A Noncollision Periodic Solution for
N-Body Problems

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Using variational minimization methods, we prove the existence of one
noncollision periodic solution for N-body type problems whose potentials are
pinched between two homogeneous potentials in \( R^k \) \((k \geq 2)\).

Key Words: N-body problems; noncollision periodic solution; variational methods.

1. INTRODUCTION AND MAIN RESULTS

The motion of N-body type problems \([1, 2, 9, 12, 19, 20]\) is related with
solving the following second order differential equations,

\[
m_i \ddot{q}_i = \frac{\partial U}{\partial q_i},
\]

where \( m_i > 0 \) is the mass of the \( i \)th body and \( q_i \in R^k \) \((k \geq 2)\) is the position
of the \( i \)th body, and the potential

\[
U(q) = U(q_1, \ldots, q_N) = \sum_{1 \leq i < j \leq N} U_{ij}(q_i - q_j),
\]

where \( U_{ij}(x) \in C^1(R^k \setminus \{0, R\}) \).

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In the last 20 years, some researchers applied variational methods to study the periodic solutions of N-body type problems [3–8, 10, 13, 14, 18, 21, 22], but they didn’t get the existence of one noncollision periodic solution for any given masses of N bodies. Observing the symmetry and choosing a suitable domain of the Lagrangian action integral for (1.1), we prove that the minimizer of the Lagrangian action integral is one noncollision periodic solution of (1.1)–(1.2) assuming the potential $U(q)$ is pinched between two homogeneous potentials.

Let $O(k)(k \geq 2)$ denote the rotational group in $\mathbb{R}^k$ and

\[ A(\theta) = \begin{pmatrix} B(\theta) & 0 \\ 0 & -I_{k-2} \end{pmatrix} \in O(k) \quad \theta \in (0, 2\pi), \tag{1.3} \]

where $I_{k-2}$ is a unit matrix with order $k - 2$ and

\[ B(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}. \tag{1.4} \]

Let

\[ H = W^{1,2}(\mathbb{R}/TZ, \mathbb{R}^k) \]

\[ H_\# = \left\{ x \in H | x(t + \frac{T}{r}) = A\left(\frac{2\pi}{r}\right)x(t), r \geq 2 \text{ an integer} \right\} \tag{1.5} \]

\[ E = \left\{ q = (q_1, \ldots, q_N) q_i - q_j \in H_\#, i, j = 1, \ldots, N \right\} \tag{1.6} \]

\[ \tilde{E} = \left\{ q = (q_1, \ldots, q_N) \in E | \sum_{i=1}^{N} m_i q_i(t) \equiv 0 \right\} \tag{1.7} \]

\[ \Lambda = \left\{ q = (q_1, \ldots, q_N) \in \tilde{E} | q_i(t) \neq q_j(t), \quad \forall t \in \mathbb{R}, 1 \leq i \neq j \leq N \right\} \tag{1.8} \]

\[ f(q) = \frac{1}{2} \int_0^T \sum_{i=1}^{N} m_i |\dot{q}_i|^2 dt + \int_0^T U(q)dt. \tag{1.9} \]

**Theorem 1.1.** Assume $U(q)$ satisfies

\[ (1) \]

\[ \frac{a}{2} \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|q_i - q_j|^\alpha} \leq U(q) \leq \frac{b}{2} \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|q_i - q_j|^\alpha}, \quad \alpha > 0 \tag{1.10} \]
for some
\[ A = A(2\pi/r) \in O(k), \]
where \( r \) will be defined later.

Then there is an integer \( r \) depending on \( \alpha \) and masses \( m_1, \ldots, m_N \) such that the minimizer of \( f(q) \) on \( \Lambda \) is one noncollision \( T \)-periodic solution for (1.1)–(1.2).

**Theorem 1.2.** If \( \alpha = 1, N = 3, m_1 = m_2 = m_3 = 1, \) and \( a = b = 1 \) then the minimizer of \( f(q) \) on \( \Lambda \) with \( r = 2 \) is one noncollision \( T \)-periodic solution.

**Remark.** The domain \( \Lambda \) for \( f(q) \) is different from the one defined by Bessi and Coti Zelati [4].

## 2. THE PROOF OF THEOREM 1.1

In order to prove Theorem 1.1, first we give some lemmas:

**Lemma 2.1.** The critical points of \( f(q) \) in \( \Lambda \) are noncollision \( T \)-periodic solutions of (1.1)–(1.2).

First, we prove that the critical point \( q \in \tilde{E} \) for \( f(q) \) restricted on \( \tilde{E} \) is also a critical point for \( f(q) \) on \( E \).

In fact, the condition that the center of masses is fixed at the origin is equivalent to
\[
g(q) = \left| \sum_{i=1}^{N} m_i q_i(t) \right|^2 = 0. \tag{2.1} \]

Hence by the Lagrangian multiplier rule, for any critical point \( q \) of \( f(q) \) on \( \tilde{E} \), we have
\[
f'(q) + \lambda g'(q) = 0; \tag{2.2} \]
that is, for any \( \varphi = (\varphi_1, \ldots, \varphi_N), \varphi_i \in W^{1,2}(\mathbb{R}/TZ, \mathbb{R}^k) \) we have
\[
\langle f'(q), \varphi \rangle + \lambda \langle g'(q), \varphi \rangle = 0 \tag{2.3} \]
\[
\langle f'(q), \varphi \rangle + 2\lambda \left| \sum_{i=1}^{N} m_i q_i \right| \sum_{i=1}^{N} (m_i \dot{q}_i, \varphi_i) = 0, \tag{2.4} \]
that is,
\[ \langle f'(q), \varphi \rangle = 0. \]  
(2.5)

Now assume \( q \in E \) is a critical point of \( f(q) \) on \( E \). Then
\[ \langle f'(q), y \rangle = 0, \quad \forall y \in E. \]  
(2.6)

Hence we have \( p = (p_1, \ldots, p_N) \in E^\perp \), where
\[ p_i \equiv \ddot{q}_i - \frac{\partial U}{\partial q_i}, \quad i = 1, \ldots, N. \]  
(2.7)

On the other hand, we notice that
\[ p_i - p_j = \ddot{q}_i - \ddot{q}_j - \frac{\partial U}{\partial q_i} + \frac{\partial U}{\partial q_j} \]
\[ = (q_i - q_j)'' - \sum_{j \neq i, j=1}^{N} \frac{aam_i m_j (q_j - q_i)}{|q_i - q_j|^{n+2}} \]
\[ + \sum_{j \neq i, i=1}^{N} \frac{aam_i m_j (q_i - q_j)}{|q_j - q_i|^{n+2}}. \]  
(2.8)

Hence \( p_i - p_j \in H_\#, p \in E \). Hence \( p \in E^\perp \cap E = \{0\} \); that is, \( q \) is a solution of (1.1).

**Lemma 2.2.** The functional \( f \) is coercive on \( \Lambda \); that is, for any \( \{q_n\} \subset \Lambda \), \( \|q_n\|_H \rightarrow +\infty, f(q_n) \rightarrow +\infty \).

**Proof.** For any \( q = (q_1, \ldots, q_N) \in \Lambda \), we have
\[ (q_i - q_j) \left( t + \frac{T}{r} \right) = q_i \left( t + \frac{T}{r} \right) - q_j \left( t + \frac{T}{r} \right) \]
\[ = A \left( \frac{2\pi}{r} \right) (q_i(t) - q_j(t)) \]
\[ = A \left( \frac{2\pi}{r} \right) (q_i - q_j)(t) \]  
(2.9)

\[ \left| (q_i - q_j) \left( t + \frac{T}{l} \right) - (q_i - q_j)(t) \right|^2 \]
\[ = \left| A \left( \frac{2\pi}{r} \right) (q_i - q_j)(t) - (q_i - q_j)(t) \right|^2 \]
\[ = \left| 2 \sin \frac{\pi}{r} \right|^2 \left| (q_i - q_j)(t) \right|^2. \]  
(2.10)
On the other hand,
\[
\left| (q_i - q_j) \left( t + \frac{T}{r} \right) - (q_i - q_j)(t) \right|^2 \\
= \left| \int_t^{t+\frac{T}{r}} (\dot{q}_i - \dot{q}_j) dt \right|^2 \\
\leq \frac{T}{r} \left( \int_t^{t+\frac{T}{r}} |(\dot{q}_i - \dot{q}_j)(t)|^2 dt \right) \\
= \frac{T}{r^2} \int_0^T |(\dot{q}_i - \dot{q}_j)(t)|^2 dt.
\]
(2.11)

Hence we have
\[
\int_0^T |(\dot{q}_i - \dot{q}_j)(t)|^2 dt \geq \frac{r^2}{T} \left| 2 \sin \frac{\pi}{r} \right|^2 |(q_i - q_j)(t)|^2 \\
\int_0^T \sum_{1 \leq i < j \leq N} m_i m_j |\dot{q}_i - \dot{q}_j|^2 dt \\
\geq \frac{r^2}{T} \left| 2 \sin \frac{\pi}{r} \right|^2 \sum_{1 \leq i < j \leq N} m_i m_j |q_i - q_j|^2 \\
M \int_0^T \sum_{i=1}^N m_i |\dot{q}_i|^2 dt \geq \frac{r^2}{T} \left| 2 \sin \frac{\pi}{r} \right|^2 M \sum_{i=1}^N m_i |q_i|^2,
\]
(2.13)
(2.14)

where
\[
M = \sum_{i=1}^N m_i.
\]
(2.15)

Hence the standard norm for \( H \) is equivalent to
\[
\| \dot{q} \|_2 = \left( \int_0^1 \sum_{i=1}^N m_i |\dot{q}_i|^2 dt \right)^{1/2}.
\]
(2.16)

Hence the definition of \( f(q) \) implies \( f \) is coercive.

**Lemma 2.3.** The system (1.1)–(1.2) has a weak \( T \)-periodic solution \( q = (q_1, \ldots, q_N) \in \mathcal{X} \) in the sense of Bari and Rabinowitz [3]:

(1°) \( q_i \in W^{1,2}(\mathbb{R}/T\mathbb{Z}, \mathbb{R}^k) \).

(2°) The collision set \( C = \{ t \in [0, T] | q_i(t) = q_j(t) \text{ for some } 1 \leq i \neq j \leq N \} \) has Lebesgue measure 0.
(3') $q_i$ is $C^2$ on $[0, T] \cap C$ and satisfies (1.1) and energy conservation,

$$\frac{1}{2} \sum_{i=1}^{N} m_i |\dot{q}_i|^2 - U(q_1, \ldots, q_N) = h. \quad (2.17)$$

**Proof.** It’s easy to prove $f(q)$ has positive lower bound and is weakly lower semi-continuous, so Lemma 2.2 implies Lemma 2.3.

In order to get a good lower bound estimate of $f(q)$ on the collision solutions, we need another lemma:

**Lemma 2.4.** Let index sets $A$ and $B$ satisfy $A \cap B = \emptyset$ and $A \cup B = \{1, 2, \ldots, N\}$. Then

$$\sum_{(i,j) \in A \times B} \frac{m_i m_j}{|q_i - q_j|^\alpha} \geq \left( \sum_{(i,j) \in (A \times B)} m_i m_j \right)^{1+\frac{\alpha}{2}} \left( \sum_{(i,j) \in (A \times B)} m_i m_j |q_i - q_j|^2 \right)^{-\frac{\alpha}{2}}. \quad (2.18)$$

For the proof of Lemma 2.4 refer to Long and Zhang [13]. In order to facilitate the reader, we repeat it. By Hölder’s inequality, we have

$$\left( \sum_{i \in A, j \in B} m_i m_j \right)^2 \leq \left( \sum_{i \in A, j \in B} \frac{m_i m_j}{|q_i - q_j|^\alpha} \right) \left( \sum_{i \in A, j \in B} m_i m_j |q_i - q_j|^2 \right)^{\frac{\alpha}{2}}. \quad (2.19)$$

By Hölder’s inequality, we have

$$\sum_{i \in A, j \in B} m_i m_j |q_i - q_j|^\alpha \leq \left( \sum_{i \in A, j \in B} m_i m_j \right)^{\frac{2\alpha}{2-\alpha}} \left( \sum_{i \in A, j \in B} m_i m_j |q_i - q_j|^2 \right)^{\frac{-\alpha}{2-\alpha}}. \quad (2.20)$$

By (2.19) and (2.20) we get (2.18).

Now we estimate the lower bound of $f(q)$ on the collision solutions. Let $S_N$ denote the group of all the permutations of $\{1, \ldots, N\}$. For $l = 2, \ldots, N$, we set

$$\partial \Lambda_l = \{ q \in E \mid \exists s \in S_N, \exists \bar{t} \in [0, T] \text{ s.t. } q_{s(i)}(\bar{t}) = \cdots = q_{s(i)}(\bar{t}) \}. \quad (2.21)$$

First, we assume $l = 2$, $s$ is the identity, and $\bar{t} = 0$. Then by the Lagrangian identity and the symmetry property $(q_i - q_j)(t + \frac{T}{2}) = A(\frac{2\pi}{T})(q_i - q_j)(t)$ we have

$$f(q) \geq g_1(q) + g_2(q) + g_3(q). \quad (2.22)$$
where

\[ g_1(q) = r \left[ \frac{1}{2M} \sum_{1 \leq i \neq j \leq 2} m_i m_j \right. \]
\[ \left. \times \int_0^{T/r} \left( \frac{1}{2} |q_i - \dot{q}_j|^2 + M a \frac{1}{|q_i - q_j|^\alpha} \right) dt \right] \quad (2.23) \]

\[ g_2(q) = r \left[ \frac{1}{2M} \sum_{3 \leq i \neq j \leq N} m_i m_j \right. \]
\[ \left. \times \int_0^{T/r} \left( \frac{1}{2} |q_i - \dot{q}_j|^2 + M a \frac{1}{|q_i - q_j|^\alpha} \right) dt \right] \quad (2.24) \]

\[ g_3(q) = r \left[ \frac{2}{2M} \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j \right. \]
\[ \left. \times \int_0^{T/r} \left( \frac{1}{2} |q_i - \dot{q}_j|^2 + M a \frac{1}{|q_i - q_j|^\alpha} \right) dt \right]. \quad (2.25) \]

Using the estimates of the Lagrangian action integral on collision solutions of two body [7] problems we have

**Lemma 2.5.**

\[ g_1(q) \geq C_1 r^{2\alpha} T^{\frac{1}{\alpha}} , \quad (2.26) \]

where

\[ C_1 = AM_2^{\frac{2}{\alpha}} \sum_{1 \leq i \neq j \leq 2} m_i m_j \quad (2.27) \]

\[ M_2 = \sum_{i=1}^2 m_i \quad (2.28) \]

\[ A = \frac{1}{2} \left( 2 + \frac{1}{\alpha} \right) (\alpha a)^{2/(\alpha + 2)} (2\pi)^{\frac{2}{\alpha}} . \quad (2.29) \]

Let

\[ B_2 = \min_{j \in S_N} \left( \sum_{1 \leq i \neq j \leq 2} \frac{m_i m_j}{\alpha/(2 + \alpha)} \right) \quad (2.30) \]

\[ C_1 = A B_2 . \quad (2.31) \]

Then

\[ \inf \{g_1(q), q \in \partial \Lambda_2 \} \geq \tilde{C}_1 r^{2\alpha} T^{\frac{1}{\alpha}} . \quad (2.32) \]
By the arguments of Degiovanni and Giannoni [7], we can get the lower bound estimate on $g_2(q)$.

$$g_2(q) \geq C_2 T^{\frac{2}{3N}},$$

where

$$C_2 = AM_{N-2}^\frac{2}{3N} \sum_{i=3}^{N} m_i m_j$$

(2.34)

$$M_{N-2} = \sum_{i=3}^{N} m_i.$$  

(2.35)

Let

$$B_{N-2} = \min_{s \in SN} \sum_{1 \leq i \leq 2}^{N} \sum_{3 \leq j \leq N} \sum_{m_{(i, j)}} m_i m_j |q_i - q_j|^2$$

(2.36)

$$\tilde{C}_2 = AB_{N-2}.$$  

(2.37)

Then

$$\inf \{ g_2(q), q \in \partial \Lambda_2 \} \geq \tilde{C}_2 T^{\frac{2}{3N}}.$$  

(2.38)

We use inequality (2.18) of Lemma 2.4, Sundman's inequality [19], and the arguments of Degiovanni and Giannoni [7] to estimate the lower bound for $g_3(q)$:

$$g_3(q) \geq \frac{1}{2M} \int_0^T \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j |\dot{q}_i - \dot{q}_j|^2 dt + a \left( \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j \right)^{1+\frac{3}{2}}$$

$$\times \left( \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j |q_i - q_j|^2 \right)^{-\alpha/2} dt$$

(2.39)

$$\geq \frac{1}{2M} \int_0^T \left[ \left( \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j |q_i - q_j|^2 \right)^{1/2} \right]^2 dt$$

$$+ a \left( \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j \right)^{1+\frac{3}{2}}$$

$$\times \int_0^T \left( \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j |q_i - q_j|^2 \right)^{-\alpha/2} dt$$

(2.40)

$$\geq \inf \left\{ \frac{1}{2M} \int_0^T |\dot{r}|^2 dt + a \left( \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j \right)^{1+\frac{3}{2}} \right.$$

$$\times \left. \int_0^T r^{-\alpha} dr, r \in W^{1,2}([0, T], R^+) \right\}$$

(2.41)
\[ T \inf \left\{ \frac{1}{2M} \left( \frac{2\pi}{T} \right)^2 R^2 + a \left( \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j \right)^{1+\gamma} \times \frac{1}{R^\gamma}, R > 0 \right\} \] (2.42)

\[ = \left( \frac{\alpha}{2} \right)^{\frac{1}{1+\gamma}} \left( 1 + \frac{2}{\alpha} \right) \frac{1}{2M} \left( \frac{2\pi}{T} \right)^{2\gamma} \left( \left( a \sum_{1 \leq i \leq 2, 3 \leq j \leq N} m_i m_j \right)^{1+\gamma} \right)^{\frac{1}{1+\gamma}} \] (2.43)

Let

\[ B = \frac{\min_{\delta \in \Sigma} \sum_{1 \leq i \neq j \leq N} m_{\delta(i)m_{\delta(j)}}}{M^{\alpha/(\alpha+2)}} \] (2.44)

\[ \tilde{C}_3 = 2AB. \] (2.45)

Then

\[ \inf\{g_\delta(q), q \in \partial \Lambda_2\} \geq \tilde{C}_3 T^{\frac{1+\gamma}{2}} \] (2.46)

\[ \inf\{f(q), q \in \partial \Lambda_2\} \geq (\tilde{C}_1 r^{\frac{1+\gamma}{2}} + \tilde{C}_2 + \tilde{C}_3) T^{\frac{1+\gamma}{2}}. \] (2.47)

It’s easy to see that for \( l > 2 \) we also have

\[ \inf\{f(q), q \in \partial \Lambda_l\} \geq (\tilde{C}_1 r^{\frac{1+\gamma}{2}} + \tilde{C}_2 + \tilde{C}_3) T^{\frac{1+\gamma}{2}}. \] (2.48)

**Remark.** The corresponding lower bound estimate in Bessi and Coti Zelati [4] is not correct since the symmetry breaks down when they move the binary collision to the origin and they work on their domain.

**Lemma 2.6 [6].** Let

\[ \rho = \sum_{1 \leq i \neq j \leq N} \frac{m_i m_j}{\sin(\pi(i-j))/N}^{\alpha} \] (2.49)

\[ \sigma = \sum_{1 \leq i \neq j \leq N} m_i m_j \left| \sin \frac{\pi(i-j)}{N} \right|^2. \] (2.50)
Then the minimizing value for Lagrangian action $f(q)$ has upper bound estimate

$$f(q) \leq \frac{1}{2} \left( \frac{1}{2} + \frac{1}{\alpha} \right) (b \alpha \frac{\pi}{2}) \rho \sigma M^{-\alpha/(\alpha+2)} T \frac{\pi}{3} \equiv \widetilde{C} T \frac{\pi}{3}.\quad (2.51)$$

Now we can prove Theorem 1.1.
Assume the minimizer $q$ for $f(q)$ on $\widetilde{\Lambda}$ has a collision time $i \in [0, T]$. Then by Lemma 2.5, we can get a lower bound estimate $(\widetilde{C}_1 r^{2a/(2+a)} + \widetilde{C}_2 + \widetilde{C}_3) T^{(2-a)/(2+a)}$ for $f(q)$, which depends on the integer $r \geq 2$. By Lemma 2.6 we can choose $r$ large enough so that

$$\widetilde{C}_1 r^{2a} + \widetilde{C}_2 + \widetilde{C}_3 > \widetilde{C}.\quad (2.52)$$

This is a contradiction.

Now we prove Theorem 1.2.
If $\alpha = 1, m_1 = m_2 = m_3 = 1, a = 1$, and $r = 2$ then we have

$$A = \frac{1}{2} (2\pi)^{2/3}$$
$$B_1 = 2^{2/3}$$
$$\widetilde{C}_1 = 3 \cdot 2^{-4/3} (2\pi)^{2/3}, \quad \widetilde{C}_1 r^{2a} = 3 \cdot 2^{-2/3} (2\pi)^{2/3}$$
$$\widetilde{C}_2 = 0$$
$$B = 2 \cdot 3^{1/3}$$
$$\widetilde{C}_3 = 2AB = 3^{2/3} (2\pi)^{2/3}$$
$$\widetilde{C}_1 r^{2a} + \widetilde{C}_2 + \widetilde{C}_3 = (3 \cdot 2^{-2/3} + 3^{2/3}) (2\pi)^{2/3}.$$

On the other hand, we compute

$$\rho = \sum_{1 \leq i \neq j \leq 3} \frac{1}{|\sin(\pi(i - j))/3|} = \frac{2}{\sqrt{3}} 6 = 4 \cdot 3^{1/2}$$
$$\sigma = \sum_{1 \leq i \neq j \leq 3} \left| \frac{\pi(i - j)}{3} \right|^2 = 6 \left| \frac{\pi}{3} \right|^2 = \frac{9}{2}$$
$$\widetilde{C} = \frac{13}{22} (2\pi)^{2/3} (4 \cdot 3^{1/2})^{2/3} \left( \frac{9}{2} \right)^{1/3} \cdot 3^{-1/3}$$
$$= \frac{1}{2} (2\pi)^{2/3} (3 \cdot 2^{-2/3} + 3^{2/3}) (2\pi)^{2/3}$$
$$= \widetilde{C}_1 r^{2a} + \widetilde{C}_2 + \widetilde{C}_3.$$
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