STABLE Diffeomorphism of Compact 4-Manifolds

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Any two smoothings of a compact orientable 4-manifold become diffeomorphic after connected sum with copies of $S^2 \times S^2$. If we also allow an $S^2 \times S^2$, the nonorientable case holds as well.


1. Introduction

A standard trick of 4-manifold theory is to prove results 'stably', up to connected sum with copies of $S^2 \times S^2$. A natural question of 4-manifold smoothing theory is then the following: Given two smoothings of a compact 4-manifold, can we make them diffeomorphic by adding enough $S^2 \times S^2$s? We show the answer is yes, provided that the manifold is orientable.

Cappell and Shaneson [1] have shown that the answer is sometimes no in the nonorientable case; a counterexample can easily be constructed from their fake $\mathbb{R}P^4$. Surprisingly, though, the answer becomes yes if we also add a copy of $S^2 \times S^2$.

M. Kreck [3] has announced similar results. His techniques, however, are much different from those presented here.

Notation. We let $S(k)$ denote the smooth manifold $\#_k S^2 \times S^2$. $\tilde{S}(k)$ denotes $S(k-1) \# S^2 \times S^2$, where the last summand is the twisted $S^2$ bundle over $S^2$. The nonorientable $S^3$ and $D^3$ bundles over $S^1$ will be denoted $S^3 \times S^1$ and $D^3 \times S^1$, respectively. Finally, if $M$ is an oriented manifold, we let $\tilde{M}$ denote $M$ with reversed orientation.

2. Statement and preliminaries

Our goal is to prove the following:
Theorem. Let $M$ be a compact 4-manifold (possibly with boundary) with smoothings $M_\alpha$ and $M_\beta$. Then for sufficiently large $k$ we have:

1) If $M$ is orientable there is an orientation-preserving diffeomorphism between $M_\alpha \neq S(k)$ and $M_\beta \neq S(k)$.

2) If $M$ is nonorientable then $M_\alpha \neq \tilde{S}(k)$ and $M_\beta \neq \tilde{S}(k)$ are diffeomorphic.

$M_\alpha$ and $M_\beta$ determine lifts of the classifying map of the tangent bundle of $M$:

$$BO_4 \to \text{fibration} \to M \to B\text{TOP}_4$$

If these lifts happen to be fiber homotopic then the result follows from work of Lashof and Shaneson [5]. (In fact, the smoothings are 'S-isotopic'.) Thus, our task is to show that for some $l$, $M_\alpha \neq S(l)$ is diffeomorphic to a new smoothing of $M \neq S(l)$ whose lift is fiber homotopic to that of $M_\beta \neq S(l)$. This involves a technique for removing obstructions (which is somewhat reminiscent of one used in [6] to prove a weak product structure theorem about compact manifolds). They key to our technique is a sequence of lemmas which construct explicit smoothings for certain manifolds.

Before discussing these lemmas, we state some well known facts which we will need. From [6] and [7] we know that the fiber $\text{TOP}_4/O_4$ of the above fibration is 2-connected, and $\pi_3(\text{TOP}_4/O_4) = \mathbb{Z}_2$. It follows that $S^3 \times \mathbb{R}$ admits exactly two fiber homotopy classes of lifts. By [4] there are now two sliced concordance classes of smoothings, distinguished by their lifts. We will call a smoothing of $S^3 \times \mathbb{R}$ good if it is sliced concordant to the standard smoothing, and bad otherwise. (Note that a smoothing $(S^3 \times \mathbb{R})_\gamma$ is good if and only if $(S^3 \times \mathbb{R})_\gamma \times \mathbb{R}$ is diffeomorphic to the standard $S^3 \times \mathbb{R}^2$; hence, we have diffeomorphism invariant.) Now let $M$ be a compact topological 4-manifold. We can smooth $M$-point [7]. Any homeomorphism $S^3 \times \mathbb{R} = \text{End}(M \text{-point})$ now gives a smoothing of $S^3 \times \mathbb{R}$. This smoothing will be good if and only if the Kirby-Siebenmann obstruction $k(M) \in H^4(M, \partial M; \mathbb{Z}_2)$ vanishes.

3. The lemmas

Consider $S^1$ to be $\mathbb{R}/\mathbb{Z}$. Now $S^3 \times (0, \frac{1}{2})$ is a subset of both $S^2 \times S^1$ and $S^3 \times S^1$, filling 'half' of each manifold. Use a disk in the other half for taking connected sums.

Lemma 1. There is a smoothing $(S^3 \times S^1 \neq S(22))_\gamma$, diffeomorphic to the standard smoothing, which induces a bad smoothing on the $S^3 \times \mathbb{R}$ embedded as $S^3 \times (0, \frac{1}{2})$.

The last claim says that the standard smoothing and $\gamma$ represent the two different classes of lifts over the 3-skeleton of $S^3 \times S^1 \neq S(22)$. 

Proof. By Freedman’s classification theorem [2] the Kummer surface $K$ splits topologically as $E_8 \neq E_8 \neq S(3)$. Since $k(E_8) \neq 0$, the $S^1 \times \mathbb{R}$ which separates the first $E_8$ from the rest must inherit a bad smoothing from $K$. Now $K \neq \bar{K}$ is well known to be diffeomorphic to $S(22)$. (To see this, note that $K \neq \bar{K}$ bounds $(K - \text{int} \, B^4) \times I$. By examining the handle structure of this, we see that its boundary must also be $S(22)$.) We have now identified a bad $S^3 \times \mathbb{R}$ in $S(22)$.

Next we do surgery on an $S^0$ consisting of one point on each side of the bad $S^3 \times \mathbb{R}$. This adds a tube diffeomorphic to $S^3 \times \mathbb{R}$. We identify the resulting smooth manifold with $S^3 \times S^1 \neq S(22)$ in such a way that the tube corresponds to $S^3 \times (0, \frac{1}{2})$. There is clearly a homeomorphism $h$ of this manifold which interchanges the good and bad $S^3 \times \mathbb{R}$’s. We define the smoothing $\gamma$ by pushing forward the standard smoothing via $h$.

Lemma 2. There is a smoothing $(S^3 \times S^1 \neq \bar{S}(2))_b$, diffeomorphic to the standard one, which induces a bad smoothing on the $S^3 \times \mathbb{R}$ embedded as $S^3 \times (0, \frac{1}{2})$ in $S^1 \times S^1$.

Proof. Let $Ch$ denote Freedman’s ‘Chern’ manifold [2], the fake $CP^2$. By Freedman’s classification theorem and additivity of the Kirby–Siebenmann obstruction under connected sum, we have $Ch \neq Ch \approx 2CP^2$ and $Ch \neq \bar{Ch} \approx CP^2 \neq C\bar{P}^2$. The manifold $2CP^2 \neq C\bar{P}^2$ now splits as $Ch \neq (Ch \neq C\bar{P}^2)$. Since $k(Ch) \neq 0$, $Ch$ is separated from the rest by a bad $S^3 \times \mathbb{R}$ which we denote by $S$. As in Lemma 1, perform $0$-surgery around $S$, but this time do it nonorientably. Call the new tube $T$. We may identify the resulting manifold with $S^3 \times S^1 \neq 2CP^2 \neq C\bar{P}^2$, so that $T$ corresponds to $S^3 \times (0, \frac{1}{2})$.

We now define a new smoothing on this manifold, with the roles of $S$ and $T$ reversed. Temporarily surger away $S$. What remains is $Ch \neq (\bar{Ch} \neq C\bar{P}^2)$. (Note that the second summand has reversed its orientation due to the twist in $T$.) This manifold smooths as $CP^2 \neq C\bar{P}^2 \neq CP^2$ with a bad smoothing induced on $T$. Now replace $S$,
but with the standard smoothing. This gives an exotic smoothing of our original manifold, with a bad smoothing on $S^3 \times (0, \frac{1}{3})$. By construction, this new smoothing is diffeomorphic to $S^3 \times S^1 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \mathbb{C}P^2$ which is diffeomorphic to the original manifold.

To complete the proof of the lemma, take the connected sum of this manifold with $\overline{\mathbb{C}P^2}$. Either smoothing is then diffeomorphic to $S^3 \times S^1 \# 2\mathbb{C}P^2 \# 2\overline{\mathbb{C}P^2} \cong S^3 \times S^1 \# 2(S^2 \times S^2) \cong S^3 \times S^1 \# \tilde{S}(2)$.

**Remark.** We may prove a similar result with $S(11)$ in place of $S(2)$, but the resulting smoothing is not diffeomorphic to the standard one. To do this, simply replace $Ch \# (Ch \# \mathbb{C}P^2)$ in the above proof by $E_8 \# (E_8 \# S(3)) \approx K$, and note that $E_8 \# \overline{E}_8 \approx S(8)$ by Freedman’s classification. This, together with Lemma 1, has the following application: Lashof and Taylor [6] prove a weak product structure theorem for 4-manifolds. The compact case works for all manifolds of the form $X \# S(k)$ for a certain fixed $k$. By the above results it easily follows that we may take $k$ to equal 22.

**Lemma 3.** a) Lemma 1 holds with $S^3 \times S^1$ replaced by $D^3 \times S^1$. That is, there is a smoothing $(D^3 \times S^1 \# S(22))_\gamma$, diffeomorphic (rel $\partial$) to the standard smoothing, whose lift is not fiber homotopic over the 3-skeleton (rel $\partial$) to that of the standard structure. b) Similarly, Lemma 2 holds with $S^3 \times S^1$ replaced by $D^3 \times S^1$.

**Proof.** We derive the first statement from Lemma 1, basically by drilling out a neighborhood of $(\text{point}) \times S^1$ in $S^3 \times S^1$. The second statement follows by the same reasoning from Lemma 2.

Consider the diffeomorphism

$$h: (S^3 \times S^1 \# S(22))_{\text{standard}} \rightarrow (S^3 \times S^1 \# S(22))_\gamma$$

which defines $\gamma$. By composing with a diffeomorphism of $(S^3 \times S^1 \# S(22))_{\text{standard}}$, we may assume that $h$ preserves orientation and induces the identity map on $\pi_1$. Now consider a circle $C = (\text{point}) \times S^1 \subset S^3 \times S^1 \# S(22)$. By Quinn’s handle straightening theorem (2.2.2 of [7]) we may isotope $h$ to $h_1$ which is smooth on a neighborhood of $C$, relative to the standard smoothing. Since $h$ induces the identity on $\pi_1$, $h_1(C)$ is homotopic to $C$. Thus, there is an ambient isotopy (smooth with respect to the standard structure) pulling $h_1(C)$ back to $C$. This gives $h_2$ (isotopic to $h_1$) which fixes $C$ and is smooth near $C$. If we identify (small disk) $\times S^1$ in $S^3 \times S^1$
with the normal bundle $\nu(C)$, we may easily isotope $h_2$ to $h_3$ which is a bundle map on $\nu(C)$. Now we define a smooth structure $\gamma^*$ by pushing forward the standard structure via $h_3$. By construction, $\gamma^*$ and $\gamma$ are isotopic. Finally, we modify $h_3$ to get $h_4$ which is the identity near $C$. The only difficulty occurs if the bundle map determined by $h_3$ on $\nu(C)$ represents the nontrivial element of $\pi_1(SO_3)$. We may correct this by composing on the right with a diffeomorphism twisting $S^3 \times S^1$ once (like a Dehn twist).

We now have a structure $\gamma^*$ isotopic to $\gamma$ and a diffeomorphism

$$h_4 : (S^3 \times S^1 \neq S(22))_{\text{standard}} \to (S^3 \times S^1 \neq S(22))_{\gamma^*}$$

which fixes (pointwise) a neighborhood of $C = (\text{point}) \times S^1$ in $S^3 \times S^1$. Now drill out a tubular neighborhood of $C$ to obtain the required smoothing $\gamma'$ of $D^3 \times S^1 \neq S(22)$. The map $h_4$ gives a diffeomorphism (rel $\partial$) of the standard structure into this, and the lift behaves as desired because of the corresponding property for $\gamma$.

4. Proof of the main theorem

Let $\tau_\alpha$ and $\tau_\beta$ denote the lifts $M \to BO_3$ corresponding to the given smoothings $\alpha$ and $\beta$, respectively. Then $\tau_\alpha = \tau_\beta$ on $\partial M$. Now $\tau_\beta$ is fiber homotopic (rel $\partial$) to $\tau_\alpha$ over the 2-skeleton of $M$. The obstruction to fiber homotopy over the 3-skeleton lies in $H^3(M, \partial M; \mathbb{Z}_2)$. If it is nonzero, we proceed in the manner of [6]: The homology class dual to the obstruction is represented by a circle smoothly embedded in $M$. This has a neighborhood diffeomorphic to $D^3 \times S^1$ or possibly to $D^3 \times S'$ in the nonorientable case. Dig this neighborhood out of $M$ and replace it with $D^3 \times S^1 \neq D(22)$ (or $D^3 \times S^1 \neq \tilde{S}(2)$ in the second case) smoothed as in Lemma 3. This gives a smoothing $\alpha'$ of $M \neq S(22)$ (or $M \neq \tilde{S}(2)$). Note that the diffeomorphism given in Lemma 3 induces a diffeomorphism $(M \neq S(22))_\alpha = M \neq S(22)$ in the oriented case. (The nonorientable case is similar; we henceforth omit reference to it.) Now compare the smoothings $(M \neq S(22))_\alpha$ and $M_\beta \neq S(22)$. Since the smoothing of Lemma 3 represents the nonstandard lift, we have removed the obstruction; the lifts $\tau_\alpha$ and $\tau_\beta$ of these two smoothings are fiber homotopic (rel $\partial$) over the 3-skeleton of $M \neq S(22)$.

The only remaining obstruction to $\tau_\alpha$ and $\tau_\beta$ being fiber homotopic is an element $\lambda \in \pi_4(TOP_3/O_3)$. (This group is presently unknown, although we may conjecture that it vanishes.) If $\lambda \neq 0$ we remove the obstruction by a trick similar to the previous one. We first modify the standard lift $S^4 \to BO_3$ by wrapping it around $\lambda$ in the fiber. By [5] this new lift is represented by an $S$-smoothing of $S^4$, i.e., a smoothing $S(k)_\lambda$ for some $k$, whose lift differs from the standard one by $\lambda$. By Wall [8] we may assume (by adding $S^2 \times S^2$'s if necessary) that $S(k)_\lambda$ is diffeomorphic to the standard structure. By Quinn [7] we may smooth this map near a point (relative to the standard structure), then isotope it to the identity near the point. Punching out a smooth $D^4$ then gives a smoothing $(D^4 \neq S(k))_\lambda$ diffeomorphic to the standard one (rel $\partial$),
whose lift differs from the standard one (rel $\partial$) by $\lambda$. Finally, gluing this into $(M \# S(22))_\alpha - (\text{smooth } \bar{D}^4)$ gives a smoothing $(M \# S(22+k))_\alpha -$, diffeomorphic to $M_\alpha \# S(22+k)$, whose lift differs by $\lambda$ from $(M \# S(22))_\alpha \# S(k)$, i.e., is fiber homotopic to that of $M_\beta \# S(22+k)$. Now [5] shows that $(M \# S(22+k))_\alpha$ and $M_\beta \# S(22+k)$ are $S$-isotopic, completing the proof.

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References