \( H: \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A} \) is a functor (called an internal horn-functor) and there exists a natural equivalence \( \gamma = (\gamma_{ABC}): \mathcal{A}(A \square B, C) \to \mathcal{A}(A, H(B, C)) \). A tensor product is a symmetric monoidal structure extendable to a structure of closed category.

**Theorem 1.1** [14]. Let \( (\mathcal{K}, V) \) be a concrete category with the following properties:

1. for any constant map \( c: VA \to VB \) there exists a \( \mathcal{K} \)-morphism \( k: A \to B \) with \( Vk = c \),
2. for any bijection \( f: VA \to X \) there exists a \( \mathcal{K} \)-isomorphism \( s: A \to B \) with \( Vs = f \),
3. there exists a \( \mathcal{K} \)-object \( A \) with \( card VA \geq 2 \).

Then for any closed structure \( (\square, G) \) on \( \mathcal{K} \) there exists a closed structure \( (\square, H) \) on \( \mathcal{K} \) isomorphic with \( (\square, G) \) with the following properties:

a) \( card VI = 1 \) where \( I \) is the unit of \( \square \),
b) \( VA \times VB \subseteq V(A \square B) \) for all \( A, B \in \mathcal{K} \),
c) for any \( r, s: A \square B \to C \), \( Vr|_{VA \times VB} = Vs|_{VA \times VB} \) implies that \( r = s \),
d) \( V(f \square g)|_{VA \times VB} = V(f \times Vg) \) for all \( f: A \to C, g: B \to D \),
e) \( VH(B, C) = \mathcal{H}(B, C) \) for all \( B, C \in \mathcal{K} \),
f) if \( \gamma: \mathcal{K}(A \square B, C) \to \mathcal{K}(A, H(B, C)) \) is the natural equivalence corresponding to \( (\square, H) \), then \( V\gamma(r) = (Vs)_r^* \) and \( V\gamma^{-1}(s)|_{VA \times VB} = (Vs)_s \) for all \( A, B, C \in \mathcal{K}, r \in \mathcal{K}(A \square B, C) \) and \( s \in \mathcal{K}(A, H(B, C)) \).

**Definition 1.2.** Let \( A \) be an infinite set, \( \mathcal{F} \) a filter on \( A \) with \( \bigcap \{ F: F \in \mathcal{F} \} = \emptyset \) and \( a \notin A \). Define a topology on \( A \cup \{a\} \) as follows: If \( U \subset A \cup \{a\} \), then \( U \) is open if and only if \( U \subset A \) or \( U - \{a\} \in \mathcal{F} \). The obtained topological space is said to be a filter space and denoted by \((A, a, \mathcal{F})\). If \( \mathcal{F} \) is an ultrafilter, then \((A, a, \mathcal{F})\) is said to be an ultraspace.

A filter space \((A, a, \mathcal{F})\) is said to be \( \mathbb{N} \)-incomplete provided that the filter \( \mathcal{F} \) is \( \mathbb{N} \)-incomplete, i.e., there exists a family \( \{ F_n: n \in \mathbb{N} \} \subset \mathcal{F} \) such that \( \bigcap \{ F_n: n \in \mathbb{N} \} = \emptyset \) (see [4]).

**Conventions 1.3.** (a) All reflective subcategories will be assumed to be full and isomorphism-closed.

(b) Since any nontrivial (i.e., containing a space with more than one element) reflective subcategory \( \mathcal{A} \) of the category \( \mathcal{T} \) satisfies (1)-(3) of Theorem 1.1 all
closed structures on $\mathcal{A}$ will be assumed to satisfy the conditions (a)-(f) of Theorem 1.1.

Let $X, Y$ be topological spaces, $UX, UY$ the underlying sets of $X, Y$ respectively. Then $X \otimes Y$ denotes the space on $UX \times UY$ given by the topology of separate continuity and $[X, Y]$ denotes the space on $\mathcal{T}(X, Y)$ given by the topology of pointwise convergence. The pair of functors $\otimes, [-, -]$ is the well-known (unique) closed structure on $\mathcal{T}$.

Let $\mathcal{A}$ be a subcategory of the category $\mathcal{T}$, $\mathcal{E}(\mathcal{A}) = \{X \in \mathcal{T}: X$ is a subspace of a product of $\mathcal{A}$-objects $\}$, $\mathcal{E}(\mathcal{A}) = \{X \in \mathcal{T}:$ there exists a monomorphism $m: X \to Y$ where $Y \in \mathcal{E}(\mathcal{A})\}$. Then (see [10]) $\mathcal{E}(\mathcal{A}) (\mathcal{E}(\mathcal{A}))$ is the epireflective (extremal-epireflective) hull of $\mathcal{A}$ in $\mathcal{T}$ and it holds:

**Theorem 1.4.** (a) [8] If $\mathcal{A}$ is a reflective subcategory of $\mathcal{T}$, $X \in \mathcal{E}(\mathcal{A})$ and $r_X: X \to A_X$ is the $\mathcal{A}$-reflection of $X$, then $r_X$ is a $\mathcal{T}$-embedding and an $\mathcal{E}(\mathcal{A})$-epimorphism.

(b) [10] If $\mathcal{E}(\mathcal{A})$ is a co-well-powered category, then $\mathcal{A}$ has a reflective hull in $\mathcal{T}$ that coincides with the epireflective hull of $\mathcal{A}$ in $\mathcal{E}(\mathcal{A})$.

(c) [19] Let $\mathcal{A}$ be a subcategory of $\mathcal{T}$ closed under the formation of limits in $\mathcal{T}$. Then the following statements are equivalent:

(i) $\mathcal{A}$ is reflective and co-well-powered,

(ii) $\mathcal{E}(\mathcal{A})$ is co-well-powered,

(iii) $\mathcal{E}(\mathcal{A})$ is co-well-powered.

2. Closed structures on reflective subcategories of the category $\mathcal{T}$

Our aim is to prove that reflective subcategories of the category $\mathcal{T}$ that contain a discrete doubleton have such properties that the method used in [3] for epireflective subcategories of $\mathcal{T}$ works also in this case.

The following assertion follows from [5] (it can be easily proved directly):

**Theorem 2.1.** Let $\mathcal{A}$ be a reflective subcategory of the category $\mathcal{T}$. If for any $A, B \in \mathcal{A}$, $[A, B] \in \mathcal{A}$, then there exists a closed structure $(\square, H)$ on $\mathcal{A}$ with $H(A, B) = [A, B]$ and $A \square B = A \otimes B$ for any $A, B \in \mathcal{A}$ where $A \otimes B$ is the $\mathcal{A}$-reflection of $A \otimes B$.

**Proposition 2.2.** Let $(\square, H)$ be a closed structure on a reflective subcategory $\mathcal{A}$ of $\mathcal{T}$ and $A, B, C \in \mathcal{A}$. Then:

(i) the map $j_{AB}: A \otimes B \to A \square B$, $(x, y) \mapsto (x, y)$, is continuous,

(ii) the map $i_{BC}: H(B, C) \to [B, C]$, $t \mapsto t$, is continuous.

**Proof.** (i) The map $j_{AB}$ is evidently separately continuous and therefore continuous. (ii) Consider the bijections

\[ \gamma: \mathcal{T}(H(B, C) \square B, C) \to \mathcal{T}(H(B, C), [B, C]), \]
\[ \delta: \mathcal{A}(H(B, C) \square B, C) \to \mathcal{A}(H(B, C), H(B, C)). \]
Denote by $1$ the identity morphism on $H(B, C)$ and put $e = \delta^{-1}(1)$. Then $e' = e\gamma_{H(B, C)B}$ belongs to $\mathcal{F}(H(B, C) \otimes B, C)$ and $i_{BC} = \gamma(e')$. □

**Convention 2.3.** In the following $\mathcal{A}$ will always denote a reflective subcategory of $\mathcal{F}$ containing a discrete doubleton.

It is obvious (see [8]) that $\mathcal{A}$ contains all zero-dimensional compact Hausdorff spaces.

**Proposition 2.4.** If there exists a closed structure $(\square, H)$ on $\mathcal{A}$, then $\mathcal{A}$ contains all discrete spaces.

**Proof.** Let $D$ be a discrete space with $D \in \mathcal{A}$ and $D_2$ the discrete space on the set $\{0, 1\}$. Let $r: D \to R(D)$ be the $\mathcal{A}$-reflection of $D$. Since $D = \bigsqcup \{\{d\}: d \in D\}$ in $\mathcal{F}$, $R(D) = \bigsqcup \{\{d\}: d \in D\}$ in $\mathcal{A}$ (any reflector preserves colimits). Then $H(R(D), D_2) = H(\bigsqcup \{\{d\}: d \in D\}, D_2) \cong \prod \{H(\{d\}, D_2): d \in D\} \cong [D, D_2]$ (the functor $H(-, D_2)$: $\mathcal{A}^{op} \to \mathcal{A}$ preserves limits) where the isomorphism $\varphi: H(R(D), D_2) \to [D, D_2]$ is given by $t \mapsto t|_D$. Clearly, $\varphi^{-1}: [D, D_2] \to H(R(D), D_2)$ is given by $s \mapsto \tilde{s}$ where $\tilde{s}$ is the unique extension of $s$. Since $D_2 \in \mathcal{A}$ $\mathcal{F}(\mathcal{A})$ contains $D$ so that we can assume that $D$ is a subspace of $R(D)$ (see Theorem 1.4(a)). Let $c \in R(D) - D$, $V_c = \{t \in H(R(D), D_2): t(c) = 1\}$. The set $V_c$ is open in $[R(D), D_2]$ and according to Proposition 2.2(ii) also in $H(R(D), D_2)$. Denote by $s$ the map $D \to D_2$ defined by $s(x) = 1$ for all $x \in D$. Evidently, $\tilde{s} = \varphi^{-1}(s)$ is given by $\tilde{s}(x) = 1$ for all $x \in R(D)$ so that $\tilde{s} \in V_c$. Let $W$ be an arbitrary element of the standard neighbourhood base of $s$ in $[D, D_2]$, i.e., there exist $x_1, \ldots, x_n \in D$ such that $W = \{t \in [D, D_2]: t(x_i) = 1$ for all $i = 1, \ldots, n\}$. Consider the element $t \in [D, D_2]$ with $t(x_i) = 1$ for $i = 1, \ldots, n$ and $t(x) = 0$ otherwise. Since $D = \{x_1, \ldots, x_n\} \sqcup D'$ in $\mathcal{F}$ and $R(\{x_1, \ldots, x_n\}) = \{x_1, \ldots, x_n, y\}$ we obtain that $R(D) = \{x_1, \ldots, x_n\} \sqcup R(D')$ ($\mathcal{A}$ contains all finite discrete spaces and the $\mathcal{A}$-coproduct of two $\mathcal{A}$-objects coincides with their $\mathcal{F}$-coproduct). Hence the map $\tilde{t}$ is given by $\tilde{t}(x_i) = 1$ for $i = 1, \ldots, n$ and $\tilde{t}(x) = 0$ otherwise so that $\tilde{t} \notin V_c$. Thus $\varphi^{-1}[W] \notin V_c$ for all elements of the standard neighbourhood base of $s$ in $[D, D_2]$, a contradiction. □

**Corollary 2.5.** If $\mathcal{A}$ admits a closed structure, then any $\mathcal{F}$-coproduct of a family of $\mathcal{A}$-objects belongs to $\mathcal{A}$ and coincides with the $\mathcal{A}$-coproduct of this family.

It is obvious that the epireflective hull $\mathcal{E}(\mathcal{D})$ of the class $\mathcal{D}$ of all discrete spaces is the category of all zero-dimensional Hausdorff spaces. Since $\mathcal{E}(\mathcal{D})$ is co-well-powered $\mathcal{D}$ has also a reflective hull $\mathcal{R}(\mathcal{D})$ which coincides with the epireflective hull of $\mathcal{D}$ in $\mathcal{E}(\mathcal{D})$ (see Theorem 1.4(b)). Now, we can prove the following:
Proposition 2.6. The reflective hull $\mathcal{R}(\mathcal{D})$ of the class $\mathcal{D}$ of all discrete spaces in $\mathcal{I}$ contains all filter spaces.

Proof. Let $K = (A, a, \mathcal{F})$ be a filter space with $K \not\in \mathcal{R}(\mathcal{D})$ and $r: K \to R(K)$ the $\mathcal{R}(\mathcal{D})$-reflection of $K$. Since $K \in \mathcal{F}(\mathcal{R}(\mathcal{D}))$, $r$ is a $\mathcal{F}$-embedding and an epimorphism in $\mathcal{F}(\mathcal{D})$ (see Theorem 1.4(a)) and $K$ can be assumed to be a subspace of $R(K)$.

Let $d \in R(K) - K$. Since the space $R(K)$ is zero-dimensional, there exist clopen neighbourhoods $V$ of $a$ and $W$ of $d$ such that $V \cap W = \emptyset$. The set $B = W \cap A$ is a clopen discrete subspace of $K$. Clearly, $B$ is also clopen in $R(K)$ ($K = B \cup R(K)$) so that $W - B = T$ is a clopen neighbourhood of $d$ with $T \cap K = \emptyset$. Hence, $r$ is not an $\mathcal{F}(\mathcal{D})$-epimorphism, a contradiction. \(\Box\)

As a consequence of Propositions 2.4 and 2.6 we obtain:

Proposition 2.7. If $\mathcal{A}$ admits a closed structure, then $\mathcal{A}$ contains all filter spaces.

Proposition 2.8. Any tensor product on $\mathcal{A}$ is uniquely determined by its values on $\mathcal{L} \times \mathcal{L}$.

Proposition 2.9. If $\mathcal{A}$ admits a closed structure and $K = (A, a, \mathcal{F})$, $L = (B, b, \mathcal{G})$ are filter spaces, then $K \otimes L$ belongs to $\mathcal{A}$.
obtain \( t \not\in \text{cl} M_3 \) in \( R(K \otimes L) \) for every \( t \in R(K \otimes L) - (K \otimes L) \) (\( g \) is continuous). Hence, if there exists \( t \in R(K \otimes L) - (K \otimes L) \) with \( t \in \text{cl} M \) in \( R(K \otimes L) \), then \( t \in \text{cl} M' \) in \( R(K \otimes L) \). It is obvious that \( M' \) is a clopen discrete subspace of \( K \otimes L \) so that \( K \otimes L = M' \sqcup X \) where \( X \) is the subspace of \( K \otimes L \) given by the set \( (K \otimes L) - M' \). Then \( R(K \otimes L) = R(M') \sqcup R(X) = M' \sqcup R(X) \). Since \( t \not\in M' \), \( t \in R(X) \) so that \( t \not\in \text{cl} M' \) in \( R(K \otimes L) \), a contradiction. \( \square \)

Now it is easy to see that we can continue in the same way as in [3], using 2.8–2.25 of [3] with only formal modifications. Namely, we change the meaning of \( \mathcal{A} \) (according to Convention 2.3) and substitute “extremal \( \mathcal{A} \)-epimorphism” by “regular \( \mathcal{A} \)-epimorphism” (in epireflective subcategories of \( \mathcal{F} \) the notions regular epimorphism and extremal epimorphism coincide).

**Remark 2.10.** In the first part of the proof of [3, 2.14] the following consideration can be used for proving the existence of \( K \in \mathcal{F} \) and \( s \in (1 \square e)_{-1}(t) \cap (N^* \square K) \) with \( s \in \text{cl} \Delta_n \) in \( N^* \square K \) (for an epireflective \( \mathcal{A} \) it can be simplified): Since for any \( n \in \mathbb{N} \), \( P_n = (N^* \square N^*) - \{(n, n)\} \) is clopen in \( N^* \square N^* \), \( P_n \in \mathcal{A} \). Then \( P = \bigcap \{P_n : n \in \mathbb{N}\} = (N^* \square N^*) - \Delta_n \) belongs to \( \mathcal{A} \) (\( \mathcal{A} \) is closed under limits) so that \( P \sqcup \Delta_n \in \mathcal{A} \) (\( \Delta_n \) with the discrete topology). If for any \( K \in \mathcal{F} \) and \( s \in (1 \square e)_{-1}(t) \cap (N^* \square K) \), \( s \not\in \text{cl} \Delta_n \) in \( N^* \square K \), then the map \( h : \bigsqcup \{\{(N^* \square K) : K \in \mathcal{F}\} \rightarrow P \sqcup \Delta_n ; \ h(x) = (1 \square e)(x) \) for all \( x \) is continuous so that \( 1 \square e \) is not a regular \( \mathcal{A} \)-epimorphism, a contradiction.

**Theorem 2.11.** If \((\square, H)\) is a closed structure on \( \mathcal{A} \), then for any \( Y, Z \in \mathcal{A} \), \( H(Y, Z) = [Y, Z] \).

**Proof.** Since for any \( K, L \in \mathcal{L} \), \( K \square L = K \otimes L \) [3, 2.25] we obtain that for any \( X, Y \in \mathcal{A} \), \( X \square Y = X \otimes_{\mathcal{A}} Y \) where \( X \otimes_{\mathcal{A}} Y \) is (the object part of) the \( \mathcal{A} \)-reflection of \( X \otimes Y \). Let there exist \( Y, Z \in \mathcal{A} \) with \( H(Y, Z) \neq [Y, Z] \) i.e. (according to Proposition 2.2(ii)) the topology of \( H(Y, Z) \) is finer than the topology of \([Y, Z]\). Then there exists a filter space \( K \) and a continuous map \( s : K \rightarrow [Y, Z] \) such that the map \( s' : K \rightarrow H(Y, Z) \) with \( s'(x) = s(x) \) for all \( x \in K \) is not continuous. Denote by \( a : K \otimes Y \rightarrow K \otimes Y \) the \( \mathcal{A} \)-reflection of \( K \otimes Y \), consider the equivalences
\[
\gamma : \mathcal{F}(K \otimes_Y Z) \rightarrow \mathcal{F}(K, [Y, Z]),
\]
\[
\delta : \mathcal{A}(K \otimes_{\mathcal{A}} Y, Z) \rightarrow \mathcal{A}(K, H(Y, Z))
\]
and put \( t = \gamma^{-1}(s) \). Let \( t' \) be the (unique) continuous map \( K \otimes_{\mathcal{A}} Y \rightarrow Z \) with \( t' \circ a = t \). Evidently, \( \delta(t) = t' \) so that \( t' \) is continuous, a contradiction. \( \square \)

**Corollary 2.12.** There exists at most one closed structure on \( \mathcal{A} \).

**Corollary 2.13.** If \( \mathcal{K} \) is a collection of closed reflective subcategories of \( \mathcal{F} \) each of which contains a discrete doubleton, and \( \mathcal{B} = \bigcap \{\mathcal{A} : \mathcal{A} \in \mathcal{K}\} \) is reflective, then \( \mathcal{B} \) is closed.

**Remark 2.14.** The case of nontrivial (Convention 1.3(b)) reflective subcategories of \( \mathcal{F} \) which do not contain a discrete doubleton remains still open. The best known
examples of such categories are those consisting of all powers of a nontrivial rigid $T_1$-space and it is not difficult to prove that they do not admit a closed structure.

Another special case is solved in the following:

**Proposition 2.15.** Let $\mathcal{B}$ be a nontrivial reflective subcategory of $\mathcal{F}$ which does not contain a discrete doubleton and $r: \mathbb{N} \to b\mathbb{N}$ the $\mathcal{B}$-reflection of $\mathbb{N}$. If $r[\mathbb{N}]$ is not closed in $b\mathbb{N}$, then $\mathcal{B}$ does not admit a closed structure.

**Proof.** We can suppose that $\mathbb{N}$ is a subspace of $b\mathbb{N}$ ($\mathbb{N} \in \mathcal{F}(\mathcal{B})$). Let $x \in \text{cl } \mathbb{N} - \mathbb{N}$ in $b\mathbb{N}$. Put $B_n = \{k \in \mathbb{N}: k \geq n\}$. Since $\mathcal{B}$ is contained in the category of all $T_1$-spaces $x \in \text{cl } B_n$ for all $n \in \mathbb{N}$. Let $(\boxtimes, H)$ be a closed structure on $\mathcal{B}$. Similarly as in the proof of Proposition 2.4 we can show that the map $\varphi: [\mathbb{N}, bD_2] \to H(b\mathbb{N}, bD_2)$ where $\varphi(s) = s$ is the (unique) extension of $s$ for each $s \in [\mathbb{N}, bD_2]$ is an isomorphism ($bD_2$ is the $\mathcal{B}$-reflection of $D_2$, $D_2 \subset bD_2$). Consider the sequence $(s_n: n \in \mathbb{N})$ in $[\mathbb{N}, bD_2]$ where $s_n: \mathbb{N} \to bD_2$, $s_n(k) = 1$ for each $k \in B_n$ and $s_n(k) = 0$ otherwise. Clearly, $(s_n)$ converges to $o \in [\mathbb{N}, bD_2]$ given by $o(k) = 0$ for all $k \in \mathbb{N}$. Therefore the sequence $(s_n)$ converges to $o$ in $H(b\mathbb{N}, bD_2)$. But for each $n \in \mathbb{N}$, $s_n(x) = 1$ and considering the neighbourhood $V = \{t \in H(b\mathbb{N}, bD_2): t(x) \neq 1\}$ of $o$ we obtain a contradiction. \(\square\)

**Corollary 2.16.** If $\mathcal{B}$ is a nontrivial reflective subcategory of $\mathcal{F}$ which does not contain a discrete doubleton, and all spaces in $\mathcal{B}$ are countably compact, then $\mathcal{B}$ does not admit a closed structure.

It is easy to see that the space $\mathbb{N}$ can be replaced by an arbitrary infinite discrete space in Proposition 2.15.

3. Applications and examples

First of all we give an example of a closed reflective subcategory of $\mathcal{F}$ that is not epireflective in $\mathcal{F}$. Recall that a $T_0$-space $X$ is said to be sober provided that for any nonempty irreducible closed subset $A$ of $X$ there exists $a \in A$ with $\text{cl } \{a\} = A$ (see [11]). The category $\mathcal{S}$ of all sober spaces is the reflective hull of the Sierpinski doubleton and it is an epireflective subcategory of the category $\mathcal{F}_0$ of all $T_0$-spaces.

**Example 3.1.** The category $\mathcal{S}$ of all sober spaces is closed.

**Proof.** Denote by $S$ the Sierpinski doubleton with elements $0, 1$ and closed sets $\emptyset$, $\{0\}$, $\{0, 1\}$. First we show that for any sober space $X$, $[X, S]$ belongs to $\mathcal{S}$. The space $[X, S]$ is a subspace of the space $S^{UX} \in \mathcal{F}$. Let $A$ be an irreducible closed subset of $[X, S]$, $A \neq \emptyset$ and $B$ the closure of $A$ in $S^{UX}$. It is easy to see that $B$ is an irreducible
closed subset of $S^{UX}$ so that there exists $t \in B$ with $\text{cl}\{t\} = B$ in $S^{UX}$. Put $V = \{x \in X: t(x) = 0\}$ and define $s: X \to S$ by $s(x) = 0$ for all $x \in \text{cl} V$ and $s(x) = 1$ otherwise. It is easy to see that $s \in A$ and $\text{cl}\{s\} = A$ in $[X, S]$. Since (a restriction of) $(\otimes, [-,-])$ is a closed structure on $\mathcal{T}_0$ the functor $[X, -]: \mathcal{T}_0 \to \mathcal{T}$ preserves products and $\mathcal{T}_0$-extremal (= $\mathcal{T}_0$-regular (see [18])) monomorphisms. If $Y \in \mathcal{F}$, then it is a $\mathcal{T}_0$-extremal subobject of a suitable power $S^I$ so that $[X, Y]$ is a $\mathcal{T}_0$-extremal subobject of $[X, S]^I$. Hence, $[X, Y] \in \mathcal{F}$. □

In the following we shall show that many reflective subcategories of $\mathcal{F}$ are not closed and therefore they are not closed under the formation of function spaces with the topology of pointwise convergence.

Denote by $\mathcal{H}$ the category of all totally disconnected spaces. It is known that $\mathcal{H}$ is the extremal-epireflective hull of the space $D_2$ (the discrete space on the set $\{0, 1\}$). Recall that the meaning of $\otimes$ is given by Convention 2.3.

**Proposition 3.2.** If $\mathcal{A}$ admits a closed structure, then $\mathcal{A}' = \mathcal{A} \cap \mathcal{H}$ is an epireflective subcategory of $\mathcal{F}$.

**Proof.** Since $D_2 \in \mathcal{H}$ and $\otimes(\mathcal{A}) = \mathcal{H}$ ($\otimes(\mathcal{A}')$ is the extremal-epireflective hull of $\mathcal{A}'$). $\mathcal{A}'$ is closed under the formation of limits in $\mathcal{F}$, $\mathcal{H}$ is co-well-powered so that by Theorem 1.4(c) and Corollary 2.13 $\mathcal{A}'$ is reflective in $\mathcal{F}$ and has a closed structure. Let $\mathcal{A}' = \mathcal{H}(\mathcal{A}')$, $X \in \mathcal{H}(\mathcal{A}') - \mathcal{A}'$ and $r: X \to R(X)$ be the $\mathcal{A}'$-reflection of $X$. We can assume that $X$ is a subspace of $R(X)$ (and $r(x) = x$ for all $x$). According to [5] ($\mathcal{A}'$ is a closed reflective subcategory of the closed category $\mathcal{F}$) $[r, 1]: [R(X), D_2] \to [X, D_2]$ is an isomorphism. Let $c \in R(X) - X$, $V_c = \{t \in [R(X), D_2]: t(c) = 1\}$. The set $V_c$ is a neighbourhood of $u': R(X) \to D_2$ where $u'(x) = 1$ for all $x \in R(X)$. Put $u = [r, 1](u')$ and choose arbitrary $x_1, \ldots, x_n \in X$. Then $W = \{s \in [X, D_2]: s(x_i) = 1\}$ and $s(x_i) = 1$ for $i = 1, \ldots, n$ is an element of the standard neighbourhood base of $u$. Since $R(X)$ is totally disconnected there exists $t \in [R(X), D_2]$ with $t(x_i) = 1$ for $i = 1, \ldots, n$ and $t(c) = 0$. Clearly, $t \not\in V_c$ and $[r, 1](t) \in W$. Hence, for any element $W$ of the standard base of neighbourhoods of $u$, $[r, 1]^{-1}[W]$ is not contained in $V_c$ so that $[r, 1]^{-1}$ is not continuous, a contradiction. □

**Corollary 3.3.** If $\mathcal{A}$ is a closed category, then $\mathcal{A}$ contains the category of all zero-dimensional Hausdorff spaces.

**Proof.** The category of all zero-dimensional Hausdorff spaces is the smallest epireflective subcategory of $\mathcal{F}$ containing $D_2$ (see [8]). □

Denote by $I$ the closed unit interval with the usual topology. Recall that a space $X$ is said to be functionally Hausdorff provided that for any $x, y \in X$ with $x \neq y$, there exists a continuous map $f: X \to I$ with $f(x) = 0$ and $f(y) = 1$. It is obvious that
the category \( \mathcal{F} \) of all functionally Hausdorff spaces is the extremal-epireflective hull of the space \( I \).

**Proposition 3.4.** Let \( \mathcal{A} \) be closed and contain a functionally Hausdorff space \( Y \) such that there exists a subspace \( I' \) of \( Y \) homeomorphic with \( I \). Then \( \mathcal{A}' = \mathcal{A} \cap \mathcal{F} \) is an epireflective subcategory of \( \mathcal{F} \).

**Proof.** Since \( \mathcal{F} \) is co-well-powered (\( \mathcal{F} \) is the extremal-epireflective hull of the category of all compact Hausdorff spaces, Theorem 1.4(c)) and \( \mathcal{F}(\mathcal{A}') = \mathcal{F} \mathcal{A}' \) is reflective and closed (Theorem 1.4(c), Corollary 2.1). Denote by \( f \) a homeomorphism \( I \to I' \) and put \( y_0 = f(0), y_1 = f(1) \). The rest of the proof is analogous as the proof of Proposition 3.2. Instead of \( D_2, 0, 1 \) we use \( Y, y_0, y_1 \), respectively. (Since \( R(X) \) is functionally Hausdorff there exists a continuous map \( t_i : R(X) \to I' \) with \( t_i(x_i) = y_0 \) for \( i = 1, \ldots, n \) and \( t_i(e) = y_1 \) so that there exists \( t \in [R(X), Y] \) with \( t(x_i) = y_0 \) for \( i = 1, \ldots, n \) and \( t(e) = y_1 \) ) \( \square \)

**Corollary 3.5.** If \( \mathcal{A} \) is closed and \( \mathcal{A} \) contains a Tychonoff space \( X \) with a subspace homeomorphic with \( I \), then \( \mathcal{A} \) contains all Tychonoff spaces.

**Corollary 3.6.** Any proper epireflective (i.e., productive and closed-hereditary) subcategory of the category of all Tychonoff spaces containing the space \( I \) is not closed.

Note that in the case of reflective subcategories of the category of topological spaces we have obtained a similar result as in [15] for varieties of algebras. But in general there are topological categories (over the category of sets) with many closed structures. In [12] Kelly and Rossi constructed for any cardinal \( a \) a (fibre small) topological category of quasitopological spaces with (at least) \( a \) different closed structures and they also showed that the category of all quasitopological spaces which is not fibre small has a collection of closed structures equivalent to a proper class. In the following example we give a very simple construction of topological categories with many closed structures.

**Example 3.7.** Let \( \mathcal{C} \) be the category of all sequential topological spaces and continuous maps (the coreflective hull of \( \mathcal{N}^* \) in \( \mathcal{F} \)). It is well known that \( \mathcal{C} \) is cartesian closed and has also a closed structure \( \otimes [\cdot, \cdot]_e \) where the tensor product \( \otimes \) is given by the topology of separate continuity and for any \( X, Y \in \mathcal{C}, [X, Y]_e \) is the \( \mathcal{C} \)-coreflection of \([X, Y]\) in \( \mathcal{F} \). Let \( a \) be a cardinal, \( a \geq 2 \). Define a category \( \mathcal{C}^a \) as follows: The objects of \( \mathcal{C}^a \) are all pairs \((X, u)\) where \( X \) is a set and \( u \) is a map \( a \to ST(X) \) where \( ST(X) \) is the set of all sequential topologies on \( X \). A \( \mathcal{C}^a \)-morphism \( f : (X, u) \to (Y, v) \) is a map \( X \to Y \) such that for each \( x \in a, f : (X, u(x)) \to (Y, v(x)) \) is a continuous map. It is easy to see that \( \mathcal{C}^a \) is a topological category. Now for any subset \( B \) of \( a \) we can define a tensor product \( \Box_B \) as follows: For any \((X, u), (Y, v) \in \mathcal{C}^a, (X, u) \Box_B (Y, v) = (X \times Y, w) \) where for each \( x \in B \) \( w(x) \) is the topology
of the space \( (X, u(x)) \otimes (Y, v(x)) \) and for each \( x \in a - B \), \( w(x) \) is the topology of \( (X, u(x)) \cap (Y, v(x)) \) (\( \cap \) denotes the product in \( \mathcal{C} \)). Denote by \( G \) the internal hom-functor corresponding to \( \cap \) in \( \mathcal{C} \). Then the internal hom-functor \( H_B \) corresponding to \( \Box_B \) is given by \( H_B((X, u), (Y, v)) = (\mathcal{C}^a((X, u), (Y, v)), t) \) where for each \( x \in B \), \( t(x) \) is the topology of the subspace of \( [(X, u(x)), (Y, v(x))]_c \) given by the subset \( \mathcal{C}^a((X, u), (Y, v)) \) and for each \( x \in a - B \), \( t(x) \) is given similarly using \( G((X, u(x)), (Y, v(x))) \). It is evident that for any \( B \subseteq a \), \( (\Box_B, H_B) \) is a closed structure on \( \mathcal{C}^a \) and for different subsets of \( a \) we obtain different closed structures. Hence, the category \( \mathcal{C}^a \) has (at least) \( 2^a \) closed structures. Now, let \( (\Box, H) \) be a closed structure on \( \mathcal{C}^a \). Then there is \( B \subseteq a \) such that \( (\Box, H) = (\Box_B, H_B) \). In fact, put \( \mathcal{C}^a_s = \{(X, u) \in \mathcal{C}^a: \text{for each } y \in a - \{x\}, u(y) \text{ is the discrete topology} \} \) for each \( x \in a \). The restriction \( (\Box_x, H_x) \) of the closed structure \( (\Box, H) \) to the subcategory \( \mathcal{C}^a_s \) is a closed structure on \( \mathcal{C}^a_s \). Since for each \( x \in a \), \( C^a_x \) is isomorphic with \( \mathcal{C}^a \) and \( \mathcal{C} \) has precisely two closed structures, \( (\Box_x, H_x) \) is isomorphic either with \( (\otimes, [-, -]_\mathcal{C}) \) or with \( (\cap, G) \). Now, let \( (X, u) \in \mathcal{C}^a, x \in a \). Denote by \( (X, u^*) \) the \( \mathcal{C}^a_s \)-object for which \( u^*(x) = u(x) \). Clearly, the map \( e: \bigoplus \{(X, u^*): x \in a\} \rightarrow (X, u) \) such that \( e|_{(X, u^*)} = 1_X \) for each \( x \in a \) is a regular epimorphism in \( \mathcal{C}^a \). This implies that the tensor product \( \Box \) is uniquely determined by its values on all subcategories \( \mathcal{C}^a_s \). Put \( B = \{x \in a: (\Box_x, H_x) \text{ is isomorphic with } (\otimes, [-, -]_\mathcal{C}) \} \). Then, obviously, \( (\Box, H) = (\Box_B, H_B) \). Thus, we obtain that the category \( \mathcal{C}^a \) has precisely \( 2^a \) (different) closed structures.

**Remark 3.8.** It is obvious that in Example 3.7 the category \( \mathcal{C} \) can be replaced by an arbitrary topological category with at least two (different) closed structures (even for different elements of \( a \) we can take different topological categories). If we replace \( a \) by a proper class \( K \) and construct a category \( \mathcal{C}^K \) in the same way as \( \mathcal{C}^a \), we obtain a topological category which is not fibre small and has a collection of closed structures equivalent to the collection of all subclasses of the class \( K \).

**References**