Existence of positive solutions for a class of semilinear and quasilinear elliptic equations with supercritical case

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Abstract

In this paper, we consider a class of semilinear elliptic Dirichlet problems in a bounded regular domain with cylindrical symmetry involving concave–convex nonlinearities with supercritical growth. Using a new Sobolev embedding theorem and variational method, we show the existence of two positive solutions of the problem. Additionally, we study the quasilinear elliptic equation and obtain a similar result.

1. Introduction

Let $\Omega$ be an open subset in $\mathbb{R}^N$ and consider the following semilinear elliptic problem

\[
\begin{cases}
-\Delta u = f_\lambda(x, u), & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\]

where $f_\lambda : \Omega \times \mathbb{R} \to \mathbb{R}^+$ is a Carathéodory mapping with $\lambda$ a real parameter. The existence, multiplicity, regularity of solutions of (P) have been extensively investigated, see [1,39] and the references therein.

When $f_\lambda$ is sublinear, for example, $f_\lambda(x, u) = \lambda u^q, 0 < q < 1$, sub-super solutions can easily provide the existence of (P) for all $\lambda > 0$.

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homology groups of $\Omega$ are nontrivial) guarantees the existence of a solution. Moreover, this nontriviality condition is only sufficient but not necessary, as some examples in [9,13,52].

When $f_\lambda$ is the sum of a sublinear term and a superlinear term, for example, $f_\lambda(x, u) = \lambda u^q + u^p$, $0 < q < 1 < p$, the so-called concave–convex nonlinearity. Ambrosetti, Brezis and Cerami [2] showed that, if $p > 1$, there exists a constant $\lambda > 0$, such that the problem has a minimal solution if $\lambda \in (0, \Lambda)$, has no solution if $\lambda > \Lambda$, and has a second solution if $p \in (1, 2^* - 1)$, $\lambda \in (0, \Lambda)$. For other results about this problem referring to [15,50,53].

When $f_\lambda$ has supercritical growth, the general variational arguments can’t be used directly because the corresponding functional is not well defined on the Sobolev space $H^1_0(\Omega)$. So we need some techniques. Merle and Peletier [16] studied the problem $f_\lambda(x, u) = u^q - \lambda u^q$, $q > p \geq 2^* - 1$, $\lambda > 0$, by defining a functional $K$ on the set $H = \{v : \nabla v \in L^2(\mathbb{R}^N), \ v \in L^q(\mathbb{R}^N)\}$ they proved that the infimum of $K$ on $H \cap \partial B$ was achieved where $\partial B = \{v \in L^{p+1}(\mathbb{R}^N) : \ f v^{p+1} = 1\}$. This contributes a solution for $\lambda$ small enough.

In fact, most researchers solved supercritical problem by other methods. One is to take advantage of ODE techniques in symmetric domains. In 1973, Joseph and Lundgree [7] showed the first result in this aspect. In 1987, Budd and Norburg [6] extended the singularity results to $N > 3$ and proved its uniqueness. In [38], Peihao Zhao and Chengkui Zhong considered the problem $f_\lambda(x, u) = \lambda u^q + u^p$, $0 < q < 1$, $p > 2^* - 1$ in a ball. They proved that there exist respectively unique constants $\lambda_*, \lambda^* > 0$, such that (P) has only one positive solution if $\lambda \in (0, \lambda_*)$; a unique singular solution and infinitely many positive solutions if $\lambda = \lambda^*$; at least two positive solutions if $\lambda \in (\lambda_*, \lambda^*)$ (where $\Lambda$ see [2]). Moreover, other results about this aspect see [5,29,41] in a ball, [18,40,54] in bounded domains, and [34] in all spaces $\mathbb{R}^N$. Also see [22,48,49] for singular solution.

The other method is to consider the effect of geometry and topology of the domain. According to [4], Rabinowitz raised that whether there are suitable conditions on the topology of $\Omega$. Some answers are given in [8,10–12].

In fact, nonexistence results of (P) hold for some $p > 2^* - 1$ in some nontrivial domains (see [10,11]); while an arbitrarily large number of solutions can be obtained in some contractible domains for all $p > 2^* - 1$ (see [8]). In [12], Passaseo considered the problem (P) for $f_\lambda(x, u) = \lambda |u|^{p-1}u$ with $p > 2^* - 1$ and $\lambda = 1$ in bounded domain $\Omega$. Several perturbations have been used to construct a contractible domain for obtaining the number of positive solutions and nodal solutions. Later, the result in [42] showed that a unique perturbation can produce an arbitrarily large number of solutions. Also see [30,31,36] for other perturbed domains. Indeed, existence results have been obtained, even in some “nearly star-shaped” domains (see the definition introduced in [44]) for $p$ sufficiently close to $\frac{N+2}{N-2}$ (see [45–47]) or for $p$ large enough (see [43]). Moreover, a different definition of “nearly star-shaped” domain is used in [14] to extend Pohožaev’s nonexistence result to nonstarshaped domains when $p$ is large enough.

Recently, Wenzhi Wang [51] establishes embedding results (see Lemma 2.1) in a cylindrically symmetric domain. He proves that functions having such symmetry and belonging to $H^1_0$ can be embedded compactly into some weighted $L^p$ spaces, with $p$ superior to the critical Sobolev exponent. Then, variational arguments is appropriate.

In this paper, we consider the semilinear elliptic equation with concave–convex nonlinearity, that is $f_\lambda(x, u) = \lambda u^q + h(x)u^p$. Our purpose is to use the embedding theorem in [51] and classical variational tools to solve the problem with supercritical growth.

This paper is organized as follows. Section 2 contains preliminaries and our main result. Section 3 gives the existence of two positive solutions for small $\lambda$. Section 4 provides the regular property for the two solutions. Section 5 proves the main result. Section 6 gives a similar existence result for the quasilinear elliptic equation.

2. Preliminary

Let $\Omega = \Omega_1 \times \Omega_2 \subset \mathbb{R}^N$, with $\Omega_1 \subset \mathbb{R}^m$, $m \geq 1$ being a bounded regular domain, and $\Omega_2$ being a $k \geq 2$ dimensional ball with radius $R$, centered at the origin.

Consider the problem

$$
\begin{cases}
-\Delta u = \lambda u^q + h(x)u^p, & x \in \Omega, \\
u > 0, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases}
(P_\lambda)
$$

where $0 < q < 1 < p < 2^* - 1 + \tau$, and $\lambda, \tau$ are positive real parameters ($\tau$ is the constant obtained in [51]). Let $h(x)$ satisfy the following conditions:

(H1) $h(x)$ is a nonnegative Hölder continuous function in $\bar{\Omega}$, radially symmetric with respect to $x_2 \in \Omega_2$, satisfying $h(x_1, 0) = 0$.

(H2) $h_\lambda > 0$, where $h_\lambda = \operatorname{sup}\{\lambda > 0 : |h(x)|/|x_2|^{2\tau} < \infty, \ x \in \Omega\}$.

Denote that

$$H^1_0(\Omega) := \{u \in H^1_0(\Omega) | u(\cdot, x_2) = u(\cdot, |x_2|), \text{ with the norm } \|u\| = (\int_{\Omega} |\nabla u|^2 \, dx)^{1/2}.$
Lemma 3.1. Indeed, for 

\[ L^p(H) := \{ u \in L^p(\Omega) \mid \int_{\Omega} h |u|^p \, dx < \infty \}, \]

with the norm \( \| u \|_{L^p} = (\int_{\Omega} h |u|^p \, dx)^{1/p}, 1 < p < \infty. \)

\( L^q(\Omega) \) is the Banach spaces for the form \( \| u \|_q = (\int_{\Omega} |u|^q \, dx)^{1/q}, 1 < q < \infty. \)

\( H^{-1}_r(\Omega) \) is the dual space of \( H^1_0(\Omega), \langle \cdot , \cdot \rangle \) denotes the pairing of \( H^1_0(\Omega) \) and \( H^{-1}_r(\Omega). \)

c, C, C_1, C_2, \ldots \) are (possibly different) positive constants.

\( \lambda_1 \) is the first eigenvalue of the equation

\[ -\Delta \phi = \lambda_1 \phi, \quad \phi \in \Omega, \]

\[ \phi = 0, \quad \phi \in \partial \Omega, \]

and \( \phi_1 \) is the positive eigenfunction associated with \( \lambda_1. \)

Lemma 2.1. (See [51].) Assume that \( h(x) \) satisfies (H1), (H2), then there exists a positive number \( \tau = \tau(h, m, k) \) such that the embedding \( H^1_0(\Omega) \hookrightarrow L^{p+1}_h(\Omega) \) is compact for all \( r \in (1, 2^* + \tau). \)

According to Lemma 2.1, the embedding mapping \( i : H^1_0(\Omega) \hookrightarrow L^{p+1}_h(\Omega) \) is compact for \( p + 1 < 2^* + \tau. \) Hence, for \( u \in H^1_0(\Omega), \) we have \( u \in L^{p+1}_h(\Omega). \)

Let

\[ f_\lambda(x, s) = \begin{cases} 
\lambda s^q + h(x) s^p, & s \geq 0, \\
0, & s < 0,
\end{cases} \]

and

\[ F_\lambda(x, u) = \int_0^u f_\lambda(x, s) \, ds. \]

Let \( u \in H^1_0(\Omega), \) \( u^+ = \max_{x \in \Omega} [u, 0], \) define

\[ l_\lambda(u) = \frac{1}{2} \| u \|^2 - \int_\Omega F_\lambda(x, u) \, dx = \frac{1}{2} \| u \|^2 - \frac{\lambda}{q+1} \int_\Omega |u^+|^{q+1} \, dx - \frac{1}{p+1} \int_\Omega h(x) |u^+|^{p+1} \, dx. \]

We know that the energy functional \( l_\lambda \) is well defined in \( H^1_0(\Omega) \) and is of \( C^1. \)

Definition 2.2. We call \( u \in H^1_0(\Omega) \) a weak solution of problem (P\( \lambda \)), if \( u \) is a critical point of \( l_\lambda. \)

Our main result is the following:

Theorem 2.3. Let \( 0 < q < 1 < p < 2^* - 1 + \tau. \) If \( h(x) \) satisfies (H1), (H2), then there exists \( \Lambda \in (0, \infty) \) such that

(1) for all \( \lambda \in (0, \Lambda), \) problem (P\( \lambda \)) has at least two classical solutions;

(2) for \( \lambda = \Lambda, \) problem (P\( \lambda \)) has at least one weak solution \( u_\Lambda \in H^1_0(\Omega) \cap L^{p+1}_h(\Omega); \)

(3) for all \( \lambda > \Lambda, \) problem (P\( \lambda \)) has no solution.

3. Existence of two solutions for \( \lambda \) small

In this section, we show existence of the first solution \( u_\lambda \) and the second solution \( v_\lambda \) of problem (P\( \lambda \)) for \( \lambda \in (0, \lambda_0). \) Moreover, we show that \( u_\lambda \) is a minimizer of \( l_\lambda(u) \) in \( H^1_0(\Omega) \) and \( v_\lambda \) is a mountain-pass type solution.

Lemma 3.1. There exist \( \lambda_0 > 0 \) and \( r_0, \rho > 0, \) such that \( l_\lambda(u) \geq \rho \) for all \( u \in H^1_0(\Omega), \) \( \| u \| = r_0 \) and all \( \lambda \in (0, \lambda_0). \)

Proof. Indeed, for \( u \in H^1_0(\Omega), \) we have

\[ l_\lambda(u) = \frac{1}{2} \| u \|^2 - \frac{\lambda}{q+1} \int_\Omega |u^+|^{q+1} \, dx - \frac{1}{p+1} \int_\Omega h(x) |u^+|^{p+1} \, dx \]

\[ = \frac{1}{2} \| u \|^2 - \frac{\lambda}{q+1} \| u^+ \|_{q+1}^q - \frac{1}{p+1} \| u^+ \|_{h,p+1}^{p+1}. \] (3.1)

Since \( 1 < q+1 < 2 < p + 1 < 2^* + \tau, \) it follows from Lemma 2.1 that the embedding \( H^1_0(\Omega) \hookrightarrow L^{p+1}_h(\Omega) \) is compact, and also \( H^1_0(\Omega) \hookrightarrow L^{q+1}_h(\Omega). \) Thus, there exists a constant \( C \) such that...
Proof. Let
\[ \|u\|_{h, p+1} \leq C\|u\|, \]  \hspace{1cm} (3.2)
\[ \|u\|_{q+1} \leq C\|u\|. \]  \hspace{1cm} (3.3)
Using (3.1), (3.2) and (3.3), we infer that
\[ I_{\lambda}(u) \geq \frac{1}{2} \|tu\|^2 - \frac{\lambda}{q+1} \int_{\Omega} |tu^+|^{q+1} - \frac{1}{p+1} \int_{\Omega} h(x)|tu^+|^{p+1} \]
\[ = \frac{1}{2} \|tu\|^2 - \frac{\lambda}{q+1} \int_{\Omega} |u^+|^{q+1} - \frac{1}{p+1} \int_{\Omega} |u^+|^{p+1} \]
\[ \geq \frac{1}{2} \|tu\|^2 - \frac{\lambda}{q+1} \int_{\Omega} |u^+|^{q+1} - \frac{1}{p+1} \int_{\Omega} |u^+|^{p+1} \]
\[ = \frac{1}{2} \|u\|^2 - \frac{\lambda}{q+1} \|u^+\|^{q+1} + (C\|u^+\|)\|u^+\|^{p+1}. \]

Since \( 0 < q < 1 < p \), we can find \( \lambda_0 \) such that for all \( 0 < \lambda \leq \lambda_0 \) there exists \( M = M(\lambda) > 0 \) satisfying
\[ M > \lambda(CM)^p + (CM)^p. \]
As a consequence, there exist \( r_0 > 0 \) and a small enough constant \( \rho > 0 \) such that
\[ I_{\lambda}(u) \geq \rho > 0 \quad \text{for every} \quad \|u\| = r_0. \]

**Lemma 3.2.** For all \( \lambda \in (0, \lambda_0) \), \( I_{\lambda} \) possesses a local minimum close to the origin.

**Proof.** Let \( \lambda \in (0, \lambda_0) \), we note that
\[ I_{\lambda}(tu) = \frac{1}{2} \|tu\|^2 - \frac{\lambda}{q+1} \int_{\Omega} |tu^+|^{q+1} - \frac{1}{p+1} \int_{\Omega} h(x)|tu^+|^{p+1} \]
\[ = \frac{1}{2} t^2 \|u\|^2 - \frac{\lambda}{q+1} \int_{\Omega} |u^+|^{q+1} - \frac{1}{p+1} t^{p+1} \|u^+\|^{p+1}. \]
Clearly, \( I_{\lambda}(tu) < 0 \) for \( t > 0 \) and any \( u \in H^1_\lambda(\Omega) \) with \( \|u^+\| \neq 0 \).

Set
\[ A = \{ u \in H^1_\lambda(\Omega) \mid \|u\| \leq r_0 \}. \]
then we have
\[ m := \min_{u \in A} I_{\lambda}(u) < 0. \]
We claim that this minimum can be achieved at some \( u_\lambda \). To see this, select a minimizing sequence \( \{u_n\}_{n=1}^\infty \), then
\[ I_{\lambda}(u_n) \to m. \]
And let \( u_n \rightharpoonup u_\lambda \) in \( H^1_\lambda(\Omega) \), we have
\[ \int_\Omega |\nabla u_\lambda|^2 \leq \liminf_{n \to \infty} \int_\Omega |\nabla u_n|^2 \]
By compact embedding theorem (Lemma 2.1), we obtain that
\[ \int_\Omega F_\lambda(x, u_n) dx \to \int_\Omega F_\lambda(x, u_\lambda) dx. \]
Thus,
\[ I_{\lambda}(u_\lambda) = \frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 dx - \int_\Omega F_\lambda(x, u_\lambda) dx \]
\[ \leq \liminf_{n \to \infty} \int_\Omega |\nabla u_n|^2 dx + \lim_{n \to \infty} \int_\Omega F_\lambda(x, u_n) dx \]
\[ = \lim_{n \to \infty} I_{\lambda}(u_n) = m. \]
Since \( u_\lambda \in A \), it follows that
\[ I_{\lambda}(u_\lambda) = m = \min_{u \in A} I(u). \]
That is, \( u_\lambda \) is a minimizer for \( I_{\lambda} \) in \( A \), and hence \( I_{\lambda}(u_\lambda) \) is a local minimum. \( \square \)
Lemma 3.5. Let $\{v_k\}_{k=1}^{\infty}$ be a sequence in $X$ such that $\Phi(v_k) \to c$ and $\Phi'(v_k) \to 0$. Then there exists a convergent subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ of $\{v_k\}_{k=1}^{\infty}$ such that $v_{k_j} \to v$ in $H^1_0(\Omega)$.

Proof. We must show that for all sequences $\{v_k\}_{k=1}^{\infty}$ in $X$ such that $\Phi(v_k) \to c$ and $\Phi'(v_k) \to 0$, there exists a convergent subsequence $\{v_{k_j}\}_{j=1}^{\infty}$ of $\{v_k\}_{k=1}^{\infty}$ such that $v_{k_j} \to v$ in $H^1_0(\Omega)$.

Combining (3.5) and (3.8), we have
\[ C \left( v_k \right) = \frac{1}{2} \int_{\Omega} |\nabla v_k|^2 \, dx - \frac{\lambda}{q+1} \int_{\Omega} |v_k|^{q+1} \, dx - \frac{1}{p+1} \int_{\Omega} h(x) |v_k|^{p+1} \, dx = c + o(1). \tag{3.5} \]

and
\[ -\Delta v_k - \lambda (v_k^+)^q - h(x)(v_k^p) = \xi_k, \quad \xi_k \to 0 \quad \text{in} \quad H^{-1}_s(\Omega). \tag{3.6} \]

Multiplying (3.6) by $v_k$ and integrating in $\Omega$, we have
\[ \int_{\Omega} |\nabla v_k|^2 \, dx - \lambda \int_{\Omega} |v_k|^{q+1} \, dx - \int_{\Omega} h(x) |v_k|^{p+1} \, dx = \langle \xi_k, v_k \rangle. \tag{3.7} \]

Taking a computation with (3.5) and (3.7), we get
\[ \frac{\lambda (q-1)}{2(q+1)} \int_{\Omega} |v_k|^{q+1} \, dx + \frac{p-1}{2(p+1)} \int_{\Omega} h(x) |v_k|^{p+1} \, dx = c - \frac{1}{2} \langle \xi_k, v_k \rangle. \]

Let $C_1 = \frac{p-1}{2(p+1)}$, $C_2 = \frac{\lambda (1-q)}{2(q+1)}$. Clearly $C_1, C_2 > 0$ for $0 < q < 1 < p$. Thus,
\[ C_1 \int_{\Omega} h(x) |v_k|^{p+1} \, dx \leq C_2 \int_{\Omega} |v_k|^{q+1} \, dx + c + \|\xi_k\|_{H^{-1}_s} \|v_k\|. \tag{3.8} \]

Combining (3.5) and (3.8), we have
\[ \frac{1}{2} \|v_k\|^2 = \frac{\lambda}{q+1} \|v_k\|^{q+1} + \frac{1}{p+1} \|v_k\|^{p+1} + c + o(1) \leq \frac{\lambda}{q+1} \|v_k\|^{q+1} + C \|v_k\|^{q+1} + c + \|\xi_k\|_{H^{-1}_s} \|v_k\| \]
\[ = C \|v_k\|^{q+1} + c + C \|\xi_k\|_{H^{-1}_s} \|v_k\|. \]

According to (3.3), it follows that
\[ \|v_k\|^2 \leq C \|v_k\|^{q+1} + C + C \|\xi_k\|_{H^{-1}_s} \|v_k\| \leq C \|v_k\|^{q+1} + C + C \|v_k\|. \]

We deduce that there exists a constant $C$, such that $\|v_k\| < C$. Consequently, $\{v_k\}_{k=1}^{\infty}$ is bounded in $H^1_0(\Omega)$, and then, there exists a weakly convergent subsequence $\{v_{k_j}\}_{j=1}^{\infty} \subset \{v_k\}_{k=1}^{\infty}$. Moreover, we have another inequality
\[ \|v_{k_j}\|_{H^{-1}_s} \leq C. \]
From above, we can deduce the following:

\[ v_{k_j} \rightharpoonup v \quad \text{weakly in } H^1_s(\Omega), \]
\[ v_{k_j} \rightarrow v \quad \text{strongly in } L^{p+1}(\Omega) \text{ for } p + 1 < 2^* + \tau, \]
\[ v_{k_j} \rightarrow v \quad \text{strongly in } L^q(\Omega) \text{ for } q < 2^*, \]
\[ v_{k_j} \rightarrow v \quad \text{a.e in } \Omega. \]

Obviously we have

\[ \lambda v^q_{k_j} + h(x)v^p_{k_j} \rightarrow \lambda v^q + v^p \quad \text{in } H^{-1}_s(\Omega). \]

From the Lax–Milgram Theorem, for each \( f_\lambda(v) \in H^{-1}_s(\Omega) \), the problem

\[
\begin{cases}
-\Delta u = f_\lambda(v) & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega
\end{cases}
\]

has a unique solution \( u \in H^1_s(\Omega) \). Writing \( u = K(f_\lambda(v)) \), so that

\[ K : H^{-1}_s(\Omega) \rightarrow H^1_s(\Omega) \]

is an isometry.

Therefore, we have

\[ K[f_\lambda(v_{k_j})] \rightarrow K[f_\lambda(v)] \quad \text{in } H^1_s(\Omega). \]

As

\[ I'_\lambda(v_{k_j}) = v_{k_j} - K(f_\lambda(v_{k_j})) \rightarrow 0 \quad \text{in } H^1_s(\Omega), \]

consequently,

\[ v_{k_j} \rightarrow v \quad \text{in } H^1_s(\Omega). \]

This completes the proof. \( \square \)

**Lemma 3.6.** For all \( \lambda \in (0, \lambda_0) \), \( I_\lambda \) has the second solution \( v_\lambda \) of mountain-pass type.

**Proof.** Let \( v \in H^1_s(\Omega) \). From Lemma 3.1, we can obtain that for all \( \lambda \in (0, \lambda_0) \), there exist constants \( r_0, \rho > 0 \) such that \( I_\lambda(v) \geq \rho \), for all \( \|v\| = r_0 \).

Next, we verify that \( I_\lambda(0) < \rho \) and there exists \( v \notin A \) such that \( I_\lambda(v) < \rho \). Clearly, \( I_\lambda(0) = 0 < \rho \). Now, fix some element \( v \in H^1_s(\Omega), v \neq 0 \). Write \( \omega := tv \) for \( t > 0 \) to be selected. From (3.1) we have

\[ I_\lambda(\omega) = \frac{1}{2} t^2 \|v\|^2 - \frac{\lambda}{q+1} t^{q+1} \|v^+\|_{q+1}^{q+1} - \frac{1}{p+1} t^{p+1} \|v^+\|_{p+1}^{p+1}. \]

Since \( q + 1 < 2 < p + 1 \), it follows that

\[ I_\lambda(\omega) = I_\lambda(tv) \rightarrow -\infty, \quad \text{as } t \rightarrow +\infty. \]

Therefore, there exists \( T > 0 \) such that

\[ \|\omega\| = \|Tv\| > r_0 \quad (\text{that is } v \notin \partial A), \]
\[ I_\lambda(\omega) = I_\lambda(tv) < \rho. \]

Consequently, there is a function \( v_\lambda \in H^1_s(\Omega), v_\lambda \neq 0 \), such that \( I_\lambda(v_\lambda) = \gamma \geq \rho > 0, I'_\lambda(v_\lambda) = 0 \). That is, \( v_\lambda \) is a nontrivial critical point of \( (P_\lambda) \). For \( I_\lambda(u_\lambda) < 0, I_\lambda(v_\lambda) > 0, v_\lambda \) is the second solution of the problem \( (P_\lambda) \). \( \square \)
4. Regularity

The solutions \( u, v \), we have found in \( H^1_s(\Omega) \) are weak solutions. In the following, we will apply bootstrap iteration [35] to improve their regularity.

**Lemma 4.1.** Let \( v \in H^1_s(\Omega) \) be a weak solution of problem \( (P_\lambda) \), then \( v \in C^{2,\alpha}(\Omega) \) for some \( \alpha \in (0, 1) \).

**Proof.** For a solution \( v \in H^1_s(\Omega) \) of \( (P_\lambda) \), we denote
\[
f_\lambda(x) = f_\lambda(x, v(x)) = \lambda v(x)^q + h(x)v(x)^p.
\]
Because of Lemma 2.1,
\[
H^1_s(\Omega) \hookrightarrow L^r(\Omega),
\]
we have,
\[
f_\lambda(x) \in L^\sigma(\Omega), \quad \sigma = \frac{r}{p}.
\]
Since \( 1 < p < 2^* - 1 + \tau \), we get
\[
\sigma > \frac{2^* + \tau}{2^* - 1 + \tau}.
\]
Thus,
\[
\sigma = \frac{2^* + \tau}{2^* - 1 + \tau} (1 + \varepsilon), \quad \text{for some } \varepsilon > 0.
\]
We can write the origin equation as a linear nonhomogeneous elliptic equation
\[
\begin{cases}
-\Delta v = f_\lambda(x), & x \in \Omega, \\
v = 0, & x \in \partial\Omega.
\end{cases}
\]
According to the boundary regularity theorem of linear nonhomogeneous elliptic equation (see [27]), and \( f_\lambda(x) \in L^\sigma(\Omega) \), we can obtain that \( v \in W^{2,\sigma}_{0}(\Omega) \). If \( 2\sigma > n \), we are done. Otherwise, from Sobolev embedding theorem, we have
\[
W^{2,\sigma}_{0}(\Omega) \hookrightarrow L^{r_1}(\Omega), \quad r_1 = \frac{N\sigma}{N - 2\sigma}.
\]
It easily follows that
\[
f_\lambda(x) \in L^{\sigma_1}(\Omega), \quad \sigma_1 = \frac{r_1}{p}.
\]
Then, \( v \in W^{2,\sigma_1}_{0}(\Omega) \).

Next, we need to show that the regularity of \( v \) has been improved, that is to show that
\[
\frac{\sigma_1}{\sigma} = \frac{r_1}{r} > 1.
\]
By computation, we obtain
\[
\frac{r_1}{r} = \frac{N(1 + \varepsilon)}{N(2^* + \tau - 1) - 2(1 + \varepsilon)(2^* + \tau)}.
\]
Thus, we only need to check
\[
\begin{cases}
N(2^* + \tau - 1) - 2(1 + \varepsilon)(2^* + \tau) > 0 \quad (a), \\
N(1 + \varepsilon) > N(2^* + \tau - 1) - 2(1 + \varepsilon)(2^* + \tau) \quad (b).
\end{cases}
\]
We can easily get from (a), (b) that
\[
\frac{N(2^* + \tau - 1)}{N + 2(2^* + \tau)} - 1 < \varepsilon < \frac{N(2^* + \tau - 1)}{2(2^* + \tau)} - 1.
\]
Clearly, we can find \( \varepsilon > 0 \) satisfying (4.1). Consequently, we indeed have
\[
\frac{\sigma_1}{\sigma} > 1.
\]
From boundary regularity theorem, not only for \( \sigma_1 > \sigma, v \in W^{2,\sigma_1}_{0}(\Omega) \), but also for any \( \sigma_k \) large enough, \( v \in W^{2,\sigma_k}_{0}(\Omega) \).

When \( 2\sigma_k > n \), from Sobolev embedding theorem, we can obtain that \( v \in C^{0,\theta}(\Omega) \), \( f_\lambda(x) \in C^{0,\theta}(\Omega) \), then \( v \in C^{2,\theta}(\Omega) \) by Schauder regularity theorem. \( \square \)
5. Existence of solutions for \( \lambda \in (0, A) \)

**Lemma 5.1.** Let \( A = \sup \{ \lambda > 0: (P_{\lambda}) \text{ has a solution} \} \), then \( A \in (0, \infty) \).

**Proof.** Let \( a > 0 \) and choose \( \Omega_1 \subset \Omega \) such that \( h(x) \geq a \) in \( \Omega_1 \). Define

\[
\eta(x) = \begin{cases} 
\phi_1(x), & x \in \Omega_1, \\
0, & x \in \Omega \setminus \Omega_1.
\end{cases}
\]

Multiplying \((P_{\lambda})\) by \( \eta(x) \) and using integrations by parts, we can get

\[
\int_{\Omega_1} \lambda_1 u \phi_1 \, dx = \int_{\Omega_1} (\lambda u^q + h(x)u^p) \phi_1 \, dx.
\]

Let \( \lambda_i \) satisfy

\[
\lambda_i t < \tilde{\lambda} t^q + at^p \quad \text{for any } t > 0.
\]

We can obtain that

\[
\int_{\Omega} (\lambda u^q + au^p) \eta(x) \, dx < \int_{\Omega_1} (\lambda u^q + h(x)u^p) \phi_1 \, dx = \int_{\Omega_1} \lambda_1 u \phi_1 \, dx < \int_{\Omega_1} (\lambda u^q + au^p) \phi_1 \, dx
\]

\[
< \int_{\Omega} (\lambda u^q + h(x)u^p) \eta(x) \, dx.
\]

Clearly, \( \lambda < \tilde{\lambda} \).

Moreover, we have obtained a solution \( u_\lambda \) of \((P_{\lambda})\) for \( \lambda \in (0, \lambda_0) \). Hence, \( 0 < \lambda_0 \leq A \leq \tilde{\lambda} < \infty \).  \( \square \)

**Lemma 5.2.** \((P_{\lambda})\) has a solution for all \( \lambda \in (0, A) \).

**Proof.** Given \( 0 < \lambda \leq \mu < A \). Let \( u_\mu \) be a solution of \((P_{\mu})\), then

\[
-\Delta u_\mu = \mu u_\mu^q + h(x)u_\mu^p > \lambda u_\mu^q + h(x)u_\mu^p,
\]

that is, \( u_\mu \) is a supersolution of \((P_{\lambda})\). Furthermore, \( \varepsilon \phi_1 \) is a subsolution of \((P_{\lambda})\), and \( \varepsilon \phi_1 < u_\mu \) for \( \varepsilon \) small enough. Therefore, there exists a solution \( u_\lambda \) of \((P_{\lambda})\) satisfying \( \varepsilon \phi_1 \leq u_\lambda \leq u_\mu \). Consequently, for all \( \lambda \in (0, A) \), \((P_{\lambda})\) has a solution.  \( \square \)

**Lemma 5.3.** For all \( \lambda \in (0, A) \), \((P_{\lambda})\) has a local minimum in the \( C^1 \) topology.

**Proof.** Fix \( \lambda \in (0, A) \). Choose \( \lambda < \lambda_1 < A \) such that \((P_{\lambda})\) has a solution \( u_1 \). Let \( u_0 \) be the unique positive solution of

\[
\begin{cases} 
-\Delta u = \lambda u^q, & x \in \Omega, \\
u = 0, & x \in \partial \Omega.
\end{cases}
\]

Because

\[
-\Delta u_1 = \lambda_1 u_1^q + h(x)u_1^p > \lambda u_1^q,
\]

\( u_1 \) is a supersolution of \((5.1)\). Moreover, \( u_0 \neq u_1 \), so \( u_0 < u_1 \) in \( \Omega \).

Set

\[
\tilde{f}_\lambda(x, s) = \begin{cases} 
f_\lambda(s), & s \leq u_0, \\
f_\lambda(u_0), & u_0 < s < u_1, \\
f_\lambda(u_1), & s \geq u_1,
\end{cases}
\]

\[
\tilde{F}_\lambda(x, u) = \int_0^u \tilde{f}_\lambda(x, s) \, ds,
\]

and the functional \( \tilde{f}_\lambda : H^1_0(\Omega) \to \mathbb{R} \) is given by

\[
\tilde{f}_\lambda(u) = \frac{1}{2} \|u\|^2 - \int_{\Omega} \tilde{F}_\lambda(x, u) \, dx.
\]
Clearly, \( I_\lambda \) is coercive and bounded below, then it achieve its global minimum at some \( u_\lambda \in H^1_\Omega \). Thus,

\[
\begin{cases}
-\Delta u_\lambda = f_\lambda(x, u_\lambda), & x \in \Omega, \\
u_\lambda > 0, & x \in \Omega, \\
u_\lambda = 0, & x \in \partial \Omega.
\end{cases}
\]

(5.2)

Since \( f_\lambda(x, u_\lambda) \geq f_\lambda(u_0) \geq \lambda u_0^q \), we get \( u_0 \leq u_\lambda \) in \( \Omega \).

Furthermore, we have

\[
\begin{cases}
-\Delta(u_\lambda - u_1) = f_\lambda(u_\lambda) - f_\lambda(u_1), & x \in \Omega, \\
u_\lambda - u_1 = 0, & x \in \partial \Omega,
\end{cases}
\]

(5.3)

the strong maximum principle yields \( u_\lambda \leq u_1 \) in \( \Omega \). Thus, \( u_0 < u_\lambda < u_1 \) in \( \Omega \).

From Lemma 4.1, \( u_\lambda \in C^{2,\alpha}(\Omega), \alpha \in (0, 1) \). For \( \| u - u_\lambda \|_{C^1} = \varepsilon \) with \( \varepsilon > 0 \) small enough, we have \( u_0 < u < u_1 \) in \( \Omega \). Hence, \( f_\lambda(u) \neq f_\lambda(u) \), and also \( f_\lambda(u) = I_\lambda(u) \) for \( u_0 < u < u_1 \). Therefore, \( u_\lambda \) is a local minimizer for \( I_\lambda \) in \( C^1 \) topology. \( \square \)

**Lemma 5.4.** Let \( \lambda \in (0, \Lambda) \). \( u_\lambda \) is a local minimizer for \( I_\lambda \) in \( H^1_\Omega \).

**Proof.** Follows from [20]. \( \square \)

In the sequel we fix \( \lambda \in (0, \Lambda) \), and hope to find a mountain-pass type solution of the form \( v_\lambda = u_\lambda + v \), where \( u_\lambda \) is the local minimizer we have found in Lemma 5.4, and \( v > 0 \) in \( \Omega \).

Let \( v \) satisfy the equation

\[
\begin{cases}
-\Delta v = f_\lambda(u_\lambda + v) - f_\lambda(u_\lambda), & x \in \Omega, \\
v > 0, & x \in \Omega, \\
v = 0, & x \in \partial \Omega.
\end{cases}
\]

(5.4)

Then, we define

\[
g_\lambda(x, s) = \begin{cases} f_\lambda(u_\lambda + s) - f_\lambda(u_\lambda), & s \geq 0, \\
0, & s < 0,
\end{cases}
\]

and

\[
G_\lambda(v) = \int_{\Omega} g_\lambda(x, s) \, ds,
\]

\[
f_\lambda(v) = \frac{1}{2} \| v \| ^2 - \int_{\Omega} G_\lambda(v) \, dx.
\]

Clearly, \( J_\lambda : H^1_\Omega \to \mathbb{R}^+ \) is a \( C^1 \) functional. Moreover, if \( v \) is a nontrivial critical point of \( J_\lambda \), then \( v_\lambda = u_\lambda + v \) is a solution of \( (P_\ast) \), and \( v_\lambda \neq u_\lambda \).

**Lemma 5.5.** Let \( \lambda \in (0, \Lambda) \). \( J_\lambda \) has a nontrivial critical point.

**Proof.** Clearly, \( J_\lambda(0) = 0 \). Moreover, it can be easily checked that \( v \equiv 0 \) is a local minimizer of \( J_\lambda \) for all \( \lambda \in (0, \Lambda) \). Recalling Section 3, we can similarly obtain that \( J_\lambda \) satisfies the \((PS)_\lambda\) condition, and \( J_\lambda(t v) \to \infty \) as \( t \to \infty \). Hence, applying the Mountain Pass Theorem, we can obtain a critical point \( v_0 \in H^1_\Omega \) with \( v_0 > 0 \) in \( \Omega \). \( \square \)

**Proof of Theorem 2.3.** (i) Lemma 5.4 and Lemma 5.5 prove point 1.

(ii) For all \( \lambda \in (0, \Lambda) \), it is easy to verify that there exists a solution \( u_\lambda \) such that \( I_\lambda(u_\lambda) < 0 \). Choose a sequence \( \{\lambda_n\} \) such that \( \lambda_n \to \Lambda \) as \( n \to \infty \). We denote the corresponding solutions to \( \lambda_n \) be \( u_{\lambda_n} \). Then, they satisfy

\[
I_{\lambda_n}(u_{\lambda_n}) < 0, \quad I'_{\lambda_n}(u_{\lambda_n}) = 0.
\]

That is

\[
\begin{align*}
\frac{1}{2} \| u_{\lambda_n} \|^2 &- \lambda_n \frac{q+1}{q+1} \| u_{\lambda_n} \|^{q+1} - \frac{1}{p+1} \| u_{\lambda_n} \|^{p+1} < 0, \\
\| u_{\lambda_n} \|^2 &- \lambda_n \| u_{\lambda_n} \|^{q+1} - \| u_{\lambda_n} \|^{p+1} = 0.
\end{align*}
\]

(5.5)

(5.6)
It follows from (5.5), (5.6) that, there exists $C > 0$ such that

$$
\|u_{\lambda_m}\| < C.
$$

Hence, there exists a convergence subsequence of $\{u_{\lambda_m}\}$, denoted by $\{u_{\lambda_m}\}$. Then,

$$
\begin{align*}
    u_{\lambda_m} \rightarrow u_A & \quad \text{in } H^1_\alpha(\Omega), \\
    u_{\lambda_m} \rightarrow u_A & \quad \text{in } L^{p+1}_h(\Omega).
\end{align*}
$$

Such a $u_A$ is a weak solution of $(P_\lambda)$ for $\lambda = \Lambda$.

(iii) Point 3 follows from the definition of $\Lambda$. □

6. Existence results for the $p$-Laplace equation

Consider the quasilinear elliptic problem

$$
\begin{align*}
    -\Delta_p u & = \text{div}(\nabla u|^{p-2}\nabla u) = f(x, u), \quad x \in \Omega, \\
    u > 0, & \quad x \in \Omega, \\
    u = 0, & \quad x \in \partial \Omega,
\end{align*}
$$

where $1 < p < N$, $p^* = \frac{Np}{N-p}$; $\lambda > 0$.

When $f_1$ is sublinear, for example, $f_1(x, u) = \lambda u^\alpha$, $0 < \alpha < p - 1$, the sub-super solutions still can provide the existence of a unique solution of $(\tilde{P})$ for all $\lambda > 0$, see [55].

When $f_1$ is the concave–convex nonlinearity, for instance, $f_1(x, u) = \lambda u^\alpha + u^\beta$, $0 < \alpha < p - 1 < \beta < p^* - 1$, see [56, 23], they showed that the local minimizers of a class of functionals in the $C^1$ topology are still their local minimizers in $W^{1,p}_0(\Omega)$. Applying this fact, they obtained that $(\tilde{P})$ has at least two solutions for all $\lambda \in (0, \Lambda)$, no solution for $\lambda > \Lambda$, at least one solution for $\lambda = \Lambda$. Such a kind of problems also has been studied in [59] by variational method and genus, in [28] by sub-super solutions.

When $f_1$ has supercritical growth, there are few papers about this aspect. The papers still take advantage of ODE techniques in balls. Zongming Guo [57,58] considered the problem $(\tilde{P})$ for $f_1(x, u) = u^\alpha - \lambda u^\beta$, $p^* - 1 \leq \alpha < \beta$. He obtained that there are at least two positive radial solutions of $(\tilde{P})$ for $\lambda$ sufficiently small, and showed their asymptotic behavior as $\lambda \rightarrow 0$. In [19], the authors studied $(\tilde{P})$ in a ball $B_R$ for $f_1(x, u) = u^\alpha + u^\beta$, where $p - 1 < \alpha < p^* - 1 < \beta$, $\lambda = 1$. They proved that there exists $R_\lambda$ such that $(\tilde{P})$ has at least two distinct radial solutions provided $R > R_\lambda$ and at least one radial solution provided $R = R_\lambda$.

In this section, we give the existence result of positive solutions for quasilinear elliptic equation with supercritical growth. In fact, we can extend the results of semilinear elliptic problem naturally to quasilinear elliptic problem by similar methods.

In the following, we study the $p$-Laplacian equation

$$
\begin{align*}
    -\Delta_p u & = \lambda u^\alpha + h(x)u^\beta, \quad x \in \Omega, \\
    u > 0, & \quad x \in \Omega, \\
    u = 0, & \quad x \in \partial \Omega,
\end{align*}
$$

where $0 < \alpha < p - 1 < \beta < p^* - 1 + \tau$, and $\lambda$, $\tau$ are positive parameters ($\tau$ is the constant obtained in [26]). $h(x)$ satisfies (H1), (H2).

We denote that

$$
W^{1,p}_{0,\beta}(\Omega) = \{ u \in W^{1,p}_0(\Omega) \mid u(\cdot, x_2) = u(\cdot, |x_2|), \forall x \in \Omega \},
$$

with the norm $\|u\|_p = (\int_\Omega |\nabla u|^p dx)^{\frac{1}{p}}$.

$\lambda_1$ is the first eigenvalue of $-\Delta_p$ in $\Omega$ with Dirichlet boundary condition, and $\phi_1$ is the associated eigenfunction such that $\phi_1 > 0$ in $\Omega$.

**Lemma 6.1.** (See [26].) Assume that $h(x)$ satisfies (H1), (H2), then there exists a positive number $\tau = \tau(h, p, m, k)$ such that the embedding $W^{1,p}_{0,\beta}(\Omega) \hookrightarrow L^p(\Omega)$ is compact for all $\tau \in (1, p^* + \tau)$.

According to Lemma 6.1, the embedding mapping $i : W^{1,p}_{0,\beta}(\Omega) \hookrightarrow L^{p+1}_h(\Omega)$ is compact for $\beta + 1 < p^* + \tau$. Hence, for $u \in W^{1,p}_{0,\beta}(\Omega)$, we have $u \in L^{p+1}_h(\Omega)$. Then we can define the functional $\tilde{I}_\lambda : W^{1,p}_{0,\beta}(\Omega) \rightarrow \mathbb{R}^+$ by

$$
\tilde{I}_\lambda(u) = \frac{1}{p} \|\nabla u\|_p - \int_\Omega \tilde{F}_\lambda(x, u) dx,
$$
where
\[
\bar{F}_\lambda(x,u) = \int_0^u \bar{f}_\lambda(x,s) \, ds \quad \text{and} \quad \bar{f}_\lambda(x,s) = \begin{cases} 
\lambda s^\alpha + h(x)s^\beta, & s \geq 0, \\
0, & s < 0.
\end{cases}
\]

We know that the energy functional \( \bar{I}_\lambda \) is of class \( C^1 \).

Now, we give the existence result for \((\bar{P}_\lambda)\):

**Theorem 6.2.** Let \( 0 < \alpha < p - 1 < \beta < p^* - 1 + \tau \). If \( h(x) \) satisfies (H1), (H2), then there exists \( \bar{A} \in (0, \infty) \), such that

1. for all \( \lambda \in (0, \bar{A}) \), problem \((\bar{P}_\lambda)\) has at least two weak solutions;
2. for \( \lambda = \bar{A} \), problem \((\bar{P}_\lambda)\) has at least one weak solution \( u_\Lambda \in W_{0,1}^{1,p}(\Omega) \cap L_0^{\beta+1}(\Omega) \);
3. for all \( \lambda > \bar{A} \), problem \((\bar{P}_\lambda)\) has no solution.

**Lemma 6.3.** Let \( v \) be a weak solution of \((\bar{P}_\lambda)\), then \( v \in C^1(\bar{\Omega}) \) for some \( \theta \in (0,1) \).

**Proof.** \( v \in W_{0,1}^{1,p}(\Omega) \) is a solution of \((\bar{P}_\lambda)\), applying Lemma 6.1, we get
\[
v \in W_{0,1}^{1,p}(\Omega) \hookrightarrow L_0^r(\Omega), \quad r = p^* + \tau.
\]

Then, we obtain
\[
f_\lambda(x) = f_\lambda(x,v(x)) = \lambda v(x)^\alpha + h(x)v(x)^\beta \in L_0^\sigma(\Omega), \quad \sigma = \frac{r}{\beta}.
\]
Since \( 1 < \beta < p^* - 1 + \tau \), we have
\[
\sigma > \frac{p^* + \tau}{p^* + \tau - 1}.
\]
We can denote
\[
\sigma = \frac{p^* + \tau}{p^* + \tau - 1}(1 + \varepsilon), \quad \text{for some } \varepsilon > 0.
\]
Clearly, we also have
\[-\text{div}(|\nabla v|^{p-2} \nabla v) \in L_0^\sigma(\Omega),
\]
and then
\[|\nabla v|^{p-1} \in W_{0,1}^{1,\sigma}(\Omega).
\]
If \( \sigma > N \), we are done. Otherwise, for \( 1 < \sigma < N \), we get from the Sobolev embedding theorem that
\[W_{0,1}^{1,\sigma}(\Omega) \hookrightarrow L_0^{s}(\Omega), \quad s = \frac{N\sigma}{N - \sigma}.
\]
Thus,
\[|\nabla v|^{p-1} \in L_0^{s}(\Omega) \quad \text{and} \quad |\nabla v| \in L_0^{s(p-1)}(\Omega),
\]
where \( s(p-1) < N \) for \( 1 < p < N \) and \( 1 < \sigma < N \). Then, we get
\[v \in W_{0,1}^{1,s(p-1)}(\Omega) \hookrightarrow L_0^{r_1}(\Omega), \quad r_1 = \frac{Ns(p-1)}{N - s(p-1)}.
\]
Clearly,
\[f_\lambda(x) \in L_0^{r_1}(\Omega), \quad \sigma_1 = \frac{r_1}{\beta}.
\]
We assert that
\[\frac{\sigma_1}{\sigma} > 1.
\]
Indeed,
\[
\frac{\sigma_1}{\sigma} = \frac{r_1}{r}, \quad \text{and} \quad r_1 > 0, \ r > 0.
\]
Thus, we only need to check
\[
r_1 > r,
\]
that is,
\[
\frac{Ns(p-1)}{N-s(p-1)} > p^* + \tau.
\]
By computation, we get
\[
\left[ N^2(p-1) + Np + p(N-p)\tau \right] \varepsilon > (N-p)^2 \tau.
\]
Since \(1 < p < N\), we have
\[
\varepsilon > \frac{(N-p)^2 \tau}{N^2(p-1) + Np + p(N-p)\tau} > 0.
\]
Because of the arbitrary of \(\varepsilon\), we conclude \(r_1 > r\), and then \(\sigma_1 > 1\).

Applying bootstrap argument, we can get \(f_\lambda(x) \in L^{\sigma_k}(\Omega)\) for \(\sigma_k\) large enough. When \(\sigma_k > n\), according to the Sobolev embedding theorem, we obtain
\[
|\nabla v|^{p-1} \in W^{1,\sigma_k}(\Omega) \hookrightarrow C^{0,\theta}(\Omega), \quad \theta \in (0, 1).
\]
It follows that
\[
|\nabla v| \in C^{0,\theta}(\Omega),
\]
and then we conclude
\[
v \in C^{1,\theta}(\Omega). \quad \Box
\]

**Remark 6.4.** In fact, the proof of Theorem 6.2 is similar to the proof of Theorem 2.3. Only the following lemma we should give a different proof. Other proofs we skip them here.

**Lemma 6.5.** Let \(\bar{\Lambda} = \sup\{\lambda > 0: (\bar{P}_\lambda) \text{ has a solution}\}\), then \(\bar{\Lambda} \in (0, \infty)\).

**Proof.** We know that \(\bar{\lambda}_1\) is isolated in bounded domain, that is, there exists \(\delta > 0\) such that for every \(\mu \in (\bar{\lambda}_1, \bar{\lambda}_1 + \delta)\), the problem
\[
\begin{cases}
-\Delta_p u = \mu |u|^{p-2} u, & x \in \Omega, \\
u = 0, & x \in \partial \Omega,
\end{cases} \quad (6.1)
\]
has no nontrivial solution.

Suppose that \(v \in W^{1,p}_0(\Omega)\) is a solution of \((\bar{P}_\lambda)\), then it follows from Lemma 6.3 that \(v \in C^{1,\theta}(\Omega)\). And there exists a small enough \(\varepsilon > 0\) such that
\[
0 < \varepsilon \phi_1 \leq v \quad \text{in} \ \Omega. \quad (6.2)
\]
Denote \(\psi = \varepsilon \phi_1\), we obtain
\[
-\Delta_p \psi = \bar{\lambda}_1 \psi^{p-1} \leq \mu \psi^{p-1}. \quad (6.3)
\]
However, let \(\bar{\lambda}\) be large enough such that for all \(\lambda > \bar{\lambda}\), we get
\[
(\bar{\lambda}_1 + \delta) v^{p-1} \leq \lambda v^\alpha + h(x) v^\beta.
\]
Thus, we have
\[
-\Delta_p v \geq (\bar{\lambda}_1 + \delta) v^{p-1} \geq \mu v^{p-1}. \quad (6.4)
\]
From (6.2), (6.3) and (6.4), we can construct a solution \(\psi \leq u \leq v\) of the problem (6.1) by sub-super solutions. But this is a contradiction. Hence, we conclude that there exists \(\bar{\lambda}\) such that \(\bar{\Lambda} \leq \bar{\lambda} < \infty\).

Moreover, we can also obtain solutions of \((\bar{P}_\lambda)\) for small \(\lambda\) similar to Section 3, then \(\bar{\Lambda} > 0\). So, \(\bar{\Lambda} \in (0, \infty)\). \(\Box\)
References


