JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **167**, 49–56 (1992)

Two Mappings in Connection to Hadamard's Inequalities

SEVER SILVESTRU DRAGOMIR

Strada Trandafirilor 60, 1600 Băile Herculane, Jud. Caraş-Severin, Romania

Submitted by J. L. Brenner

Received November 27, 1990

1. INTRODUCTION

Recall that the inequalities

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \leqslant \frac{f(a)+f(b)}{2} \tag{1}$$

which hold for all convex mappings $f\colon [a, b] \to \mathbb{R}$ are known in the literature as Hadamard's inequalities. We note that J. Hadamard was not the first who discovered them. As is pointed out by D. S. Mitrinović and I. B. Lačković [3] the inequalities (1) are due to C. Hermite who obtained them in 1883, ten years before J. Hadamard.

In this paper, by the use of mappings (2) and (4), we shall establish some refinements of (1). For other inequalities in connection to Hadamard's result see [1, 2, 5], where further references are given.

2. THE MAIN RESULTS

Now, for a given convex mapping $f\colon [a, b] \to \mathbb{R}$, let $H\colon [0, 1] \to \mathbb{R}$ be defined by

$$H(t) = \frac{1}{b-a}\int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right) dx. \tag{2}$$

The following theorem holds.

0022-247X/92 $5.00

THEOREM 1. *Let $f\colon [a, b] \to \mathbb{R}$ be as above. Then*

(i) *H is convex on $[0, 1]$;*

(ii) *we have*

$$\inf_{t \in [0,1]} H(t) = H(0) = f\left(\frac{a+b}{2}\right),$$

$$\sup_{t \in [0,1]} H(t) = H(1) = \frac{1}{b-a}\int_a^b f(x)\,dx;$$

(iii) *H increases monotonically on $[0, 1]$.*

Proof. (i) Let $\alpha, \beta \geqslant 0$ with $\alpha + \beta = 1$ and $t_1, t_2 \in [0, 1]$. Then

$$\begin{aligned} H(\alpha t_1 + \beta t_2) &= \frac{1}{b-a}\int_a^b f\left(\alpha\left(t_1 x + (1-t_1)\frac{a+b}{2}\right)\right. \\ &\qquad \left. + \beta\left(t_2 x + (1-t_2)\frac{a+b}{2}\right)\right) dx \\ &\leqslant \alpha \frac{1}{b-a}\int_a^b f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) dx \\ &\qquad + \beta \frac{1}{b-a}\int_a^b f\left(t_2 x + (1-t_2)\frac{a+b}{2}\right) dx \\ &= \alpha H(t_1) + \beta H(t_2) \end{aligned}$$

which shows that H is convex in $[0, 1]$.

(ii) We shall prove the following inequalities:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \leqslant H(t) &\leqslant t \cdot \frac{1}{b-a}\int_a^b f(x)\,dx + (1-t) \cdot f\left(\frac{a+b}{2}\right) \\ &\leqslant \frac{1}{b-a}\int_a^b f(x)\,dx \end{aligned} \tag{3}$$

for all t in $[0, 1]$.

By Jensen's integral inequality we have

$$H(t) \geqslant f\left(\frac{1}{b-a}\int_a^b \left[tx + (1-t)\frac{a+b}{2}\right] dx\right) = f\left(\frac{a+b}{2}\right).$$

Now, using the convexity of f we get

$$H(t) \leqslant \frac{1}{b-a} \int_a^b \left[tf(x) + (1-t) f\left(\frac{a+b}{2}\right)\right] dx$$
$$= t \cdot \frac{1}{b-a} \int_a^b f(x)\, dx + (1-t) \cdot f\left(\frac{a+b}{2}\right)$$

and the second inequality in (3) is also proven.

The last inequality is obvious because the mapping

$$g(t) := t \cdot \frac{1}{b-a} \int_a^b f(x)\, dx + (1-t) \cdot f\left(\frac{a+b}{2}\right)$$

is increases monotonically on $[0, 1]$.

(iii) Let $t_1, t_2 \in (0, 1)$ with $t_2 > t_1$. Then, H being convex on $(0, 1)$,

$$\frac{H(t_2) - H(t_1)}{t_2 - t_1}$$
$$\geqslant H'_+(t_1) = \frac{1}{b-a} \int_a^b f'_+ \left(t_1 x + (1-t_1)\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) dx.$$

Since f is convex on $[a, b]$, we deduce that

$$f\left(\frac{a+b}{2}\right) - f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right)$$
$$\geqslant t_1 f'_+ \left(t_1 x + (1-t_1)\frac{a+b}{2}\right)\left(\frac{a+b}{2} - x\right)$$

for every x in $[a, b]$. Thus

$$\frac{1}{b-a} \int_a^b f'_+ \left(t_1 x + (1-t_1)\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) dx$$
$$\geqslant \frac{1}{t_1}\left[\frac{1}{b-a} \int_a^b f\left(t_1 x + (1-t_1)\frac{a+b}{2}\right) dx - f\left(\frac{a+b}{2}\right)\right]$$
$$= \frac{1}{t_1}\left[H(t_1) - f\left(\frac{a+b}{2}\right)\right] \geqslant 0.$$

Consequently $H(t_2) - H(t_1) \geqslant 0$ for $1 \geqslant t_2 > t_1 \geqslant 0$ which shows that H increases monotonically on $[0, 1]$.

The proof is finished.

COROLLARY. *In the above assumptions we have*

$$0\leqslant\frac{1}{b-a}\int_a^b f\left(\frac{2x+a+b}{4}\right)dx-f\left(\frac{a+b}{2}\right)$$
$$\leqslant\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{1}{b-a}\int_a^b f\left(\frac{2x+a+b}{4}\right)dx.$$

Applications. (1) Let $p\geqslant 1$ and $0\leqslant a\leqslant b$. Then

$$\left(\frac{a+b}{2}\right)^p\leqslant\frac{1}{t(p+1)(b-a)}\left[\left(\frac{a+b}{2}+t\left(\frac{b-a}{2}\right)\right)^{p+1}\right.$$
$$\left.-\left(\frac{a+b}{2}-t\left(\frac{b-a}{2}\right)\right)^{p+1}\right]$$
$$\leqslant\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\leqslant\frac{a^p+b^p}{2}$$

for all t in $(0, 1]$.

(2) Let $0<a<b$. Then

$$\frac{2}{a+b}\leqslant\frac{1}{t(b-a)}\cdot\ln\frac{[(a+b)/2+t((b-a)/2)]}{[(a+b)/2-t((b-a)/2)]}\leqslant\frac{\ln b-\ln a}{b-a}$$

for all t in $(0, 1]$.

Remark 1. In paper [4] B. Ostle and H. L. Terwilliger prove that the logarithmic mean $L(a, b)=(b-a)/(\ln b-\ln a)$ satisfies the inequality

$$L(a,b)<\frac{a+b}{2},\qquad\text{i.e.,}\quad\frac{\ln b-\ln a}{b-a}>\frac{2}{a+b}.$$

Consequently, our result from Application (2) gives a refinement of this classic fact.

Now, we shall define the second mapping in connection with Hadamard's inequalities. Let $f\colon [a, b]\to\mathbb{R}$ be a convex function on $[a, b]$. Put

$$F\colon [0,1]\to\mathbb{R},\qquad F(t)=\frac{1}{(b-a)^2}\int_a^b\int_a^b f(tx+(1-t)\,y)\,dx\,dy.\tag{4}$$

The following theorem holds.

THEOREM 2. *Let $f\colon [a, b]\to\mathbb{R}$ be as above. Then*:

(i) *$F(\tau+1/2)=F(1/2-\tau)$ for all τ in $[0, 1/2]$;*

(ii) *F is convex on $[0, 1]$;*

(iii) *we have*

$$\sup_{t \in [0,1]} F(t) = F(0) = F(1) = \frac{1}{b-a} \int_a^b f(x)\,dx,$$

$$\inf_{t \in [0,1]} F(t) = F\left(\frac{1}{2}\right) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dx\,dy;$$

(iv) *The following inequality is valid*:

$$f\left(\frac{a+b}{2}\right) \leqslant F\left(\frac{1}{2}\right);$$

(v) *F decreases monotonically on* $[0, 1/2]$ *and increases monotonically on* $[1/2, 1]$;

(vi) *We have the inequality*

$$H(t) \leqslant F(t) \qquad \text{for all} \quad t \in [0, 1].$$

Proof. (i) Let $\tau \in [0, 1/2]$. We have

$$\begin{aligned} F\left(\tau + \frac{1}{2}\right) &= \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\left(\tau + \frac{1}{2}\right) x + \left(\frac{1}{2} - \tau\right) y\right) dx\,dy \\ &= \frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\left(\frac{1}{2} - \tau\right) x + \left(\tau + \frac{1}{2}\right) y\right) dx\,dy \\ &= F\left(\frac{1}{2} - \tau\right). \end{aligned}$$

(ii) The argument is similar to that in the proof of Theorem 1(i) and we omit it.

(iii) For all x, y in $[a, b]$ and t in $(0, 1]$ we have

$$f(tx + (1-t)\,y) \leqslant tf(x) + (1-t)\,f(y).$$

Integrating this inequality in $[a, b] \times [a, b]$ we get

$$\begin{aligned} &\int_a^b \int_a^b f(tx + (1-t)\,y)\,dx\,dy \\ &\qquad \leqslant \int_a^b \int_a^b [tf(x) + (1-t)\,f(y)]\,dx\,dy \\ &\qquad = (b-a) \int_a^b f(x)\,dx \end{aligned}$$

which shows that $F(t) \leqslant F(0) = F(1)$ for all t in $[0, 1]$.

Since f is convex on $[a, b]$, for all $t \in [0, 1]$ and x, y in $[a, b]$, we have

$$\frac{1}{2}[f(tx+(1-t)y)+f(ty+(1-t)x)] \geqslant f\left(\frac{x+y}{2}\right).$$

Integrating this inequality in $[a, b] \times [a, b]$ we deduce

$$\begin{aligned}
&\int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dx\, dy \\
&\quad \leqslant \frac{1}{2} \int_a^b \int_a^b [f(tx+(1-t)y)+f(ty+(1-t)x)]\, dx\, dy \\
&\quad = \int_a^b \int_a^b f(tx+(1-t)y)\, dx\, dy
\end{aligned}$$

which implies that $F(1/2) \leqslant F(t)$ for all t in $[0, 1]$ and the statement is proven.

(iv) Using Jensen's inequality for double integrals, we have

$$\begin{aligned}
&\frac{1}{(b-a)^2} \int_a^b \int_a^b f\left(\frac{x+y}{2}\right) dx\, dy \\
&\quad \geqslant f\left(\frac{1}{(b-a)^2} \int_a^b \int_a^b \left(\frac{x+y}{2}\right) dx\, dy\right).
\end{aligned}$$

Since a simple computation shows that

$$\frac{1}{(b-a)^2} \int_a^b \int_a^b \left(\frac{x+y}{2}\right) dx\, dy = \frac{a+b}{2}$$

the proof is finished.

(v) Since F is convex on $(0, 1)$ we have for $t_2 > t_1$, $t_2, t_1 \in (1/2, 1)$,

$$\begin{aligned}
&\frac{F(t_2)-F(t_1)}{t_2-t_1} \\
&\quad \geqslant F'_+(t_1) = \frac{1}{(b-a)^2} \int_a^b \int_a^b f'_+(t_1 x+(1-t_1)y)(x-y)\, dx\, dy.
\end{aligned}$$

By the convexity of f on $[a, b]$ we deduce

$$\begin{aligned}
&f\left(\frac{x+y}{2}\right) - f(t_1 x + (1-t_1)y) \\
&\quad \geqslant f'_+(t_1 x+(1-t_1)y)\, \frac{(x-y)(1-2t_1)}{2}
\end{aligned}$$

for all x, y in $[a, b]$ and $t_1 \in (1/2, 1)$, which is equivalent to

$$(x-y) f'_+(t_1 x + (1-t_1) y) \geqslant \frac{2}{2t_1 - 1}\left[f(t_1 x + (1-t_1) y) - f\left(\frac{x+y}{2}\right)\right].$$

Integrating on $[a, b] \times [a, b]$ we obtain

$$F'_+(t_1) \geqslant \frac{2}{2t_1 - 1}\left(F(t_1) - F\left(\frac{1}{2}\right)\right) \geqslant 0, \qquad t_1 \in (1/2, 1)$$

which shows that F increases monotonically on $[1/2, 1]$.

The fact that F increases monotonically on $[0, 1/2]$ follows from the above conclusion using statement (i).

(vi) A simple computation shows that

$$H(t) = \frac{1}{b-a}\int_a^b f\left(\int_a^b [tx + (1-t) y]\, dx\right) dx.$$

Using Jensen's integral inequality we derive

$$H(t) = \frac{1}{b-a}\int_a^b f\left(\int_a^b [tx + (1-t) y]\, dy\right) dx.$$

for all t in 0, 1 and the proof is finished.

Applications. (1) Let $p \geqslant 1$ and $0 \leqslant a < b$. Then

$$\begin{aligned}\left(\frac{a+b}{2}\right)^p &\leqslant \frac{4}{(p+1)(p+2)}\left[b^{p+2} - 2\left(\frac{a+b}{2}\right)^{p+2} + b^{p+2}\right] \\ &\leqslant \frac{1}{(p+1)(p+2)\, t(1-t)} \\ &\quad \times [b^{p+2} - ((1-t)a + tb)^{p+2} - ((1-t)b + ta)^{p+2} + b^{p+2}] \\ &\leqslant \frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \leqslant \frac{a^p + b^p}{2}\end{aligned}$$

for all t in $(0, 1)$.

(2) Let $0<a<b$. Then

$$\begin{aligned}\frac{2}{a+b} &\leqslant 4\left[b\ln b-2\left(\frac{a+b}{2}\right)\ln\left(\frac{a+b}{2}\right)+a\ln a\right]\\ &\leqslant \frac{1}{t(1-t)}\,[b\ln b-((1-t)a+tb)\ln((1-t)a+tb)\\ &\quad -((1-t)b+ta)\ln((1-t)b+ta)+a\ln a]\\ &\leqslant \frac{\ln b-\ln a}{b-a};\end{aligned}$$

for all t in $(0, 1)$.

Remark 2. The above inequality gives another refinement of the Ostle–Terwilliger inequality in the form

$$\frac{\ln b-\ln a}{b-a}>\frac{2}{a+b}.$$

References

1. H. Alzer, A note on Hadamard's inequalities, *C.R. Math. Rep. Acad. Sci. Canada* **11** (1989), 255–258.
2. S. S. Dragomir, J. E. Pečarić, and J. Sándor, A note on the Jensen–Hadamard inequality, *Anal. Numér. Théor. Approx.*, in press.
3. D. S. Mitrinović and I. B. Lačković, Hermite and convexity, *Aequationes Math.* **28** (1985), 225–232.
4. B. Ostle and H. L. Terwilliger, A companion of two means, *Proc. Montana Acad. Sci.* **17** (1957), 69–70.
5. J. Sándor, Some integral inequalities, *Elem. Math.* **43** (1988), 177–180.