JOURNAL OF ALGEBRA 45, 465–515 (1977)

Linear Groups Containing an Involution with Two Eigenvalues —1

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Communicated by Walter Feit

Received May 9, 1975

1. INTRODUCTION

The main theorem of this paper describes quasiprimitive linear groups G which contain a matrix with two eigenvalues -1 and the remaining eigenvalues 1. This is a special case of a linear group containing a unimodular matrix with a trivial eigenspace of codimension 2. If a linear group contains a unimodular matrix with trivial eigenspace of codimension 2 other than this, the group is known by [1], [12], or [8], as is described in [8]. In a later paper [9], we treat linear groups containing a matrix with any eigenspace of codimension 2. Of course, there we refer to this work. Linear groups containing a matrix with eigenspace of codimension 1 were determined in [14] in 1914.

We prove the following theorem.

MAIN THEOREM. *Suppose G is a finite quasiprimitive linear group of degree $n \geqslant 8$ and X is the corresponding representation. Suppose further that G contains an involution τ for which $X(\tau)$ has trace $n - 4$ (i.e., $X(\tau)$ has exactly 2 eigenvalues -1 and exactly $n - 2$ eigenvalues 1). Then G* mod *the maximal solvable normal subgroup is known and G satisfies one of the following two conditions*:

* Sponsored in part by the Army Research Office, Durham, under Grant DA-ARO-D-31-124-72-G171.

† Sponsored in part by NSF GP-35678.

ISSN 0021-8693

(1) *There is an element* γ *of* G *for which* $X(\gamma)$ *has one eigenvalue* ω, *one eigenvalue* $\bar{\omega}$, *and* $n - 2$ *eigenvalues* 1. *The group is known by* [8]. *Here* $\omega = e^{2\pi i/3}$.

(2) *The product of any two elements* τ_1, τ_2 *with* $X(\tau_1)$ *and* $X(\tau_2)$ *similar to* $X(\tau)$ *has order* 2, 3, 4, *or* 5. *If* $\tau_1\tau_2$ *has order* 4, *either* $(\tau_1\tau_2)^2$ *is in* $O_2(G)$ *or* $X((\tau_1\tau_2)^2)$ *is similar to* $X(\tau)$. *Here* G mod *the maximal solvable subgroup is known by* [15]. *Also* $O_2(G)$ *is the maximal normal 2-group of* G.

We note that in G the group generated by all conjugates in G of τ is a normal subgroup H. Either $X \mid H$ is irreducible or as in [9, Theorem 3], $X \mid H = 2 \cdot X_1$ where X_1 has degree $n/2$ and $X_1(\tau)$ is a reflection. For this latter case, as in [9, Theorem 3], we use [10] to show $G/Z(G) \cong K \times A$ where K is generated by reflections and so is listed in [14], and $A \cong A_4$, S_4, or A_5. Also $X(G)$ is a subgroup of $Y \otimes Z$ where Y is a projective representation of K of degree $n/2$, and Z is a projective representation of A of degree 2.

The proof is organized as follows. We assume G does not satisfy either condition 1 or 2 and so has elements τ_1 and τ_2 for which $X(\tau_1)$ and $X(\tau_2)$ are similar to $X(\tau)$. Also $\tau_1\tau_2$ has order $2m$ where $m \geqslant 2$ and if $m = 2$, $X(\tau_1\tau_2)^2$ is not similar to $X(\tau)$ and is not in $O_2(G)$ where $O_2(G)$ is the largest normal 2-group of G. By considering various restrictions to subgroups containing τ_1 and τ_2 we show in Section 3 that m is 2. In Section 4 we show that the product of any two distinct elements of $X(G)$ similar to $X(\tau)$ is of order 2, 3, 4, or 5. In Section 5 we find the possible subgroups generated by τ_1, τ_2, and another involution τ_3 for which $X(\tau_3)$ is similar to $X(\tau)$; $\tau_1\tau_2$ has order 4, but $X((\tau_1\tau_2)^2)$ is not similar to $X(\tau)$. In Section 6 we show that this last case is impossible. This last section involves generators and relations for appropriate subgroups as well as actual matrices for appropriate subgroups.

The notation is as follows. The group G is a quasiprimitive linear group of degree n which does not satisfy the Main Theorem. We let X be the faithful representation of G acting on the n-dimensional vector space V. There is an element τ in G such that $X(\tau)$ has two eigenvalues -1, and $n - 2$ eigenvalues 1. Denote by D the set of involutions σ of G such that $X(\sigma)$ is similar to $X(\tau)$. An element γ of G is called a special element if $X(\gamma)$ has eigenvalues ϵ, $\bar{\epsilon}$, and $n - 2$ eigenvalues 1. If ϵ is a primitive rth root of unity, γ is called a special r-element. Note that elements of D are special 2-elements. Elements arising in case 1 of the Main Theorem are special 3-elements. The group G contains no special 3-elements as we assume condition 1 of the Main Theorem does not hold. Also G contains no special r-elements for $r \geqslant 4$ by [1, 8, 12]. A representation X of a group G is called quasiprimitive if X is irreducible, and for every $H \lhd G$, $X \mid H$ breaks into similar constituents. By [4, (9.11)] if X is not quasiprimitive it is induced from a proper subgroup. The term Blichfeldt refers to [1, p. 96]. If Y is a monomial representation of a group H we assume the matrices are in monomial form and speak of the associated permutation of the elements of H. This permutation naturally is the one obtained by replacing the unique nonzero element in each row and column by 1. For typographical convenience we let $\operatorname{diag}(d_1, \ldots, d_n)$ denote the $n \times n$ diagonal matrix whose (i, i) entry is d_i.

The remaining notation is standard as in [6, pp. 4–6].

2. Properties of the Small Dimensional Quasiprimitive Groups

In this section we gather together some properties of the small-dimensional primitive linear groups. These groups are known to dimension 7 by [1, 3, 11, 16, 17]. They are listed in [5, Sect. 8.5]. Since the properties we need can be found by inspection, we just sketch some of the details.

Lemma 2.1. *Suppose H is a subgroup of G generated by special involutions and $X \mid H = Y \oplus \xi \oplus (n - r - 1)\, 1_H$, Y is primitive of degree r, ξ is linear, and 1_H is the trivial character of H. Assume r is 5, 6, or 7. Then ξ is trivial and the product of any two special 2-elements in H has order 1, 2, 3, 4, 5, or 6. If it is 4, the square is again special. If it is 6, $r = 5$, and $H \cong S_5$.*

Proof. Note first that ξ is trivial; otherwise the matrix $Y(\tau)$, τ in D, has one eigenvalue -1, the rest are eigenvalue 1. These groups are described in [14] and all have a special 3-element in the commutator. This would be a special 3-element in G. As H is generated by special 2-elements, $Y(H)$ is unimodular and so H is listed in [5, Sect. 8.5]. We refer to this notation.

If $r = 7$ we note the groups A_8, S_8, $Sp_6(2)$ all have special 3-elements. The involutions in I_7, $PSL_2(13)$, $PSL_2(8)$ are not special. In $G_2(2)$ (case VI) there are two classes of involutions. Those outside $U_3(3)$ are not special; those inside satisfy condition 2 of the main theorem. The same holds for $PSL_2(7)$ and $PGL_2(7)$.

If $r = 6$ the groups II, XI, XII have no special 2-elements. The groups of I could not be generated by special 2-elements as such elements would be $Y(\tau) = A(a) \otimes B(b)$ where A, B have degree 3, 2, respectively, and $B(b)$ must be a scalar. The groups VI, XIII have special 3-elements. The groups V, VIII, XV, XVII have centers of order 6 which contradict Blichfeldt's theorem. Also, XVI has an element of order 6 with three eigenvalues $-\omega$, and three $-\bar{\omega}$; this contradicts Blichfeldt. The group $U_3(3)$ or its extension, XIV, is handled as in the case when $r = 7$, as are the groups in IX. For case X, $SL_2(7)$ has only one involution which is not special. In $GL_2(7)$ there is one class of involutions not in $SL_2(7)$ represented by the matrix $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Each normalizes an element of order 7 and so is not special. The cases remaining are III, IV, VII.

To handle case III note that an element of order 5 is conjugate to all its powers so that the character is 1. The subgroup of index 2 must be $\tilde{A}_5$ as the center cannot split by the quasiprimitivity. Note that A_5 has no irreducible representation of degree 6 and that a sum of two identical ones of degree 3 has the wrong trace. As $\tilde{A}_5 \simeq SL_2(5)$, there are no special involutions in this subgroup. The special involutions, then, all correspond in S_5 to 2-cycles. The product of two in S_5 has order 2 or 3 and so in $\tilde{S}_5$ the product can only be 2, 4, 3, or 6. If it is 4 or 6, its square or cube would be central with six eigenvalues -1. This is impossible for a product of special involutions.

Cases IV and VII remain. From inspection it can be seen that the involutions in $\tilde{A}_6$ satisfy condition 2 of the theorem and in the extension of $\tilde{A}_6$ in case IV there are no special involutions outside $\tilde{A}_6$ which leaves only case VII to be considered. Note that the orders of products of special elements have the values 1, 2, 3, 4, 5, 6.

To handle case VII, suppose Y acts on the irreducible 6-dimensional space U. Let τ be an element of D such that $X(\tau)$ moves U. Now $X \mid \langle H, \tau\rangle = Y_1 \oplus (n-8)\mid_{\langle H,\tau\rangle}$. Suppose Y_1 is irreducible and primitive. By [13], $7^2 \nmid |\langle H, \tau\rangle|$ and so $|\langle H, \tau\rangle|$ has 7 to the first power only. By Brauer, an element with six eigenvalues ω and two eigenvalues 1 cannot centralize an element of order 7 with trace 1 [2, II]. If Y_1 permutes 2-dimensional subspaces, an element of order 7 is block diagonal and $Y_1 \mid H$ cannot have an irreducible 6-dimensional constituent. If Y_1 is monomial, an element of order 7 must be a 7-cycle and cannot centralize an element with exactly six eigenvalues ω. This means $Y_1 = Y_2 \oplus \xi$ where Y_2 is irreducible of degree 7. If Y_2 is primitive we contradict the above proof when $r = 7$. If Y_2 is monomial we get a contradiction as above.

To handle the case $r = 5$ note that A_6, S_6, and $O_5(3)$ have a special 3-element and the involutions in A_5 and I_5 satisfy condition 2. For $PSL_2(11)$ we must adjoin to H another special involution which moves the invariant subspace corresponding to Y. This group has a 6- or a 7-dimensional irreducible constituent containing an element of order 11. By examining the groups in [5, Sect. 8.5] one sees that this is impossible. This leaves S_5. By consulting the character table of S_5 one sees that there is a unique irreducible 5-dimensional representation in which involutions in S_5–A_5 are special 2-elements. The product of two involutions, one in A_5, the other in S_5–A_5, has order 6.

3. $|\tau_1\tau_2| = 2k,\ k \geqslant 3$

In this section we assume that there are two special involutions whose product has order $2k$, $k \geqslant 3$, and we reach a contradiction. We prove the following theorem.

THEOREM 3.1. *If τ_1 and τ_2 are distinct special involutions in G, $|\tau_1\tau_2| = 2$, 4, or odd.*

Before proving this theorem we require some preliminary notation and lemmas. The lemmas describe in certain situations how X, restricted to certain subgroups containing τ_1 and τ_2, breaks into irreducible constituents.

Suppose τ_1 and τ_2 are special involutions. As $\langle\tau_1, \tau_2\rangle$ is dihedral and each $X(\tau_i)$ has an $n-2$-dimensional fixed space, $X \mid \langle\tau_1, \tau_2\rangle = X_1 \oplus X_2 \oplus (n-4)\, 1_{\langle\tau_1,\tau_2\rangle}$ where each X_i has degree 2 and may be reducible. If τ_1 and τ_2 do not commute, either X_1 or X_2 is irreducible. If X_1 is irreducible and X_2 reducible, $X_2((\tau_1\tau_2)^2) = I$. Now $(\tau_1\tau_2)^2$ is a special element. As G has only special r-elements for $r = 2$, $X_1((\tau_1\tau_2)^2) = -I$, and so $\tau_1\tau_2$ has order 4. In general, if τ_1 and τ_2 do not commute, let K_i be the kernel of X_i and suppose τ is an element in K_i. As $X_i(\tau) = I$, τ is a special element and so τ has order 1 or 2 and $X_j(\tau) = \pm I$. This means $|K_i| \leqslant 2$ and K_i is in the center of $\langle\tau_1, \tau_2\rangle$. Assuming τ_1 and τ_2 do not commute, the center of $\langle\tau_1, \tau_2\rangle$ is cyclic and so at most one K_i is nontrivial. If X_1 and X_2 are both irreducible, one must be faithful; the other could have a kernel of order 1 or 2.

We say that two special involutions are bad of order m if $|\tau_1\tau_2| = m$ where $m = 2k$, $k \geqslant 3$. To prove Theorem 3.1 we must show there are no bad pairs of special involutions. Suppose then that τ_1, τ_2 are a bad pair. Now $X \mid \langle\tau_1, \tau_2\rangle = X_1 \oplus X_2 \oplus (n-4)\, 1_{\langle\tau_1,\tau_2\rangle}$ where X_i are both irreducible. Assume X_1 is faithful.

By examining the dihedral group D_{4k}, we can if necessary replace τ_1 and τ_2 by special elements for which the order of $\tau_1\tau_2$ is 8 or $2p$ with p an odd prime. We assume then that $k = 4$ or p. Let X_1 act on U_1 and X_2 act on U_2. We let $X_i(\tau_1\tau_2)$ have eigenvalues α_i and $\bar{\alpha}_i$. Note that α_1 is a primitive $2k$th root of 1, $\alpha_1 \neq \alpha_2$ or $\bar{\alpha}_2$ or $X(\tau_1\tau_2)$ would contradict Blichfeldt and so U_1 and U_2 are unique. In a series of lemmas we show that some subgroups of G containing $\langle\tau_1, \tau_2\rangle$ are restricted.

LEMMA 3.2. *Let H be a subgroup of G containing $\langle\tau_1, \tau_2\rangle$ and generated by special involutions. Suppose $X \mid H = Y \oplus \xi \oplus (n-7)\, 1_H$ where Y is irreducible of degree 6. Then one of the following holds.*

(i) *Y is monomial*

$$Y(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Y(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \bar{\alpha}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

or

$$Y(\tau_2) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{d} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $d = (\alpha_1)^2$ and the permutation group contains no 3-cycles. The second form of $Y(\tau_2)$ occurs only if $k \leqslant 5$.

(ii) *Y is not monomial*

$$Y(\tau_1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Y(\tau_2) = \begin{bmatrix} 0 & 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{\alpha}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Y permutes 2-dimensional subspaces, $| \tau_1\tau_2 | = 8$, and X_1, X_2 are faithful on both U_1 and U_2.

Proof. We first remark that except for the treatment of the nonmonomial case this proof also works when $| \tau_1\tau_2 | = p$, 9, or 15 with $p \geqslant 7$. This will be dealt with in Lemma 4.2.

Note that if Y is primitive, Lemma 2.1 gives a contradiction. Otherwise, $Y(H)$ permutes 1-, 2-, or 3-dimensional subspaces. An involution interchanging two 3-dimensional subspaces has trace 0 and cannot be special. As H is generated by special involutions, $Y(H)$ cannot interchange two 3-dimensional subspaces. Suppose first that $Y(H)$ is not monomial and so permutes 2-dimensional subspaces but not 1-dimensional subspaces. Let these spaces be V_1, V_2, and V_3. As H is generated by special involutions, there must be special involutions μ_1 and μ_2 such that $Y(\mu_1)$ interchanges V_1 and V_2 and $Y(\mu_2)$ interchanges V_2 and V_3. By choosing an appropriate basis we can assume $Y(\mu_1)$ is the permutation matrix corresponding to (1, 3)(2, 4) and $Y(\mu_2)$ is the permutation matrix corresponding to (3, 5)(4, 6). Then $\mu_3 = (\mu_1)^{\mu_2}$ will correspond to (1, 5)(2, 6).

We examine the possible permutation actions of $Y(\tau_1)$ and $Y(\tau_2)$ on V_1, V_2, V_3. Suppose all are fixed. By reordering V_1, V_2, V_3 and rechoosing the basis we can assume $Y(\mu_i)$, where $i = 1, 2, 3$ are unchanged and

$$Y(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad Y(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 \\ 0 & 0 & \bar{\alpha}_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Now $Y[(\tau_1\tau_2), \mu_3] = \text{diag}(\alpha_1, \bar{\alpha}_1, 1, 1, \bar{\alpha}_1, \alpha_1)$ contradicts Blichfeldt as α_1 is a primitive $2k$th root of 1. Suppose then $Y(\tau_i)$ $i = 1$ or 2 fixes all three V_j. By reordering and changing the basis we can assume

$$Y(\tau_1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad Y(\tau_2) = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & I_2 \end{bmatrix},$$

where A, B are 2×2 matrices of order 2 and $A = \operatorname{diag}(\pm 1, \pm 1)$. If $A = \pm I_2$, $Y(\tau_1\tau_2)$ has order 4, a contradiction, and we can assume $A = \operatorname{diag}(1, -1)$. Note $B \neq I_2$ as $\xi \mid \langle \tau_1, \tau_2 \rangle$ is trivial. Replace μ_1 by τ_1 and change the basis of V_3 so that $Y(\mu_2)$, $Y(\mu_3)$ are the permutation matrices as above and we have

$$Y(\tau_2^{\mu_3}) = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A \end{bmatrix}, \qquad Y(\tau_2^{\mu_2}) = \begin{bmatrix} A & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & B \end{bmatrix},$$

so

$$Y(\tau_2\tau_2^{\mu_3}\tau_2^{\mu_2}) = \begin{bmatrix} I_4 & 0 \\ 0 & AB \end{bmatrix}.$$

This is a special r-element where $r \geqslant 3$ and hence impossible unless $AB = \pm I_2$. In each case B is determined and $\tau_1\tau_2$ has order 2 or 4, a contradiction.

We have shown that both $Y(\tau_1)$ and $Y(\tau_2)$ interchange two of V_1, V_2, V_3. If they interchange different ones the product $\tau_1\tau_2$ has order 3, contradicting our assumptions. We may assume

$$Y(\tau_1) = \begin{bmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_2 \end{bmatrix} \qquad Y(\tau_2) = \begin{bmatrix} 0 & A & 0 \\ A^{-1} & 0 & 0 \\ 0 & 0 & I_2 \end{bmatrix}.$$

By rechoosing a basis for V_1 and V_2 and if necessary replacing α_i with $\bar{\alpha}_i$, we can assume $A = \operatorname{diag}(\alpha_1, \alpha_2)$.

Let K be the normal subgroup of H such that $Y(K)$ fixes V_1, V_2, V_3. Clearly $H = K\langle \tau_1, \mu_2 \rangle$ and $\langle \tau_1, \mu_2 \rangle \simeq S_3$. Let $Y \mid K = R_1 \oplus R_2 \oplus R_3$ where R_i acts irreducibly on V_i. As we are assuming Y is not monomial, R_i is not monomial. This follows as if R_1 were induced from a subgroup K_1 of K of index 2, Y would be induced from $\langle K_1, \mu_2 \rangle$ of index 6 in H. Note $\tau_1\tau_2 \in K$ and $R_3(\tau_1\tau_2) = I_2$. The groups $R_i(K)/Z(R_i(K))$ are isomorphic as groups and must be A_5, S_4, or A_4 by Blichfeldt [1].

Suppose the cyclic group $\det(R_i(K))$ has order d. If L is $SL_2(5)$, $GL_2(3)$, or $SL_2(3)$, then $R_i(K)$ is a subgroup of $L \circ D$ where D is cyclic of order $2d$. Here $R_i(K)/Z(R_i(K))$ covers $L \circ D/Z(L \circ D)$. The matrix $R_1(\tau_1\tau_2) = \operatorname{diag}(\alpha_1, \alpha_2)$. As $\alpha_1 \neq \alpha_2$ by Blichfeldt, $R_1(\tau_1\tau_2)$ is a noncentral element. As an element of $L \circ D$, $R_1(\tau_1\tau_2) = XY$ where $X \in L$, $Y \in D$. As $R_1(\tau_1\tau_2)$ is noncentral, X is noncentral in L. The commutators of $R_1(\tau_1\tau_2)$ by elements of $R_1(K)$ generate a normal subgroup of $L \circ D$ containing at least the quaternion group Q_8.

It follows that in K' is an element γ for which $R_1(\gamma) = -I_2$, $R_2(\gamma) = \pm I_2$, $R_3(\gamma) = I_2$. By conjugating with $\langle \tau_1, \mu_2 \rangle$ we obtain an element τ of K' with $R_1(\tau) = R_2(\tau) = -I_2$, $R_3(\tau) = I_2$. Now if $k \neq 4$ and $\alpha_1 \neq \pm\alpha_2$, there is an element of order k in $R_1(K)/Z(R_1(K))$ and so k is 3 or 5. If it is 5, a high commutator contains an element with eigenvalues on V of $-\omega, -\bar{\omega}, 1, 1, 1, 1$; $-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1, 1$; or $-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, -\omega, -\bar{\omega}$. This contradicts Blichfeldt. If $k = 3$, $\tau(\tau_1\tau_2)^2$ contradicts Blichfeldt with eigenvalues on V of $-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1, 1$. If $\alpha_1 = \pm\alpha_2$, $\alpha_1 = -\alpha_2$ by Blichfeldt, and $\tau(\tau_1\tau_2)^2$ has eigenvalues on V of $-\alpha_1^2, -\alpha_1^2, -\bar{\alpha}_1^2, -\bar{\alpha}_1^2, 1, 1$, contradicting Blichfeldt. Finally, if $k = 4$ and α_2 is not a primitive 8th root of 1, $R_1(K)/Z(R_1(K))$ has an element of order 8 which is a contradiction. This means that X_1 and X_2 are faithful and gives case ii completing the nonmonomial case.

Assume now that Y is monomial. Suppose first that $Y(\tau_1)$ and $Y(\tau_2)$ as permutations are both pairs of disjoint transpositions moving the same set of four letters. We can assume by reordering and rescaling that $Y(\tau_i)$ have the same form as (i) in the lemma. Note that if $k > 5$, the second form of $Y(\tau_2)$ cannot occur or $(\tau_1\tau_2)^2$ contradicts Blichfeldt. Suppose there is an element s of H for which $Y(s)$ acts as a 3-cycle. As H is generated by special involutions which cannot interchange two 3-dimensional subspaces, the permutation group $Y(H)$ must be transitive. This means it is A_6 or S_6. If t is an element of H such that $Y(t)$ represents the 3-cycle (1, 5, 6), $Y([(\tau_1\tau_2)^2, t]) = \mathrm{diag}(\mu, 1, 1, 1, \bar{\mu}, 1)$ where μ is a primitive kth root of 1. This is a special k-element contradicting our assumptions. The lemma is proved now if $Y(\tau_1)$ and $Y(\tau_2)$ act on the same four letters both as products of two disjoint transpositions.

Suppose $Y(\tau_1)$ and $Y(\tau_2)$ both act as products of two disjoint transpositions. If they are transitive on five letters, $Y(\tau_1\tau_2)$ has order 5 contrary to assumptions. If they are transitive on four letters and interchange the other two, $Y(\tau_1\tau_2)$ has order 4. If they move six letters and act like (12)(34), (15)(36), $\tau_1\tau_2$ has order 3. By reordering and rescaling and recalling that $Y \mid \langle \tau_1, \tau_2 \rangle$ has no nontrivial linear constituent, we can now assume

$$Y(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad Y(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The product $\tau_1\tau_2$ must have order 6 and so we can assume $\alpha_1 = -\omega$ where $\omega = e^{2\pi i/3}$. The permutation group is A_6 or S_6 as $Y((\tau_1\tau_2)^2)$ is a 3-cycle. As $Y((\tau_1\tau_2)^3) = \mathrm{diag}(-1, -1, 1, 1, 1, 1)$ and $Y(H)$ as a permutation group is A_6 or S_6 we easily get $-I_6$ in $Y(H)$. Let S be a Sylow 3-group such that $Y(S)$ as a permutation group is $\langle (123), (456) \rangle$. If S is nonabelian $Z(S) \cap S'$ contains $\mathrm{diag}(\omega, \omega, \omega, \bar{\omega}, \bar{\omega}, \bar{\omega})$, $\mathrm{diag}(\omega, \omega, \omega, 1, 1, 1)$, $\mathrm{diag}(1, 1, 1, \omega, \omega, \omega)$, or ωI_6. In the first three cases, by conjugating with an element representing (34)(56) one gets a special 3-element. In the last case $-\omega I_6$ is in $Y(H')$, an element contradicting Blichfeldt. If S is abelian an element s representing (123) must be scalar on $\{4, 5, 6\}$ or it would not commute with an element representing (4, 5, 6). However, $Y((\tau_1\tau_2)^2)$ is not scalar on the points it fixes. This case is therefore impossible.

If $Y(\tau_1)$ is diagonal, $Y(\tau_1\tau_2)$ has order at most 4. The only possibility not dealt with is that $Y(\tau_i)$ is a transposition for $i = 1$ or 2. As $X \mid \langle\tau_1, \tau_2\rangle$ has two irreducible constituents of degree 2, $Y \mid \langle\tau_1, \tau_2\rangle$ also has two irreducible constituents of degree 2. The eigenvalues of $Y(\tau_1\tau_2)$ are $\alpha_1, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_2$ where α_1 is a primitive $2k$th root of 1 and α_2 is a primitive $2k$th or kth root of 1. The only possibility for $Y(\tau_1)$ and $Y(\tau_2)$ after reordering and rescaling is now

$$Y(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Y(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

or

$$Y(\tau_2) = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where α_1 is a primitive 8th root of 1. If the permutation group contains a 3-cycle, the permutation group is S_6, a conjugate σ of τ_2 represents either (1, 2) or (2, 3) and $(\sigma\tau_2)^2$ is a special 3-element. This means the permutation group contains no 3-cycles. In the first case of $Y(\tau_2)$, the 2-cycles present so far are (1, 2) and (3, 4). As the permutation group is transitive there must also be (5, 6). As there are no 3-cycles these are the totality of transpositions in the permutation group. As the permutation group is generated by special 2-elements and transitive on the sets $\{1, 2\}$, $\{3, 4\}$, and $\{5, 6\}$ there is a special two element τ interchanging the sets $\{3, 4\}$ and $\{5, 6\}$. This of course acts trivially on the first two coordinates and so

$$Y(\tau_2{}^\tau) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \pm 1 \end{bmatrix}.$$

Now $(\tau_1\tau_2{}^\tau)^2 = \operatorname{diag}((\bar{\alpha}_1)^2, (\alpha_1)^2, 1, 1, 1, 1)$ a special 4-element.

In the second case of $Y(\tau_2)$ the transpositions obtained so far are (2, 3) and (1, 4). Again (5, 6) must be in the group. Again there must be a special 2-element τ_3 such that $Y(\tau_3)$ is the permutation matrix corresponding to (15)(46). Now $((\tau_2\tau_3)^2\tau_1)^2\tau_2$ is a special 4-element.

LEMMA 3.3. *Let H be a subgroup of G containing $\langle\tau_1, \tau_2\rangle$ and generated by special involutions. Suppose $X \mid H = Y \oplus \xi \oplus (n-8)\, 1_H$ where Y is irreducible of degree 7. Then Y is monomial, the permutation group contains no 3-cycles, no 2-cycles, and has no element of order 5. In an appropriate basis*

$$Y(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Y(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & \bar{\alpha}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

or

$$Y(\tau_2) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{d} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \textit{where} \quad d = (\alpha_1)^2.$$

The second form of $Y(\tau_2)$ occurs only if $k \leqslant 5$.

Proof. This proof again works in the case $|\tau_1\tau_2| = p$, 9, or 15 with $p \geqslant 7$ which will be handled in Lemma 4.3.

By Lemma 2.1, Y cannot be primitive and so must be monomial. If $Y(H)$ contains a 3-cycle, the permutation group is A_7 or S_7 as these are the only transitive subgroups of S_7 containing 3-cycles. It follows as in Lemma 3.2 that $Y(\tau_1)$ and $Y(\tau_2)$ are both products of disjoint 2-cycles moving the same four points. The form for $Y(\tau_1)$ and $Y(\tau_2)$ after reordering and rescaling is as specified. As in Lemma 3.2 the permutation group contains no 3-cycles and so no 2-cycles. It has no elements of order 5 as a transitive subgroup of S_7 containing an element of order 5 is A_7 or S_7.

LEMMA 3.4. *There can be no subgroup H of G such that H contains $\langle\tau_1, \tau_2\rangle$ and $X \mid H = Y \oplus \xi \oplus (n-6)\, 1_H$ where Y is irreducible of degree 5.*

Proof. This proof again works in the case $|\tau_1\tau_2| = p$, 9, or 15 with $p \geqslant 7$ to be dealt with in Lemma 4.4.

Replace H by the normal subgroup generated by all conjugates of τ_1 and τ_2. There is still an irreducible constituent of degree 5 as otherwise, by Clifford's theorem [6, Theorem 3.4.1], τ_1 and τ_2 would commute. As $X \mid \langle \tau_1, \tau_2 \rangle$ has only trivial linear constituents, $\xi(\tau_1) = \xi(\tau_2) = 1$ and so ξ is now trivial.

As X is irreducible there must be a special 2-element τ in G such that $X(\tau)$ does not fix the 5-dimensional space U on which Y acts. As $X(\tau)$ has an $n-2$-dimensional fixed space and $X(H)$ has an $n-5$-dimensional fixed space, $X(\langle H, \tau \rangle)$ has an $n-7$-dimensional fixed space and satisfies the hypothesis of either Lemma 3.2 or 3.3. The groups in Lemma 3.3 have no subgroup which has an irreducible constituent of degree 5 as any elements of order 5 would be in the diagonal abelian subgroup.

This means $X \mid \langle H, \tau \rangle$ has an irreducible imprimitive constituent of degree 6. Again replace $\langle H, \tau \rangle$ with the normal subgroup K generated by H and all conjugates in $\langle H, \tau \rangle$ of τ_1 and τ_2. This contains H and again by Clifford's theorem this group has an irreducible constituent of degree of at least 5 and so has an irreducible constituent of degree 6. Now $X \mid K = Y \oplus (n-6)\, 1_K$. For some special 2-element σ in K, $X(\sigma)$ must move U and $X \mid \langle H, \sigma \rangle = Y_1 \oplus (n-6)\, 1_{\langle H, \sigma \rangle}$ where Y_1 acts irreducibly on U_1 of dimension 6. As X is irreducible there is a special 2-element σ_1 for which $X(\sigma_1)$ moves U_1. Either $X(\sigma_1)$ or $X(\sigma_1{}^{\sigma})$ moves U as well. Assume $X(\sigma_1)$ does. As above, we may replace $X(\sigma_1)$ by a special 2-element $X(\sigma_2)$ moving U and U_1, such that $X \mid \langle H, \sigma_2 \rangle = Y_2 \oplus (n-6)\, 1_{\langle H, \sigma_2 \rangle}$, where Y_2 acts on U_2. As $U_1 \neq U_2$, $X \mid \langle H, \sigma, \sigma_2 \rangle$ satisfies the hypothesis of Lemma 3.3 a contradiction as no such group has a subgroup with an irreducible constituent of degree 5. This completes the proof of the lemma.

LEMMA 3.5. *There is no subgroup K of G generated by* 3 *special* 2-*elements* τ_1, τ_2, τ *with the following special form.* $X \mid \langle \tau_1, \tau_2, \tau \rangle = T_1 \oplus T_2 \oplus (n-5)\, 1_K$

$$T_1(\tau_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad T_1(\tau_2) = \begin{bmatrix} 0 & -\omega \\ -\bar{\omega} & 0 \end{bmatrix}, \qquad T_1(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$T_2(\tau_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T_2(\tau_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \qquad T_2(\tau) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Here $\omega = e^{2\pi i/3}$.

Proof. Let T_1 act on V_1, T_2 on V_2, and let σ be a special 2-element for which $X(\sigma)$ moves V_1. Let $H = \langle \tau_1, \tau_2, \tau, \sigma \rangle$. Note that T_1 and T_2 are irreducible.

Note first that $X \mid H$ has an $n-7$-dimensional fixed space. If there is an irreducible 6- or 7-dimensional constituent Y, apply Lemma 3.2 or 3.3. Since $|\tau_1\tau_2| = 6$ the group is monomial and $Y(\tau_1)$ acts as the permutation (12) (34). Now $Y(\tau)$ must permute the letters 1, 2, 3, 4 among themselves as τ and τ_1

commute. However, now $Y \mid \langle \tau_1, \tau_2, \tau \rangle$ has two irreducible constituents of degree 2, while the rest are linear. This conflicts with T_1 and T_2 being irreducible. Consequently $X \mid H$ has constituents of degree at most 5. If there is a constituent Y of degree 5 the remaining constituents are linear as Y must act on $V_1 + V_2$ as $X(\sigma)$ moves V_1 and $X(\sigma)$ is special. This means H must satisfy the hypothesis of Lemma 3.4 and so this is impossible. We conclude the constituents have degree at most 4.

Suppose $X \mid H = S_1 \oplus S_2 \oplus (n-7)\, 1_H$ where S_1 acts irreducibly on $V_1{}^*$ and $V_1 \subseteq V_1{}^*$. There are three cases to consider:

(i) S_1 has degree 4 for some σ.

(ii) S_1 has degree 3 and S_2 is irreducible of degree 4 for some σ.

(iii) S_1 has degree 3 and S_2 has a linear constituent for all σ.

These are the only possibilities as S_1 cannot have degree 2 since $X(\sigma)$ moves V_1 and T_2 has degree 3.

In case (i) above S_1 must be imprimitive, by Blichfeldt. Suppose S_1 permutes 2-dimensional subspaces. As $S_1(\tau_i)$ are reflections, we have

$$S_1(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad S_1(\tau_2) = \begin{bmatrix} 0 & -\omega & 0 & 0 \\ -\bar{\omega} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_1(\sigma) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now $S_2(\sigma)$ is trivial and $(S_1 \oplus S_2)\,[\tau_1\tau_2, \sigma] = \operatorname{diag}(-\omega, -\bar{\omega}, -\bar{\omega}, -\omega, 1, 1, 1)$ contradicting Blichfeldt. If S_1 is monomial we obtain the same forms as $S_1(\tau)$ is trivial and so $S_1(\langle \tau_1, \tau_2, \sigma \rangle)$ must be transitive.

In case (ii), S_1 is again imprimitive by Blichfeldt and so must be monomial. We can assume

$$S_1(\tau_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad S_1(\tau_2) = \begin{bmatrix} 0 & -\omega & 0 \\ -\bar{\omega} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad S_1(\sigma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and so $S_1(H) = DS$ where D are diagonal matrices and $S \cong S_3$. A Sylow 3-group of D has order 3^2. If $3^2 \nmid |S_2(H)|$, there is a special 3-element in $\ker S_2$. Suppose S_2 is primitive. The groups $S_2(H)$ are listed in [14] or [5, Sect. 8.5]. All have elementary abelian Sylow 3-groups of order at most 9. Note that $\tilde{O}_5(3)$ is not generated by reflections. Consider the group $H_1 = \langle \tau_1, \tau_2, \sigma \rangle$.

As $S_1(H_1)$ contains a full Sylow 3-group of $S_1(H)$, $S_2(H_1)$ must also contain a full Sylow 3-group of $S_2(H)$, or H_1 would contain a special 3-element. Now $S_2(H_1)$ is generated by three reflections and so $S_2 \mid H_1$ has a 1-dimensional fixed space. A Sylow 3-group of $S_2(H_1)$ being elementary abelian must now contain an element with eigenvalues ω, ω, ω, 1. This is impossible according to [14] or inspection of the groups in [5, Sect. 8.5]. If S_2 permutes 2-dimensional subspaces, then $S_2(\tau_i)$ are block diagonal. As $[\tau_1, \tau] = 1$, $S_2(\tau)$ is block diagonal and so T_2 is reducible. Finally, S_2 must be monomial. We may assume

$$S_2(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad S_2(\tau_2) = \begin{bmatrix} 0 & \omega & 0 & 0 \\ -\bar{\omega} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

or

$$S_2(\tau_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As $[\tau_1, \tau] = 1$, $S_2(\tau)$ permutes the first two coordinates. As T_2 is irreducible $S_2(\tau_2)$ must be the permutation matrix. Now $S_2(H) = D_1 S^*$ where D_1 are diagonal matrices and $S^* \simeq S_4$. As $3^2 \mid | S_2(H)|$, $3 \mid | D_1 |$. Now $S_1(H'')$ is diagonal and $S_2(H'')$ is a diagonal group with $V = \langle (12)(34), (14)(23) \rangle$ acting. As $3 \mid | D_1 |$, there is an element γ in H'' with $S_2(\gamma) = \operatorname{diag}(\omega, \omega, \omega, 1)$. Now conjugating by an element γ_1 of H'', for which $S_2(\gamma_1)$ acts as (12) (34), gives a special 3-element.

In the final case $S_2 = R \oplus \xi$ where R is irreducible of degree 3 and ξ is linear. Let R act on V_2^*. Here V_2^* must be the irreducible 3-dimensional space T_2 acts on. Again by Blichfeldt, S_1 is monomial. If $R(\sigma)$ is trivial $(\tau_1\sigma)^2$ must be a special 3-element. It follows that ξ is trivial. This shows that if $X(\sigma)$ moves V_1, $X(\sigma)$ fixes V_2^*, acts nontrivially on V_2^*, and $V_1^* = \langle V_1, X(\sigma) V_1 \rangle$ is an invariant subspace of dimension 3. Relabel V_1^* as W_3, $\sigma = \sigma_3$, $H_3 = \langle \tau_1, \tau_2, \tau, \sigma_3 \rangle$. Note $W_3 = \langle W_2, X(\sigma_3) W_2 \rangle$, where $W_2 = V_1$. Suppose $\sigma_3, \ldots, \sigma_i$ special 2-elements have been chosen so that $H_i = \langle \tau_1, \tau_2, \tau, \sigma_3, \ldots, \sigma_i \rangle = \langle H_{i-1}, \sigma_i \rangle$, $W_i = \langle W_{i-1}, X(\sigma_i) W_{i-1} \rangle$, $X(\sigma_i)(V_2^*) = V_2^*$, $X(\sigma_i) \mid V_2^*$ is not trivial, and $X(H_i) \mid W_i$ is irreducible. Choose σ such that $X(\sigma) W_i \neq W_i$. There is some conjugate σ_{i+1} of σ by an element τ' of H_i such that $X(\sigma_{i+1})$ moves V_1. Then $W_{i+1} = \langle X(\sigma_{i+1}) W_i, W_i \rangle$ is an irreducible subspace for $H_{i+1} = \langle H_i, \sigma_{i+1} \rangle$ of dimension $i + 1$ and $X(\sigma_{i+1})(V_2^*) = V_2^*$. Continuing until $i = n - 2$ we obtain a contradiction.

We now turn to two lemmas which demonstrate how an arbitrary special 2-element τ_3 interacts with τ_1 and τ_2. In particular we show that, except for very special situations, $\tau_1\tau_3$ and $\tau_2\tau_3$ have order 3.

LEMMA 3.6. *Suppose τ_3 is a special 2-element such that $X(\tau_3)$ moves both $U_1 \oplus U_2$ and U_1. Then $|\tau_1\tau_3|$ is 3 or 4 and if 4, $X(\tau_1\tau_3)$ has eigenvalues $i, -i, -1, -1$, the rest being 1. Also, if $|\tau_1\tau_3| = 4$, $X \mid \langle\tau_1, \tau_2, \tau_3\rangle = Y \oplus (n-6)\, 1_{\langle\tau_1,\tau_2,\tau_3\rangle}$ where Y satisfies the hypothesis of Lemma 3.2(i) with $Y(\tau_2)$ representing (1, 3) (2, 4) and $Y(\tau_3)$, the permutation matrix (1, 5) (2, 6), $|\tau_1\tau_2| = 8$, and X_2 is faithful.*

Proof. We divide the proof into cases according to how $X \mid \langle\tau_1, \tau_2, \tau_3\rangle$ breaks into irreducible constituents. As τ_1, τ_2, τ_3 are special 2-elements there will always be an $n-6$ dimensional fixed space. Let $H = \langle\tau_1, \tau_2, \tau_3\rangle$.

Case A. $X \mid H = Y_1 \oplus Y_2 \oplus (n-6)\, 1_H$ where Y_1 is irreducible of degree 3 and $Y_1 \mid \langle\tau_1, \tau_2\rangle$ contains X_1 as a constituent.

As $Y_1 \mid \langle\tau_1, \tau_2\rangle$ contains X_1 as a constituent and $X \mid \langle\tau_1, \tau_2\rangle = X_1 \oplus X_2 \oplus (n-4)\, 1_{\langle\tau_1,\tau_2\rangle}$, $Y_1 \mid \langle\tau_1, \tau_2\rangle = X_1 \oplus 1_{\langle\tau_1,\tau_2\rangle}$. Now Y_1 must be imprimitive as otherwise $Y_1(\tau_1\tau_2)$ contradicts Blichfeldt. We may assume

$$Y_1(\tau_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Y_1(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 \\ \bar{\alpha}_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Y_1(\tau_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

This implies $Y_1(\tau_1\tau_3)$ has order 3 and so if $|\tau_1\tau_3| \neq 3$, $|\tau_1\tau_3| = 6$. In this case $Y_2(\tau_1\tau_3)$ must have order 6.

If Y_2 is irreducible it is again monomial by Blichfeldt. In order that $Y_2(\tau_1\tau_3)$ have order 6 we must have

$$Y_2(\tau_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Y_2(\tau_3) = \begin{bmatrix} 0 & -\omega & 0 \\ -\bar{\omega} & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Y_2(\tau_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

But now $\tau_1, \tau_2, (\tau_1\tau_3)^3$ satisfy the hypothesis of Lemma 3.5 and this is impossible.

Suppose Y_2 is reducible. If it is monomial, $Y_2((\tau_1\tau_2)^2)$ and $Y_2((\tau_1\tau_3)^2)$ are diagonal. Let $[(\tau_1\tau_2)^2, (\tau_1\tau_3)^2]$ be x. Now $Y_1[x, \tau_1] = \operatorname{diag}(\bar{\alpha}_1^6, \alpha_1^6, 1)$, $Y_2[x, \tau_1]$ is trivial and this is a special element not allowed unless $k = 3$. If $k = 3$, $Y_2((\tau_1\tau_3)^3) = -I_2 \oplus 1$ and $\tau_1\tau_2(\tau_1\tau_3)^3$ contradicts Blichfeldt. This means $Y_2(H)/Z(Y_2(H)) \cong A_4$, S_4, or A_5. It cannot be A_5 as then H'' contains a Blichfeldt element. It follows that $\tau_1\tau_2$ must have order 6 and the Sylow 3-group of $Y_2(H)$ is of order 3. As the Sylow 3-group S of $Y_1(H)$ is nonabelian, class 3, of order 27, there is a special 3-element in S in the kernel of Y_2. This contradiction eliminates case A.

Case B. $X \mid H = Y_1 \oplus Y_2 \oplus (n-6)\, 1_H$ where Y_1 is irreducible of degree 4 and $Y_1 \mid \langle\tau_1, \tau_2\rangle$ contains X_1 as a constituent.

As $U_1 \oplus U_2$ is not fixed by $X(\tau_3)$, $Y_1 \mid \langle \tau_1, \tau_2 \rangle = X_1 \oplus 2 \cdot 1_{\langle \tau_1, \tau_2 \rangle}$, and $Y_2 \mid \langle \tau_1, \tau_2 \rangle = X_2$. Now Y_1 is imprimitive as otherwise $Y_1(\tau_1\tau_2)$ contradicts Blichfeldt. If Y_1 permutes 2-dimensional subspaces or is monomial the basis can be chosen as follows.

$$Y_1(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Y_1(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Y_2(\tau_3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Note $Y_2(\tau_3) = I$. Now $[\tau_1 \tau_2, \tau_3]$ contradicts Blichfeldt.

Case C. $X \mid H = Y_1 \oplus \xi \oplus (n-6)\, 1_H$ where Y_1 is irreducible of degree 5. This case is impossible by Lemma 3.4.

Case D. $X \mid H = Y \oplus (n-6)\, 1_H$ where Y is irreducible of degree 6.

By Lemma 3.2, if Y is not monomial and so permutes 2-dimensional subspaces, $Y(\tau_1)$ and $Y(\tau_2)$ interchange the same subspaces. Now to make Y irreducible, $Y(\tau_3)$ must permute one of these to the third and $Y(\tau_1\tau_3)$ would have order 3. We can assume then that Y is monomial and has the form specified by Lemma 3.2. In order that the permutation group be transitive on 6 letters, $Y(\tau_2)$ must have the second form. If $Y(\tau_1\tau_3)$ as a permutation has cycle type (3, 3), its cube must be trivial as such a matrix has the wrong eigenvalue structure to be of order 6. After reordering we may now assume that $Y(\tau_3)$ as a permutation is (1, 5) (2, 6) as other inequivalent choices give $Y(\tau_1\tau_3)$ of type (3, 3). Now $Y([(\tau_1\tau_2)^2, (\tau_1\tau_3)^2]) = \operatorname{diag}(\bar{d}^2, d^2, 1, 1, 1, 1)$ which is a special element. This means $d^2 = -1$ and α_1 is an 8th root of 1. Now $|\tau_1\tau_2| = 8$, the eigenvalues of $Y(\tau_1\tau_3)$ are as specified, and X_1 and X_2 are both faithful.

LEMMA 3.7. *There is no special 2-element τ_3 such that $X(\tau_3)$ moves $U_1 \oplus U_2$ but fixes U_1.*

Proof. Again let $H = \langle \tau_1, \tau_2, \tau_3 \rangle$. As $X(\tau_3)$ leaves U_1 invariant $X \mid H = R_1 \oplus R_2 \oplus (n-6)\, 1_H$ where R_1 acts on U_1. Here either R_2 is irreducible or $R_2 = S \oplus \xi$ where ξ is linear and S is irreducible. We consider first the latter case.

Suppose that R_1 is monomial. If S is primitive and $S(H)$ is not solvable there is a special 3-element in H^∞. This means $S(H)$ is primitive and solvable and so is one of the groups listed under [5, Sect. 8.5].

Note that as the Sylow 3-group of $R_1(H)$ is unimodular, it is cyclic. As the

Sylow 3-group of $S(H)$ is nonabelian of order at least 27 there is a special 3-element in ker R_1. This means S is monomial. We may assume

$$S(\tau_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad S(\tau_2) = \begin{bmatrix} 0 & \alpha_2 & 0 \\ \bar{\alpha}_2 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \text{or} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

In the first case, τ_3 must represent the permutation (2, 3). If $R_1(\tau_3)$ is diagonal, $[\tau_1\tau_2, \tau_3]$ or its square is a special k-element. Consequently, $R_1(\tau_1\tau_3)$ is diagonal and $S(\tau_1\tau_3)$ is a 3-cycle. Now let $\sigma = [\tau_1\tau_2, \tau_1\tau_3]$. We see $R_1(\sigma) = I_2$, $\xi(\sigma) = 1$ and $S(\sigma) = \text{diag}((\alpha_2)^2, \bar{\alpha}_2, \bar{\alpha}_2)$. Now $[\sigma, \tau_1]$ gives an element γ for which R_1 and ξ are trivial and $S(\gamma) = \text{diag}(\bar{\alpha}_2{}^3, \alpha_2{}^3, 1)$. This is special unless α_2 is a cube or a sixth root in which case $k = 3$. Now the Sylow 3-group of $R_1(H)$ is cyclic, the Sylow 3 group of $S(H)$ is nonabelian of order 27, and there is a special 3-element in ker R_1.

In the second case for $S(\tau_2)$, $S(\tau_1\tau_2)$ has order 3 and so $|\tau_1\tau_2| = 6$. Let D be the normal subgroup consisting of elements γ for which $S(\gamma)$ is diagonal. If there is a nonscalar element of order 3, a Sylow 3-group of $S(H)$ must contain the nonbelian exponent 3-group of order 27 and there is a special 3-element in ker R_1. Otherwise let A be a subgroup of D for which $S(A)$ is elementary abelian for some prime $p \neq 3$ and $[S(A), S(\tau_1\tau_2)] = S(A)$. This is possible by [6, Theorem 5.2.3] as $S(D)$ is not scalar or S would be reducible. As $R_1(\tau_1\tau_2)$ is diagonal by taking commutators of A with $\tau_1\tau_2$ sufficiently often one obtains a subgroup A_1 for which $R_1(A_1)$ and $\xi(A_1)$ are trivial and $[S(A_1), S(\tau_1\tau_2)] = S(A_1)$. Now if $p = 2$, A is $Z_2 \times Z_2$, and τ_1, τ_2, together with an element of A_1 contradict Lemma 3.5. If $p \neq 2$, there is either a special p-element in A, or conjugating by τ_1 gives one. We conclude R_1 is not monomial.

We note this implies ξ is trivial. Let τ_4 be a special 2-element for which $X(\tau_4)$ moves U_1 and let $K = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle = \langle H, \tau_4 \rangle$. As in Lemma 3.6 we divide the argument according to how $X \mid K$ breaks into irreducible constituents.

Case A. $X \mid K = T_1 \oplus T_2 \oplus (n - 7)\, 1_K$ where T_1 is irreducible of degree 3 and T_1 acts on a subspace containing U_1.

By Blichfeldt, T_1 is monomial. We may assume

$$T_1(\tau_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad T_1(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 \\ \bar{\alpha}_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As the 2-dimensional space U_1 is unique $U_1 = \langle v_1, v_2 \rangle$ where here v_i is the ith coordinate vector. As $X(\tau_3)$ leaves U_1 invariant it follows that $T_1(H)$ is monomial contrary to the above.

Case B. $X \mid K = T_1 \oplus T_2 \oplus (n-7)\, 1_K$ where T_1 is irreducible of degree 4 and acts on a subspace containing U_1.

Again T_1 is imprimitive by Blichfeldt. We see

$$T_1(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad T_1(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As $T_1(\tau_3)$ must act nontrivially on U_1,

$$T_1(\tau_3) = \left[\begin{array}{c|c} * & 0 \\ \hline 0 & I_2 \end{array}\right].$$

This means

$$T_1(\tau_4) = \begin{bmatrix} 0 & I_2 \\ I_2 & 0 \end{bmatrix}.$$

Now $[\tau_1\tau_2, \tau_4]$ has eigenvalues contradicting Blichfeldt.

Case C. $X \mid K = T \oplus \xi \oplus (n-6)\, 1_K$ where T is irreducible of degree 5. This contradicts Lemma 3.4.

Case D. $X \mid K = T \oplus \xi \oplus (n-7)\, 1_K$ where T is irreducible of degree 6.

Note that Lemma 3.2 applies. As $X \mid H$ has irreducible constituents of degree 3 and 2 and only trivial linear constituents, T cannot permute 2-dimensional subspaces. This means that T is monomial and $T(\tau_1)$ and $T(\tau_2)$ have the form described in Lemma 3.2. If $T(\tau_1)$ and $T(\tau_2)$ both act as (1, 2) (3, 4), the first two coordinates span U_1. As $X(\tau_3)$ leaves U_1 fixed, R_1 is monomial. In the remaining case $T(\tau_1)$ and $T(\tau_2)$ act as (1, 2) (3, 4) and (1, 3) (2, 4). As there are no 3-cycles in the permutation group, and $X \mid H$ has the irreducible constituents R_1 and S, $T(\tau_1)$, $T(\tau_2)$, $T(\tau_3)$ must be transitive on six letters. Now the diagonal subgroup of $T(\langle\tau_1, \tau_2, \tau_3\rangle)$ has six nontrivial linear characters which is impossible in this case since $X \mid \langle\tau_1, \tau_2, \tau_3\rangle = R_1 \oplus S \oplus (n-5)\, 1_{\langle\tau_1,\tau_2,\tau_3\rangle}$.

Case E. $X \mid K = T \oplus (n-7)\, 1_K$ where T is irreducible of degree 7.

In this case Lemma 3.3 applies and can be handled as in Case D. This final contradiction shows that R_2 must be irreducible of degree 4. As $R_2(\tau_3)$ extends U_2 to an irreducible 4-dimensional subspace, say V_2, $R_2(\tau_3) \mid V_2$ cannot be a reflection. This means that $R_1(\tau_3)$ is trivial and $R_1(H) = R_1(\langle\tau_1, \tau_2\rangle)$ which is dihedral. If R_2 is primitive the groups are listed by [14] or [5, Sect. 8.5]. Note that $R_2(\tau_1)$ is a reflection. All of these groups in our situation contain either special 3-elements or elements contradicting Blichfeldt. This checking is facilitated by noting that if $R_2(H)$ is nonsolvable, $R_1(H^\infty)$ is trivial; there is a special 3-element in $R_2(H^\infty)$, and so H contains a special 3-element. The remaining possibilities for $R_2(H)$ all have Sylow 3-groups S of order 9 and centers of order 2. These groups are generated by $(\omega, \bar{\omega}, 1, 1)$ and $\mathrm{diag}(1, 1, \omega, \bar{\omega})$. Now S contains s for which $R_2(s) = (\omega, \bar{\omega}, \omega, \bar{\omega})$ and H contains z for which $R_2(z) = -I$, $R_1(z) = I$. Also $R_1(s) = \mathrm{diag}(\omega, \bar{\omega})$ or I_2, $R_1(\tau_1\tau_2)^3 = -I_2$, and $R_2((\tau_1\tau_2)^3) = I_4$. Now $(\tau_1\tau_2)^3 sz$ or sz contradicts Blichfeldt.

This means R_2 is imprimitive. Suppose R_2 permutes 2-dimensional subspaces. Then

$$R_1(\tau_1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad R_1(\tau_2) = \begin{bmatrix} 0 & \alpha_1 \\ \bar{\alpha}_1 & 0 \end{bmatrix}, \qquad R_1(\tau_3) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$R_2(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R_2(\tau_2) = \begin{bmatrix} 0 & \alpha_2 & 0 & 0 \\ \bar{\alpha}_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_2(\tau_3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Then $R_1 \oplus R_2([[\tau_1\tau_2, \tau_3], \tau_2]) = \mathrm{diag}(1, 1, \bar{\alpha}_2^2, \alpha_2^2, 1, 1)$ which is a special k element unless $k = 4$. In this case $R_1 \oplus R_2((\tau_1\tau_2)^2 [[\tau_1\tau_2, \tau_3], \tau_2]) = \mathrm{diag}((\bar{\alpha}_1)^2, (\alpha_1)^2, 1, 1, 1, 1)$ a special 4-element. This means that R_2 is monomial and the representations for $R_2(\tau_1)$, $R_2(\tau_2)$, $R_2(\tau_3)$ do not have the form above. The only possibilities are that R_1 is as above and

$$R_2(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R_2(\tau_2) = \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \mp 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R_2(\tau_3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

or

$$R_2(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad R_2(\tau_2) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In the first case, $k = 4$ and $(\tau_1(\tau_2)^{\tau_3})^2$ is a special 4-element. In the final case we have $k = 3$. If $R_2(\tau_3)$ is a 2-cycle it can by conjugation be assumed to be $(3, 4)$

and now $(\tau_2\tau_3)^2$ is a special 3-element. This means that $R_2(\tau_3)$ is a product of disjoint 2-cycles. After conjugating if necessary by $\tau_1\tau_2$ and rescaling we can assume

$$R_2(\tau_3) = \begin{bmatrix} 0 & \beta & 0 & 0 \\ \bar{\beta} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

As R_2 is irreducible, $\beta \neq 1$. Now $(\tau_1\tau_3)^2$ is a special m-element $m \geqslant 3$ unless $\beta = -1$ or $\pm i$. If $\beta = \pm i$, $\tau_1, \tau_2, (\tau_1\tau_3)^2$ contradict Lemma 3.5. This means $\beta = -1$. However, if v_i is the ith coordinate vector, $\langle v_1 + v_2 + v_3 - v_4 \rangle$ is invariant and R_2 is reducible. This case is therefore impossible and Lemma 3.7 is proven.

Proof of Theorem 3.1. We now proceed directly to the proof. Suppose first that $k \geqslant 7$. We have chosen τ_1 and τ_2 to be bad with $| \tau_1\tau_2 | = 2k$. Suppose τ_3 is any other special involution for which $| \tau_1\tau_3 | = 2k$. Further assume that under the isomorphism sending $\tau_1 \to \tau_1$ and $\tau_2 \to \tau_3$, $X | \langle \tau_1, \tau_2 \rangle$ is similar to $X | \langle \tau_1, \tau_3 \rangle$. By Lemmas 3.6 and 3.7 as $| \tau_1\tau_3 | \neq 3$ or 4, $X(\tau_3)$ must fix $U_1 \oplus U_2$. Since $X | \langle \tau_1, \tau_i \rangle$ for $i = 2, 3$ has only trivial linear constituents $X | \langle \tau_1, \tau_2, \tau_3 \rangle = Z \oplus (n-4)\, 1_{\langle \tau_1, \tau_2, \tau_3 \rangle}$. Suppose Z is irreducible. Let τ_4 be a special involution for which $X(\tau_4)$ moves $U_1 \oplus U_2$. Then by Lemmas 3.2 and 3.4, and as $| \tau_1\tau_2 | \neq 8$, $X | \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle = R \oplus (n-6)\, 1_{\langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle}$. Here R is irreducible and monomial and $R(\tau_1)$, $R(\tau_2)$ have one of the forms described in Lemma 3.2. As $k \geqslant 7$, $R(\tau_2)$ represents the permutation (1, 2) (3, 4). As $X(\tau_1\tau_3)$ has order $2k$ with eigenvalues the same as $X(\tau_1\tau_2)$ as a permutation it must also act as (1, 2) (3, 4) or a contradiction arises. We see that $X(\tau_3)$ acts on U_1 and U_2, and in particular Z is reducible.

Let $Z = Z_1 \oplus Z_2$ with Z_i acting on U_i. As the eigenvalues of $Z_i(\tau_1\tau_2)$ and $Z_i(\tau_1\tau_3)$ are primitive kth or $2k$th roots of 1, each Z_i is monomial. Now $Z_i((\tau_1\tau_j)^2)$ for $i = 1, 2; j = 1, 2$ are diagonal and unimodular. This means $\langle (\tau_1\tau_2)^2, (\tau_1\tau_3)^2 \rangle$ is abelian of order k or k^2. In the latter case one obtains a special k-element a contradiction. This means $\langle (\tau_1\tau_2)^2 \rangle = \langle (\tau_1\tau_3)^2 \rangle$.

We now define $\gamma(\tau_1) = \langle (\tau_1\tau_2)^2 \rangle$. This is an important definition for our subsequent work. The lemmas we have proved so far have been designed to determine properties of $\gamma(\tau_1)$. Note that the argument of the above paragraph shows that if one replaces τ_2 in the definition by any τ_3 for which $X | \langle \tau_1, \tau_3 \rangle$ is similar to $X | \langle \tau_1, \tau_2 \rangle$ under the isomorphism sending $\tau_1 \to \tau_1$ and $\tau_2 \to \tau_3$, $\gamma(\tau_1)$ is the same group of order k. The definition is thus independent of the choice of τ_2.

Suppose now that σ_1 is any special involution for which there is a special involution σ_2 for which $| \sigma_1\sigma_2 | = 2k$ and under the isomorphism sending $\tau_1 \to \sigma_1$, $\tau_2 \to \sigma_2$, $X | \langle \tau_1, \tau_2 \rangle$ is similar to $X | \langle \sigma_1, \sigma_2 \rangle$. Define $\gamma(\sigma_1) =$

$\langle(\sigma_1\sigma_2)^2\rangle$. Note that τ_2 is such an involution using $\langle\tau_2, \tau_1\rangle$ and $\gamma(\tau_1) = \gamma(\tau_2)$. Note also that any conjugate of τ_1 is such an involution and $(\gamma(\tau_1))^g = \langle(\tau_1\tau_2)^2\rangle^g = \langle(\tau_1{}^g\tau_2{}^g)^2\rangle = \gamma(\tau_1{}^g)$.

Let $\tau_3 = \tau_1{}^g$, $\tau_4 = \tau_2{}^g$ be conjugates of τ_1 and τ_2 by the same group element g. Our goal is to show that $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute. Once this has been done, the group generated by all conjugates of $\gamma(\tau_1)$ forms a noncentral normal abelian subgroup contrary to the supposed quasiprimitivity of X.

To this end suppose $|\tau_1\tau_3| = |\tau_1\tau_4| = 3$. As $|\tau_1\tau_3| = 3$, $(\tau_1)^{\tau_3} = (\tau_3)^{\tau_1}$. Now $(\gamma(\tau_3))^{\tau_1} = \gamma((\tau_3)^{\tau_1}) = \gamma((\tau_1)^{\tau_3}) = (\gamma(\tau_1))^{\tau_3}$. Similarly $(\gamma(\tau_1))^{\tau_4} = (\gamma(\tau_4))^{\tau_1} = (\gamma(\tau_3))^{\tau_1}$ as $\gamma(\tau_4) = \gamma(\tau_3)$. Now $(\gamma(\tau_1))^{\tau_3\tau_4} = \gamma(\tau_1)$ and so $\tau_3\tau_4 \in N(\gamma(\tau_1))$ and $\gamma(\tau_3) \in N(\gamma(\tau_1))$. As $|\gamma(\tau_1)| = k$, a prime, this implies that $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute.

Suppose now that $\gamma(\tau_1)$ and $\gamma(\tau_3)$ do not commute. By the above argument one of $|\tau_1\tau_3|$, $|\tau_1\tau_4|$ is not 3 as is the case with $|\tau_2\tau_3|$, $|\tau_2\tau_4|$; $|\tau_1\tau_3|$, $|\tau_2\tau_3|$; and $|\tau_1\tau_4|$, $|\tau_2\tau_4|$. Now $X \mid \langle\tau_3, \tau_4\rangle$ acts nontrivially on $V_1 = X(g^{-1})\, U_1$ and $V_2 = X(g^{-1})\, U_2$, and $X(\tau_1)$ and $X(\tau_2)$ fix $V_1 \oplus V_2$ by Lemmas 3.6 and 3.7. Also $X(\tau_3)$ and $X(\tau_4)$ fix $U_1 \oplus U_2$. Now $X \mid \langle\tau_1, \tau_2, \tau_3, \tau_4\rangle = Y \oplus Y_1 \oplus (n-8)\, 1_{\langle\tau_1,\tau_2,\tau_3,\tau_4\rangle}$ where Y acts on $U_1 \oplus U_2$. If Y_1 acts on $V_1 \oplus V_2$, then $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute since all the action is on complementary subspaces. If Y_1 has only two linear constituents, all U_1, U_2, V_1, V_2 are fixed. As 2-dimensional primitive groups have no noncentral elements of order k, $\gamma(\tau_1)$ and $\gamma(\tau_3)$ are diagonal and so commute. This means Y_1 is trivial. If Y is reducible again $\gamma(\tau_1)$ and $\gamma(\tau_3)$ are diagonal on U_1 and U_2 and so commute. If Y is irreducible adjoin a special involution τ_5 for which $X(\tau_5)$ moves $U_1 \oplus U_2$. Then by Lemmas 3.2 and 3.4, $X \mid \langle\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\rangle$ has a 6-dimensional irreducible monomial constituent R. Again $R((\tau_1\tau_2)^2)$ and $R((\tau_3\tau_4)^2)$ are diagonal and we have $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commuting.

It now follows that the group generated by all conjugates of $\gamma(\tau_1)$ is an abelian normal noncentral subgroup. This contradicts the quasiprimitivity of X. We have shown $k = 3$, 4, or 5.

Suppose first $k = 3$ or 5. Again let τ_3 be a special 2-element for which $|\tau_1\tau_3| = 2k$ and under the isomorphism $\tau_1 \to \tau_1$, $\tau_2 \to \tau_3$, $X \mid \langle\tau_1, \tau_2\rangle$ is similar to $X \mid \langle\tau_1, \tau_3\rangle$. We want to show $\langle(\tau_1\tau_2)^2\rangle = \langle(\tau_1\tau_3)^2\rangle$. Again as $|\tau_1\tau_3| = 6$ or 10, $X(\tau_3)$ leaves $U_1 \oplus U_2$ invariant and so $X \mid \langle\tau_1, \tau_2, \tau_3\rangle = Z \oplus (n-4)\, 1_{\langle\tau_1,\tau_2,\tau_3\rangle}$. If Z is irreducible, let τ_4 be a special 2-element for which $X(\tau_4)$ moves $U_1 \oplus U_2$. Again by Lemmas 3.2 and 3.4, and as $|\tau_1\tau_2| \neq 8$, $X \mid \langle\tau_1, \tau_2, \tau_3, \tau_4\rangle = R \oplus (n-6)\, 1_{\langle\tau_1,\tau_2,\tau_3,\tau_4\rangle}$ where R is irreducible and monomial. As $X \mid \langle\tau_1, \tau_2, \tau_3\rangle$ has an irreducible 4-dimensional constituent one of $X(\tau_2)$, $X(\tau_3)$ must act as the permutation (1, 3) (2, 4) or (1, 4) (2, 3). Now $Z((\tau_1\tau_2)^2)$ or $Z((\tau_1\tau_3)^2)$ has nontrivial eigenvalues d, $\bar{d}$, $\bar{d}$, d and so both $Z((\tau_1\tau_2)^2)$ and $Z((\tau_1\tau_3)^2)$ have these eigenvalues. Consequently $X \mid \langle(\tau_1\tau_2)^2, (\tau_1\tau_3)^2\rangle$ has at most 2-dimensional constituents. This is of course true also if Z is reducible.

Now let $A = \langle(\tau_1\tau_2)^2, (\tau_1\tau_3)^2\rangle$ and $X \mid A = T_1 \oplus T_2 \oplus (n-4)\, 1_A$ where

T_1 and T_2 have degree 2. If A is abelian and $\langle(\tau_1\tau_2)^2\rangle \neq \langle(\tau_1\tau_3)^2\rangle$, A contains a special k-element. If A is nonabelian assume T_1 is irreducible. As it is generated by elements of order k it is primitive. If $k = 5$, $T_1(A)/Z(T_1(A)) \cong A_5$ and in a high commutator of A are elements contradicting Blichfeldt. This means $k = 3$ and to avoid elements contradicting Blichfeldt, $T_1(A)/Z(T_1(A)) \cong A_4$. If T_2 is reducible there is a special 4-element in A'. If T_2 is irreducible, $T_2(A)/Z(T_2(A)) \cong A_4$ and again there is an element v for which $X(v)$ has eigenvalues $(-1, -1, -1, -1, 1,..., 1)$. Now $X(v(\tau_1\tau_2)^2)$ contradicts Blichfeldt.

We have shown $\langle(\tau_1\tau_2)^2\rangle = \langle(\tau_1\tau_3)^2\rangle$. Again let $\gamma(\tau_1) = \langle(\tau_1\tau_2)^2\rangle$. This definition is independent of the particular choice of τ_2 and we have the properties of γ obtained above. Again extend the definition to all special involutions σ_1 for which there is a σ_2 for which $|\sigma_1\sigma_2| = 2k$ and under the map $\sigma_1 \to \tau_1$, $\sigma_2 \to \tau_2$, $X \mid \langle\sigma_1, \sigma_2\rangle$ is similar to $X \mid \langle\tau_1, \tau_2\rangle$.

As above, we again let $\tau_3 = \tau_1^g$, $\tau_4 = \tau_2^g$ for some $g \in G$. Suppose $k = 5$ and $\gamma(\tau_1)$ does not commute with $\gamma(\tau_3)$. The argument above provides a contradiction unless $X \mid \langle\gamma(\tau_1), \gamma(\tau_3)\rangle$ has some two-dimensional constituents projectively representing A_5. Unless there are four such constituents, a high commutator contains an element contradicting Blichfeldt. If there are four, $U_1 \oplus U_2$ and $V_1 \oplus V_2$ are complementary. Here $\gamma(\tau_1)$ and $\gamma(\tau_3)$ act nontrivially on complementary subspaces and so commute.

Consider now $k = 3$. We assume that $\gamma(\tau_1)$ and $\gamma(\tau_3)$ do not commute. As above, the pair $\{|\tau_1\tau_3|, |\tau_1\tau_4|\}$ cannot both be 3 nor can $\{|\tau_2\tau_3|, |\tau_2\tau_4|\}$, $\{|\tau_1\tau_3|, |\tau_2\tau_3|\}$, or $\{|\tau_1\tau_4|, |\tau_2\tau_4|\}$. It follows from Lemmas 3.6 and 3.7 that $X(\tau_3)$ and $X(\tau_4)$ both fix $U_1 \oplus U_2$ and $X(\tau_1)$ and $X(\tau_2)$ both fix $X(g^{-1})(U_1 \oplus U_2) = V_1 \oplus V_2$. If $U_1 \oplus U_2$ and $V_1 \oplus V_2$ are complementary, $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute. If $U_1 \oplus U_2 = V_1 \oplus V_2$, let τ_5 be a special 2-element for which $X(\tau_5)$ moves $U_1 \oplus U_2$. If $X(\langle\tau_1, \tau_2, \tau_3, \tau_4\rangle) \mid U_1 \oplus U_2$ is irreducible, Lemmas 3.2 and 3.4 show $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute. If it is reducible, $X \mid \langle\gamma(\tau_1), \gamma(\tau_3)\rangle = Y_1 \oplus Y_2 \oplus (n-4)\, 1_{\langle\gamma(\tau_1),\gamma(\tau_3)\rangle}$. As $[\gamma(\tau_1), \gamma(\tau_3)] \neq 1$, we may assume Y_1 is irreducible and hence primitive. If Y_2 is reducible $\langle\gamma(\tau_1), \gamma(\tau_3)\rangle'$ contains a special 4-element. In any other case there is an element z with $Y_1 \oplus Y_2(z) = \mathrm{diag}(-1, -1, -1, -1)$ and $z(\tau_1\tau_2)^2$ contradicts Blichfeldt. In the remaining case $X \mid \langle\tau_1, \tau_2, \tau_3, \tau_4\rangle$ is a sum of three 2-dimensional constituents. Now $X \mid \langle\gamma(\tau_1), \gamma(\tau_3)\rangle$ has one 2-dimensional constituent, the rest being linear. A commutator contains a special 4-element. This shows $k \neq 3$.

The final case remaining is $k = 4$. This time we define $\gamma(\tau_1) = (\tau_1\tau_2)^4$. Again, we wish γ to be independent of our choice of τ_2 and so we let τ_3 be a special 2-element such that $X \mid \langle\tau_1, \tau_2\rangle$ is similar to $X \mid \langle\tau_1, \tau_3\rangle$ under the usual isomorphism. By Lemmas 3.6 and 3.7 $X \mid \langle\tau_1, \tau_2, \tau_3\rangle = Y \oplus (n-4)_{\langle\tau_1,\tau_2,\tau_3\rangle}$. If X_1 and X_2 are both faithful then $X((\tau_1\tau_2)^4) = X((\tau_1\tau_3)^4) = \mathrm{diag}(-1, -1, -1, -1, 1, 1,..., 1)$. Suppose X_2 is not faithful. If Y is irreducible let τ_4 be a special 2-element such that $X(\tau_4)$ moves $U_1 \oplus U_2$. Now by Lemma 3.4 $X \mid \langle\tau_1, \tau_2, \tau_3, \tau_4\rangle$ satisfies the hypothesis of Lemma 3.2 with a monomial irreducible constituent of degree 6 as X_2 is not faithful. However, again as X_2 is unfaithful, each of τ_1, τ_2, τ_3 represent the same permutations and Y is reducible. This means $X \mid \langle \tau_1, \tau_2, \tau_3 \rangle = Y_1 \oplus Y_2$ where Y_i acts on U_i for $i = 1, 2$. If $X((\tau_1\tau_2)^4) \neq X((\tau_1\tau_3)^4)$, $(\tau_1\tau_2)^2 (\tau_1\tau_3)^4 = \text{diag}(i, -i, 1, 1, 1, \ldots, 1)$, a special 4-element. It follows then that $(\tau_1\tau_2)^4 = (\tau_1\tau_3)^4 = \gamma(\tau_1)$ and γ is independent of the choice of τ_2. Again extend γ to other possible special involutions.

Again let $\tau_3 = \tau_1^g$, $\tau_4 = \tau_2^g$ be conjugates of τ_1, τ_2 and assume $\gamma(\tau_1)$ does not commute with $\gamma(\tau_3)$. Suppose first X_2 is not faithful. Again the various pairs of orders cannot both be 3 and we see that $X(\tau_3)$ and $X(\tau_4)$ leave $U_1 \oplus U_2$ invariant and $X(\tau_1)$ and $X(\tau_2)$ leave $X(g^{-1})(U_1 \oplus U_2) = V_1 \oplus V_2$ invariant. Again if $V_1 \oplus V_2$ and $U_1 \oplus U_2$ are complementary, $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute. If $V_1 \oplus V_2 \neq U_1 \oplus U_2$, $X \mid \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$ acts nontrivially on U_1, U_2, V_1, V_2 and as $\gamma(\tau_1)$ and $\gamma(\tau_3)$ are scalar on each they must commute. This leaves $V_1 \oplus V_2 = U_1 \oplus U_2$. Now $X \mid \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle = Y \oplus (n-4)\, 1_{\langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle}$. If Y is reducible $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute as each is scalar on U_i. If Y is irreducible let τ_5 be a special 2-element for which $X(\tau_5)$ moves $U_1 \oplus U_2$. The usual contradiction follows from Lemma 3.2 as here the irreducible constituent is monomial.

We are left with the case in which X_2 is faithful. If $X(\tau_3)$, $X(\tau_4)$ leave $U_1 \oplus U_2$ invariant, and $X(\tau_1)$, $X(\tau_2)$ leave $V_1 \oplus V_2$ invariant, the argument above applies and provides a contradiction. This follows as the unfaithfulness of X_2 was only used when $U_1 \oplus U_2 = V_1 \oplus V_2$ and if X_2 is faithful, $\gamma(\tau_1)$ and $\gamma(\tau_3)$ are both scalar on $U_1 \oplus U_2$ and trivial elsewhere. We can assume then that $X(\tau_3)$ moves $U_1 \oplus U_2$ and that τ_1, τ_2, τ_3 satisfy the hypothesis of Lemma 3.6 where $|\tau_1\tau_3| = 4$. From Lemma 3.6 we see

$$Y(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Y(\tau_2) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & d & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \bar{d} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Y(\tau_3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\tau = (\tau_1\tau_3)^2$. Then $Y(\tau)$ is the permutation matrix corresponding to (1, 2) (5, 6) and $Y((\tau_1\tau_2)^2) = \text{diag}(d, \bar{d}, \bar{d}, d, 1, 1)$. Now let $\tilde{\tau} = [(\tau_1\tau_2)^2, \tau]$, $\tau^* = (\tilde{\tau})^{\tau_2}$. Computing we see $Y(\tilde{\tau}) = \text{diag}(-1, -1, 1, 1, 1, 1)$, $Y(\tau^*) = \text{diag}(1, 1, -1, -1,$

1, 1) and $\langle \tilde{\tau}\tau^* \rangle = \gamma(\tau_1)$. Note that τ^* and τ_3 commute and $|\tilde{\tau}\tau_3| = 4$ with eigenvalues i, i, $-i$, $-i$, 1, 1. It follows that $X(\tau^*)$ and $X(\tilde{\tau})$ do not move $V_1 \oplus V_2$ and so $\tau^*\tilde{\tau}$ commutes with $\gamma(\tau_3)$; therefore, $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute. This final contradiction shows $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute and the proof of Theorem 3.1 is finished.

4. $|\tau_1\tau_2| = 2, 3, 4$, OR 5

In this section we show that the product of two distinct special involutions must have order 2, 3, 4, or 5. This uses and considerably improves upon Theorem 3.1.

THEOREM 4.1. *If τ_1 and τ_2 are distinct special involutions in G, $|\tau_1\tau_2| = 2, 3,$ 4, or 5.*

The proof of this theorem is the same in spirit as the proof of Theorem 3.1 and in fact could have been included in its proof. However, we felt Theorem 3.1 was complicated enough as it stands and included in this section the extra details needed to prove this stronger version.

Suppose that τ_1 and τ_2 are special involutions not satisfying the hypothesis of Theorem 4.1. This means $|\tau_1\tau_2|$ is odd. Replace τ_1 and τ_2 by special involutions for which $|\tau_1\tau_2| = p$, 9, 15, or 25 where $p \geqslant 7$. This can be done by rechoosing special elements from $\langle \tau_1, \tau_2 \rangle$. If $|\tau_1\tau_2| = 25$, some power contains an element contradicting Blichfeldt or [12, Theorem 2] and so $|\tau_1\tau_2| = p$, 9, or 15. Note that $X \mid \langle \tau_1, \tau_2 \rangle = X_1 \oplus X_2 \oplus (n-4)\, 1_{\langle \tau_1, \tau_2 \rangle}$ where X_i acts on U_i, $i = 1, 2$. To avoid special elements X_1 and X_2 must be faithful and X_1 not similar to X_2. This means U_1 and U_2 are unique. We prove analogs of Lemmas 3.2, 3.3, 3.4, and 3.6.

LEMMA 4.2. *Let H be a subgroup of G containing $\langle \tau_1, \tau_2 \rangle$ and generated by special involutions. Suppose $X \mid H = Y \oplus \xi \oplus (n-7)\, 1_H$ where Y is irreducible of degree 6. Then Y is monomial and in an appropriate basis*

$$Y(\tau_1) = \begin{bmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, \qquad Y(\tau_2) = \begin{bmatrix} 0&\alpha_1&0&0&0&0\\ \bar{\alpha}_1&0&0&0&0&0\\ 0&0&0&\alpha_2&0&0\\ 0&0&\bar{\alpha}_2&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}$$

where $X_i(\tau_1\tau_2) = \operatorname{diag}(\alpha_i, \bar{\alpha}_i)$ and $i = 1, 2$. The permutation group $Y(H)$ contains no 3-cycles.

Proof. The proof is identical to the proof of Lemma 3.2 except in the treatment of the nonmonomial case. Note that the second form of $Y(\tau_2)$ in Lemma 3.2 cannot occur since otherwise $\tau_1\tau_2$ has even order. Assume that Y permutes 2-dimensional subspaces. As in Lemma 3.2, it can be shown

$$Y(\tau_1) = \begin{bmatrix} 0 & I_2 & 0 \\ I_2 & 0 & 0 \\ 0 & 0 & I_2 \end{bmatrix}, \qquad Y(\tau_2) = \begin{bmatrix} 0 & A & 0 \\ A^{-1} & 0 & 0 \\ 0 & 0 & I_2 \end{bmatrix},$$

where $A = \text{diag}(\alpha_1, \alpha_2)$. Again let K be the normal subgroup which fixes each of the three 2-dimensional subspaces V_1, V_2, V_3, and $Y \mid K = R_1 \oplus R_2 \oplus R_3$. Again $R_i(H)/Z(R_i(H))$ is projectively A_4, A_5, or S_4, and R_i is primitive. As $\alpha_2 \neq \alpha_1$ and R_i is primitive, $|\alpha_1| \neq p$ where $p \geqslant 7$. However, $|\alpha_1|$ could be 9 or 15 with $\alpha_1\bar{\alpha}_2$ a cube or fifth root of 1. If $R_i(H)$ represents A_5, a high commutator contains an element contradicting Blichfeldt. There must be an element z in K' for which $R_i(z) = \text{diag}(-1, -1)$. Conjugating and taking products we obtain either diag$(-1, -1, -1, -1, 1, 1)$ or diag$(-1, -1, -1, -1, -1, -1)$. If $h = (\tau_1\tau_2)^5$ or $h = (\tau_1\tau_2)^3$, $Y(h) = \text{diag}(\omega, \bar{\omega}, \bar{\omega}, \omega, 1, 1)$. If we have diag$(-1, -1, -1, -1, 1, 1)$, multiplying by $Y(h)$ contradicts Blichfeldt. Otherwise, if μ_2 is as in Lemma 3.2, $Y([h, \mu_2])$ diag$(-1, -1, -1, -1, -1, -1)$ has all eigenvalues $-\omega$ or $-\bar{\omega}$ contradicting Blichfeldt.

LEMMA 4.3. *Let H be a subgroup of G containing $\langle \tau_1, \tau_2 \rangle$ and generated by special involutions. Suppose $X \mid H = Y \oplus \xi \oplus (n-8)\, 1_H$ where Y is irreducible of degree 7. Then Y is monomial and in an appropriate basis*

$$Y(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

and

$$Y(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 & 0 & 0 \\ 0 & 0 & \bar{\alpha}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Also $Y(H)$ contains no 3-cycles.

Proof. This is the same as Lemma 3.3. The second form for $Y(\tau_2)$ does not occur as $|\tau_1\tau_2|$ is odd.

LEMMA 4.4. *There can be no subgroup H of G such that H contains $\langle\tau_1, \tau_2\rangle$ and $X \mid H = Y \oplus \xi \oplus (n-6)\, 1_H$ where Y is irreducible of degree 5.*

Proof. This is the same as Lemma 3.4.

We now proceed to a lemma analogous to Lemma 3.6.

LEMMA 4.5. *Suppose τ_3 is a special 2-element such that $X(\tau_3)$ moves both $U_1 \oplus U_2$ and U_1. Then $|\tau_1\tau_3| = 3$.*

Proof. We again divide the proof into cases according to how $Y \mid H$ breaks into irreducible constituents where $H = \langle\tau_1, \tau_2, \tau_3\rangle$.

Case A. $X \mid H = Y_1 \oplus Y_2 \oplus (n-6)\, 1_H$ where Y_1 is irreducible of degree 3 and acts on a space containing U_1.

In this case Y_1 is monomial as $Y_1(\tau_1\tau_2)$ is a Blichfeldt element. As $|X(\tau_1\tau_2)| = |\tau_1\tau_2| \neq 3$ we can assume

$$Y_1(\tau_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Y_1(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 \\ \bar{\alpha}_1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Y_1(\tau_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and so $|Y_1(\tau_1\tau_3)| = 3$. To avoid special elements using Theorem 3.1 we see $|Y_2(\tau_1\tau_3)| = 3$ and $|\tau_1\tau_3| = 3$.

Case B. $X \mid H = Y_1 \oplus Y_2 \oplus (n-6)\, 1_H$ where Y_1 is irreducible of degree 4 acting on a space containing U_1.

Again Y_1 is imprimitive and we see

$$Y_1(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Y_1(\tau_2) = \begin{bmatrix} 0 & \alpha_1 & 0 & 0 \\ \bar{\alpha}_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Y_1(\tau_3) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

Now $[\tau_1\tau_2, \tau_3]$ contradicts Blichfeldt.

Case C. $X \mid H = Y \oplus \xi \oplus (n-6)\, 1_H$ where Y is irreducible of degree 5. This case is impossible by Lemma 4.4.

Case D. $X \mid H = Y \oplus (n-6)\, 1_H$ where Y is irreducible of degree 6. By

Lemma 4.2, Y is monomial with $Y(\tau_1)$ and $Y(\tau_2)$ representing the same permutation (1, 2) (3, 4). But now adjoining the permutation $Y(\tau_3)$ cannot make the permutation group transitive on six letters and Y must be reducible. This proves the lemma.

As in the proof of Theorem 3.1 we now wish to define $\gamma(\tau_1)$. We do this differently for the different values of $|\tau_1\tau_2|$. If $|\tau_1\tau_2| = p$, a prime, define $\gamma(\tau_1) = \langle\tau_1\tau_2\rangle$. If $|\tau_1\tau_2| = 15$, define $\gamma(\tau_1) = \langle(\tau_1\tau_2)^5\rangle$, and if $|\tau_1\tau_2| = 9$, define $\gamma(\tau_1) = \langle(\tau_1\tau_2)^3\rangle$. We again wish to show that this definition does not depend on the choice of τ_2. Let τ_3 be any other special involution such that $X \mid \langle\tau_1, \tau_2\rangle$ is similar to $X \mid \langle\tau_1, \tau_3\rangle$ under the isomorphism sending $\tau_1 \to \tau_1$ and $\tau_2 \to \tau_3$. By Lemma 4.5, as $|\tau_1\tau_3| \neq 3$, $X(\tau_3)$ fixes one of $U_1 \oplus U_2$ or U_1. Since $X_i \mid \langle\tau_1, \tau_2\rangle$ is faithful on U_i, Lemma 4.5 holds when U_1 is replaced by U_2, and so $X(\tau_3)$ fixes one of $U_1 \oplus U_2$ or U_2. In any case $X(\tau_3)$ now fixes $U_1 \oplus U_2$. So $X \mid \langle\tau_1, \tau_2, \tau_3\rangle = Y \oplus (n-4)\, 1_{\langle\tau_1,\tau_2,\tau_3\rangle}$ If Y is irreducible let τ_4 be a special involution such that $X(\tau_4)$ moves $U_1 \oplus U_2$. By Lemmas 4.2 and 4.4, $X \mid \langle\tau_1, \tau_2, \tau_3, \tau_4\rangle$ has an irreducible monomial 6-dimensional constituent R. Now using the form for $R(\tau_i)$ for $i = 1, 2, 3$ we see Y is reducible. This means $Y = Y_1 \oplus Y_2$ where Y_i acts on U_i for $i = 1, 2$. As $Y_i(\tau_1\tau_2) = \text{diag}(\alpha_i, \bar{\alpha}_i)$, Y_i is imprimitive and so $\tau_1\tau_2$ and $\tau_1\tau_3$ commute. If $\langle\tau_1\tau_2\rangle \neq \langle\tau_1\tau_3\rangle$ there is a special element. We see $\gamma(\tau_1)$ does not depend on the choice of τ_2 and the properties of γ needed will apply. As in Section 3, extend the definition of γ to all appropriate special involutions.

Now let $\tau_3 = \tau_1^g$ and $\tau_4 = \tau_2^g$. We will obtain a proof of Theorem 4.1 by showing $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute. If $\gamma(\tau_1)$ and $\gamma(\tau_3)$ do not commute we again find that $|\tau_1\tau_3|$ and $|\tau_2\tau_3|$ are not both 3 and so $X(\tau_3)$ fixes $U_1 \oplus U_2$. Here we use the fact that if $|\tau_1\tau_2| = 9$ we have chosen $\gamma(\tau_1) = \langle(\tau_1\tau_2)^3\rangle$ rather than $\langle\tau_1\tau_2\rangle$. Similarly $X(\tau_i)$ for $i = 1, 2, 3, 4$ fix $U_1 \oplus U_2$ and $X(g^{-1})(U_1 \oplus U_2) = V_1 \oplus V_2$. If $U_1 \oplus U_2$ and $V_1 \oplus V_2$ are complementary, $\gamma(\tau_1)$ and $\gamma(\tau_3)$ act nontrivially on complementary spaces and so commute. If $U_1 \oplus U_2 \neq V_1 \oplus V_2$, all $X(\tau_i)$ leave U_1, U_2, V_1, V_2 invariant and on each such subspace the representation is imprimitive. Consequently $\gamma(\tau_1)$ and $\gamma(\tau_3)$ commute. If $U_1 \oplus U_2 = V_1 \oplus V_2$ and U_1, U_2 are left invariant, $\gamma(\tau_1)$ and $\gamma(\tau_3)$ must commute as above. If $U_1 \oplus U_2 = V_1 \oplus V_2$ and U_1 is not fixed by $X(\langle\tau_3, \tau_4\rangle)$ let τ_5 be a special involution for which $X(\tau_5)$ moves $U_1 \oplus U_2$. Applying Lemma 4.2 we see $\gamma(\tau_1)$ and $\gamma(\tau_3)$ are diagonal and so commute. This completes the proof of Theorem 4.1.

5. Bad Involutions Whose Product Has Order 4

In this section we reduce the main theorem to one final case that is completed in Section 6. We call two special involutions bad of order 4 if their product has order 4 and their square is not special. In a counterexample to the main theorem, in view of Theorems 3.1 and 4.1, there must be some bad pair of order 4 whose square is not in $O_2(G)$. We prove some preliminary results and then describe the subgroups generated by three special involutions τ_1, τ_2, τ_3 where τ_1, τ_2 are bad of order 4. The object is to show that if τ_1, τ_2 are bad of order 4 and τ_3, τ_4 are bad of order 4, then $\langle(\tau_1\tau_2)^2, (\tau_3\tau_4)^2\rangle$ is a 2-group, which by Baer's theorem [6, Theorem 3.8.2] will complete the proof of the main theorem. Theorem 5.4 reduces the problem to the final case done in Section 6.

LEMMA 5.1. *Let H be a subgroup generated by special 2-elements such that $X \mid H = X_1 \oplus \xi \oplus (n-8)\, 1_H$ where X_1 is irreducible and monomial acting on a basis $v_1, \ldots, v_7$, and ξ is linear. Then $X_1(H)$ acts as the permutation group $PSL_2(7)$ on $\langle v_1\rangle, \ldots, \langle v_7\rangle$.*

Proof. Special 2-elements acting on $\langle v_1\rangle, \ldots, \langle v_7\rangle$ either fix each subspace, or act as a 2-cycle, or as a product of disjoint 2-cycles. As X_1 is irreducible, $X_1(H)$ is transitive on $\langle v_1\rangle, \ldots, \langle v_7\rangle$.

We first show that a transitive permutation group on seven letters generated by elements of the form (a, b) or $(a, b)\,(c, d)$ is $PSL_2(7)$, A_7, or S_7. Let L be such a group. If $(a, b) \in L$, $L \cong S_7$ as is well known. We may assume the generators of L have the form $(a, b)\,(c, d)$. So $L \subseteq A_7$. Assume L is solvable. Let L_1 be a minimal normal elementary abelian subgroup of L. A 7-cycle does not normalize a subgroup with a fixed point and so L_1 is not a 2-group, 3-group, or 5-group. As $(ab)\,(cd)$ does not normalize a 7-cycle, L_1 could not exist. If $L \neq A_7$, then L contains A_5, $PSL_2(7)$, or $PSL_2(8)$ as a composition factor. The latter is impossible as A_7 has no subgroup of index 5. The first is impossible as $|A_5| \,\big|\, |L|$ and $7 \,\big|\, |L|$ implies $2^2 \cdot 3 \cdot 5 \cdot 7 \,\big|\, |L|$ but A_7 has no subgroup of index 6, 3, or 2. As $PSL_2(7)$ has index 15 in A_7, it is maximal, and so $L \cong PSL_2(7)$.

We now assume $X_1(H)$ acts as the permutation group A_7 or S_7. Let H_1 be a Sylow 3-subgroup of the diagonal group of $(X_1 \oplus \xi)(H)$. Let $K = \Omega_1(H_1)$. Assume K is nontrivial. Then as $\xi \mid K = 1_K$, $X_1(K)$ has no nontrivial scalar matrices. Let $g \in H$ be a 7-element with $X_1(g)$ a 7-cycle. Then $g \in N_H(K)$ and $\langle g, K\rangle$ has order $3^a 7^b$ where $|K| = 3^a$. As $[g, K] \neq 1$ because $X_1(K)$ is not all scalars, $\langle g, K\rangle$ has more than one Sylow 7-group. So for some $c \leqslant a$ and $c \geqslant 1$, $3^c \equiv 1 \bmod 7$. Hence, $a \geqslant 6$ and we have a special 3-element in K, a contradiction. So H_1 is trivial, and in particular the Sylow 3-group of H is $Z_3 \times Z_3$. Such a group is $\langle g, h\rangle$ and must be

$$X_1(g) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad X_1(h) = \begin{bmatrix} \omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where $\omega = e^{2\pi i/3}$.

As H is generated by special 2-elements, there is a special 2-element $\tau \in H$ with

$$X_1(\tau) = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1\\ 0&0&0&0&0&1&0 \end{bmatrix}.$$

So $g^{-1}g^{\tau}$ is a special 3-element, a contradiction.

LEMMA 5.2. *Let τ_1, τ_2, τ_3 be special involutions such that $X((\tau_1\tau_2)^2)$ has eigenvalues $-1, -1, -1, -1, 1, 1, \ldots$. Then $|\tau_1\tau_3| \neq 5$.*

Proof. Assume $|\tau_1\tau_3| = 5$. Then $X \mid \langle\tau_1, \tau_2\rangle = X_1 \oplus X_2 \oplus (n-4)\, 1_{\langle\tau_1,\tau_2\rangle}$ and $X \mid \langle\tau_1, \tau_3\rangle = Y_1 \oplus Y_2 \oplus (n-4)\, 1_{\langle\tau_1,\tau_3\rangle}$ where X_i, Y_i are faithful and irreducible of degree 2. Let X_i act on U_i. Assume first that τ_3 fixes $U_1 \oplus U_2$. Then $X \mid \langle\tau_1, \tau_2, \tau_3\rangle = Y \oplus (n-4)\, 1_{\langle\tau_1,\tau_2,\tau_3\rangle}$ and $Y((\tau_1\tau_2)^2) = \text{diag}(-1, -1, -1, -1)$. So $\tau = \tau_3(\tau_1\tau_2)^2$ is a special involution and $\tau_1\tau$ has order 10, a contradiction to Theorem 3.1. Without loss of generality we may assume $X(\tau_3)\, U_1 \not\subseteq U_1 \oplus U_2$. Let $H = \langle\tau_1, \tau_2, \tau_3\rangle$. We examine four cases.

Case A. $X \mid H = T_1 \oplus T_2 \oplus (n-6)\, 1_H$ where T_1 is irreducible of degree 3 acting on a subspace containing U_1. If T_1 is monomial, $T_1(\langle\tau_1, \tau_2\rangle)$ must fix one of the basis vectors. As T_1 is irreducible, $T_1(\tau_3)$ must move that vector; hence $T_1(\tau_1\tau_3)$ is a 2-cycle or 3-cycle contradicting $|T_1(\tau_1\tau_3)| = 5$. So T_1 is primitive. As $5 \mid |T_1(H)|$, $T_1(H)$ is projectively A_5 or $\tilde{A}_6$. In the first case $T_1(H) \cong Z_2 \times A_5$, which has no elements of order 4, a contradiction. In the second case $T_1(H')$ must be the nonsplitting central extension of Z_3 by A_6. As T_1 is nonunimodular, $Z(T_1(H)) > Z(T_1(H'))$ and so $|Z(T_1(H))| = 6$. By [7, Theorem 5.5.1] as G has no special 3-elements, T_2 is irreducible with $T_1(H) \cong T_2(H)$ with $\ker T_i \subseteq Z(H)$. In any case we get an element with eigenvalues $-\omega, -\omega, -\omega, -\omega, -\omega, -\omega, 1, 1, \ldots$, or $-\omega, -\omega, -\omega, -\bar{\omega}, -\bar{\omega}, -\bar{\omega}, 1, 1, \ldots$, contradicting Blichfeldt.

Case B. $X \mid H = T_1 \oplus T_2 \oplus (n-6)\, 1_H$ where T_1 is irreducible of degree 4 acting on a subspace containing U_1. By assumption $T_2(\langle\tau_1, \tau_2\rangle)$ is irreducible. But then $T_2(\tau_3)$ must be trivial as $T_1(\tau_i)$ cannot all have exactly one eigenvalue -1 in order for T_1 to be irreducible. So $T_2(\langle\tau_1, \tau_3\rangle)$ has a nontrivial linear constituent, a contradiction.

Case C. $X \mid H = T \oplus \xi \oplus (n-6)\, 1_H$ where T is irreducible of degree 5. As $\langle\tau_1, \tau_2\rangle$, $\langle\tau_1, \tau_3\rangle$ have no nontrivial linear constituents, $\xi = 1_H$. Let τ be a special involution moving the subspace V_0 on which T acts. Let $K = \langle H, \tau\rangle$. Then $X \mid K = R \oplus (n-7)\, 1_K$.

Suppose R is irreducible. So R is monomial by Lemma 2.1 in some basis $v_1, \dots, v_7$. As a permutation group of $\langle v_1\rangle, \dots, \langle v_7\rangle$, $R(K)$ is $PSL_2(7)$ by Lemma 5.1. We may assume τ_1 acts as the permutation (1, 2) (3, 4) or is diagonal. In the latter case $|\tau_1\tau_3| \neq 5$. As τ_1, τ_2 are bad of order 4, the only possibility is for $R(\tau_2)$ to fix $\langle v_5\rangle$, $\langle v_6\rangle$, and $\langle v_7\rangle$. As T is irreducible τ_3 must move one of $\langle v_5\rangle$, $\langle v_6\rangle$, and $\langle v_7\rangle$, making $\tau_1\tau_3$ a 5-cycle. As $PSL_2(7)$ has no 5-elements, we have a contradiction.

So $R = R_1 \oplus \xi$ where R_1 is irreducible of degree 6. If H^τ acts invariantly on V_0, $H\tau V_0 = \tau V_0$ and so $\tau V_0 = V_0$, a contradiction. So $R_1 \mid \langle H, H^\tau\rangle$ is irreducible and $\xi \mid \langle H, H^\tau\rangle$ is trivial. Let $\tilde{\tau}$ be a special involution which moves the subspace on which R_1 acts. If $\tilde{\tau}$ fixes V_0, then $\tilde{\tau}^g$ does not for some $g \in \langle H, H^\tau\rangle$. Replace $\tilde{\tau}$ by $\tilde{\tau}^g$. If $X \mid \langle H, \tilde{\tau}\rangle$ has an irreducible constituent of degree 7, we argue as in the preceding paragraph. If not, the $n - 6$ dimensional subspaces on which $(n-6)\,1_{\langle H, H^\tau\rangle}$ and $(n-6)\,1_{\langle H, H^{\tilde{\tau}}\rangle}$ act intersect in a subspace of dimension $n - 7$. Hence $X \mid \langle H, H^\tau, H^{\tilde{\tau}}\rangle = S \oplus (n-7)\,1_{\langle H, H^\tau, H^{\tilde{\tau}}\rangle}$ where S is irreducible. We obtain a contradiction as in the preceding paragraph.

Case D. $X \mid H = T \oplus (n-6)\,1_H$ where T is irreducible of degree 6. By Lemma 2.1 T is imprimitive. Suppose T permutes 2-dimensional spaces. As T is irreducible, at most one $T(\tau_i)$ is block diagonal. As $|T(\tau_1\tau_3)| = 5$, we may assume

$$T(\tau_1) = \begin{bmatrix} 0 & A & 0 \\ B & 0 & 0 \\ 0 & 0 & I_2 \end{bmatrix}, \qquad T(\tau_3) = \begin{bmatrix} 0 & C & 0 \\ D & 0 & 0 \\ 0 & 0 & I_2 \end{bmatrix}.$$

But then T is irreducible implies

$$T(\tau_2) = \begin{bmatrix} I_2 & 0 & 0 \\ 0 & 0 & E \\ 0 & F & 0 \end{bmatrix} \quad \text{or} \quad T(\tau_2) = \begin{bmatrix} 0 & 0 & E \\ 0 & I_2 & 0 \\ F & 0 & 0 \end{bmatrix}.$$

Now $3 \mid |\tau_1\tau_2|$, a contradiction.

This means that T is monomial in some basis $v_1, \dots, v_6$. If both $T(\tau_1)$ and $T(\tau_3)$ are not products of two disjoint 2-cycles, then either T is reducible or $|\tau_1\tau_3| \neq 5$ a contradiction. As τ_1, τ_2 are bad of order 4, $T_{(\tau_1)}$ and $T_{(\tau_2)}$ both fix 5 and 6. Since T is irreducible, $T(\tau_3)$ is $(a, 5)\,(b, 6)$ contradicting $|\tau_1\tau_3| = 5$.

THEOREM 5.3. *Let τ_1, τ_2, τ_3 be special involutions such that $X \mid \langle \tau_1, \tau_2\rangle = X_1 \oplus X_2 \oplus (n-4)\,1_{\langle \tau_1, \tau_2\rangle}$ where the X_i are irreducible of degree 2 with $X((\tau_1\tau_2)^2)$ having eigenvalues $-1, -1, -1, -1, 1, 1, 1, \dots$. Let $H = \langle \tau_1, \tau_2, \tau_3\rangle$. Then, by ordering τ_1, τ_2 correctly, one of the following occurs.*

I. *$X \mid H = Y \oplus \xi \oplus (n-5)\,1_H$ where Y has degree 4 and ξ is linear.*

II. *$X \mid H = Y_1 \oplus Y_2 \oplus \xi_1 \oplus \xi_2 \oplus (n-6)\,1_H$ where Y_1, Y_2 are irreducible of degree 2 and ξ_1, ξ_2 are nontrivial; $H \cong \langle \tau_1, \tau_2\rangle \times \langle \tau_3\rangle \cong D_8 \times Z_2$.*

III. $X \mid H = Y_1 \oplus Y_2 \oplus (n-6)\, 1_H$ *where* Y_1, Y_2 *are irreducible of degree* 3 *with*

$$(Y_1 \oplus Y_2)(\tau_1) = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 1 & 0 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & 1 & 0 \\ & 0 & & 1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{array}\right], \quad (Y_1 \oplus Y_2)(\tau_3) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 0 & 1 & 0 & & & \\ \hline & & & 1 & 0 & 0 \\ & 0 & & 0 & 0 & 1 \\ & & & 0 & 1 & 0 \end{array}\right].$$

The following possibilities hold for τ_2:

$$\text{A.} \quad (Y_1 \oplus Y_2)(\tau_2) = \left[\begin{array}{ccc|ccc} 0 & \mp i & 0 & & & \\ \pm i & 0 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & \mp i & 0 \\ & 0 & & \pm i & 0 & 0 \\ & & & 0 & 0 & 1 \end{array}\right];$$

$$H \cong (Z_4 \times Z_4) * S_3$$

B. $(Y_1 \oplus Y_2)(\tau_2) = \operatorname{diag}(-1, 1, 1, -1, 1, 1)$; $\quad H \cong Z_2 \times S_4$.

C. $(Y_1 \oplus Y_2)(\tau_2) = \operatorname{diag}(-1, 1, 1, 1, -1, 1)$;

$$H \cong (Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2) * S_3.$$

IV. $X \mid H = Y_1 \oplus Y_2 \oplus (n-6)\, 1_H$ *where* Y_1 *is irreducible of degree* 4 *and* Y_2 *is irreducible of degree* 2 *such that*

$$(Y_1 \oplus Y_2)(\tau_1) = \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & & \\ 1 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & \\ \hline & & 0 & & 0 & 1 \\ & & & & 1 & 0 \end{array}\right],$$

$$(Y_1 \oplus Y_2)(\tau_2) = \operatorname{diag}(-1, 1, 1, 1, -1, 1),$$

$$(Y_1 \oplus Y_2)(\tau_3) = \left[\begin{array}{cccc|cc} 0 & 0 & 1 & 0 & & \\ 0 & 0 & 0 & 1 & & 0 \\ 1 & 0 & 0 & 0 & & \\ 0 & 1 & 0 & 0 & & \\ \hline & & 0 & & 1 & 0 \\ & & & & 0 & 1 \end{array}\right], \quad H \cong (Z_2 \times Z_2 \times Z_2 \times Z_2 \times Z_2) * D_8.$$

Here $A * B$ *is a semidirect product of* A *by* B.

Proof. Let $X_1 \oplus X_2$ act on U. If τ_3 fixes U, then clearly I or II are the only possibilities. So we assume τ_3 does not fix U. We obtain the following cases.

Case A. $X \mid H = Y_1 \oplus Y_2 \oplus (n-6)\, 1_H$ where Y_1 is irreducible of degree 3. Then either Y_2 is irreducible or $Y_2 = Y_3 \oplus \xi$ where Y_3 is irreducible of degree 2.

Assume first that Y_1 is primitive. By [1] or [5, Sect. 8.5] we note that $Y_1(H)$ is not a subgroup of an extra special group extended by $SL_2(3)$ as none of these groups are generated by involutions. $Y_1(H)$ is not projectively A_5 since then $Y_1(H) \simeq A_5 \times Z_2$ which has no elements of order 4. If $Y_1(H)$ is projectively $PSL_2(7)$ or $\tilde{A}_6$, in order to avoid special 3-elements, Y_2 is irreducible. If $Y_1(H)$ is projectively $\tilde{A}_6$, $Y_1(H) \simeq \tilde{A}_6 \times Z_2$; avoiding special 3-elements implies $Y_2(H) \simeq \tilde{A}_6 \times Z_2$. Then H contains an element with eigenvalues $-\omega, -\omega, -\omega, -\omega, -\omega, -\omega, 1, 1, \ldots$, or $-\bar{\omega}, -\bar{\omega}, -\bar{\omega}, -\omega, -\omega, -\omega, 1, 1, \ldots$, contradicting Blichfeldt. So $Y_1(H) \simeq PSL_2(7) \times Z_2$. In order to avoid special 3-elements, $Y_2(H) \simeq PSL_2(7) \times Z_2$. Choose a special involution τ_4 which moves the subspace on which $Y_1 \oplus Y_2$ acts. Let $K = \langle H, \tau_4 \rangle$. Then $X \mid K = T \oplus (n-8)\, 1_K$. The following could happen:

(i) $T = T_1 \oplus T_2$ where T_1 is irreducible of degree 4. As $PSL_2(7)$ is simple, T_1 is primitive. By [3, II, p. 426] and [1] or [5, Sect. 8.5], elements centralizing a 7-element are scalars, contradicting the forms of H.

(ii) $T = T_1 \oplus T_2$ where T_1 is irreducible of degree 5 or 6. As $PSL_2(7)$ is simple, T_1 is primitive, contradicting Lemma 2.1.

(iii) $T = T_1 \oplus \xi$ where T_1 is irreducible of degree 7.

If T_1 is monomial, the 7-element is a 7-cycle which could not be centralized by a nonscalar element, a contradiction. So T_1 is primitive, a contradiction to Lemma 2.1.

(iv) T is irreducible. As in (iii) T is not monomial. As $PSL_2(7)$ is simple, T cannot permute 2-dimensional subspaces. So T is primitive and by [13], $7^2 \nmid |K|$. By [2], the centralizer of a 7-element never has an element with eigenvalues $-1, -1, -1, -1, -1, -1, 1, 1$, a contradiction.

So Y_1 is monomial. By ordering τ_1, τ_2 correctly, we may assume

$$Y_1(\tau_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Y_1(\tau_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

and

$$Y_1(\tau_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & \pm i & 0 \\ \mp i & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Suppose $Y_2 = Y_3 \oplus \xi$ where Y_3 is irreducible of degree 2. As $|\, Y_1(\tau_1\tau_3)| = 3$, $|\, \tau_1\tau_3 \,| = 3$. Then if $\tau = (\tau_1\tau_2)^2$ and $z = \tau^{\tau_3}\tau^{\tau_3\tau_1}\tau$, $(Y_1 \oplus Y_3)\,(z) = \operatorname{diag}(1, 1, 1, -1, -1)$. But $\tau_3 z$ is a special involution and $|\, \tau_1(\tau_3 z)| = 6$, a contradition to Theorem 3.1.

So Y_2 is irreducible and hence also monomial. In order to avoid special 3-elements,

$$Y_2(\tau_1) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad Y_2(\tau_3) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

The possibilities for $Y_1 \oplus Y_2(\tau_2)$ are

$$\left[\begin{array}{ccc|ccc} 0 & \pm i & 0 & & & \\ \mp i & 0 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & \pm i & 0 \\ & 0 & & \mp i & 0 & 0 \\ & & & 0 & 0 & 1 \end{array}\right], \quad \left[\begin{array}{ccc|ccc} 0 & \pm i & 0 & & & \\ \mp i & 0 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & \pm 1 & 0 & 0 \\ & 0 & & 0 & \mp 1 & 0 \\ & & & 0 & 0 & 1 \end{array}\right],$$

$$\left[\begin{array}{ccc|ccc} \pm 1 & 0 & 0 & & & \\ 0 & \mp 1 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & \pm i & 0 \\ & 0 & & \mp i & 0 & 0 \\ & & & 0 & 0 & 1 \end{array}\right], \quad \operatorname{diag}(-1, 1, 1, -1, 1, 1),$$

$\operatorname{diag}(1, -1, 1, 1, -1, 1)$, $\operatorname{diag}(-1, 1, 1, 1, -1, 1)$, and $\operatorname{diag}(1, -1, 1, -1, 1, 1)$.

In the second and third cases, $(\tau_2\tau_3)^4$ is a special 3-element. Replacing τ_2 by $\tau_2^{\tau_1}$, cases 4 and 6 are equivalent to 5 and 7, respectively.

Case B. $X \mid H = Y_1 \oplus Y_2 \oplus (n-6)\, 1_H$ where Y_1 is irreducible of degree 4 and Y_2 is irreducible of degree 2. Then $Y_2(\tau_3)$ is trivial and $Y_2(H)$ is dihedral of order 8. Assume Y_1 is primitive. Then in order to avoid special 3-elements there is an element $z \in H''$ with $Y_1(z) = \operatorname{diag}(-1, -1, -1, -1)$. Suppose there are special involutions $\tau, \tilde{\tau} \in H$ such that $|\, \tau\tilde{\tau} \,| = 3$ or 5. Then $\tilde{\tau} z$ is a special involution and $|\, \tau(\tilde{\tau} z)| = 6$ or 10, a contradiction. So by [6, Theorem 3.8.2], H is a 2-group contradicting the primitivity of Y_1. If Y_1 permutes 2 dimensional spaces, $Y_1(\tau_i)$ for $i = 1, 2$ are block diagonal and $Y_1(\tau_3)$ permutes the blocks, implying Y_1 is monomial.

So

$$(Y_1 \oplus Y_2)(\tau_1) = \left[\begin{array}{c|c} \begin{array}{cc} A & 0 \\ 0 & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \end{array} & 0 \\ \hline 0 & A \end{array}\right], \quad (Y_1 \oplus Y_2)(\tau_3) = \left[\begin{array}{c|c} \begin{array}{cccc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} & 0 \\ \hline 0 & \begin{array}{cc} 1 & \\ & 1 \end{array} \end{array}\right],$$

and

$$(Y_1 \oplus Y_2)(\tau_2) = \left[\begin{array}{c|c} \begin{array}{cc} B & 0 \\ 0 & \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \end{array} & 0 \\ \hline 0 & B \end{array}\right].$$

By a change of basis, we get conclusion IV.

Case C. $X \mid H = Y \oplus \xi \oplus (n-6)\, 1_H$ where Y is irreducible of degree 5. By Lemma 2.1, Y is monomial. $Y(H)$ can have no 2-cycles as $Y(H)$ would be an abelian diagonal group acted upon by S_5 and would contain a special 3-element. By ordering τ_1, τ_2 correctly,

$$Y(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Y(\tau_2) = \begin{bmatrix} \pm 1 & 0 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 & 0 \\ 0 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} 0 & \pm i & 0 & 0 & 0 \\ \mp i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \pm i & 0 \\ 0 & 0 & \mp i & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$Y(\tau_3)$ must act like a permutation $(a, b)\,(c, 5)$. The permutation group must contain a dihedral subgroup of order 10 containing τ_1. In this subgroup there is a conjugate τ of τ_1 such that $\mid \tau\tau_1 \mid = 5$, contradicting Lemma 5.2.

Case D. $X \mid H = Y \oplus (n-6)\, 1_H$ where Y is irreducible of degree 6. Y is not primitive by Lemma 2.1. Suppose Y permutes two-dimensional spaces.

As Y is irreducible, by ordering τ_1, τ_2 correctly and in an appropriate basis,

$$Y(\tau_1) = \left[\begin{array}{c|c|c} 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 \\ \hline \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} \end{array}\right] \quad \text{and} \quad Y(\tau_3) = \left[\begin{array}{c|c|c} \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} \\ \hline 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 \end{array}\right].$$

If

$$Y(\tau_2) = \left[\begin{array}{c|c|c} 0 & A & 0 \\ \hline B & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} \end{array}\right], \qquad B = A^{-1}$$

and we can change basis without changing the form of $Y(\tau_1)$, $Y(\tau_2)$, so that A is diagonal. If

$$Y(\tau_2) = \left[\begin{array}{c|c|c} A & 0 & 0 \\ \hline 0 & B & 0 \\ \hline 0 & 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} \end{array}\right],$$

we may assume A is diagonal. If

$$A = \pm \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \qquad B = \mp \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \qquad \text{If } A = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix},$$

then

$$Y(\tau_2 \tau_2^{\tau_3\tau_1\tau_3} \tau_2^{\tau_3}) = \left[\begin{array}{c|c|c} \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 & 0 \\ \hline 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix} & 0 \\ \hline 0 & 0 & AB \end{array}\right]$$

and in order to avoid special elements, $AB = \pm \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$. So $B = \pm A$ and in all cases Y is really monomial.

Assume Y is monomial in the basis $v_1, \ldots, v_6$. By the irreducibility of Y, not both $Y(\tau_1)$ and $Y(\tau_2)$ represent transpositions. By ordering τ_1, τ_2 correctly and by scaling and ordering $v_1, \ldots, v_6$ correctly, we may assume

$$Y(\tau_1) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As Y is irreducible $Y(\tau_2)$ is not diagonal and cannot represent the permutations $(1, 2)$, $(3, 4)$, or $(1, 2)(3, 4)$. If it is a 2-cycle, we may assume by correctly scaling and ordering the basis,

$$Y(\tau_2) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

But then $|\tau_1\tau_2| = 8$, a contradiction. So $Y(\tau_2)$ is a product of disjoint 2-cycles and as $|\tau_1\tau_2| = 4$, we may scale and order the basis correctly so that

$$Y(\tau_2) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$Y(\tau_3)$ must represent the permutation $(a, 5)(b, 6)$ and we may assume $a < b$. By conjugating by τ_1, τ_2 or $\tau_1\tau_2$ we may assume $Y(\tau_3)$ to represent $(1, 5)(2, 6)$, $(1, 5)(3, 6)$, or $(1, 5)(4, 6)$. Interchanging τ_1, τ_2 and v_2, v_3 and rescaling, we may assume it is $(1, 5)(2, 6)$ or $(1, 5)(4, 6)$. So by scaling v_5, v_6 correctly,

$$Y(\tau_3) = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

In the first case $\langle v_1 + v_2, v_3 - v_4, v_5 + v_6\rangle$ is invariant, and in the second case $\langle v_1 + iv_4, v_2 + iv_3, v_5 + iv_6\rangle$ is invariant. This proves the theorem.

THEOREM 5.4. *Let τ_1, τ_2 be bad of order 4, and let τ_3, τ_4 also be bad of order 4. Let $H = \langle\tau_1, \tau_2, \tau_3, \tau_4\rangle$ and $X \mid H = Y \oplus (n-7) 1_H$. Then $\langle(\tau_1\tau_2)^2, (\tau_3\tau_4)^2\rangle$ is a 2-group.*

Proof. First assume that Y is monomial in some basis $v_1, ..., v_7$. As τ_1, τ_2 are bad of order 4, by ordering $v_1, ..., v_7$ correctly, $Y(\tau_1)$ and $Y(\tau_2)$ are trivial on v_5, v_6, and v_7. As $Y((\tau_1\tau_2)^2)$ has trace -1, then $Y((\tau_1\tau_2)^2)$ is diagonal. Similarly, $Y((\tau_3\tau_4)^2)$ is diagonal and the result holds.

Now assume Y permutes three two-dimensional spaces and acts linearly on a one-dimensional space. Then as τ_1, τ_2 are bad of order 4, $Y(\langle\tau_1, \tau_2\rangle)$ acts trivially on one of the two-dimensional spaces and the one-dimensional space. So $Y((\tau_1\tau_2)^2)$ acts as a scalar on each of the spaces. A similar result holds for $T((\tau_3\tau_4)^2)$ and so the theorem holds.

We now examine several cases:

Case A. Y is irreducible or $Y = Y_1 \oplus \xi$ where Y_1 is irreducible of degree 6. By the preceding arguments Y is primitive in the first case and Y_1 is primitive in the second, contradicting Lemma 2.1.

Case B. $Y = Y_1 \oplus Y_2$ where Y_1 is irreducible of degree 5. If Y_1 is primitive it contains no special 4-elements by [8], and so Y_2 is trivial, which contradicts Lemma 2.1. This means Y_1 is monomial. As Y_1 is irreducible, one of $Y_2(\langle\tau_1, \tau_2\rangle)$ or $Y_2(\langle\tau_3, \tau_4\rangle)$ is trivial. This means Y_2 is monomial and so Y is monomial, and the result holds.

Case C. $Y = Y_1 \oplus Y_2$ where Y_1 is irreducible of degree 4.

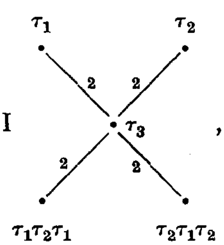
If $Y_2(\langle\tau_1, \tau_2\rangle)$ is trivial, then $Y_1((\tau_1\tau_2)^2)$ is scalar and $[(\tau_1\tau_2)^2, (\tau_3\tau_4)^2] = 1$. Using the same argument with $Y_2(\langle\tau_3, \tau_4\rangle)$ we may assume $Y_1(\tau_j)$ has eigenvalues 1, 1, 1, -1 for each j. If $Y_i((\tau_1\tau_2)^2) \in O_2(Y_i(H))$ for both $i = 1$ and 2, the result holds as $O_2(H) = O_2(Y_1(H)) \cap O_2(Y_2(H))$. If $Y_1((\tau_1\tau_2)^2) \in O_2(Y_1(H))$ but $Y_2((\tau_1\tau_2)^2) \notin O_2(Y_2(H))$, there must be an element $k \in \langle\{(\tau_1\tau_2)^2\}^H\rangle$ of order 3 such that $Y_2(k)$ is not scalar. As $Y_1(k)$ is trivial, k is a special 3-element.

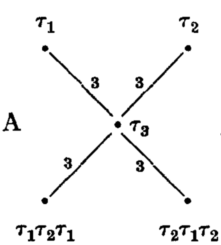
So we may assume $Y_1((\tau_1\tau_2)^2) \notin O_2(Y_1(H))$. By the irreducibility of Y_1, Y_1 is primitive. By examining [1, 14], the group $Y_1(H)/Z(Y_1(H))$ has the following orders and is one of the groups listed in parentheses from Blichfeldt's list [1, pp. 139–173]: $2^3 \cdot 3 \cdot 5(H, G)$; $2^5 \cdot 3^2(2°, 3°, 10°)$; $2^6 \cdot 3^2(5°, 8°, 9°)$; $2^7 \cdot 3^2(12°)$; $2^7 \cdot 3 \cdot 5(18°, 19°)$; $2^8 \cdot 3^2 \cdot 5(21°)$; $2^5 \cdot 3^2 \cdot 5^2(11°)$. As an element with eigenvalues 1, 1, 1, -1 is not the tensor product of two 2×2 matrices, 2°, 3°, 5° are impossible as they are subgroups of tensor products of two dimensional groups; as 8°–12° are extensions of index 2 of groups which are subgroups of tensor products, $Y_1(\tau_1)$, $Y_1(\tau_2)$ are not in the tensor product and hence $Y_1(\tau_1\tau_2)$ is. If we have cases 8°, 9°, 10°, or 12°, the tensor product involved is projectively a subgroup of $S_4 \times S_4$. In $S_4 \times S_4$ squares of 2-elements lie in $O_2(S_4 \times S_4)$ and so $Y_1((\tau_1\tau_2)^2) \in O_2(Y_1(H))$, a contradiction. In case 11°, the tensor product involved is projectively $A_5 \times A_5$, and so $Y_1((\tau_1\tau_2)^2) \in Z(Y_1(H))$, a contradiction. The remaining groups of order $2^3 \cdot 3 \cdot 5$, $2^7 \cdot 3 \cdot 5$, and $2^8 \cdot 3^2 \cdot 5$ are projectively S_5, an extension of an extra special group by S_5, and an extension of an extra special group by S_6, respectively. None of the 2- or 3-dimensional groups have S_5 or S_6 as a section. Thus, by the subdirect product theorem [7, Theorem 5.5.1], the kernel of $Y_2(H)$ contains at least either A_5 or $SL_2(5)$ in the first case or an extra special group of order 32 extended by A_5 in the latter two cases. The first case gives a special 3-element, and the latter cases give either a special 3-element or an element with eigenvalues $-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1, 1, \ldots$, a contradiction.

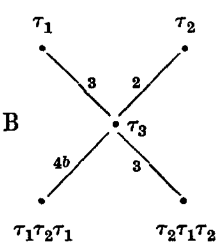
Case D. $Y = Y_1 \oplus Y_2$ where Y_1 is irreducible of degree 3 and Y_2 has a constituent of degree at most 3. If Y_2 has an irreducible constituent of degree 3, $Y_2 = Y_3 \oplus 1_H$ and as in the previous theorem, Y_1 and Y_3 are monomial (i.e., Y is monomial). So $Y_2 = Y_3 \oplus Y_4$ where Y_3, Y_4 are of degree 2. If Y_1 is primitive, $Y_1(H)$ is $PSL_2(7) \times Z_2$ or $\tilde{A}_6 \times Z_2$, and we obtain a special 3-element in kernel Y_2 as $Y_2(H)$ and $Y_1(H)$ cannot have common nontrivial homomorphic images. So Y_1 is monomial and $Y_1((\tau_1\tau_2)^2)$, $Y_1((\tau_3\tau_4)^2)$ are diagonal. As $Y_j((\tau_1\tau_2)^2)$, $Y_j((\tau_3\tau_4)^2)$ are scalar for $j = 3, 4$, $[Y((\tau_1\tau_2)^2), Y((\tau_3\tau_4)^2)] = 1$.

6. Final Case

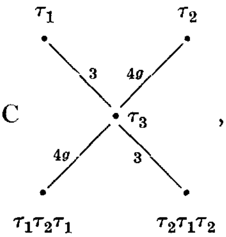
We now introduce notation describing the three generator groups. Suppose τ_1, τ_2, τ_3 are special involutions such that τ_1, τ_2 are bad of order 4. The special involutions in $\langle \tau_1, \tau_2 \rangle$ are τ_1, τ_2, $\tau_1\tau_2\tau_1$, and $\tau_2\tau_1\tau_2$. The notation $\tau \overset{r}{\bullet\!\!-\!\!\bullet} \tilde{\tau}$ will mean τ, $\tilde{\tau}$ are special involutions whose product has order r. If r is $4g$ or $4b$, τ and $\tilde{\tau}$ have product of order 4 and in the first case $(\tau\tilde{\tau})^2$ is special and in the second $X((\tau\tilde{\tau})^2)$ has eigenvalues $-1, -1, -1, -1, 1, 1, \ldots$. We now examine cases II–IV of Theorem 5.3 and describe certain of the 3 generator groups by the orders between some of the special 2-elements. We obtain

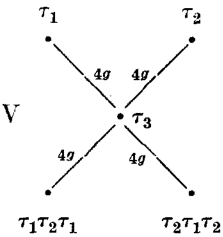
II 

, III A 

, III B 

,

III C 

, and IV 

Suppose that H is a group generated by special involutions such that $X \mid H = Y \oplus (n - 8)\, 1_H$ where Y contains no trivial linear characters. Then if τ_1, τ_2, τ_3 are special involutions such that τ_1, τ_2 are bad of order 4, and if $H = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$ for some special involution τ_4, then $\langle \tau_1, \tau_2, \tau_3 \rangle$ is one of the groups in cases II–IV of Theorem 5.3, because case I could not occur as Y has no trivial linear characters. We notice that if we have the order of τ_3 with any two generators of $\langle \tau_1, \tau_2 \rangle$ we have the case determined. We now extend the previous theorem using a computer program for coset enumeration.[1]

THEOREM 6.1. *Let τ_1, τ_2, τ_3, τ_4 be special involutions such that τ_1, τ_2 and τ_3, τ_4 are bad of order 4. Let $H = \langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle$. If $X \mid H = Y \oplus (n - 8)\, 1_H$ where Y contains no trivial linear constituents, then $\langle (\tau_1\tau_2)^2, (\tau_3\tau_4)^2 \rangle$ is a 2-group.*

Proof. We work with several cases and rename τ_1, τ_2, τ_3, τ_4 by A, B, C, D respectively to simplify notation.

Case A. Suppose all subgroups of the form $\langle A, B, F \rangle$ where $F \in \{C, D\}$ are case II or III B of Theorem 5.3. If we have

$$A \bullet\overset{2}{\text{——}}\underset{C}{\bullet}\overset{2}{\text{——}}\bullet B \qquad \text{and} \qquad A \bullet\overset{2}{\text{——}}\underset{D}{\bullet}\overset{2}{\text{——}}\bullet B,$$

then $H \cong D_8 \times D_8$, and the result holds. By ordering C, D correctly and replacing A, B by other generators of $\langle A, B \rangle$, we may assume

$$A \bullet\overset{3}{\text{——}}\underset{C}{\bullet}\overset{2}{\text{——}}\bullet B.$$

We now have 5 possibilities.

(i) $A \bullet\overset{2}{\text{——}}\underset{D}{\bullet}\overset{2}{\text{——}}\bullet B \Rightarrow C \bullet\overset{4b}{\text{——}}\underset{ABA}{\bullet}\overset{2}{\text{——}}\bullet D$, a contradiction.

(ii) $A \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{2}{\text{——}}\bullet B \Rightarrow C \bullet\overset{4b}{\text{——}}\underset{ABA}{\bullet}\overset{4b}{\text{——}}\bullet D$, a contradiction.

[1] The program was written for us by Chris Landauer.

(iii) $A\ \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{4b}{\text{——}}\bullet\ B \Rightarrow C\ \bullet\overset{4b}{\text{——}}\underset{ABA}{\bullet}\overset{2}{\text{——}}\bullet\ D,$ a contradiction.

(iv) $A\ \bullet\overset{2}{\text{——}}\underset{D}{\bullet}\overset{3}{\text{——}}\bullet\ B \Rightarrow C\ \bullet\overset{3}{\text{——}}\underset{A}{\bullet}\overset{2}{\text{——}}\bullet\ D,\ C\ \bullet\overset{2}{\text{——}}\underset{B}{\bullet}\overset{3}{\text{——}}\bullet\ D,$

$C\ \bullet\overset{4b}{\text{——}}\underset{ABA}{\bullet}\overset{3}{\text{——}}\bullet\ D.$

Using the program for coset enumeration with generators and relations derived from the above diagrams, we obtain a faithful permutation group on 24 letters in which $\langle (AB)^2, (CD)^2 \rangle$ is a 2-group.

(v) $A\ \bullet\overset{4b}{\text{——}}\underset{D}{\bullet}\overset{3}{\text{——}}\bullet\ B.$

Let $C_1 = C^B$, $D_1 = D^B$. Then $\langle A, B, C, D \rangle = \langle A, B, C_1, D_1 \rangle$ and $\{(AB)^2, (C_1D_1)^2\} = \{(AB)^2, (CD)^2\}^B$ implying $\langle (AB)^2, (CD)^2 \rangle \cong \langle (AB)^2, (C_1D_1)^2 \rangle$ and $(AB)^2 (CD)^2$ is a 2-element if and only if $(AB)^2 (C_1D_1)^2$ is a 2-element. We have

$$A\ \bullet\overset{3}{\text{——}}\underset{C_1}{\bullet}\overset{2}{\text{——}}\bullet\ B,\ A\ \bullet\overset{2}{\text{——}}\underset{D_1}{\bullet}\overset{3}{\text{——}}\bullet\ B,$$

which gives (iv).

Case B. Suppose all subgroups of the form $\langle A, B, F \rangle$ where $F \in \{C, D\}$ satisfy case II, III A, or III B of Theorem 5.3. By case A, we may assume

$$A\ \bullet\overset{3}{\text{——}}\underset{C}{\bullet}\overset{3}{\text{——}}\bullet\ B.$$

We now have six possibilities:

(i) $A\ \bullet\overset{2}{\text{——}}\underset{D}{\bullet}\overset{2}{\text{——}}\bullet\ B \Rightarrow C\ \bullet\overset{3}{\text{——}}\underset{A}{\bullet}\overset{2}{\text{——}}\bullet\ D,$

$C\ \bullet\overset{3}{\text{——}}\underset{B}{\bullet}\overset{2}{\text{——}}\bullet\ D \Rightarrow A\ \bullet\overset{4b}{\text{——}}\underset{CDC}{\bullet}\overset{4b}{\text{——}}\bullet\ B,$ a contradiction.

(ii) $A\ \bullet\overset{2}{\text{——}}\underset{D}{\bullet}\overset{3}{\text{——}}\bullet\ B \Rightarrow C\ \bullet\overset{3}{\text{——}}\underset{A}{\bullet}\overset{2}{\text{——}}\bullet\ D,\ C\ \bullet\overset{3}{\text{——}}\underset{B}{\bullet}\overset{3}{\text{——}}\bullet\ D,$

$C\ \bullet\overset{3}{\text{——}}\underset{BAB}{\bullet}\overset{4b}{\text{——}}\bullet\ D \Rightarrow A\ \bullet\overset{4b}{\text{——}}\underset{CDC}{\bullet}\overset{3}{\text{——}}\bullet\ B.$

Using the program for coset enumeration on the subgroup $\langle A, B, C\rangle$ and generators A, B, C, D with relations from the above diagrams, we obtain a faithful permutation group on 80 letters in which the result holds.

(iii) $A \bullet\overset{4b}{\text{——}}\underset{D}{\bullet}\overset{3}{\text{——}}\bullet B.$ Let $C_1 = C^B$, $D_1 = D^B$. As in case A(v).

$\langle (AB)^2, (CD)^2\rangle \cong \langle (AB)^2, (C_1D_1)^2\rangle$ and we have

$$A \bullet\overset{3}{\text{——}}\underset{C_1}{\bullet}\overset{3}{\text{——}}\bullet B \quad \text{and} \quad A \bullet\overset{2}{\text{——}}\underset{D_1}{\bullet}\overset{3}{\text{——}}\bullet B,$$

which is the group in (ii).

(iv) $A \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{2}{\text{——}}\bullet B$. Interchanging A and B gives ii.

As $(AB)^2 = (BA)^2$ and $(BA)^2 (CD)^2$ is a 2-element, so is $(AB)^2 (CD)^2$.

(v) $A \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{4b}{\text{——}}\bullet B$. Interchanging A and B gives iii and the arguement is as in iv.

(vi) $A \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{3}{\text{——}}\bullet B \Rightarrow C \bullet\overset{3}{\text{——}}\underset{F}{\bullet}\overset{3}{\text{——}}\bullet D$

where $F \in \{A, B, ABA, BAB\}$.

Also using the relations given we get $(B^{CA}C)^2 = (BABA)^{CA} \Rightarrow B^{CA}$ and C are bad of order 4. As $\langle B^{CA}, C, D, A\rangle = \langle A, B, C, D\rangle$, and as

$$D \bullet\overset{4b}{\text{——}}\bullet C \text{ we get } B^{CA} \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{4b}{\text{——}}\bullet C.$$

Using the program for coset enumeration on the subgroup $\langle A, B, C\rangle$ where relations between generators A, B, C, and D come from the above diagrams, we obtain a faithful representation on 864 letters in which the result holds.

Case C. Suppose all subgroups of the form $\langle A, B, F\rangle$ where $F \in \{C, D\}$ satisfy cases II or III of Theorem 5.3. By the preceding cases, we may assume

$$A \bullet\overset{3}{\text{——}}\underset{C}{\bullet}\overset{4a}{\text{——}}\bullet B.$$

We have $X \mid \langle A, B, C\rangle = T \oplus (n-6)\, 1_{\langle A,B,C\rangle}$ where

$$T(A) = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & & & \\ 1 & 0 & 0 & & 0 & \\ 0 & 0 & 1 & & & \\ \hline & & & 0 & 1 & 0 \\ & 0 & & 1 & 0 & 0 \\ & & & 0 & 0 & 1 \end{array}\right], \qquad T(B) = \mathrm{diag}(-1, 1, 1, 1, -1, 1),$$

$$T(C) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & & & \\ 0 & 0 & 1 & & 0 & \\ 0 & 1 & 0 & & & \\ \hline & & & 1 & 0 & 0 \\ & 0 & & 0 & 0 & 1 \\ & & & 0 & 1 & 0 \end{array}\right] \Rightarrow T(B^C B^{CA} B^A) = \mathrm{diag}(-1, 1, 1, -1, 1, 1).$$

Now $(AB)^2 = (A(B^C B^{CA} B^A))^2$ and if $B^* = B^C B^{CA} B^A$, $A \bullet\overset{3}{—}\underset{C}{\bullet}\overset{2}{—}\bullet B^*$.

If $Y(\langle A, B^*, C, D\rangle)$ has a trivial linear character, Theorem 5.4 gives the result. So assume our hypothesis holds if B^* replaces B. By case A, we may assume $\langle A, B^*, D\rangle$ has form III A, III C, or IV of Theorem 5.3. If it has form III A, we may interchange C and D to get case B iv. We now have three possibilities:

(i) $A \bullet\overset{3}{—}\underset{D}{\bullet}\overset{4g}{—}\bullet B^* \Rightarrow C \bullet\overset{2}{—}\underset{B^*}{\bullet}\overset{4g}{—}\bullet D$, a contradiction.

(ii) $A \bullet\overset{4g}{—}\underset{D}{\bullet}\overset{3}{—}\bullet B^* \Rightarrow C \bullet\overset{3}{—}\underset{A}{\bullet}\overset{4g}{—}\bullet D$.

Replace D by D^* as earlier so that $C \bullet\overset{3}{—}\underset{A}{\bullet}\overset{2}{—}\bullet D^*$. We may assume $Y(\langle A, B^*, C, D^*\rangle)$ has no nontrivial linear constituents. We have $A \bullet\overset{2}{—}\underset{D^*}{\bullet}\overset{?}{—}\bullet B^*$ and so $D^* \bullet\overset{2}{—}\bullet B^*$ or $D^* \bullet\overset{3}{—}\bullet B^*$ which is covered in case A.

(iii) $A \bullet\overset{4g}{—}\underset{D}{\bullet}\overset{4g}{—}\bullet B^* \Rightarrow C \bullet\overset{2}{—}\underset{B^*}{\bullet}\overset{4g}{—}\bullet D$, a contradiction.

Case D. We may now assume by interchanging C, D if necessary that $A \bullet\overset{4g}{—}\underset{C}{\bullet}\overset{4g}{—}\bullet B$. We have eight possibilities:

(i) $A \bullet\overset{2}{\text{——}}\underset{D}{\bullet}\overset{?}{\text{——}}\bullet B \Rightarrow C \bullet\overset{4g}{\text{——}}\underset{A}{\bullet}\overset{2}{\text{——}}\bullet D,$ a contradiction.

(ii) $A \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{2}{\text{——}}\bullet B \Rightarrow C \bullet\overset{4g}{\text{——}}\underset{B}{\bullet}\overset{2}{\text{——}}\bullet D,$ a contradiction.

(iii) $A \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{4b}{\text{——}}\bullet B \Rightarrow C \bullet\overset{4g}{\text{——}}\underset{ABA}{\bullet}\overset{2}{\text{——}}\bullet D,$ a contradiction.

(iv) $A \bullet\overset{4b}{\text{——}}\underset{D}{\bullet}\overset{3}{\text{——}}\bullet B \Rightarrow C \bullet\overset{4g}{\text{——}}\underset{BAB}{\bullet}\overset{2}{\text{——}}\bullet D,$ a contradiction.

(v) $A \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{3}{\text{——}}\bullet B \Rightarrow C \bullet\overset{4g}{\text{——}}\underset{A}{\bullet}\overset{3}{\text{——}}\bullet D, C \bullet\overset{4g}{\text{——}}\underset{B}{\bullet}\overset{3}{\text{——}}\bullet D.$ If we interchange A with D and B with C, we get case C.

(vi) $A \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{4g}{\text{——}}\bullet B.$ As in case C, choose B^* such that $(AB)^2 = (AB^*)^2$ where

$$A \bullet\overset{3}{\text{——}}\underset{D}{\bullet}\overset{2}{\text{——}}\bullet B^*.$$

We may assume $Y(\langle A, B^*, C, D\rangle)$ has no trivial linear constituents. So examining all possibilities for $A \bullet\text{——}\underset{C}{\bullet}\text{——}\bullet B^*$, we get cases A, B, C, or D ii with B^* replacing B.

(vii) $A \bullet\overset{4g}{\text{——}}\underset{D}{\bullet}\overset{3}{\text{——}}\bullet B.$ Interchange A and B to get vi.

(viii) $A \bullet\overset{4g}{\text{——}}\underset{D}{\bullet}\overset{4g}{\text{——}}\bullet B \Rightarrow C \bullet\overset{4g}{\text{——}}\underset{F}{\bullet}\overset{4g}{\text{——}}\bullet D$ where $F \in \{A, B, ABA, BAB\}$ and $A \bullet\overset{4g}{\text{——}}\underset{F}{\bullet}\overset{4g}{\text{——}}\bullet B$ where $F \in \{CDC, DCD\}$.

This is the only remaining case. We need the following lemma to complete the proof of the theorem.

LEMMA 6.2. *Let A, B, C, D be special involutions satisfying the diagram*

$$A \bullet\overset{4g}{\text{——}}\underset{C}{\bullet}\overset{4g}{\text{——}}\bullet B, \quad A \bullet\overset{4g}{\text{——}}\underset{D}{\bullet}\overset{4g}{\text{——}}\bullet B, \quad \text{and} \quad C \bullet\overset{4g}{\text{——}}\underset{B}{\bullet}\overset{4g}{\text{——}}\bullet D.$$

Then either $[(AB)^2, (CD)^2] = 1$ *or if* $F = ACAC$, *then* $X \mid \langle F, B, C, D\rangle = R \oplus (n-7)\, 1_{\langle F,B,C,D\rangle}$ *where*

$$R(F) = \begin{bmatrix} 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 1&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1 \end{bmatrix}, \qquad R(B) = \begin{bmatrix} 0&1&0&0&0&0&0\\ 1&0&0&0&0&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1 \end{bmatrix},$$

$$R(C) = \mathrm{diag}(1, -1, 1, -1, 1, 1, 1)$$

and

$$R(D) = \begin{bmatrix} 1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0\\ 0&0&1&0&0&0&0\\ 0&0&0&0&0&0&1\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&1&0\\ 0&0&0&1&0&0&0 \end{bmatrix}.$$

Proof. From Theorem 5.3, $X \mid \langle A, B, C\rangle = Y_1 \oplus Y_2 \oplus (n-6)\, 1_{\langle A,B,C\rangle}$ where

$$(Y_1 \oplus Y_2)(A) = \left[\begin{array}{cccc|cc} 0&1&0&0&&\\ 1&0&0&0&&0\\ 0&0&1&0&&\\ 0&0&0&1&&\\ \hline &&0&&0&1\\ &&&&1&0 \end{array}\right],$$

$$(Y_1 \oplus Y_2)(B) = \left[\begin{array}{cccc|cc} -1&0&0&0&&\\ 0&1&0&0&&0\\ 0&0&1&0&&\\ 0&0&0&1&&\\ \hline &&0&&-1&0\\ &&&&0&1 \end{array}\right],$$

and

$$(Y_1 \oplus Y_2)(C) = \left[\begin{array}{cccc|cc} 0&0&1&0&&\\ 0&0&0&1&&0\\ 1&0&0&0&&\\ 0&1&0&0&&\\ \hline &&0&&1&0\\ &&&&0&1 \end{array}\right],$$

and Y_1 acts on the 4-dimensional space V_1. Then $X \mid \langle F, B, C\rangle = T \oplus \xi \oplus (n-5)\,1_{\langle F,B,C\rangle}$ where T acts irreducibly on V_1. So $X \mid \langle F, B, C, D\rangle = R \oplus (n-7)\,1_{\langle F,B,C,D\rangle}$. We examine the possible cases.

Case A. $R = R_1 \oplus R_2$ where R_1 is irreducible of degree 4. Then R_1 acts on V_1 and $R_1((CD)^2) = \mathrm{diag}(-1, -1, -1, -1)$ as C, D are bad of order 4. As $X(A)$ and $X(B)$ act on this subspace, $[(AB)^2, (CD)^2] = 1$.

Case B. $R = R_1 \oplus R_2$ where R_1 is irreducible of degree 5. As $R_2(C)$ is trivial, so is $R_2(D)$ because C, D are bad of order 4. But $R_2(F)$ is trivial and hence R_2 is reducible, contradicting $C \bullet\!\overset{4g}{\text{——}}\!\underset{B}{\bullet}\!\overset{4g}{\text{——}}\!\bullet D$ and Theorem 5.3 IV.

Case C. $R = R_1 \oplus \xi$ where R_1 is irreducible of degree 6. As BB^F is special and BB^F, C are bad of order 4, R_1 is not primitive. Suppose R_1 permutes two dimensional spaces. As BB^F, C are bad, we may assume $R_1(\langle BB^F, C\rangle)$ looks like

$$\left[\begin{array}{c|c} * & 0 \\ \hline 0 & \begin{matrix}1 & 0\\ 0 & 1\end{matrix}\end{array}\right]$$

where [*] is 4 dimensional and acts on V_1. As $R_1(D)$ moves V_1, we may assume

$$R_1(D) = \left[\begin{array}{c|c|c} \begin{matrix}1&0\\0&1\end{matrix} & 0 & 0 \\ \hline 0 & 0 & \begin{matrix}1&0\\0&1\end{matrix} \\ \hline 0 & \begin{matrix}1&0\\0&1\end{matrix} & 0 \end{array}\right].$$

As C, D are a bad pair, $R_1(C) = \mathrm{diag}(1, 1, -1, -1, 1, 1)$. But for all possibilities of $R_1(B)$, we have either $B \bullet\!\overset{2}{\text{——}}\!\bullet C$ or $B \bullet\!\overset{4b}{\text{——}}\!\bullet C$, a contradiction.

So R_1 is monomial in some basis $v_1, \ldots, v_6$. We may assume $R_1(\langle BB^F, C\rangle)$ acts trivially on v_5, v_6 and so $V_1 = \langle v_1, \ldots, v_4\rangle$. So we may assume

$$R_1(C) = \begin{bmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix}, \quad \begin{bmatrix} 1&0&0&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&-1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},$$

or

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

As $R_1(D)$ moves V_I, it moves one of v_5, v_6; as C, D are bad of order 4, $R_1(C)$ is not the first choice. Assume $R_1(C)$ is diagonal. Because $B \stackrel{4g}{\bullet\!\!-\!\!-\!\!\bullet} C$

$$R_1(B) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mp 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

As $[F, C] = 1$, $R_1(F)$ is diagonal,

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mp 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{or} \quad \begin{bmatrix} \pm 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \mp 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since T is irreducible, $R_1(F)$ is the second case. As C, D are bad, then by ordering v_5, v_6 correctly, $R_1(D)$ must represent the permutations (1, 2) (4, 5), (2, 3) (4, 5), (1, 4) (2, 5), (2, 5) (3, 4), or (2, 5) (4, 6). As C, D are bad and $5 \nmid |FD|$ by Lemma 5.2, $R_1(D)$ must represent (2, 5) (4, 6). However in that case $3 \mid |BD|$ or $5 \mid |BD|$, a contradiction. So $R_1(C)$ is the third choice.

$$\text{As } B \stackrel{4g}{\bullet\!\!-\!\!-\!\!\bullet} C,\ R_1(B) = \left[\begin{array}{cccc|c} 0 & i & 0 & 0 & \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \\ 0 & 0 & 0 & 1 & \\ \hline & & 0 & & * \end{array}\right],$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \\ 0 & 0 & 1 & 0 & \\ \hline & & 0 & & * \end{array}\right], \quad \text{or} \quad \left[\begin{array}{cccc|cc} 0 & -1 & 0 & 0 & & \\ -1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 1 & & \\ 0 & 0 & 1 & 0 & & \\ \hline & & 0 & & 1 & 0 \\ & & & & 0 & 1 \end{array}\right].$$

Since $[F, C] = 1$, $R_1(F)$ maps $\langle v_1, v_2\rangle$ into itself and $\langle v_3, v_4\rangle$ into itself. This implies T is reducible, a contradiction.

Case D. R is irreducible. By Lemmas 2.1 and 5.1 R is monomial and has no 2-cycles. Arguing as in case C and because there are no 2-cycles,

$$R(C) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R(B) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R(F) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

where V_1 is the span of the first four vectors. $R(D)$ must move v_7. Since C, D is bad, $R(D)$ represents (2, 7) (4, c) where $c \in \{1, 3, 5, 6\}$, or $(2, d)$ (4, 7) where $d \in \{1, 3, 5, 6\}$. As $3 \nmid |BD|$ and $R \mid \langle B, C, D\rangle$ has irreducible constituents of degree 4, 2, and 1, the first is out. By Lemma 5.2, $5 \nmid |FD|$ which implies $d \neq 1$ or 3. Interchanging v_5, v_6 if necessary, we may assume $d = 5$. The lemma is proved.

With this lemma we can now construct the group $Y(\langle A, B, C, D\rangle)$. We assume $\langle (AB)^2, (CD)^2\rangle$ is not a 2-group. Let Y act on V_1.

We have $Y \mid \langle ACAC, B, C, D\rangle = R \oplus 1_{\langle ACAC,B,C,D\rangle}$ where R is monomial in some basis such that

$$R(ACAC) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad R(B) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$R(C) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad R(D) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

We notice that by Lemma 5.1, as R is irreducible $\langle ACAC, B, D\rangle \simeq PSL_2(7)$ and $R \mid \langle ACAC, B, D\rangle = S \oplus 1_{\langle ACAC,B,D\rangle}$. Let R act on the space U. The vector in U, generating the space on which $1_{\langle ACAC,B,D\rangle}$ acts, is $v_7 = (1, 1, 1, 1, 1, 1, 1)^T$ where T denotes the transpose. Let $\langle v_7, v_8\rangle \subseteq V_1$ be the space on which $2 \cdot 1_{\langle ACAC,B,D\rangle}$ acts. Then if $e_1 = (1, -1, 0, 0, 0, 0, 0)^T$, $e_2 = (1, 0, -1, 0, 0, 0, 0)^T$, $e_3 = (1, 0, 0, -1, 0, 0, 0)^T$, $e_4 = (1, 0, 0, 0, -1, 0, 0)^T$, $e_5 = (1, 0, 0, 0, 0, -1, 0)^T$, $e_6 = (1, 0, 0, 0, 0, 0, -1)^T$, S acts on $\langle e_1, \dots, e_6\rangle$. In the basis $v_8, v_7, e_1, e_2, \dots, e_6$ we get by calculating $Y(g)\, v$ where v is a basis element and $g \in \{ACAC, B, C, D\}$:

$$Y(ACAC) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$Y(B) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$Y(C) = \begin{bmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ b & \frac{3}{7} & \frac{2}{7} & 0 & \frac{2}{7} & 0 & 0 & 0 \\ c & \frac{10}{7} & \frac{-5}{7} & 0 & \frac{2}{7} & 0 & 0 & 0 \\ d & \frac{-4}{7} & \frac{2}{7} & 1 & \frac{2}{7} & 0 & 0 & 0 \\ e & \frac{10}{7} & \frac{2}{7} & 0 & \frac{-5}{7} & 0 & 0 & 0 \\ f & \frac{-4}{7} & \frac{2}{7} & 0 & \frac{2}{7} & 1 & 0 & 0 \\ g & \frac{-4}{7} & \frac{2}{7} & 0 & \frac{2}{7} & 0 & 1 & 0 \\ h & \frac{-4}{7} & \frac{2}{7} & 0 & \frac{2}{7} & 0 & 0 & 1 \end{bmatrix}, \quad Y(D) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now interchange A with C and B with D in the preceding Lemma. So $Y \mid \langle CACA, D, A, B\rangle = R_1 \oplus 1_{\langle CACA,D,A,B\rangle}$ where in some basis of the space on which R_1 acts, we have $R_1(CACA)$, $R_1(D)$, $R_1(A)$, $R_1(B)$ look like the matrices $R(ACAC)$, $R(B)$, $R(C)$, $R(D)$ in Eq. (1), respectively. As $\langle A, C\rangle$ is dihedral of order 8, $ACAC = CACA$. Let R_1 act on the subspace U_1. Then $R_1 \mid \langle ACAC, B, D\rangle = S \oplus 1_{\langle ACAC,B,D\rangle}$; the vector in U_1 generating the space on which $1_{\langle ACAC,B,D\rangle}$ acts is $v_7{}^* = (1, 1, 1, 1, 1, 1, 1)^T$. Let $\langle v_7{}^*, v_8{}^*\rangle \subseteq V_1$ be the space on which $2 \cdot 1_{\langle ACAC,B,D\rangle}$ acts (i.e., $\langle v_7{}^*, v_8{}^*\rangle = \langle v_7, v_8\rangle\cdot$) Let $e_1{}^* = (1, -1, 0, 0, 0, 0, 0)^T$, $e_2{}^* = (1, 0, -1, 0, 0, 0, 0)^T$, $e_3{}^* = (1, 0, 0, -1, 0, 0, 0)^T$, $e_4{}^* = (1, 0, 0, 0, -1, 0, 0)^T$, $e_5{}^* = (1, 0, 0, 0, 0, -1, 0)^T$, and $e_6{}^* = (1, 0, 0, 0, 0, 0, -1)^T$. In the basis $v_8{}^*, v_7{}^*, e_1{}^*, \ldots, e_6{}^*$, $Y(ACAC)$, $Y(D)$, $Y(A)$, $Y(B)$ looks like the matrices $Y(ACAC)$, $Y(B)$, $Y(C)$, $Y(D)$ of (2), respectively (where $Y(A)$ may have different unknowns that $Y(C)$, of course).

We have $\langle e_1, \ldots, e_6\rangle = \langle e_1{}^*, \ldots, e_6{}^*\rangle$; as Y is irreducible $\langle v_7\rangle \neq \langle v_7{}^*\rangle$. So we may choose $v_8 = v_7{}^*$ and $v_8{}^* = v_7$. We want to find $Y(A)$ in the basis $v_8, v_7, e_1, \ldots, e_6$. So we need to find a linear transformation S with $Sv_8 = v_8{}^*$, $Sv_7 = v_7{}^*$, $Se_i = e_i{}^*$. If T is a linear transformation of V_1, and $m_1(T)$ and $m_2(T)$ are the matrices of T in the basis $v_8, v_7, e_1, \ldots, e_6$ and $v_8{}^*, v_7{}^*, e_1{}^*, \ldots, e_6{}^*$, respectively, then we have $m_1(S)^{-1}\, m_1(T)\, m_1(S) = m_2(T)$. We need to find a linear transformation S such that $m_1(T)\, m_1(S) = m_1(S)\, m_2(T)$ where T ranges over $Y(ACAC)$, $Y(B)$, $Y(D)$. But we know $m_1(Y(ACAC)) = m_2(Y(ACAC))$, $m_1(Y(B)) = m_2(Y(D))$, and $m_1(Y(D)) = m_2(Y(B))$. These results plus $v_8{}^* = v_7$, $v_7{}^* = v_8$, give

$$m_1(S) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & s & 0 & 0 & s & s \\ 0 & 0 & 0 & -s & -s & 0 & 0 & -s \\ 0 & 0 & -s & -s & 0 & -s & 0 & 0 \\ 0 & 0 & -s & 0 & -s & 0 & -s & 0 \\ 0 & 0 & 0 & 0 & 0 & -s & -s & -s \\ 0 & 0 & s & 0 & s & s & 0 & s \end{bmatrix} \quad \text{where} \quad s \neq 0.$$

Also

$$m_1(S)^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & t & 0 & 0 & t & t \\ 0 & 0 & 0 & -t & -t & 0 & 0 & -t \\ 0 & 0 & -t & -t & 0 & -t & 0 & 0 \\ 0 & 0 & -t & 0 & -t & 0 & -t & 0 \\ 0 & 0 & 0 & 0 & 0 & -t & -t & -t \\ 0 & 0 & t & 0 & t & t & 0 & t \end{bmatrix} \quad \text{where} \quad t = \frac{1}{2s}.$$

Hence

$$m_1(Y(A)) = \begin{bmatrix} \frac{3}{7} & b^* & 0 & 0 & 0 & \frac{-2}{7}t & \frac{2}{7}t & \frac{2}{7}t \\ 0 & a^* & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-2s}{7} & c^* & 0 & -1 & 0 & \frac{-4}{7} & \frac{-3}{7} & \frac{-3}{7} \\ \frac{-2s}{7} & d^* & -1 & 0 & 0 & \frac{-4}{7} & \frac{-3}{7} & \frac{-3}{7} \\ \frac{-2s}{7} & e^* & 1 & 1 & 1 & \frac{3}{7} & \frac{4}{7} & \frac{4}{7} \\ \frac{-16s}{7} & f^* & 0 & 0 & 0 & \frac{3}{7} & \frac{4}{7} & \frac{4}{7} \\ \frac{12s}{7} & g^* & 0 & 0 & 0 & \frac{3}{7} & \frac{4}{7} & \frac{-3}{7} \\ \frac{12s}{7} & h^* & 0 & 0 & 0 & \frac{3}{7} & \frac{-3}{7} & \frac{4}{7} \end{bmatrix}.$$

Replacing v_8 by tv_8, we may assume $s = \frac{1}{2}$ and $t = 1$ in the above. As $Y(A)$, $Y(C)$ have trace 4, $a = a^* = 1$. Also $Y(A)$, $Y(C)$ each have eigenvectors in $\langle v_8, v_7 \rangle$ corresponding to the eigenvalue 1. If these eigenvectors are $(\alpha, \beta, 0,\ldots, 0)^T$ and $(\gamma, \delta, 0,\ldots, 0)$, respectively, by the forms obtained so far, $m_1(Y_1(A))\,(1, 0,\ldots, 0)^T \neq (1, 0,\ldots, 0)^T$ and $m_1(Y(C))\,(0, 1, 0,\ldots, 0)^T \neq (0, 1, 0,\ldots, 0)^T$. We may assume $\beta = 1$ and $\gamma = 1$.

Calculating $m_1(Y(A))\,(\alpha, 1, 0,\ldots, 0)^T$ and $m_1(Y(C))\,(1, \delta, 0,\ldots, 0)^T$ gives $b^* = \frac{4}{7}\alpha$, $c^* = d^* = e^* = \frac{1}{7}\alpha$, $f^* = \frac{8}{7}\alpha$, $g^* = h^* = \frac{-6}{7}\alpha$, $b = \frac{4}{7}\delta$, $c = e = -\frac{10}{7}\delta$, and $d = f = g = h = \frac{4}{7}\delta$. The (2, 8)-entry of $m_1(Y(ACAC))$ is 0 from above. Calculating the (2, 8)-entry of $(m_1(Y(A))\,m_1(Y(C)))^2$ gives $\delta = -\frac{1}{4}$. Calculating the (1, 2) entry of $(m_1(Y(C))\,m_1(Y(A)))^2 = m_1(Y(CACA)) = m_1(Y(ACAC))$ which also is 0 gives $\alpha = -\frac{1}{2}$. Thus $Y(\langle A, B, C, D \rangle)$ is now determined. Inside this group we verify that $(AB)^2\,(CD)^2$ has order 4 contradicting our assumption that $\langle (AB)^2, (CD)^2 \rangle$ was not a 2-group.

This completes the proof of the main theorem by Baer's theorem [6, Theorem 3.8.2] as it shows $(\tau_1\tau_2)^2$ is in $O_2(G)$ for any bad pair of order 4.

Appendix

It seems of interest to determine explicitly the group $\langle A, B, C, D \rangle = G$ found at the end of Section 6. The matrices $Y(B)$ and $Y(D)$ are given in (2), Section 6.

$$Y(A) = \frac{1}{14}\begin{bmatrix} 6 & -4 & 0 & 0 & 0 & -4 & 4 & 4 \\ 0 & 14 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -1 & 0 & -14 & 0 & -8 & -6 & -6 \\ -2 & -1 & -14 & 0 & 0 & -8 & -6 & -6 \\ -2 & -1 & 14 & 14 & 14 & 6 & 8 & 8 \\ -16 & -8 & 0 & 0 & 0 & 6 & 8 & 8 \\ 12 & 6 & 0 & 0 & 0 & 6 & 8 & -6 \\ 12 & 6 & 0 & 0 & 0 & 6 & -6 & 8 \end{bmatrix}$$

$$Y(C) = \frac{1}{14}\begin{bmatrix} 14 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 6 & 4 & 0 & 4 & 0 & 0 & 0 \\ 5 & 20 & -10 & 0 & 4 & 0 & 0 & 0 \\ -2 & -8 & 4 & 14 & 4 & 0 & 0 & 0 \\ 5 & 20 & 4 & 0 & -10 & 0 & 0 & 0 \\ -2 & -8 & 4 & 0 & 4 & 14 & 0 & 0 \\ -2 & -8 & 4 & 0 & 4 & 0 & 14 & 0 \\ -2 & -8 & 4 & 0 & 4 & 0 & 0 & 14 \end{bmatrix}$$

It has been found that G is an extension of an extra special group H of order 128 by $(O^+(6, 2)) \cong L_4(2) \cong A_8$. This was determined by explicitly showing that the conjugates of $(AB)^2 (CD)^2$ generate H. As H admits automorphisms from $K = \langle ACAC, C, B, D\rangle$, $H \cong D_8 \circ D_8 \circ D_8$, $Y(H)$ is irreducible, and $G/H \cong$ subgroup of $\mathrm{Out}(H)$. It was found A fused some orbits in $H/Z(H)$ which K did not. From inspection, $G/H \cong A_8$. It is found that $(AB)^2$ and $(CD)^2$ are in H and G satisfies part 2 of the main theorem.

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