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Highly irregular graphs with extreme numbers of edges

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Abstract

A simple connected graph is highly irregular if each of its vertices is adjacent only to vertices with distinct degrees. In this paper we find: (1) the greatest number of edges of a highly irregular graph with n vertices, where n is an odd integer (for n even this number is given in [1]), (2) the smallest number of edges of a highly irregular graph of given order.

1. Introduction

This paper has been inspired by Paul Erdős's question, which was asked during Second Kraków Conference of Graph Theory (1994), concerning extreme sizes of highly irregular graphs of given order. The notion of a highly irregular graph is defined in [1] as follows:

... For a vertex v of a graph H we denote the set of all vertices adjacent to v by $N(v)$. We define a connected graph H to be *highly irregular*, if for every vertex v,

$$u, w \in N(v),\ u \neq w, \text{ implies that } \deg_H(u) \neq \deg_H(w). \ \ldots$$

For the sake of convenience such a graph will be called a *HI-graph*.

In [1] it is proved that the size of a HI-graph of order n is at most $\frac{1}{8}n(n+2)$, with equality possible for n even. We prove that if n is odd, $n \geqslant 9$, then the greatest size of a HI-graph of order n is equal to $\frac{1}{8}(n-1)(n+1)\lfloor\frac{1}{10}(n+1)\rfloor$. We also find the smallest number of edges of a HI-graph of order n, where $n \in N\backslash\{3,5,7\}$ (for $n = 3,5,7$ there does not exist HI-graph of order n, see [1]). We show that all HI-graphs of order n, $n \in N\backslash\{6,11,12,13\}$, with minimum number of edges are trees. For every $n \geqslant 16$ we give a construction of HI-trees of order n with maximum degree 4.

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2. The greatest number of edges in a HI-graph of given order

In [2], the degree sequences of highly irregular graphs are described. There it is also showed that every degree sequence of a HI-graph with maximum degree m is of the form:

$$a = (\underbrace{m,\dots,m}_{n_m},\dots,\underbrace{i,\dots,i}_{n_i},\dots,\underbrace{1,\dots,1}_{n_1})$$

$$\begin{aligned}&\text{or shorter } a = (m^{n_m},\dots,i^{n_i},\dots,1^{n_1}), \text{where}\\ &n_m \text{ and } \sum_{i=1}^{m} i\cdot n_i \text{ are even positive integers,}\\ &n_i \geqslant n_m \quad \text{for } i = 1,2,\dots,m.\end{aligned} \tag{1}$$

In [1] a construction of a HI-graph H of order n with $\frac{1}{8}n(n+2)$ edges is given. The graph H has the order $n = 2m$ and realizes the sequence $(m^2,\dots,i^2,\dots,2^2,1^2)$. We will prove that for $n = 2m+1$ and $m \geqslant 4$ there exists a HI-graph which realizes the sequence $(m^2,\dots,(r+1)^2,r^3,(r-1)^2,\dots,1^2)$, where $r = 2\lfloor\frac{1}{5}(m+1)\rfloor$.

Proposition 1. *If $n = 2k+1$ and $k \geqslant 4$, then every HI-graph of order n has at most $\frac{1}{2}k(k+1) + \lfloor\frac{1}{5}(k+1)\rfloor$ edges.*

Proof. Let G be a HI-graph of order n with maximum degree m. Then the number of all edges of G is equal to $\frac{1}{2}\sum_{i=1}^{m} i\cdot n_i$ and $m \leqslant k$. It is not difficult to notice that among all sequences of the form (1) with the length n, the number $\sum_{i=1}^{m} i\cdot n_i$ is maximal if $m = k$. Then $n_i = 2$ for all except one i, $1 \leqslant i \leqslant k$.

Let V_j be the set of all vertices of G of degree j and let $|V_i| = 3$ for some $i \in \{1,2,\dots,k\}$. We will prove that $i \leqslant k/2$. Assume, a contrario, that $i > k/2$. Every vertex of V_1 is joined only with a vertex of V_k, every vertex of $V_1 \cup V_2$ is joined only with vertices of $V_k \cup V_{k-1}$, etc. Every vertex of the set $V_1 \cup \dots \cup V_{k-i}$ is joined only with vertices of $V_k \cup \dots \cup V_{i+1}$. Hence, because $i > k-i$, every vertex of V_i can be joined only with vertices of the set $V_k \cup \dots \cup V_{k-i+1}$. Note that $|V_k \cup \dots \cup V_{k-i+1}| = 2i+1$, but $\sum_{v\in V_i} \deg(v) = 3i$, hence we have a contradiction. Thus, $i \leqslant k/2$. In the similar way as above it is possible to show that in the case $i \leqslant k/2$ each vertex of the set $V_1 \cup \dots \cup V_{i-1}$ cannot be joined with a vertex of V_i. Hence the number of vertices which may be joined with the vertices of V_i is equal to $2(k-i+1)$. Thus, we have: $3i \leqslant 2(k-i+1)$ and

$$i \leqslant \tfrac{1}{5}(2k+2). \tag{2}$$

Then in the HI-graph G there is at most $e = \frac{1}{2}k(k+1) + \frac{1}{2}i_0$ edges, where i_0 is the greatest even integer satisfying inequality (2), Thus,

$$e \leqslant \tfrac{1}{2}k(k+1) + \lfloor\tfrac{1}{5}(k+1)\rfloor. \qquad \square$$

Proposition 2. *If $n = 2m + 1$, $m \geqslant 4$, then there exists a HI-graph of order n in which the number of edges is equal to* $\frac{1}{2}m(m+1) + \lfloor \frac{1}{5}(m+1) \rfloor$.

Proof. We will construct a HI-graph G which realizes the sequence $(m^2, \ldots, (r+1)^2, r^3, (r-1)^2, \ldots, 1^2)$, where $r = 2 \cdot \lfloor \frac{1}{5}(m+1) \rfloor$. We will use the graph H which realizes the sequence $(m^2, (m-1)^2, \ldots, 2^2, 1^2)$ (it has been given in [1]). Namely, $H = (V, E)$, where

$$V = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_m\}, \qquad E = \bigcup_{i=1}^{m} \{(v_i, u_j);\ m - i + 1 \leqslant j \leqslant m\}.$$

Note: $(v_i, u_j) \in E \Leftrightarrow (v_j, u_i) \in E$, and $\deg(v_i) = \deg(u_i) = i$ for $i = 1, 2, \ldots, m$. Let $A = \{v_{r+1}, v_{r+2}, \ldots, v_{m-r}\}$ and $B = \{u_{r+1}, u_{r+2}, \ldots, u_{m-r}\}$. Note that $|A| = |B| = m - 2r$ and the graph H has no edges (x, y) such that $x \in A \cup B$, $y \in \{u_r, v_r\}$. By the assumption concerning the number r it follows that

$$m - 2r \geqslant \tfrac{1}{2}r - 1.$$

Let E_1 be a set of $(\frac{1}{2}r - 1)$ vertex-disjoint edges joining A and B, and let $A' = V_{E_1} \cap A$, $B' = V_{E_1} \cap B$, where V_{E_1} denotes the set of ends of all edges belonging to E_1. Moreover, let C be any subset of the set $\{u_{m-r+1}, u_{m-r+2}, \ldots, u_m\}$ containing $\frac{1}{2}r$ elements. Then the graph G which realizes the sequence $(m^2, \ldots, (r+1)^2, r^3, (r-1)^2, \ldots, 1^2)$ we define as follows:

(1) to the set V we add a new vertex v_r^*,

(2) from the graph H we remove: all edges of the set E_1 and all edges that join the vertex v_r with the vertices of C,

(3) we join: the vertex v_r with all vertices of B'; the vertex v_r^* with all vertices of $A' \cup C$; the vertices v_r and v_r^*. □

From Propositions 1 and 2, it follows immediately

Theorem 1. *If n is an odd integer and $n \geqslant 9$, then the maximum number of edges of a HI-graph with n vertices is equal to*

$$\tfrac{1}{8}(n-1)(n+1) + \lfloor \tfrac{1}{10}(n+1) \rfloor.$$

3. The smallest number of edges of a HI-graph with given number of vertices

It is clear that a highly irregular graph of order n has at least $n - 1$ edges. We show that for every $n \in N \backslash \{3, 5, 6, 7, 11, 12, 13\}$ there exists a tree of order n being a HI-graph (a HI-tree). To prove it we use the following criterion (see [2]):

Let $a = (\underbrace{m, \ldots, m}_{n_m}, \ldots, \underbrace{i, \ldots, i}_{n_i}, \ldots, \underbrace{1, \ldots, 1}_{n_1})$ be a sequence of positive integers satisfying the conditions: the numbers n_m and $\sum_{i=1}^{m} i \cdot n_i$ are even positive integers, and $n_i \geqslant n_m$

for $i = 1, 2, \ldots, m$. The sequence a is the degree sequence of some graph in which each component is a HI-graph if and only if for every $I, J \subseteq \{1, 2, \ldots, m\}$ we have

$$\sum_{i \in I, j \in J} c_{ij}^* \geqslant \sum_{i \in I} i \cdot n_i + \sum_{j \in J} j \cdot n_j - \sum_{r=1}^{m} r \cdot n_r, \tag{3}$$

where

$$c_{ij}^* = \begin{cases} \min\{n_i, n_j\} & \text{for } i \neq j, \\ 2 \cdot \lfloor \frac{1}{2} \cdot n_i \rfloor & \text{for } i = j. \end{cases}$$

Proposition 3. *For $n = 6, 11, 12, 13$ a HI-tree of order n does not exist.*

Proof. For $n = 6, 11, 12, 13$, only $(3^2, 2^5, 1^4)$, $(3^2, 2^6, 1^4)$, $(3^2, 2^7, 1^4)$ are n-element sequences of the form (1) for which the sum of all members is equal to $2(n-1)$. For these sequences condition (3) is not satisfied (it is sufficient to put $I = J = \{2, 3\}$). $\square$

For $n = 6, 12, 13$ there exist unicyclic HI-graphs (see Fig. 1).

Proposition 4. *A unicyclic HI-graph of order* 11 *does not exist.*

Proof. It is sufficient to note that $(3^2, 2^7, 1^2)$ is the only sequence satisfying (1) for which the sum of members is equal to 22. For this sequence condition (3) does not hold (it is sufficient to put $I = J = \{2, 3\}$). $\square$

In Fig. 1 a 2-cyclic HI-graph G_{11}, with 11 vertices is given.

Proposition 5. *For each $n \in N \setminus \{3, 5, 6, 7, 11, 12, 13\}$ there exists a HI-tree of order n.*

Proof. In Fig. 2 HI-trees of order at most 17 are presented. $\square$

For $n \geqslant 18$ we give a construction of a HI-tree of order n. This construction is based on the tree G_{18} presented in Fig. 3.

To shorten the description of the steps of the construction we accept the following definitions:

Let G and H be graphs such that $V(G) \cap V(H) = \emptyset$ and let $y \in V(G)$, $x \in V(H)$. By $G(y\,\mathrm{id}\,x)H$ the graph obtained from G and H by identification of the vertices y and x will be denoted.

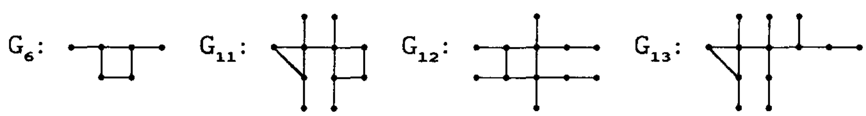

Fig. 1.

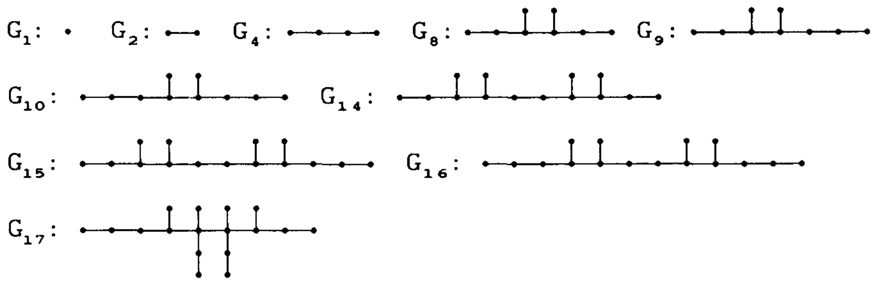

Fig. 2.

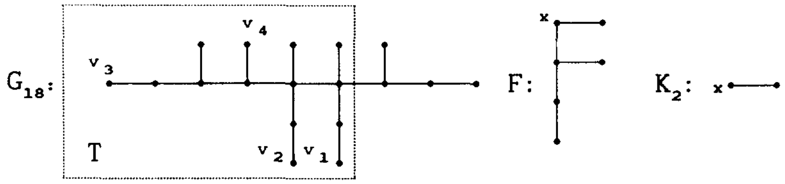

Fig. 3.

If H is a subgraph of a graph G, then a vertex y of G will be called *$(G-H)$-pendant* if: (1) $y \notin V(H)$, (2) y is a pendant vertex in G, and (3) y is joined in G with a vertex of degree 2.

Let T be the subgraph of G_{18} marked out in Fig. 3 by the dotted rectangle and let F and K_2 be remaining graphs in this figure.

Step 0: $G_{19} = G_{18}(y\,\mathrm{id}\,x)K_2$, where y is $(G_{18} - T)$-pendant

$$G_{19+j} = \begin{cases} G_{19}(v_j\,\mathrm{id}\,x)K_2 & \text{for } j = 1,2,3,4, \\ G_{19}(y\,\mathrm{id}\,x)F, \text{ where } y \text{ is } (G_{19} - T)\text{-pendant} & \text{for } j = 5. \end{cases}$$

Step k: $G_{19+6k} = G_{19+6k-1}(y\,\mathrm{id}\,x)K_2$, where y is $(G_{19+6k-1} - T)$-pendant

$$G_{19+6k+j} = \begin{cases} G_{19+6k}(v_j\,\mathrm{id}\,x)K_2 & \text{for } j=1,2,3,4, \\ G_{19+6k}(y\,\mathrm{id}\,x)F, \text{ where } y \text{ is } (G_{19+6k} - T)\text{-pendant} & \text{for } j = 5. \end{cases}$$

It is easy to check that the graphs G_n obtained by above construction, are HI-trees.

Note that each of the graphs G_n, $n \geqslant 18$, has exactly two vertices of degree 4 and maximum degree of them is 4, too. In a similar way as above we can construct a sequence of HI-trees of order n, $n \geqslant 2^m$, with given maximum degree $m > 4$.

From Propositions 3, 4 and 5 we obtain

Theorem 2. *If n is a positive integer different from* 3, 5, 7, *then the minimal number of edges of a HI-graph with given order n is equal to*:

(i) $n - 1$ *for* $n \neq 6, 11, 12, 13$,

(ii) n *for* $n = 6, 12, 13$,

(iii) $n + 1$ *for* $n = 11$.

References

[1] Y. Alavi, G. Chartrand, F.R.K. Chung, Paul Erdős, R.L. Graham and O.R. Oellermann, Highly irregular graphs, J. Graph Theory 11(2) (1987) 235–249.

[2] Z. Majcher and J. Michael, Degree sequences of highly irregular graphs, Discrete Mathematics 164 (this Vol.) (1997) 225–236.