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Differentiable functions of quaternion variables

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Abstract

We investigate differentiability of functions defined on regions of the real quaternion field and obtain a noncommutative version of the Cauchy–Riemann conditions. Then we study the noncommutative analog of the Cauchy integral as well as criteria for functions of a quternion variable to be analytic. In particular, the quaternionic exponential and logarithmic functions are being considered. Main results include quaternion versions of Hurwitz' theorem, Mittag-Leffler's theorem and Weierstrass' theorem.

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1. Introduction

Since the time of the investigation of quaternions by Hamilton [17,40] there were a lot of attempts to develop an analysis of functions of quaternion variables (see, for example, [1, 8,12–15,30,31,34,38] and references therein), but they have operated with narrow classes of regular in some sense functions of quaternion variables and instead of line integral they have used an integration over three-dimensional surfaces, that does not permit to get such well properties as in the complex case. Hamilton itself had tried to develop such theory, but in his time mathematical analysis of real and complex variables had been much less developed, general and algebraic topologies had not been existing and,

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moreover, he had died soon after the invention of quaternions. In his lectures [17,34] he had outlined a way for such activity: differentiation of functions of quaternion variables along real straight lines and then definition of differentiation by quaternion variables on the base of it, a line integral of differentials of differentiable in such a way functions of quaternion variables. But this way was abandoned in works of his followers. Then it was a development of the theory of Clifford algebras generalizing quaternions (see, for example, [25,34] and references therein). There was not a great advance in a development of theory of functions of quaternion variables, because even formulas for calculations of roots of polynomials of quaternion variables are not quite well developed [20]. In physics and geometry mainly algebraic properties of quaternions were used, functions of quaternion variables were treated by means of a tool of functions of real and complex variables (see, for example, [4,10,25] and references therein). There are a lot of works on quaternion manifolds, but strictly speaking they are complex manifolds with an additional structure in the tangent bundle (see, for example, [32,41] and references therein). Using our definition of quaternion holomorphic functions and results on them below we can say, that a manifold modelled on a quaternion Banach space with quaternion holomorphic connecting functions of charts is a quaternion manifold. Each complex manifold can be embedded into a corresponding quaternion manifold, moreover, each quaternion manifold defined earlier in the literature can be presented as a quotient of a corresponding quaternion manifold in our sense.

Our research of functions of quaternion variables differ from that of preceding authors and permit to work with larger families of functions. Then we have defined the quaternion line integral on a space of continuous functions resembling the main properties analogous to that of the Cauchy complex line integral and we have obtained many specific properties of functions of quaternion variables. We believe that our results can be applied in modern physical theories. In mathematics they can be applied to develop a theory of quaternion manifolds, operator theory, etc.

The noncommutativity of the quaternion field **H** obstructs the immediate application of the theory of analytic and meromoprhic complex functions to functions of quaternion arguments. The latter may be thought of as functions of two noncommuting complex variables, but we shall adopt matrix notation representing the standard generators of the quaternions by their Pauli-matrices. This allows a rather elegant introduction of differentiable functions of a quaternion variable, integrals of functions along curves in **H**, residues of a function, ... The new results contained in this paper provide noncommutative analogs of the Cauchy–Riemann conditions for superdifferentiable functions as well as basic properties of the related noncommutative integrals, the argument principle, etc. .... The quaternionic residue theory depends on the definition and description of the exponential and logarithm functions of quaternion variables. An explicit description of the exponential is obtained in Proposition 3.2 allowing to view it as an epimorphism from a set of imaginary quaternions to the three dimensional quaternionic unit sphere. The relation between the quaternion version of holomorphicity and local analyticity is investigated, in particular we obtain in Theorem 3.10 that for a continuous function on an open subset U of **H**, the property of being locally analytic follows from the integral holomorphicity. Section 3 also contains the quaternionic version of the classical theorems of Cauchy, Liouville and Morera. Although the analytic theory of functions of a quaternion variable, or more general of functions of noncommuting variables with suitable "commutation" rules, has an interest in its own right, we were more motivated by the connection with noncommutative geometry, the analytic structure induced on **H** modulo a **Z**-lattice, and the quaternionic version of arithmetical functions like the zeta-function. We hope to return to these applications in forthcoming work.

Though some results in noncommutative geometry are concerned with function families [3,7,9,22,33] they are rather general and do not take into account the particular quaternion case and its specific features. It is necessary to note, that we use a weaker superdifferentiability condition, compared to, for example, [3,9,22]. Traditionally one uses the condition, that a right derivative is right superlinear on a superalgebra, which causes severe restrictions on these classes of functions (see Theorems I.1.4 and I.2.3 in [22]). This is too restrictive in the particular quaternion case as it does not permit to describe an **H**-algebra of quaternion holomorphic functions on an open subset U in $\mathbf{H}^n$ extending that of complex holomorphic functions. We have withdrawn the condition of right superlinearity of a superderivative on a superalgebra, supposing only that it is additive on $\mathbf{H}^n$ and **R**-homogeneous. Nevertheless, it also satisfies distributivity and associativity laws relative to the multiplication from the right on (scalar) quaternions $\lambda \in \mathbf{H}$ and there are also distributivity and associativity laws relative to a left multiplication on $\lambda \in \mathbf{H}$ (see §2.1). That is, we have considered Frechét differentiable functions on the Euclidean space $\mathbf{R}^{4n}$ with some additional conditions on increments of functions, taking into account a superalgebra structure. This permits to encompass classes of all analytic functions on a region in $\mathbf{H}^n$, in particular, all polynomial functions. Moreover, this approach permits to investigate an analog of functions having Laurent series expansions. We have proved, that for each complex holomorphic function f on a region V open in $\mathbf{C}^n$ there exists a quaternion holomorphic function F on a suitable region U in $\mathbf{H}^n$ such that a restriction of F on V coincides with f. The theory of complex holomorphic functions turns out to be rather different from a theory of quaternion holomorphic functions. The quaternion field **H** has nontrivial algebraic structure and identities, so there are different ways to define not only function spaces, but also their differentiations. A differentiation is not only analytic, it also has algebraic properties. In some sense the notion of the family of all quaternion holomorphic functions unifies together complex holomorphic and antiholomorphic functions. On the other hand, weaker differentiability, for example, "pointwise" as defined in the classical case by Gatô [23] yields a too poor algebraic structure of function spaces not taking into account the gradation of a superalgebra.

In previous works [26–29] the first author investigated loop and diffeomorphism groups of complex manifolfds and quasi-invariant measures and stochastic processes on them. Complex manifolds also have the structure of supermanifolds, since the field **C** can be considered as a graded algebra over **R**. The graded structure of the quaternion field over the reals is more complicated. Conceivably, the investigations in this work allow to continue this work for quaternion manifolds, loop and diffeomorphism groups of these manifolds, quasi-invariant measures and stochastic processes on them, as well as their associated representations including irreducible ones.

2. Differentiability of functions of quaternion variables

To avoid misunderstandings we first introduce notations. We write **H** for the skewfield of quaternions over the real field **R**. This skewfield can be represented as a subring of the ring $\mathbf{M}_2(\mathbf{C})$ of all 2×2 complex matrices by representing the classical quaternion basis 1, i, j, k, by the Pauli-matrices I, J, K, L defined as follows:

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix};\qquad J=\begin{pmatrix}i&0\\0&-i\end{pmatrix};\qquad K=\begin{pmatrix}0&1\\-1&0\end{pmatrix};\qquad L=\begin{pmatrix}0&i\\i&0\end{pmatrix},$$

where $i=(-1)^{1/2}$. Hence, each quaternion z is written as a 2×2 matrix over **C** having matrix elements $z_{1,1}=\bar z_{2,2}=:t$, $z_{1,2}=-\bar z_{2,1}=:u$, where t and $u\in\mathbf{C}$ such that $t=v+iw$ and $u=x+iy$, v, w, x and y are in the field **R** of real numbers. The quaternion skewfield **H** has an anti-automorphism η of order two induced in **H** by the Hermite conjugation in $\mathbf{M}_2(\mathbf{C})$, that is, $\eta: z\mapsto\tilde z$, where $\tilde z_{1,1}=\bar t$ and $\tilde z_{1,2}=-u$. There is a norm in **H** such that $|z|=(|t|^2+|u|^2)^{1/2}$, hence $\det(z)=|z|^2$ and $\tilde z=|z|^2z^{-1}$. The noncommutative field **H** is the $\mathbf{Z}_2$-graded **R**-algebra $\mathbf{H}=\mathbf{H}_0+\mathbf{H}_1$, where elements of $\mathbf{H}_0$ are *even* and elements of $\mathbf{H}_1$ are *odd* (see, for example, [6,24,40]).

In view of noncommutativity of **H** a polynomial function $P:U\to\mathbf{H}$ may have several different representations

$$\breve P(z,\tilde z)=\sum_k b_{k,1}\hat z^{k_1}\dots b_{k,m}\hat z^{k_m},$$

where $b_{k,j}\in\mathbf{H}$ are constants, $k=(k_1,\dots,k_m)$, $m\in\mathbf{N}$, $k_j=(k_{j,1},\dots,k_{j,2n})$, $k_{j,l}\in\mathbf{Z}$, $\hat z^{k_j}:={}^1z^{k_{j,1}}\,{}^1\tilde z^{k_{j,n+1}}\dots{}^nz^{k_{j,n}}\,{}^n\tilde z^{k_{j,2n}}$, ${}^lz^0:=1$, ${}^l\tilde z^0=1$, U is an open subset of $\mathbf{H}^n$. Each term $b_{k,1}\hat z^{k_1}\dots b_{k,m}\hat z^{k_m}=:\omega(b_k,z,\tilde z)\neq 0$ we consider as a word of length $\xi(\omega)=\sum_{j,l}\delta(k_{j,l})+\sum_j\kappa(b_{k,j})$, where $\delta(k_{j,l})=0$ for $k_{j,l}=0$ and $\delta(k_{j,l})=1$ for $k_{j,l}\neq 0$, $\kappa(b_{k,j})=j$ for $b_{k,j}=1$, $\kappa(b_{k,j})=j+1$ for $b_{k,j}\in\mathbf{H}\setminus\{0,1\}$. A polynomial P is considered as a phrase $\breve P$ of a length $\xi(\breve P):=\sum_k\xi(\omega(b_k,z,\tilde z))$. Using multiplication of constants in **H**, commutativity of vI with each ${}^lz$ and ${}^l\tilde z$, and ${}^lz^a\,{}^lz^b={}^lz^{a+b}$ and ${}^l\tilde z^a\,{}^l\tilde z^b={}^l\tilde z^{a+b}$, ${}^lz\,{}^l\tilde z={}^l\tilde z\,{}^lz$, it is possible to consider representations of P as phrases $\breve P$ of a minimal length $\xi(\breve P)$. We choose one such $\breve P$ of a minimal length. If $f:U\to\mathbf{H}$ is a function presented by a convergent by z and $\tilde z$ series $f(z,\tilde z)=\sum_n P_n(z,\tilde z)$, where $P_n(vz,v\tilde z)=v^nP_n(z,\tilde z)$ for each $v\in\mathbf{R}$ is a **R**-homogeneous polynomial, $n\in\mathbf{Z}$, then we consider among all representations of f such for which $\xi(\breve P_n)$ is minimal for each $n\in\mathbf{Z}$. We may use this convention separately for families of functions f having (a) z-series decompositions, (b) $\tilde z$-series decompositions, (c) $(z,\tilde z)$-series decompositions (that is, by indicated variables). The corresponding families of locally analytic functions on U are denoted by $C^\omega_z(U,\mathbf{H})$, $C^\omega_{\tilde z}(U,\mathbf{H})$, $C^\omega_{z,\tilde z}(U,\mathbf{H})$. If each P_n for f has a decomposition of a particular left type

$$\breve P(z,\tilde z)=\sum_{k,p}b_{k,p}z^k\tilde z^p,$$

where $0\leqslant k$, $p\in\mathbf{Z}$, $b_{k,p}\in\mathbf{H}$, then the space of all such locally analytic functions on U is denoted by ${}_lC^\omega_{z,\tilde z}(U,\mathbf{H})$, for z-series or $\tilde z$-series decompositions only the corresponding spaces are denoted by ${}_lC^\omega_z(U,\mathbf{H})$ and ${}_lC^\omega_{\tilde z}(U,\mathbf{H})$ respectively. They are proper subspaces of that of given above. Spaces of locally analytic functions f having right type decompositions for each P_n

$$\check{P}(z,\tilde{z}) = \sum_{k,p} z^k \tilde{z}^p b_{k,p}$$

are denoted by ${}_rC^{\omega}_{z,\tilde{z}}(U,\mathbf{H})$, etc.

2.1. Definition. Consider an open region U in $\mathbf{H}^n$, the n-fold product of copies of $\mathbf{H}$, and let $f:U\to\mathbf{H}$ be a function. Then f is said to be (right) superdifferentiable at a point $({}^1z,\dots,{}^nz) = e_1{}^1z + \cdots + e_n{}^nz \in U$ (with respect to a chosen (right) $\mathbf{H}$-basis for $\mathbf{H}^n$, $\{e_1,\dots,e_n\}$), if it can be written in the form

$$f(z+h) = f(z) + \sum_{j=1}^n A_j{}^jh + \varepsilon(h)|h|$$

for each $h\in\mathbf{H}^n$ such that $z+h\in U$, where A_j is an $\mathbf{H}$-valued additive $\mathbf{R}$-homogeneous operator of h-variable, in general it is non-linear for each $j=1,\dots,n$ and A_j is denoted by $\partial f(z)/\partial{}^jz$, that is, there exists a (right) derivative $f'(z)$ such that a (right) differential is given by

$$D_zf(z).h := f'(z).h := \sum_{j=1}^n \bigl(\partial f(z)/\partial{}^jz\bigr){}^jh,$$

where $\varepsilon(h)$ is a function continuous at zero such that $\varepsilon(0)=0$, $e_j=(0,\dots,0,1,0,\dots,0)$ is the vector in $\mathbf{H}^n$ with 1 on jth place,

$$D_zf(z).h =: (Df)(z;h)$$

such that $(Df)(z;h)$ is additive in h and $\mathbf{R}$-homogeneous, that is,

$$(Df)(z;h_1+h_2) = (Df)(h_1)+(Df)(h_2) \quad\text{and}\quad (Df)(z;vh) = v(Df)(z;h)$$

for each h_1, h_2 and $h\in\mathbf{H}^n$, $v\in\mathbf{R}$. There are imposed conditions:

$$D_{\tilde{z}}z=0,\quad D_z\tilde{z}=0,\quad (D_zz).h=h,\quad D_z1=0,\quad (D_{\tilde{z}}\tilde{z}).h=h,\quad D_{\tilde{z}}1=0,$$
$$\text{also}\quad \bigl(D_z(fg)\bigr).h = \bigl((D_zf).h\bigr)g + f(D_zg).h$$

for a product of two superdifferentiable functions f and g and each $h\in\mathbf{H}^n$. We also have distributivity and associativity laws relative to multiplication from the right by (scalar) quaternions $\lambda\in\mathbf{H}$:

$$\bigl(D(f+g)\bigr)(z;h\lambda) = (Df)(z;h\lambda)+(Dg)(z;h\lambda),$$
$$(Df)\bigl(z;h(\lambda_1+\lambda_2)\bigr) = (Df)(z;h\lambda_1)+(Df)(z;h\lambda_2),$$
$$(Df)\bigl(z;(h\lambda_1)\lambda_2\bigr) = (Df)\bigl(z;h(\lambda_1\lambda_2)\bigr)$$

for each superdifferentiable functions f and g at z and each λ, λ_1 and $\lambda_2\in\mathbf{H}$. There are also left distributive and associative laws:

$$\bigl(D\lambda(f+g)\bigr)(z;h) = \lambda(Df)(z;h)+\lambda(Dg)(z;h),$$

$$\big(D(\lambda_1+\lambda_2)f\big)(z;h)=\lambda_1(Df)(z;h)+\lambda_2(Df)(z;h),$$
$$\big(D(\lambda_1\lambda_2)f\big)(z;h)=\lambda_1(D\lambda_2 f)(z;h).$$

That is, we consider Frechét differentiable functions on the Euclidean space $\mathbf{R}^{4n}$ with some additional conditions on increments of functions, taking into account a superalgebra structure. Quite analogously we defined the notion of (right) superdifferentiability by $\tilde z$ and by their pair $(z,\tilde z)$.

Notation. We write f as a 2×2 complex matrix with entries $f_{i,j}$ such that $f_{i,j}=g_{i,j}+ih_{i,j}$ and $g_{i,j}, h_{i,j}$ being real-valued functions. For $n=1$ we also write ${}^1z$ without its superscript. We may write a function $f(z,\tilde z)$ in variables (v,w,x,y) as $F(v,w,x,y)=f\circ\sigma(v,w,x,y)$, where $\sigma({}^lv,{}^lw,{}^lx,{}^ly)=({}^lz,{}^l\tilde z)$ is a bijective mapping.

2.2. Proposition. *A function $f:U\to\mathbf{H}$ is (right) superdifferentiable at a point $a\in U$ if and only if F is Frechét differentiable at a and*

$$D_{\tilde z}f(z)|_{z=a}=0. \tag{2.1}$$

If in addition $f'(a)$ is right superlinear on the superalgebra $\mathbf{H}^n$, then f is superdifferentiable at a if and only if F is Frechét differentiable at a and satisfies the following equations:

$$\begin{aligned}
&\partial G_{1,1}/\partial\,{}^jv=\partial H_{1,1}/\partial\,{}^jw, && \partial G_{1,1}/\partial\,{}^jw=-\partial H_{1,1}/\partial\,{}^jv,\\
&\partial G_{1,2}/\partial\,{}^jv=-\partial H_{1,2}/\partial\,{}^jw, && \partial G_{1,2}/\partial\,{}^jw=\partial H_{1,2}/\partial\,{}^jv,\\
&\partial G_{1,1}/\partial\,{}^jw=-\partial H_{1,2}/\partial\,{}^jx, && \partial G_{1,1}/\partial\,{}^jx=\partial H_{1,2}/\partial\,{}^jw,\\
&\partial G_{1,2}/\partial\,{}^jw=-\partial H_{1,1}/\partial\,{}^jx, && \partial G_{1,2}/\partial\,{}^jx=\partial H_{1,1}/\partial\,{}^jw,\\
&\partial G_{1,1}/\partial\,{}^jx=-\partial H_{1,1}/\partial\,{}^jy, && \partial G_{1,1}/\partial\,{}^jy=\partial H_{1,1}/\partial\,{}^jx,\\
&\partial G_{1,2}/\partial\,{}^jx=\partial H_{1,2}/\partial\,{}^jy, && \partial G_{1,2}/\partial\,{}^jy=-\partial H_{1,2}/\partial\,{}^jx,
\end{aligned} \tag{2.2}$$

or shortly in matrix notation:

$$\partial F/\partial\,{}^jv=(\partial F/\partial\,{}^jw)J^{-1}=(\partial F/\partial\,{}^jx)K^{-1}=(\partial F/\partial\,{}^jy)L^{-1} \tag{2.3}$$

for each $j=1,\dots,n$.

Proof. Verify that z and $\tilde z$ are independent variables. Suppose contrary that there exists $\gamma\in\mathbf{H}$ such that $z+\gamma\tilde z=0$ for each $z\in\mathbf{H}$. This is equivalent to a system of two linear equations $\gamma_{1,1}\bar t-\gamma_{1,2}u=-t$ and $\gamma_{1,1}\bar u+\gamma_{1,2}t=-u$. If $z\neq 0$, then $\gamma_{1,1}=-(t^2+u^2)/(|t|^2+|u|^2)$ and $\gamma_{1,2}=(t\bar u-\bar tu)/(|t|^2+|u|^2)$, hence $\partial\gamma/\partial t\neq 0$ and $\partial\gamma/\partial u\neq 0$. Therefore, there is not any $\gamma\in\mathbf{H}$ such that $z+\gamma\tilde z=0$ for each $z\in\mathbf{H}$.

For each canonical closed compact set U in $\mathbf{H}$ the set of all polynomial by z and $\tilde z$ functions is dense in the space of all continuous on U Frechét differentiable functions on $\mathrm{Int}(U)$. In particular functions of the form of series $f=\sum_{l_1}f\dots{}_{l_n}f$ converging on U together with its superdifferential on $\mathrm{Int}(U)$ such that each ${}_lf$ is (right) superlinearly superdifferentiable on $\mathrm{Int}(U)$ relative to the superalgebra $\mathbf{H}^n$ is dense in the $\mathbf{R}$-linear space of (right) superdifferentiable functions. From conditions of §2.1 it follows, that the superdifferentiability conditions are defined uniquely on space of polynomials. Such that the superdifferentiability of a polynomial P on U means that it is expressible through a sum of products of ${}^jz$ and constants from $\mathbf{H}$, that is, without terms containing $\tilde{z}$. Suppose that f is superdifferentiable at a point a. To each $f'(z)$ there corresponds a $\mathbf{R}$-linear operator on the Euclidean space $\mathbf{R}^{4n}$. Moreover, we have the distributivity and associativity laws for $(Df)(z;h)$ relative to the right multiplication on quaternions $\lambda \in \mathbf{H}$ (see §2.1). Then $f(a+h) - f(a) = D_a f(a,\tilde{a}).h + D_{\tilde{a}} f(a,\tilde{a}).\tilde{h} + \varepsilon(h)|h| = D_a f(a,\tilde{a}).h + \varepsilon(h)|h|$, where $\varepsilon(h)$ is continuous by h and $\varepsilon(0) = 0$, therefore, $D_{\tilde{a}} f = 0$. Vice versa, if F is Frechét differentiable, then expressing vI, wJ, xK and yL through linear combinations of z and $\tilde{z}$ with constant coefficients we get the increment of f as above which is independent of $\tilde{h}$ if and only if $D_{\tilde{a}} f = 0$.

Consider now the particular case, when f' is right superlinear on the superalgebra $\mathbf{H}^n$. In this case $f'(a)$ is right $\mathbf{H}$-linear. Using the definition of the right superderivative and that there is a bijective correspondence between z and (v,w,x,y) we consider a function $f = f(z,\tilde{z}) = F(v,w,x,y)$ (right) superdifferentiable by z and $\tilde{z}$, hence it is also differentiable by $(v,w,x,y) = (b_1,\dots,b_4)$ and we obtain the expressions:

$$\partial F/\partial\, {}^jb_l = (\partial F/\partial\, {}^jz).\partial\, {}^jz/\partial\, {}^jb_l + (\partial F/\partial\, {}^j\tilde{z}).\partial\, {}^j\tilde{z}/\partial\, {}^jb_l,$$

since $\partial\, {}^jz/\partial\, {}^kb_l = 0$ and $\partial\, {}^j\tilde{z}/\partial\, {}^kb_l = 0$ for each $k \neq l$. From $D_{\tilde{z}} f = 0$ and $\partial\, {}^jz/\partial\, {}^jv = I$, $\partial\, {}^jz/\partial\, {}^jw = J$, $\partial\, {}^jz/\partial\, {}^jx = K$, $\partial\, {}^jz/\partial\, {}^jy = L$ we get Eqs. (2.3). Substituting J^{-1} for the equation with pair of variables (v,w) in (2.3) we get $\partial F_{1,1}/\partial\, {}^jv = -i\partial F_{1,1}/\partial\, {}^jw$ and $\partial F_{1,2}/\partial\, {}^jv = i\partial F_{1,2}/\partial\, {}^jw$, substituting $K^{-1}J = L$ for Eq. (2.3) with the pair of variables (w,x) we get: $\partial F_{1,1}/\partial\, {}^jw = i\partial F_{1,2}/\partial\, {}^jx$ and $\partial F_{1,1}/\partial\, {}^jx = -i\partial F_{1,2}/\partial\, {}^jw$, substituting $L^{-1}K = J$ for Eq. (2.3) with pair of variables (x,y) we get $\partial F_{1,1}/\partial\, {}^jx = i\partial F_{1,1}/\partial\, {}^jy$ and $\partial F_{1,2}/\partial\, {}^jx = -i\partial F_{1,2}/\partial\, {}^jy$. Using the equality $F_{l,j} = G_{l,j} + iH_{l,j}$ we get Eqs. (2.2) from the latter equations.

Let now F be differentiable at a and let F be satisfying conditions (2.2). Then

$$\begin{aligned} f(z) - f(a) = \sum_{l=1}^{n} \bigl\{ &(\partial F/\partial\, {}^lv)\Delta\, {}^lv + (\partial F/\partial\, {}^lw)\Delta\, {}^lw + (\partial F/\partial\, {}^lx)\Delta\, {}^lx \\ &+ (\partial F/\partial\, {}^ly)\Delta\, {}^ly \bigr\} + \varepsilon(z-a)|z-a|, \end{aligned}$$

where $\Delta({}^lv, {}^lw, {}^lx, {}^ly) = \sigma^{-1}({}^lz) - \sigma({}^la)$ for each $l = 1,\dots,n$. From conditions (2.3) equivalent to (2.2) we get

$$\begin{aligned} f(z) - f(a) &= \sum_{l=1}^{n} \bigl\{ (\partial F/\partial\, {}^lv)I\Delta\, {}^lv + (\partial F/\partial\, {}^lv)J\Delta\, {}^lw + (\partial F/\partial\, {}^lv)K\Delta\, {}^lx \\ &\qquad + (\partial F/\partial\, {}^lv)L\Delta\, {}^ly \bigr\} + \varepsilon(z-a)|z-a| \\ &= (\partial F/\partial\, {}^lv)I\Delta\, {}^lz + \varepsilon(z-a)|z-a|, \end{aligned}$$

where ε is a function continuous at 0 and $\varepsilon(0) = 0$. Therefore, f is superdifferentiable by z at a such that $f'(a)$ is right superlinear. □

2.3. Remark. A function f superdifferentiable at each point $a \in U$ (by either z or $\tilde{z}$ or $(z,\tilde{z})$) is called superdifferentiable in U (by either z or $\tilde{z}$ or $(z,\tilde{z})$ respectively). The first pair of equations in (2.2) yields the usual Cauchy–Riemann conditions for complex-valued differentiable functions. On the other hand, we restrict z by $\left(\begin{smallmatrix} t & 0 \\ 0 & \tilde{t} \end{smallmatrix}\right)$, then (2.1) also yields the usual Cauchy–Riemann condition in complex form $D_{\tilde{t}} f_{1,1}(t,0) = 0$.

If a series $f(z) = \sum_{l_1,\dots,l_n} {}_{l_1}f(z)\dots{}_{l_n}f(z)$ uniformly converges on U together with its superdifferential, then

$$D_z f(z).h = \sum_j \sum_{l_1,\dots,l_n} {}_{l_1}f(z)\dots{}_{l_{j-1}}f(z)\bigl(D_z\, {}_{l_j}f(z).h\bigr)_{l_{j+1}}f(z)\dots{}_{l_n}f(z),$$

where each ${}_l f$ is supposed to be superdifferentiable. A similar equality holds for $D_{\tilde{z}} f$. This illustrates, that in general a product of $\mathbf{H}$-valued functions need not have a right superlinear superdifferential on the superalgebra $\mathbf{H}^n$ even if each ${}_l f$ has that property. Nevertheless such functions f satisfy the superdifferentiability conditions of Definition 2.1.

2.4. Corollary. *Let f be a continuously superdifferentiable function by z with a right superlinear superdifferential on the superalgebra $\mathbf{H}^n$ in an open subset U in $\mathbf{H}^n$ and let F be twice continuously differentiable by (v, w, x, y) in U, then certain components of F are harmonic functions by pairs of variables (v, w), (w, x), (x, y) and (v, y), namely*:

$$\begin{aligned}
&\Delta_{l_v,l_w} G_{1,1} = 0, \quad \Delta_{l_v,l_w} H_{1,1} = 0, \quad \Delta_{l_v,l_w} G_{1,2} = 0, \quad \Delta_{l_v,l_w} H_{1,2} = 0,\\
&\Delta_{l_w,l_x} G_{1,1} = 0, \quad \Delta_{l_w,l_x} H_{1,1} = 0, \quad \Delta_{l_w,l_x} G_{1,2} = 0, \quad \Delta_{l_w,l_x} H_{1,2} = 0,\\
&\Delta_{l_x,l_y} G_{1,1} = 0, \quad \Delta_{l_x,l_y} H_{1,1} = 0, \quad \Delta_{l_x,l_y} G_{1,2} = 0, \quad \Delta_{l_x,l_y} H_{1,2} = 0,\\
&\Delta_{l_v,l_y} G_{1,1} = 0, \quad \Delta_{l_v,l_y} H_{1,1} = 0, \quad \Delta_{l_v,l_y} G_{1,2} = 0, \quad \Delta_{l_v,l_y} H_{1,2} = 0
\end{aligned} \tag{2.4}$$

for each $l = 1,\dots,n$, where $\Delta_{l_v,l_w} G_{1,1} := \partial^2 G_{1,1}/\partial\, {}^l v^2 + \partial^2 G_{1,1}/\partial\, {}^l w^2$.

Proof. From the first row of (2.2) and in view of the twice continuous differentiability of F it follows, that $\partial^2 G_{1,1}/\partial\, {}^l v^2 = \partial^2 H_{1,1}/\partial\, {}^l v\partial\, {}^l w = \partial^2 H_{1,1}/\partial\, {}^l w\partial\, {}^l v = -\partial^2 G_{1,1}/\partial\, {}^l w^2$. Analogously, from the remaining rows of (2.2) we deduce the other equations in (2.4). The latter equations follow from $\Delta_{l_v,l_y} = \Delta_{l_v,l_w} - \Delta_{l_w,l_x} + \Delta_{l_x,l_y}$.

2.5. Note and Definition. Let U be an open subset in $\mathbf{H}$ and let $f: U \to \mathbf{H}$ be a function defined on U such that

$$f(z,\tilde{z}) = f^1(z,\tilde{z})\dots f^l(z,\tilde{z}), \tag{2.5i}$$

where each function $f^s(z,\tilde{z})$ is presented by a Laurent series

$$f^s(z,\tilde{z}) = \sum_{n=n_0}^{\infty} \sum_{m=m_0}^{\infty} f^s_{n,m}(z-\zeta)^n(\tilde{z}-\tilde{\zeta})^m \tag{2.5ii}$$

converging on U, where $f^s_{n,m} \in \mathbf{H}$, $z \in U$, $\zeta \in \mathbf{H}$ is a marked point, $m \in \mathbf{Z}$, $n \in \mathbf{Z}$, if $\min(n_0, m_0) < 0$, then $\zeta \notin U$. Consider the case $f^s_{-1,m} = 0$ for each m and s. The case with terms $f^s_{-1,m} \neq 0$ will be considered later.

Let $[a,b]$ be a segment in $\mathbf{R}$ and $\gamma: [a,b] \to \mathbf{H}$ be a continuous function. Consider a partitioning P of $[a,b]$, that is, P is a finite subset of $[a,b]$ consisting of an increasing sequence of points $a = c_0 < \dots < c_k < c_{k+1} < \dots < c_q = b$, then the norm of P is defined as $|P| := \max_k(x_{k+1} - x_k)$ and the P-variation of γ as $v(\gamma; P) := \sum_{k=0}^{q-1} |\gamma(c_{k+1}) - \gamma(c_k)|$, where $q = q(P) \in \mathbf{N}$. The total variation (or the length) of γ is defined as $V(\gamma) = \sup_P v(\gamma; P)$. Suppose that γ is rectifiable, that is, $V(\gamma) < \infty$. For f having decomposition (2.5) with $f^s_{-1,m} = 0$ for each m and s and a rectifiable path $\gamma : [a, b] \to U$ we define a (noncommutative) quaternion line integral by the formula:

$$\int_\gamma f(z, \tilde{z})\, dz := \lim_P I(f, \gamma; P), \tag{2.6}$$

where

$$I(f, \gamma; P) := \sum_{k=0}^{q-1} \hat{f}(z_{k+1}, \tilde{z}_{k+1}).(\Delta z_k), \tag{2.7}$$

$\hat{f}(z, \tilde{z}).h := (\partial g(z, \tilde{z})/\partial z).h$ for each $h \in \mathbf{H}$ and each s, where $\Delta z_k := z_{k+1} - z_k$, $z_k := \gamma(c_k)$ for each $k = 0, \dots, q$, and where without loss of generality we suppose, that g is a function such that $(\partial g(z, \tilde{z})/\partial z).I = f(z, \tilde{z})$ for each $z \in U$. In a similar way we define $\int_\gamma f(z, \tilde{z})\, d\tilde{z}$. We may write shortly $\int_\gamma f(z)\, dz$ or $\int_\gamma f(z)\, d\tilde{z}$ also for such integrals due to the bijective correspondence between z and $\tilde{z}$.

This definition is justified by the following proposition.

2.6. Proposition. *Let f be a function as in* §2.5 *and suppose that there are two constants r and R such that the Laurent series* (2.5) *converges in the set* $B(a, r, R, \mathbf{H}) := \{z \in \mathbf{H}$: $r \leqslant |z - a| \leqslant R\}$ *for each $s = 1, \dots, l$, let also γ be a rectifiable path contained in $U \cap B(a, r', R', \mathbf{H})$, where $r < r' < R' < R$. Then the quaternion line integral exists.*

Proof. Since each f^s converges in $B(a, r, R, \mathbf{H})$, then

$$\overline{\lim_{n>0,\ m>0}} |f^s_{n,m}|^{1/(n+m)} R \leqslant 1, \quad \text{hence}$$

$$\|f\|_\omega := \prod_{s=1}^{l} \Bigl(\sup_{n+m<0} |f^s_{n,m}| r^{n+m}, \sup_{n+m\geqslant 0} |f^s_{n,m}| R^{n+m}\Bigr) < \infty$$

and inevitably

$$\|f\|_{1,\omega,B(a,r',R',\mathbf{H})} := \prod_{s=1}^{l} \biggl[\biggl(\sum_{n+m<0} |f^s_{n,m}| r'^{n+m}\biggr) + \biggl(\sum_{n+m>0} |f^s_{n,m}| R'^{n+m}\biggr)\biggr] < \infty.$$

For each locally $(z, \tilde{z})$-analytic function f in U and each z_0 in U there exists a ball of radius $r > 0$ with center z_0 such that f has a decomposition analogous to (2.5i,ii) in this ball with all n_j and m_j nonnegative, $j = 1, \dots, l$. Consider two quaternion $(z, \tilde{z})$-locally analytic functions f and q on U such that f and q noncommute. Let $f^0 := f$, $q^0 := q$, $q^{-n} := q^{(n)}$, $\partial(q^n)/\partial z =: q^{n-1}$ and $q^{-k-1} = 0$ for some $k \in \mathbf{N}$, then

(i) $(fq)^1 = f^1 q - f^2 q^{-1} + f^3 q^{-2} + \dots + (-1)^k f^{k+1} q^{-k}$. In particular, if $f = az^n$, $q = bz^k$, with $n > 0$, $k > 0$, $b \in \mathbf{H} \setminus \mathbf{R}I$, then $f^p = [(n+1)\dots(n+p)]^{-1} a z^{n+p}$ for each $p \in \mathbf{N}$, $q^s = (k-1)\dots(k-s+1) b z^{k-s}$. Also

(ii) $(fq)^1 = fq^1 - f^{-1}q^2 + f^{-2}q^3 + \cdots + (-1)^p f^{-p} q^{p+1}$. Apply (i) for $n \geqslant m$ and (ii) for $n < m$ to solve the equation $(\partial g(z, \tilde{z})/\partial z).I = f(z, \tilde{z})$ for each $z \in U$. If f and q have series converging in $\mathrm{Int}(B(z_0, r, \mathbf{H}))$, then these formulas show that there exists a $(z, \tilde{z})$-analytic function $(fq)^1$ with series converging in $\mathrm{Int}(B(0, r, \mathbf{H}))$, since $\lim_{n\to\infty}(nr^n)^{1/n} = r$, where $0 < r < \infty$. Consider the equation $BA = AC$, where A, B and C are quaternions. Therefore, for each quaternion locally $(z, \tilde{z})$-analytic function f there exists the operator $\hat{f}$. Considering a function G of real variables corresponding to g we get that all solutions g differ on quaternion constants, hence $\hat{f}$ is unique for f. If $A \neq 0$, then $C = A^{-1}BA$, hence $|C| = |B|$. If $B \neq 0$, then $C = BD$, where $D = B^{-1}C$ and $|D| = 1$. Therefore,

$$\begin{aligned} &f^s_{n,m}(z_{j+1} - a)^k(\Delta z_j)(z_{j+1} - a)^{n-k}(\tilde{z}_{j+1} - \tilde{a})^m \\ &\quad = f^s_{n,m}(z_{j+1} - a)^n(\tilde{z}_{j+1} - \tilde{a})^m C(n-k, m; z_{j+1}, z_j, a)(\Delta z_j), \end{aligned} \tag{2.8}$$

where $C(p, m; z_{j+1}, z_j, a) \in \mathbf{H}$ and $|C(p, m; z_{j+1}, z_j, a)| = 1$ for each $z_{j+1} \neq z_j$, $z_{j+1} \neq a$, for each p and m. From Eq. (2.8) it follows, that $|I(f, \gamma; P)| \leqslant \|f\|_{1,\omega,B(a,r',R',\mathbf{H})} v(\gamma; P)$, for each P, and inevitably

$$\bigl|I(f, \gamma; P) - I(f, \gamma; Q)\bigr| \leqslant w(\hat{f}; P)V(\gamma) \tag{2.9}$$

for each $Q \supset P$, where

$$w(\hat{f}; P) := \max_{(z,\zeta \in \gamma([c_j, c_{j+1}]))} \bigl\{\bigl\|\hat{f}(z, \tilde{z}) - \hat{f}(\zeta, \tilde{\zeta})\bigr\|: z_j = \gamma(c_j), c_j \in P\bigr\}, \tag{2.10}$$

$\|\hat{f}(z, \tilde{z}) - \hat{f}(\zeta, \tilde{\zeta})\| := \sup_{h\neq 0} |\hat{f}(z, \tilde{z}).h - \hat{f}(\zeta, \tilde{\zeta}).h|/|h|$. Since $\lim_{n\to\infty}(n)^{1/n} = 1$, then $\lim_P \omega(\hat{f}, P) = 0$. From $\lim_P w(\hat{f}; P) = 0$ the existence of $\lim_P I(f, \gamma; P)$ now follows.

2.7. Theorem. *Let γ be a rectifiable path in U, then the quaternion line integral has a continuous extension on the space $C^0_b(U, \mathbf{H})$ of bounded continuous functions $f : U \to \mathbf{H}$. This integral is an $\mathbf{R}$-linear and left-$\mathbf{H}$-linear functional on $C^0_b(U, \mathbf{H})$.*

Proof. Since γ is continuous and $[a, b]$ is compact, then there exists a compact canonical closed subset V in $\mathbf{H}$, that is, $\mathrm{cl}(\mathrm{Int}(V)) = V$, such that $\gamma([a, b]) \subset V \subset U$. Let $f \in C^0_b(U, \mathbf{H})$, then in view of the Stone–Weierstrass theorem for a function $F(v, w, x, y) = f \circ \sigma(v, w, x, y)$ and each $\delta > 0$ there exists a polynomial T such that $\|F - T\|_0 < \delta$, where $\|f\|_0 := \sup_{z\in U} |f(z)|$. This polynomial takes values in $\mathbf{H}$, hence it has the form: $T_{1,1} = \overline{T}_{2,2} = \alpha^{1,1}_{i_1,i_2,i_3,i_4} v^{i_1} w^{i_2} x^{i_3} y^{i_4}$ and $T_{1,2} = -\overline{T}_{2,1} = \alpha^{1,2}_{i_1,i_2,i_3,i_4} v^{i_1} w^{i_2} x^{i_3} y^{i_4}$, where summation is accomplished by repeated upper and lower indices. There are relations

$$\begin{aligned} &J^2 = K^2 = L^2 = -I, \qquad JK = -KJ = L, \\ &KL = -LK = J, \qquad LJ = -JL = K, \end{aligned}$$

consequently,

$$\begin{aligned} &zJ = vJ - wI - xL + yK, \qquad zK = vK + wL - xI - yJ, \\ &zL = vL - wK + xJ - yI, \end{aligned}$$

where $z = vI + wJ + xK + yL$ is in $\mathbf{H}$, v,w, x and y are in $\mathbf{R}$. Therefore,

$$\tilde{z} = vI - wJ - xK - yL, \qquad J\tilde{z} = vJ + wI - xL + yK,$$
$$K\tilde{z} = vK + wL + xI - yJ, \qquad L\tilde{z} = vL - wK + xJ + yI,$$

hence

$$(z+\tilde{z})/2 = vI, \qquad (J\tilde{z} - zJ)/2 = wI,$$
$$(K\tilde{z} - zK)/2 = xI, \qquad (L\tilde{z} - zL)/2 = yI.$$

From this it follows, that T can be expressed in z and $\tilde{z}$ such that

$$T = \big[\mathrm{Re}\big(\alpha^{1,1}_{i_1,i_2,i_3,i_4}\big)I + \mathrm{Im}\big(\alpha^{1,1}_{i_1,i_2,i_3,i_4}\big)J + \mathrm{Re}\big(\alpha^{1,2}_{i_1,i_2,i_3,i_4}\big)K + \mathrm{Im}\big(\alpha^{1,1}_{i_1,i_2,i_3,i_4}\big)L\big]$$
$$\times \big[(z+\tilde{z})/2\big]^{i_1}\big[(J\tilde{z} - zJ)/2\big]^{i_2}\big[(K\tilde{z} - zK)/2\big]^{i_3}\big[(L\tilde{z} - zL)/2\big]^{i_4}.$$

This polynomial can be rewritten in a form similar to f in §2.5 (see formulas (2.5i,ii)), since z and $\tilde{z}$ commute. Two variables $(z+\tilde{z})/2 = vI$ and $(z-\tilde{z})/2 = wJ + xK + yL$ commute for each $z \in \mathbf{H}$. Therefore, the $\mathbf{R}$-linear space of functions on U having decomposition (2.5i,ii) is dense in $C^0_b(U, \mathbf{H})$.

Consider a function $g(z, \tilde{z})$ on U, suppose that $q(z, \tilde{\zeta})$ is another function on U^2 such that $q(z, \tilde{\zeta})|_{z=\zeta} = g(z, \tilde{z})$. Let $q(z, \tilde{\zeta})$ be superdifferentiable by z for a fixed $\zeta \in U$, then $\partial q(z, \tilde{\zeta})/\partial z$ for $z = \zeta$ is denoted by $\partial g(z, \tilde{z})/\partial z$. Consider a space of all such that g on U for which $(\partial g(z, \tilde{z})/\partial z).S$ is a bounded continuous function on U for each $S \in \{I, J, K, L\}$, it is denoted by $C^1_b(U, \mathbf{H}) = C^{1,0}_b(U, \mathbf{H})$ and it is supplied with the norm $\|g\|_{C^1_b} := \|g\|_{C^0_b} + \sum_{S\in\{I,J,K,L\}} \|(\partial g(z, \tilde{z})/\partial z).S\|_{C^0_b}$, where $\|g\|_{C^0_b} := \sup_{z\in U} |g(z)|$. In view of Proposition 2.2 for each $g \in C^1_b(U, \mathbf{H})$ the corresponding function $Q(z, \zeta)$ satisfies condition (2.1) by z. This entails, that $\partial Q/\partial v$, $\partial Q/\partial w$, $\partial Q/\partial x$ and $\partial Q/\partial y$ are in $C^0_b(U^2, \mathbf{H})$ (see §2.1). Consequently, imposing the condition $z = \zeta$: $(\partial g/\partial z).J$, $(\partial g/\partial z).K$ and $(\partial g/\partial z).L$ are also continuous bounded functions, hence $(\partial g/\partial z).h \in C^0_b(U \times B(0, 0, 1, \mathbf{H}), \mathbf{H})$, where $h \in B(0, 0, 1, \mathbf{H})$. Therefore, there exists a positive constant C such that

$$\sup_{h\neq 0}\big|(\partial g/\partial z).h\big|/|h| \leqslant C \sum_{S\in\{I,J,K,L\}} \big\|(\partial g/\partial z).S\big\|_{C^0_b}, \tag{2.11}$$

since $h = v_h I + w_h J + x_h K + y_h L$ for each $h \in \mathbf{H}$ and $(\partial g/\partial z)$ is $\mathbf{R}$-linear and $(\partial g/\partial z).(h_1 + h_2) = (\partial g/\partial z).h_1 + (\partial g/\partial z).h_2$ for each h_1 and $h_2 \in \mathbf{H}$, where v_h, w_h, x_h and y_h are real numbers, $G(v, w, x, y) := g \circ \sigma(v, w, x, y)$ is Frechét differentiable on an open subset $U_\sigma \subset \mathbf{R}^4$ such that $\sigma(U_\sigma) = U$.

In §2.6 it was shown that the equation $(\partial g(z, \tilde{z})/\partial z).I = f(z, \tilde{z})$ has a solution in a class of quaternion locally $(z, \tilde{z})$-analytic functions on U. The subset $C^\omega_{(z,\tilde{z})}(U, \mathbf{H})$ is dense in the uniform space $C^0_b(U, \mathbf{H})$.

If $g = g^1 \dots g^l$ is a product of functions $g^s \in C^{1,0}_b(U, \mathbf{H})$, then $(\partial g/\partial z).h = \sum_{s=1}^l g^1(z, \tilde{z}) \dots g^{s-1}(z, \tilde{z})[(\partial g^s/\partial z).h]g^{s+1}(z, \tilde{z}) \dots g^l(z, \tilde{z})$ for each $h \in \mathbf{H}$. Consider the space $\widehat{C}^0_b(U, \mathbf{H}) := \{((\partial g/\partial z).I, (\partial g/\partial z).J, (\partial g/\partial z).K, (\partial g/\partial z).L)\colon g \in C^{1,0}_b(U, \mathbf{H})\}$. It has an embedding ξ into $C^0_b(U, \mathbf{H})$ and $\|g\|_{C^{1,0}_b} \geqslant \sum_{S\in\{I,J,K,L\}} \|(\partial g/\partial z).S\|_{C^0_b}$. In view of inequality (2.11) the completion of $\widehat{C}_b^0(U,\mathbf{H})$ relative to $\|*\|_{C_b^0(U,\mathbf{H})}$ coinsides with $C_b^0(U,\mathbf{H})$.

Let $\{f^j\colon j\in\mathbf{N}\}$ be a sequence of functions having decomposition (2.5) and converging to f in $C_b^0(U,\mathbf{H})$ relative to the metric $\rho(f,q):=\sup_{z\in U}|f(z,\tilde z)-q(z,\tilde z)|$ such that

$$f^j=\xi\bigl((\partial g^j/\partial z).I,(\partial g^j/\partial z).J,(\partial g^j/\partial z).K,(\partial g^j/\partial z).L\bigr)$$

for some $g^j\in C_b^{1,0}(U,\mathbf{H})$. Relative to this metric $C_b^0(U,\mathbf{H})$ is complete. We have the equality

$$\partial\left(\int_0^s F(v_0+\phi h_v,w_0+\phi h_w,x_0+\phi h_x,y_0+\phi h_y)\,d\phi\right)\Big/\partial s=F(v,w,x,y)$$

for each continuous function F on U_σ, where $v=v_0+sh_v$, $w=w_0+sh_w$, $x=x_0+sh_x$ and $y=y_0+sh_y$, $(v_0,w_0,x_0,y_0)+\phi(h_v,h_w,h_x,h_y)\in U_\sigma$ for each $\phi\in\mathbf{R}$ with $0\leqslant\phi\leqslant s$, h_v, h_w, h_x and $h_y\in\mathbf{R}^4$. Let z_0 be a marked point in V. There exists $R>0$ such that γ is contained in the interior of the parallelepiped $V:=\{z\in\mathbf{H}\colon z=vI+wJ+xK+yL, |v-v_0|\leqslant R, |w-w_0|\leqslant R, |x-x_0|\leqslant R, |y-y_0|\leqslant R\}$.

If V is not contained in U consider a continuous extension of a continuous function F from $V\cap U_0$ on V, where U_0 is a closed subset in U such that $\mathrm{Int}(U_0)\supset\gamma$ (about the theorem of a continuous extension see [11]). Therefore, suppose that F is given on V. Then the function $F_1(v,w,x,y):=\int_{v_0}^v\int_{w_0}^w\int_{x_0}^x\int_{y_0}^y F(v_1,w_1,x_1,y_1)\,dv_1\,dw_1\,dx_1\,dy_1$ is in $C^1(V,\mathbf{H})$ (with one sided derivatives on ∂V from inside V). Consider a foliation of V by three dimensional C^0-manifolds Υ_z such that $\Upsilon_z\cap\Upsilon_{z_1}=\emptyset$ for each $z\neq z_1$, where $z,z_1\in\gamma$, $\bigcup_{z\in\gamma}\Upsilon_z=V_1$, V_1 is a canonical closed subset in $\mathbf{H}$ such that $\gamma\subset V_1\subset V$. Choose this foliation such that to have decomposition of a Lebesgue measure dV into the product of measures $d\nu(z)$ along γ and $d\Upsilon_z$ for each $z\in\gamma$. In view of the Fubini theorem there exists $\int_V f(v_1,\dots,y_1)\,dV=\int_\gamma(\int_{\Upsilon_z}f(z,\tilde z)\,d\Upsilon_z)\,d\nu(z)$. If γ is a straight line segment then $\int_\gamma f(z,\tilde z)\,dz$ is in $L^1(\Upsilon,\mathbf{H})$. Let $U_{\mathbf{R}}$ be a real region in $\mathbf{R}^4$ corresponding to U in $\mathbf{H}$.

Consider the Sobolev space $W_2^s(U_{\mathbf{R}},\mathbf{R}^4)$ of functions $h:U_{\mathbf{R}}\to\mathbf{R}^4$ for which $D^\alpha h\in L^2(U_{\mathbf{R}},\mathbf{R}^4)$ for each $|\alpha|\leqslant s$, where $0\leqslant s\in\mathbf{Z}$. In view of Theorem 18.1.24 [19] (see also the notation there) if $A\in\Psi^m$ is a properly supported pseudodifferential elliptic operator of order m in the sence that the principal symbol $a\in S^m(T^*(X))/S^{m-1}(T^*(X))$ has an inverse in $S^{-m}(T^*(X))/S^{-m-1}(T^*(X))$, then one can find $B\in\Psi^{-m}$ properly supported such that $BA-I\in\Psi^{-\infty}$, $AB-I\in\Psi^{-\infty}$. One calls B a parametrix for A. In view of Proposition 18.1.21 [19] each $A\in\Psi^m$ can be written as a sum $A=A_1+A_0$, where $A_1\in\Psi^m$ is properly supported and the kernel of A_0 is in C^∞. In particular we can take a pseudodifferential operator with the principal symbol $a(x,\xi)=(b+|\xi|^2)^{s/2}$, where $b>0$ is a constant and $s\in\mathbf{Z}$, which corresponds to $b+\Delta$ for $s=1$ up to minor terms, where $\Delta=\nabla^2$ is the Laplacian (see also Theorem 3.2.13 [16] about its parametrix family). For estimates of a solution there may be also applied Theorem 3.3.2 and Corollary 3.3.3 [16] concerning parabolic pseudodifferential equations for our particular case corresponding to $(\partial g/\partial z).I=f$ rewritten in real variables.

Due to the Sobolev theorem (see [36,37]) there exists an embedding of the Sobolev space $W_2^3(V,\mathbf{H})$ into $C^0(V,\mathbf{H})$ such that $\|g\|_{C^0}\leqslant C\|g\|_{W_2^3}$ for each $g\in W_2^3$, where C

is a positive constant independent of g. If $h \in W_2^{k+1}(V, \mathbf{H})$, then $\partial h/\partial b_j \in W_2^k(V, \mathbf{H})$ for each $k \in \mathbf{N}$ and in particular for $k = 3$ and each $j = 1, \ldots, 4$ (see [36]). On the other hand $\|h\|_{L^2(V,\mathbf{H})} \leqslant \|h\|_{C^0(V,\mathbf{H})}(2R)^2$ for each $h \in L^2(V, \mathbf{H})$. Therefore, $\|A^{-k}h\|_{W_2^k(V,\mathbf{H})} \leqslant C\|h\|_{C^0(V,\mathbf{H})}(2R)^{k+2}$ for each $k \in \mathbf{N}$, where $C = \text{const} > 0$, A is an elliptic pseudodifferential operator such that A^2 corresponds to $(1 + \Delta)$. From Eqs. (2.6), (2.7) and inequality (2.11) it follows, that

$$\bigl|I(f - q, \gamma; P)\bigr| \leqslant \rho(f, q)V(\gamma)C_1 \exp\bigl(C_2R^6\bigr) \tag{2.12}$$

for each partitioning P, where C_1 and C_2 are positive constants independent of R, f and q (for this estimate the Gronwall Lemma is used, see, for example, Section 3.3.1 [2]). In view of formulas (2.9), (2.10) $\{\int_\gamma f^j(z, \tilde{z})\,dz\colon\ j \in \mathbf{N}\}$ is a Cauchy sequence in $\mathbf{H}$ and the latter is complete as the metric space. Therefore, there exists $\lim_j \lim_P I(f^j, \gamma; P) = \lim_j \int_\gamma f^j(z, \tilde{z})\,dz$, which we denote by $\int_\gamma f(z, \tilde{z})\,dz$. As in §2.6 we get that all solutions g differ on quaternion constants on each connected component of U, consequently, the functional $\int_\gamma$ is uniquely defined on $C_b^0(U, \mathbf{H})$. The functional $\int_\gamma : C_b^0(U, \mathbf{H}) \to \mathbf{H}$ is continuous due to formula (2.12) and evidently it is $\mathbf{R}$-linear, since $\lambda z = z\lambda$ for each $\lambda \in \mathbf{R}$ and each $z \in \mathbf{H}$, that is,

$$\int_\gamma \bigl(\lambda_1 f_1(z, \tilde{z}) + \lambda_2 f_2(z, \tilde{z})\bigr)\,dz = \int_\gamma \bigl(f_1(z, \tilde{z})\lambda_1 + f_2(z, \tilde{z})\lambda_2\bigr)\,dz$$
$$= \lambda_1 \int_\gamma f_1(z, \tilde{z})\,dz + \lambda_2 \int_\gamma f_2(z, \tilde{z})\,dz$$

for each λ_1 and $\lambda_2 \in \mathbf{R}$, f_1 and $f_2 \in C_b^0(U, \mathbf{H})$. Moreover, it is left-$\mathbf{H}$-linear, that is,

$$\int_\gamma \bigl(\lambda_1 f_1(z, \tilde{z}) + \lambda_2 f_2(z, \tilde{z})\bigr)\,dz = \lambda_1 \int_\gamma f_1(z, \tilde{z})\,dz + \lambda_2 \int_\gamma f_2(z, \tilde{z})\,dz$$

for each λ_1 and $\lambda_2 \in \mathbf{H}$, f_1 and $f_2 \in C_b^0(U, \mathbf{H})$, since $I(f, \gamma; P)$ is left-$\mathbf{H}$-linear.

2.8. Remark. Let η be a differential form on open subset U of the Euclidean space $\mathbf{R}^{4m}$ with values in $\mathbf{H}$, then it can be written as

$$\eta = \sum_\Upsilon \eta_\Upsilon\, db^{\wedge\Upsilon}, \tag{2.13}$$

where $b = ({}^1b, \ldots, {}^mb) \in \mathbf{R}^{4m}$, ${}^jb = ({}^jb_1, \ldots, {}^jb_4)$, ${}^jb_i \in \mathbf{R}$, $\eta_\Upsilon = \eta_\Upsilon(b) : \mathbf{R}^{4m} \to \mathbf{H}$ are s times continuously differentiable $\mathbf{H}$-valued functions with $s \in \mathbf{N}$, $\Upsilon = (\Upsilon(1), \ldots, \Upsilon(m))$, $\Upsilon(j) = (\Upsilon(j, 1), \ldots, \Upsilon(j, 4)) \in \mathbf{N}^4$ for each j, $db^{\wedge\Upsilon} = d^1b^{\wedge\Upsilon(1)} \wedge \cdots \wedge d^mb^{\wedge\Upsilon(m)}$, $d^jb^{\wedge\Upsilon(j)} = d^jb_1^{\Upsilon(j,1)} \wedge \cdots \wedge d^jb_4^{\Upsilon(j,4)}$, where $d^jb_i^0 = 1$, $d^jb_i^1 = d^jb_i$, $d^jb_i^k = 0$ for each $k > 1$. If $s \geqslant 1$, then there is defined an (external) differential

$$d\eta = \sum_{\Upsilon,(j,i)} (\partial\eta_\Upsilon/\partial\,{}^jb_i)(-1)^{\alpha(j,i)} db^{\wedge(\Upsilon + e(j,i))},$$

where $e(j,i) = (0,\dots,0,1,0,\dots,0)$ with 1 on the $4(j-1)+i$th place,

$$\alpha(j,i) = \left(\sum_{l=1}^{j-1}\sum_{k=1}^{4}\Upsilon(l,k)\right) + \sum_{k=1}^{i-1}\Upsilon(j,k).$$

These differential forms have matrix structure themselves. Consider basic matrices $S = ({}^1S,\dots,{}^mS)$ and their ordered product $S^{\to\Upsilon} := {}^1S^{\to\Upsilon(1)}\dots{}^mS^{\to\Upsilon(m)}$, where ${}^jS = ({}^jS_1,\dots,{}^jS_4) = (I,J,K,L)$, ${}^jS^{\to\Upsilon(j)} = J^{\Upsilon(j,2)}K^{\Upsilon(j,3)}L^{\Upsilon(j,4)}$, $S^0 = I$. Then Eq. (2.13) can be rewritten in the form:

$$\eta = \sum_{\Upsilon}\xi_\Upsilon\, d(Sb)^{\wedge\Upsilon}, \tag{2.14}$$

where $Sb = ({}^1S_1{}^1b_1,\dots,{}^mS_m{}^mb_m) \in \mathbf{H}^{4m}$, $d{}^jS_k{}^jb_k = {}^jS_k d{}^jb_k$, $\xi_\Upsilon := \eta_\Upsilon(S^{\to\Upsilon})^{-1}$. Relative to the external product $I d{}^jb_1$ anticommutes with others basic differential 1-forms ${}^jS_k d{}^jb_k$; for $k = 2,3,4$ these forms commute with each other relative to the external product. This means that the algebra of quaternion differential forms is graded relative to the external product.

From §2.7 it follows, that $(dz + d\tilde{z})/2 = I\,dv$, $J\,dw = -(d\tilde{z} + J(dz)J)/2$, $K\,dx = -(d\tilde{z} + K(dz)K)/2$, $L\,dy = -(d\tilde{z} + L(dz)L)/2$. Therefore, the right side of Eq. (2.14) can be rewritten with d^jz, $d^j\tilde{z}$, $J\,d^jz\,J$, $K\,d^jzK$ and $L\,d^jzL$ on the right side. From the latter 5 differential 1-forms 4 linearly independent ones can be chosen, since summing these forms we have: $dz = -2\,d\tilde{z} - J\,dz\,J - K\,dz\,K - L\,dz\,L$, hence $L\,dz\,L = -(dz + 2\,d\tilde{z} + J\,dz\,J + K\,dz\,K)$. These 1-forms do neither commute nor anticommute, since they are not pure elements of the graded algebra. For example,

$$(dz\wedge dz)_{1,1} = \overline{(dz\wedge dz)}_{2,2} = -du\wedge d\bar{u},$$

$$(dz\wedge dz)_{1,2} = -\overline{(dz\wedge dz)}_{2,1} = dt\wedge du - d\bar{t}\wedge du;$$

$$(dz\wedge d\tilde{z})_{1,1} = \overline{(dz\wedge d\tilde{z})}_{2,2} = dt\wedge d\bar{t} + du\wedge d\bar{u},$$

$$(dz\wedge d\tilde{z})_{1,2} = -\overline{(dz\wedge d\tilde{z})}_{2,1} = -2\,dt\wedge du;$$

$$(d\tilde{z}\wedge dz)_{1,1} = \overline{(d\tilde{z}\wedge dz)}_{2,2} = d\bar{t}\wedge dt + du\wedge d\bar{u},$$

$$(d\tilde{z}\wedge dz)_{1,2} = -\overline{(d\tilde{z}\wedge dz)}_{2,1} = 2\,d\bar{t}\wedge du;$$

$$(dz\wedge J\,dz\,J)_{1,1} = \overline{(dz\wedge J\,dz\,J)}_{2,2} = -du\wedge d\bar{u},$$

$$(dz\wedge J\,dz\,J)_{1,2} = -\overline{(dz\wedge J\,dz\,J)}_{2,1} = dt\wedge du + d\bar{t}\wedge du;$$

$$(J\,dz\,J\wedge dz)_{1,1} = \overline{(JdzJ\wedge dz)}_{2,2} = -du\wedge d\bar{u},$$

$$(J\,dz\,J\wedge dz)_{1,2} = -\overline{(J\,dz\,J\wedge dz)}_{2,1} = -dt\wedge du - d\bar{t}\wedge du;$$

$$(dz\wedge K\,dz\,K)_{1,1} = \overline{(dz\wedge K\,dz\,K)}_{2,2} = -dt\wedge d\bar{t},$$

$$(dz\wedge K\,dz\,K)_{1,2} = -\overline{(dz\wedge K\,dz\,K)}_{2,1} = -dt\wedge d\bar{u} + dt\wedge du;$$

$$(K\,dz\,K\wedge dz)_{1,1} = \overline{(K\,dz\,K\wedge dz)}_{2,2} = dt\wedge d\bar{t},$$

$$(K\,dz\,K\wedge dz)_{1,2} = -\overline{(K\,dz\,K\wedge dz)}_{2,1} = -d\bar{t}\wedge du + d\bar{t}\wedge d\bar{u}.$$

On the other hand Eq. (2.13) can be rewritten using the identities: $(dz + d\tilde{z})/2 = I\,dv$, $I\,dw = (J\,d\tilde{z} - dz\,J)/2$, $I\,dx = (K\,d\tilde{z} - dz\,K)/2$, $I\,dy = (L\,d\tilde{z} - dz\,L)/2$, where $dz = \begin{pmatrix} dt & du \\ -d\bar{u} & d\bar{t} \end{pmatrix}$ in 2×2 complex matrix notation.

Consider a C^1-function f on U with values in $\mathbf{H}$, then

$$D_{A\,{}^j\tilde{z}} f = \bigl(\partial f/\partial(A^j\tilde{z})\bigr).A^j\tilde{z} = (\partial f/\partial\,{}^j\tilde{z}).(A^{-1}A^j\tilde{z}) = D_{{}^j\tilde{z}} f$$

for each $A \in \mathbf{H}$, also

$$D_{{}^jzA} f = \bigl(\partial f/\partial({}^jzA)\bigr).{}^jzA = (\partial f/\partial\,{}^jz).({}^jz)AA^{-1} = D_{{}^jz} f$$

for each $A \in \mathbf{H}$, where $D_h f := (\partial f/\partial h).h$. We apply this also in particular to Iv, wJ, xK and yL.

There is the standard embedding of the algebra of complex $n \times n$ matrices A into the algebra of real $2n \times 2n$ matrices B such that in its block form $B_{1,1} = \mathrm{Re}(A)$, $B_{2,2} = \mathrm{Re}(A)$, $B_{1,2} = \mathrm{Im}(A)$, $B_{2,1} = -\mathrm{Im}(A)$, where $B_{i,j}$ are $n \times n$ blocks. Therefore, quaternion differential forms can be embedded into the algebra of differential forms over the algebra of real 4×4 matrices. This shows, that the exterior differentiation operator ${}_{\mathbf{H}}d$ for $\mathbf{H}$-valued differential forms over $\mathbf{H}$ and that of for their real matrix realization ${}_{\mathbf{R}}d$ coincide and their common operator is denoted by d. Consider the equality

$$\begin{aligned}(\partial\eta_\Upsilon/\partial\,{}^jb^l)\,{}^jb^l \wedge db^\Upsilon &= \bigl[(\partial\eta_\Upsilon/\partial\,{}^jz).(\partial\,{}^jz/\partial\,{}^jb^l)\bigr]\,{}^jb^l \wedge db^\Upsilon \\ &\quad + \bigl[(\partial\eta_\Upsilon/\partial\,{}^j\tilde{z}).(\partial\,{}^j\tilde{z}/\partial\,{}^jb_l)\bigr]\,{}^jb^l \wedge db^\Upsilon.\end{aligned}$$

Applying it to $l = 1, \ldots, 4$ and summing left anf right parts of these equalities we get $d\eta(z,\tilde{z}) = ((\partial\eta/\partial z).d^jz) \wedge db^\Upsilon + ((\partial\eta/\partial\tilde{z}).d^j\tilde{z}) \wedge db^\Upsilon$, hence the external differentiation can be presented in the form

$$d = \partial_z + \partial_{\tilde{z}}, \tag{2.15}$$

where ∂_z and $\partial_{\tilde{z}}$ are external differentiations by variables z and $\tilde{z}$ respectively.

Certainly for an external product $\eta_1 \wedge \eta_2$ there is not (in general) an $\lambda \in \mathbf{H}$ such that $\lambda\eta_2 \wedge \eta_1 = \eta_1 \wedge \eta_2$, if η_1 and η_2 are not pure elements (even or odd) of the graded algebra.

2.9. Definition. A Hausdorff topological space X is said to be n-connected for $n \geqslant 0$ if each continuous map $f : S^k \to X$ from the k-dimensional real unit sphere into X has a continuous extension over $\mathbf{R}^{k+1}$ for each $k \leqslant n$. A 1-connected space is also said to be simply connected.

2.10. Remark. In accordance with Theorem 1.6.7 [35] a space X is n-connected if and only if it is path connected and $\pi_k(X, x)$ is trivial for every base point $x \in X$ and each k such that $1 \leqslant k \leqslant n$.

Denote by $\mathrm{Int}(U)$ an interior of a subset U in a topological space X, by $\mathrm{cl}(U) = \overline{U}$ a closure of U in X. For a subset U in $\mathbf{H}$, let $\pi_{1,t}(U) := \{u\colon z \in U,$ where $z_{1,1} = \bar{z}_{2,2} = t, z_{1,2} = -\bar{z}_{2,1} = u\}$ for a given $t \in \mathbf{C}$; $\pi_{2,u}(U) := \{t\colon z \in U,$ where $z_{1,1} = \bar{z}_{2,2} = t, z_{1,2} = -\bar{z}_{2,1} = u\}$ for a given $u \in \mathbf{C}$, that is, geometrically $\pi_{1,t}(U)$ and $\pi_{2,u}(U)$ are projections on complex planes $\mathbf{C}_2$ and $\mathbf{C}_1$ of intersections of U with planes $\tilde{\pi}_{1,t} \ni \begin{pmatrix} t & 0 \\ 0 & \bar{t} \end{pmatrix}$ and $\tilde{\pi}_{2,u} \ni \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}$ parallel to $\mathbf{C}_2$ and $\mathbf{C}_1$ respectively.

2.11. Theorem. *Let U be a domain in $\mathbf{H}$ such that $\emptyset \ne \mathrm{Int}(U) \subset U \subset \mathrm{cl}(\mathrm{Int}(U))$ and U is 3-connected; $\pi_{1,t}(U)$ and $\pi_{2,u}(U)$ are simply connected in $\mathbf{C}$ for each t and $u \in \mathbf{C}$ for which there exists $z \in U$. Let f be a bounded continuous function from U into $\mathbf{H}$ which satisfies condition (2.1) on an open domain W such that $W \supset U$. Then for each rectifiable closed path γ in U a quaternion line integral $\int_\gamma f(z)\,dz = 0$ is equal to zero.*

Proof. For a path γ there exists a compact canonical closed subset in $\mathbf{H}$: $W \subset \mathrm{Int}(U)$ such that $\gamma([0,1]) \subset W$, since γ is rectifiable and $\mathbf{H}$ is locally compact. In view of Theorem 2.7 for each sequence of functions $f_n \in C^{1,1}(U,\mathbf{H})$ converging to f in $C^0_b(U,\mathbf{H})$ such that $f_n(z) = (\partial g_n(z,\tilde z)/\partial \tilde z).I$ with $g_n \in C^{2,1}(U,\mathbf{H})$ and each sequence of paths $\gamma_n : [0,1] \to U$ C^3-continuously differentiable and converging to γ relative to the total variation $V(\gamma - \gamma_n)$ there exists $\lim_n \int_{\gamma_n} f_n(z)\,dz = \int_\gamma f(z)\,dz$. Therefore, it is sufficient to consider the case of $f \in C^{1,1}(U,\mathbf{H})$ such that $f(z) = (\partial g(z,\tilde z)/\partial \tilde z).I$ on U, and continuously differentiable γ. Denote the integral $\int_\gamma f(z)\,dz$ by Q, then $Q = 0$ if and only if $Q\widetilde{Q} = 0$. On the other hand, $\widetilde{Q} = \lim_P \tilde I(f,\gamma;P)$ and $Q = \int_\gamma (\partial g(z,\tilde z)/\partial z).dz$, hence $\widetilde{Q} = \int_\gamma d\tilde z(\partial_L \tilde g(z,\tilde z)/\partial \tilde z) = \int_{\tilde\gamma} dz(\partial_L \tilde g(z,\tilde z)/\partial z)$, where $(\partial_L q(z,\tilde z)/\partial z)$ is the left derivative, $\tilde q(z,\tilde z) := \tilde a$, where $a = q(z,\tilde z)$. We can write this integral in the form $Q = \int_0^1 (\partial g(z,\tilde z)/\partial z).\gamma'(t)\,dt$. Denoting components of $(\partial g(z,\tilde z)/\partial z)$ as the complex 2×2 matrix with entries $\hat f_{i,j}$ with i and $j \in \{1,2\}$, we get

$$Q_{1,1} = \overline{Q}_{2,2} = \int_0^1 \hat f_{1,1}\big(t(\theta),u(\theta)\big)\gamma'_{1,1}(\theta)\,d\theta - \int_0^1 \hat f_{1,2}\big(t(\theta),u(\theta)\big)\bar\gamma'_{1,2}(\theta)\,d\theta;$$

$$Q_{1,2} = -\overline{Q}_{2,1} = \int_0^1 \hat f_{1,1}\big(t(\theta),u(\theta)\big)\gamma'_{1,2}(\theta)\,d\theta + \int_0^1 \hat f_{1,2}\big(t(\theta),u(\theta)\big)\bar\gamma'_{1,1}(\theta)\,d\theta,$$

where $t(\theta) = \gamma_{1,1}(\theta)$, $u(\theta) = \gamma_{1,2}(\theta)$. Evidently $\gamma(1) = \gamma(0)$ if and only if two equalities are satisfied $\gamma_{1,1}(1) = \gamma_{1,1}(0)$ and $\gamma_{1,2}(1) = \gamma_{1,2}(0)$. That is, paths $\gamma_{1,1}$ and $\gamma_{1,2}$ are closed in the corresponding complex planes $\mathbf{C}_1 = \mathbf{C}$ and $\mathbf{C}_2 = \mathbf{C}$ embedded into $\mathbf{H}$. In view of the conditions of the theorem, $\gamma_{1,1}$ for each u and $\gamma_{1,2}$ for each t corresponding to $z \in \mathbf{H}$ are contained in subsets $\pi_{2,u}(U)$ and $\pi_{1,t}(U)$ respectively which are simply connected. Hence subsets $\Omega_{1,1}$ and $\Omega_{1,2}$ exist in $\mathbf{C}_1$ and in $\mathbf{C}_2$ such that $\partial\Omega_{1,1} = \gamma_{1,1}$ and $\partial\Omega_{1,2} = \gamma_{1,2}$ and $\Omega_{1,1} \subset \pi_{2,u}(U)$ and $\Omega_{1,2} \subset \pi_{1,t}(U)$ for each t and u corresponding to $z \in U$ such that $\Omega_{1,1}$ and $\Omega_{1,2}$ are simply connected in $\mathbf{C}_1$ and in $\mathbf{C}_2$ respectively. It may easily be seen, taking into account §2.8, that this integral can be considered as the integral of a real differential 1-form along the path γ in $\mathbf{R}^4$. To these integrals Q and $\widetilde{Q}$ the classical (generalized) Stokes theorem can be applied (see Theorem V.1.1 [39]). In view of the Hurewicz isomorphism theorem (see §7.5.4 [35]) $H_q(U,x) = 0$ for each $x \in U$ and each $q < 4$, hence $H^l(U,x) = 0$ for each $l \geqslant 1$.

If $f : Y \to V$ is continuous, then $r \circ f : Y \to \Omega$ is continuous, if f is onto V, then $r \circ f$ is onto Ω, where $r : V \to \Omega$ is a retraction, V, Y and Ω are topological spaces. The topological space U is metrizable, hence for each closed subset Ω in U there exists a canonical closed subset $V \subset U$ such that $V \supset \Omega$ and Ω is a retraction of V, that is, there exists a continuous mapping $r : V \to \Omega$, $r(z) = z$ for each $z \in \Omega$ (see [11] and Theorem 7.1 [21]). Therefore, if V is a 3-connected canonical closed subset of U and Ω is a two dimensional C^0-manifold such that Ω is a retraction of V, then Ω is simply connected, since each continuous mapping $f : S^k \to \Omega$ with $k \leqslant 1$ has a continuous extension $f : \mathbf{R}^{k+1} \to V$ and $r \circ f : \mathbf{R}^{k+1} \to \Omega$ is also a continuous extension of f from S^k on $\mathbf{R}^{k+1}$.

From 3-connectedness of U it follows, that there are two dimensional real differentiable manifolds Ω_j contained in U such that $\partial\Omega_j = \gamma$. This may be lightly seen by considering partitions Z_n of U by $S^n_{l,k} \cap U$ and taking $n \to \infty$, where $S^n_{l,k}$ are parallelepipeds in $\mathbf{H}$ with ribs of length n^{-1}, l, k and $n \in \mathbf{N}$, two dimensional faces ${}_1S^n_l$ and ${}_2S^n_k$ of $S^n_{l,k} = {}_1S^n_l \times {}_2S^n_k$ are parallel to $\mathbf{C}_1$ or $\mathbf{C}_2$ respectively such that there exists a sequence of paths γ_n converging to γ relative to $|*|_{\mathbf{H}}$ and a sequence of (continuous) two dimensional C^0-manifolds Ω^n_j with $\partial\Omega^n_j = \gamma^n$, $\Omega^n_j \subset \bigcup_{l,k}[(\partial\,{}_1S^n_l) \times (\partial\,{}_2S^n_k)]$. Choose Ω_1 and Ω_2 orientable and of class C^3 as Riemann manifolds such that taking their projections on $\mathbf{C}_1$ and $\mathbf{C}_2$ the corresponding paths $\gamma_{1,1}$, $\gamma_{1,2}$ and regions $\Omega^j_{1,1}$ and $\Omega^j_{1,2}$ in $\mathbf{C}_1$ and in $\mathbf{C}_2$ satisfy the conditions mentioned above in this proof, where $j = 1$ or $j = 2$ for Ω_1 or Ω_2 respectively. In this situation the abstract Stokes theorem is applicable. In view of the Fubini theorem we obtain $Q\widetilde{Q} = \int_{\Omega_1\times\Omega_2} \eta(z_1, \tilde z_1) \wedge \tilde\eta(z_2, \tilde z_2)$, where $\eta = d((\partial g(z, \tilde z)/\partial z).dz)$ is the 2-differential form, $z_1 \in \Omega_1$, $z_2 \in \Omega_2$. The function g is in $C^{2,1}(U, \mathbf{H})$ and $p\tilde p = \tilde p p$ for each $p \in \mathbf{H}$, hence $(\partial^2 g(z, \tilde z)/\partial z\partial\tilde z).(h_1, h_2) := (\partial[(\partial g(z, \tilde z)/\partial z).h_1]/\partial\tilde z).h_2 = (\partial^2 g(z, \tilde z)/\partial\tilde z\partial z).(h_2, h_1)$ for each h_1 and h_2 in $\mathbf{H}$, in particular for $h_1 = h_2 = I$. Due to condition (2.1) there is the equality $(\partial f(z, \tilde z)/\partial\tilde z) = 0$, hence $(\partial^2 g(z, \tilde z)/\partial z\partial\tilde z) = (\partial^2 g(z, \tilde z)/\partial\tilde z\partial z) = 0$ and inevitably $g = p(z) + q(\tilde z)$, where functions p and q are of class $C^{2,1}$ such that $\partial p/\partial\tilde z = 0$ and $\partial q(\tilde z)/\partial z = 0$. This is evident in the class of polynomial functions, that is dense in $C^{2,1}(W, \mathbf{H})$ for each compact canonical closed set W contained in $\mathbf{H}$ such that $W \subset U$. Hence it is true in $C^{2,1}$ also. Therefore, $\partial_z g(z, \tilde z) = \partial_z p(z)$. This means, that $\partial_z g(z, \tilde z) = dp(z)$, since $\partial_{\tilde z} p = 0$ (see Eq. (2.15)). Then $\eta = d^2(p) = 0$, since $d = {}_{\mathbf{R}}d$ and $d^2 = 0$. Therefore, $Q = 0$.

2.12. Definitions. A continuous function on a domain U in $\mathbf{H}$ such that $\emptyset \neq \mathrm{Int}(U) \subset U \subset \mathrm{cl}(\mathrm{Int}(U))$ and $\int_\gamma f\,dz = 0$ for each rectifiable closed path γ in U, then f is called quaternion integral holomorphic (on U).

If f is a superdifferentiable function on U such that it satisfies condition (2.1), then it is called quaternion holomorphic (on U).

Let $B(a, 0, R, \mathbf{H})$ be a disk in $\mathbf{H}^n$, then the completion of the space of all functions having decomposition (2.5i,ii) with respect to z only, and with $n_0 \geqslant 0$ relative to the norm $\|*\|_\omega$ from §2.6, is denoted by $C^\omega_z(B(a, 0, R, \mathbf{H}), \mathbf{H})$. It is $\mathbf{R}$-linear. Then $C^\omega_z(U, \mathbf{H})$ denotes the space of all continuous functions f on U with values in $\mathbf{H}$ such that for each $a \in U$ there are $R = R(f) > 0$ and $g \in C^\omega_z(B(a, 0, R, \mathbf{H}), \mathbf{H})$ with the restriction $g|_U = f$. If $f \in C^\omega_z(U, \mathbf{H})$, then it is called quaternion locally z-analytic (on U).

2.13. Corollary. *Let f be a quaternion holomorphic function on an open 3-connected domain U in $\mathbf{H}$ such that $\pi_{1,t}(U)$ and $\pi_{2,u}(U)$ are simply connected in $\mathbf{C}_1$ for each t and $u \in \mathbf{C}$ for which there exists $z \in U$, then f is quaternion integral holomorphic.*

This follows immediately from Theorem 2.11.

2.14. Definition. Let U be a subset of $\mathbf{H}$ and $\gamma_0:[0,1]\to\mathbf{H}$ and $\gamma_1:[0,1]\to\mathbf{H}$ be two continuous paths. Then γ_0 and γ_1 are called homotopic relative to U, if there exists a continuous mapping $\gamma:[0,1]^2\to U$ such that $\gamma([0,1],[0,1])\subset U$ and $\gamma(t,0)=\gamma_0(t)$ and $\gamma(t,1)=\gamma_1(t)$ for each $t\in[0,1]$.

2.15. Theorem. *Let W be an open subset in $\mathbf{H}$ and f be a quaternion holomorphic function on W with values in $\mathbf{H}$. Suppose that there are two rectifiable paths γ_0 and γ_1 in W with common initial and final points ($\gamma_0(0)=\gamma_1(0)$ and $\gamma_0(1)=\gamma_1(1)$) homotopic relative to U, where U is a 3-connected subset in W such that $\pi_{1,t}(U)$ and $\pi_{2,u}(U)$ are simply connected in $\mathbf{C}$ for each t and $u\in\mathbf{C}$ for which there exists $z\in\mathbf{U}$. Then $\int_{\gamma_0} f\,dz=\int_{\gamma_1} f\,dz$.*

Proof. A homotopy of γ_0 with γ_1 realtive to U implies homotopies of $(\gamma_0)_{1,j}$ with $(\gamma_1)_{1,j}$ relative to $\pi_{1,t}(U)$ and $\pi_{2,u}(U)$ in $\mathbf{C}$ for $j=1$ and $j=2$ respectively for each t and $u\in\mathbf{C}$ for which there exists $z\in U$. Consider a path ζ such that $\zeta(t)=\gamma_0(2t)$ for each $0\leqslant t\leqslant 1/2$ and $\zeta(t)=\gamma_1(2-2t)$ for each $1/2\leqslant t\leqslant 1$. Then ζ is a closed path contained in a U. In view of Theorem 2.11 $\int_\zeta (z)\,dz=0$. On the other hand, $\int_\zeta f(z)\,dz=\int_{\gamma_0} f(z)\,dz-\int_{\gamma_1} f(z)\,dz$, consequently, $\int_{\gamma_0} f(z)\,dz=\int_{\gamma_1} f(z)\,dz$.

2.16. Theorem. *Let f be a quaternion locally z-analytic function on an open domain U in $\mathbf{H}^n$, then f is quaternion holomorphic on U.*

Proof. From the definition of the superdifferential we get $(\partial z^n/\partial z).h=\sum_{k=0}^{n-1} z^k h z^{n-k-1}$ and $\partial z^n/\partial\tilde{z}=0$. Using the formula of the superdifferential for a product of functions, from §2.7 we obtain, that each f of the form (2.5i, ii) is superdifferentiable by z when $n_0\geqslant 0$ and $m=0$ in (2.5ii) and hence satisfies condition (2.1). Using the norm $\|*\|_\omega$-convergence of series with respect to z for a given $f\in C^\omega(U,\mathbf{H})$ we obtain for each $a\in U$, that there exists its neighbourhood W, where f is quaternion holomorphic, hence f is quaternion holomorphic on U.

2.17. Note. In the next section it is shown that a quaternion holomorphic function is infinite differentiable; furthermore, under suitable conditions equivalences between the properties of quaternion holomorphicity, quaternion integral holomorphicity and quaternion local z-analyticity, will be proved there too. Integral (2.6) may be generalized for a continuous function $q:U\to\mathbf{H}$ such that $V(q\circ\gamma)<\infty$. Substituting Δz_k on $q(z_{k+1})-q(z_k)=:\Delta q_k$ in (2.7) we get

$$\int_\gamma f(z,\tilde{z})\,dq(z):=\lim_P I(f,q\circ\gamma;P), \tag{2.6'}$$

where

$$I(f,q\circ\gamma;P)=\sum_{k=0}^{q-1}\hat{f}(z_{k+1},\tilde{z}_{k+1}).(\Delta q_k). \tag{2.7'}$$

In particular, if $\gamma \in C^1$ and q is quaternion holomorphic on U, also $f(z,\tilde{z}) = (\partial g/\partial z).I$, where $g \in C^{1,0}(U,\mathbf{H})$, then

$$\int_\gamma f(z,\tilde{z})\,dq(z) = \int_0^1 (\partial g/\partial z).\big((D_z q(z)|_{z=\gamma(s)}).\gamma'(s)\big)\,ds$$

and $V(\gamma) \leqslant \int_0^1 |\gamma'(s)|\,ds$.

Let $f : U \to \mathbf{H}$, where U is an open subset of $\mathbf{H}^n$. If there exists a quaternion holomorphic function $g : U \to \mathbf{H}$ such that $g'(z).I = f(z)$ for each $z \in U$, then g is called a primitive of f.

2.18. Proposition. *Let U be an open connected subset of $\mathbf{H}^n$ and g be a primitive of f on U, then a set of all primitives of f is*: $\{h\colon\ h = g + C,\ C = \text{const} \in \mathbf{H}\}$.

Proof. Suppose $h'(z) = 0$ for each $z \in U$, then consider $q(s) := h((1-s)a + sz)$ for each $s \in [0,r]$, where a is a marked point in U and $B(a,r,\mathbf{H})$ is a ball contained in U, $r > 0$, $z \in B(a,r,\mathbf{H})$. Then q is correctly defined and $q(0) = q(1)$. Therefore, the set $V := \{z \in U\colon\ h(z) = h(a)\}$ is open in U, since with each point a it contains its neighbourhood. On the other hand, it is closed due continuity of h, hence $V = U$, since U is connected, consequently, $h = \text{const}$ on U.

3. Meromorphic functions and their residues

At first we define and describe the exponential and the logarithmic functions of quaternion variables and then apply them to the investigation of quaternionic residues.

3.1. Note and Definition. Let $z \in \mathbf{H}$, then

$$\exp(z) := \sum_{n=0}^{\infty} z^n/n!. \tag{3.1}$$

This definition is correct, since real numbers commute with quaternions. If $|z| \leqslant R < \infty$, then the series (3.1) converges, since $|\exp(z)| \leqslant \sum_{n=0}^{\infty} |z^n/n!| \leqslant \exp(R) < \infty$. Therefore, exp is the function defined on $\mathbf{H}$ with values in $\mathbf{H}$. The restriction of exp on the subset $\mathbf{Q_d} := \{z\colon\ z \in \mathbf{H},\ z = \begin{pmatrix} t & 0 \\ 0 & \tilde{t}\end{pmatrix},\ t \in \mathbf{C}\}$ is commutative, but in general two quaternions z_1 and z_2 do not commute and on $\mathbf{H}^2$ the function $\exp(z_1 + z_2)$ does not coincide with $\exp(z_1)\exp(z_2)$.

3.2. Proposition. *Let $z \in \mathbf{H}$ be written in the form $z = vI + s(wJ + xK + yL)$ with real v, w, x, y and s with $w^2 + x^2 + y^2 \neq 0$, then*

$$\exp(z) = \exp(v)\left\{\cos(s\phi)I + i\big(\sin(s\phi)/\phi\big)\begin{pmatrix} w & (y - ix) \\ (y + ix) & -w \end{pmatrix}\right\}, \tag{3.2}$$

where $\phi := (w^2 + x^2 + y^2)^{1/2}$.

Proof. Since the unit matrix I commutes with J, K and L, then $\exp(vI + z) = \exp(v)\exp(z)$ for each $v \in \mathbf{R}$ and each $z \in \mathbf{H}$. Consider $z = s(w, J + xK + yL)$ with real s, w, x and y such that $w^2 + x^2 + y^2 = 1$, then

$$z = is\begin{pmatrix} w & (y-ix) \\ (y+ix) & -w \end{pmatrix},$$

where $i = (-1)^{1/2}$. Denote $y - ix =: m$, then

$$\begin{pmatrix} w & m \\ \bar{m} & -w \end{pmatrix}^{2k} = \left(w^2 + |m|^2\right)^k I$$

for each $k \in \mathbf{N}$. On the other hand $|m|^2 = x^2 + y^2$, hence $w^2 + |m|^2 = 1$, consequently,

$$\sum_{k=0}^{\infty} z^k/k! = \sum_{k=0}^{\infty}(-1)^k s^{2k} I/(2k)! + i\sum_{k=0}^{\infty}(-1)^k s^{2k+1}\begin{pmatrix} w & m \\ \bar{m} & -w \end{pmatrix}/(2k+1)!$$
$$= \cos(s)I + i\sin(s)\begin{pmatrix} w & m \\ \bar{m} & -w \end{pmatrix}.$$

The particular case $s = 0$ corresponds to $\exp(0) = 1$. From this formula (3.2) follows.

3.3. Corollary. *If $z \in \mathbf{H}$ is written in the form $z = vI + wJ + xK + yL$ with real v, w, x and y, then $|\exp(z)| = \exp(v)$.*

Proof. If $w^2 + x^2 + y^2 = 0$ this is evident. Suppose $w^2 + x^2 + y^2 \neq 0$. In view of formula (3.2)

$$\exp(z) = \exp(v)A, \quad \text{where}$$
$$A = \begin{pmatrix} (\cos(\phi) + i\sin(\phi)w) & \sin(\phi)(x+iy) \\ \sin(\phi)(-x+iy) & (\cos(\phi) - i\sin(\phi)w) \end{pmatrix}. \tag{3.3}$$

Since $A \in \mathbf{H}$, then $|A|^2 = \det(A) = 1$ and inevitably $|\exp(z)| = \exp(v)$.

3.4. Corollary. *The function $\exp(z)$ on the set $\mathbf{H}_i := \{z\colon z \in \mathbf{H},\ \mathrm{Re}(z_{1,1}) = 0\}$ is periodic with three generators of periods J, K and L such that $\exp(z(1 + 2\pi n/|z|)) = \exp(z)$ for each $0 \neq z \in \mathbf{H}_i$ and each integer number n. If $z \in \mathbf{H}$ is written in the form $z = 2\pi sM$, where $M = wJ + xK + yL$, with real w, x and y such that $w^2 + x^2 + y^2 = 1$, then $\exp(z) = 1$ if and only if $s \in \mathbf{Z}$.*

Proof. In view of formula (3.2) $\exp(z) = 1$ for a given $z \in \mathbf{H}_i$ if and only if $\cos(s\phi) = 1$ and $\sin(s\phi) = 0$, that is equivalent to $s \in \{2\pi n\colon n \in \mathbf{Z}\}$, since $\phi = 1$ by the hypothesis of this corollary. The particular cases of formula (3.2) are either $w \neq 0$, $x = y = 0$; or $w = y = 0$ and $x \neq 0$; or $w = x = 0$ and $y \neq 0$, hence J, K and L are the three generators for the periods of exp.

3.5. Corollary. *The function exp is the epimorphism from $\mathbf{H}_i$ on the three-dimensional quaternion unit sphere $S^3(0,1,\mathbf{H}) := \{z\colon z \in \mathbf{H},\ |z| = 1\}$.*

Proof. In view of Corollary 3.3 the image $\exp(\mathbf{H}_i)$ is contained in $S^3(0,1,\mathbf{H})$. The sphere $S^3(0,1,\mathbf{H})$ is characterized by the condition $v_1^2+w_1^2+x_1^2+y_1^2=1$. In view of formula (3.2) we have $v_1=\cos(s)$, $w_1=\sin(s)w$, $x_1=\sin(s)x$ and $y_1=\sin(s)y$, where $s\in\mathbf{R}$ and $w^2+x^2+y^2=1$. Vice versa let $z_1\in S^3(0,1,\mathbf{H})$. For each $v_1\in[-1,1]$ there exists $s=\arcsin(v_1)$ such that $v_1=\cos(s)$ and $w_1^2+x_1^2+y_1^2=\sin^2(s)$. The case $\sin(s)=0$ corresponds to $v_1=1$ and others coordinates equal to zero, hence $z_1=\exp(0)$. If $\sin(s)\neq 0$ there are $w=w_1/\sin(s)$, $x=x_1/\sin(s)$ and $y=y_1/\sin(s)$, consequently, $\exp(z)=z_1$ in this case too. Therefore, exp is an epimorphism of $\mathbf{H}_i$ on $S^3(0,1,\mathbf{H})$.

3.6. Corollary. *Each quaternion has a polar decomposition*

$$z=\rho\exp\bigl(2\pi(\phi_1 J+\phi_2 K+\phi_3 L)\bigr), \tag{3.4}$$

where $\phi_j\in[-1,1]$ *for each* $j=1,2,3$, $\phi_1^2+\phi_2^2+\phi_3^2=1$, $\rho:=|z|$.

Proof. This follows from formula (3.2) and Corollary 3.5.

3.7. Note. In the noncommutative quaternion case there is the following relation for exp and its (right) derivative:

$$\exp(z)'.h=\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}z^k h z^{n-k-1}/n!, \tag{3.5}$$

where z and $h\in\mathbf{H}$. In particular,

$$\exp(z)'.vI=v\exp(z) \tag{3.6}$$

for each $v\in\mathbf{R}$, but generally not for all $h\in\mathbf{H}$. In view of §2.6 the derivative (3.5) reduces to the form of the definition of superdifferentiability given in §2.1. The function exp is periodic on $\mathbf{H}$, hence the inverse function denoted by Ln is defined only locally. Consider the space $\mathbf{R}^3$ of all variables w,x and y for which exp is periodic on $\mathbf{H}$. The condition $w^2+x^2+y^2=1$ defines in $\mathbf{R}^3$ the unit sphere S^2. The latter has a central symmetry element C for the transformation $C(w,x,y)=(-w,-x,-y)$. Consider a subset $P=\bigcup_{k=1}^4 P_k$ of S^2 of all points characterized by the conditions: $P_1:=\{(w,x,y)\in S^2\colon\ w\leqslant 0,\ x\leqslant 0,\ y\leqslant 0\}$, $P_2:=\{(w,x,y)\in S^2\colon\ w\geqslant 0,\ x\leqslant 0,\ y\leqslant 0\}$, $P_3:=\{(w,x,y)\in S^2\colon\ w\leqslant 0,\ x\geqslant 0,\ y\leqslant 0\}$, $P_4:=\{(w,x,y)\in S^2\colon\ w\leqslant 0,\ x\leqslant 0,\ y\geqslant 0\}$, then $P\cup CP=S^2$ and the intersection $P\cap CP$ is one dimensional over $\mathbf{R}$. This sphere S^2 corresponds to the embedding $\theta_1:(w,x,y)\hookrightarrow(0,w,x,y)\in\mathbf{R}^4$. Consider the embedding of $\mathbf{R}^4$ into $\mathbf{H}$ given by $\theta_2:(v,w,x,y)\hookrightarrow vI+wJ+xK+yL\in\mathbf{H}$. This yields the embedding $\theta:=\theta_2\circ\theta_1$ of S^2 in $\mathbf{H}$. Each unit circle with the center 0 in $\mathbf{H}$ intersects the equator $\theta(S^2)$ of the unit sphere $S^3(0,1,\mathbf{H})$. Join each point $(wJ+xK+yL)$ on $\theta(S^2)$ with the zero point in $\mathbf{H}$ by a line $\{s(wJ+xK+yL)\colon\ s\in\overline{\mathbf{R}}_+\}$, where $\overline{\mathbf{R}}_+:=\{s\in\mathbf{R}\colon\ s\geqslant 0\}$. This line crosses a circle embedded into $S^3(0,1,\mathbf{H})$, which is a trace of a circle $\{\exp(2\pi s(wJ+xK+yL))\colon\ s\in[0,1]\}$ of radius 1 in $\mathbf{H}$. Therefore, $\psi(s):=\exp(vI+2\pi s(wJ+xK+yL))$ as a function of (v,s) for fixed $(w,x,y)\in S^2$ defines a bijection of the domain $X\setminus\{s(wJ+xK+yL)\colon\ s\in\overline{\mathbf{R}}_+\}$ onto its image, where X is $\mathbf{R}^2$ embedded as $(v,s)\hookrightarrow(vI+s(wJ+xK+yL))\in\mathbf{H}$. This means, that $\mathrm{Ln}(z)$

is correctly defined on each subset $X\setminus\{s(wJ+xK+yL)\colon\ s\in\overline{\mathbf{R}}_+\}$ in $\mathbf{H}$. The union $\bigcup_{(w,x,y)\in P}\{s(wJ+xK+yL)\colon\ s\in\overline{\mathbf{R}}_+\}$ produces the three dimensional (over $\mathbf{R}$) subset $Q:=\bigcup_{k=1}^4 Q_k$, where $Q_k:=\theta(S_k)$, $S_1:=\{(w,x,y)\in\mathbf{R}^3\colon\ w\leqslant 0,\ x\leqslant 0,\ y\leqslant 0\}$, $S_2:=\{(w,x,y)\in\mathbf{R}^3\colon\ w\geqslant 0,\ x\leqslant 0,\ y\leqslant 0\}$, $S_3:=\{(w,x,y)\in\mathbf{R}^3\colon\ w\leqslant 0,\ x\geqslant 0,\ y\leqslant 0\}$, $S_4:=\{(w,x,y)\in\mathbf{R}^3\colon\ w\leqslant 0,\ x\leqslant 0,\ y\geqslant 0\}$, $\overline{\mathbf{R}}_+:=[0,\infty)$. Then, on the domain $\mathbf{H}\setminus Q$, the function $\exp(z)$ defines a bijection with image $\exp(\mathbf{H}\setminus Q)$ and its inverse function $\mathrm{Ln}(z)$ is correctly defined on $\mathbf{H}\setminus\exp(Q)$. By rotating $\mathbf{H}\setminus Q$ one may produce other domains on which Ln can be defined as the univalued function (that is, $\mathrm{Ln}(z)$ is one point in $\mathbf{H}$), but not on the entire $\mathbf{H}$. This means that $\mathrm{Ln}(z)$ is a locally bijective function. We have elementary identities $\cos(2\pi-\phi)=\cos(\phi)$ and $\sin(2\pi-\phi)=-\sin(\phi)$ for each $\phi\in\mathbf{R}$. If $0<\phi<2\pi$, then $w_1\sin(\phi)/\phi=w_2\sin(2\pi-\phi)/(2\pi-\phi)$ if and only if $w_1=-\phi w_2/(2\pi-\phi)$. To exclude this ambiguity we put in formula (3.2) $\phi\geqslant 0$ such that $\phi=(w^2+x^2+y^2)^{1/2}$ is the nonegative (arithmetical) branch of the square root function on $\overline{\mathbf{R}}_+$ and $w\geqslant 0$. Therefore, $\mathrm{Ln}(\exp(z))=z$ on $\mathbf{H}\setminus Q$, hence using formulas (3.3), (3.4) we obtain the multivalued function

$$\mathrm{Ln}(z)=\ln(|z|)+\mathrm{Arg}(z),\quad \text{where } \mathrm{Arg}(z):=\arg(z)+2\pi sM \tag{3.7}$$

on $\mathbf{H}\setminus\{0\}$, where ln is the usual real logarithm on $(0,\infty)$, $s\in\mathbf{Z}$,

$$|z|\exp\bigl(2\pi\arg(z)\bigr)=z,\quad \arg(z):=w_zJ+x_zK+y_zL,\quad (w_z,x_z,y_z)\in\mathbf{R}^3,$$

$w_z^2+x_z^2+y_z^2<1$, $w_z\geqslant 0$, $M=wJ+xK+yL$ is any unit vector (that is, $|M|=1$) in $\mathbf{H}$ commuting with $\arg(z)\in\mathbf{H}$, $\arg(z)$ is uniquely defined by such restriction on (w_z,x_z,y_z), for example, $M=\zeta\arg(z)$ for any $\zeta\in\mathbf{R}$, when $\arg(z)\neq 0$.

For each fixed $M=wJ+xK+yL$ $\exp(sM)$ is a one-parameter family of special unitary transformations (that is, $\det(\exp(sM))=1$) of $\mathbf{H}$ (that induces rotations of the Euclidean space $\mathbf{R}^4$), that is, $\exp(sM)\eta\in\mathbf{H}$ for each $\eta\in\mathbf{H}$, where $\mathbf{H}$ as the linear space over $\mathbf{R}$ is isomorphic with $\mathbf{R}^4$. On the other hand, there are special unitary transformations of $\mathbf{H}$ for which $s=\pi/2+\pi k$, but M is variable with $|M|=1$, where $k\in\mathbf{Z}$, then $\exp(z)=(-1)^kA$, where $A=\left(\begin{smallmatrix} iw & (x+iy)\\ (-x+iy) & -iw\end{smallmatrix}\right)$ (see formula (3.3)). To each closed curve γ in $\mathbf{H}$ there corresponds a closed curve $P_\xi(\gamma)$ in a $\mathbf{R}$-linear subspace $\xi\ni 0$, where P_ξ is a projection on ξ, for example,

$$P_{\mathbf{R}I\oplus\mathbf{R}J}(z)=(z-JzJ)/2=vI+wJ$$

for $\xi=\mathbf{R}I\oplus\mathbf{R}J$,

$$P_{\mathbf{R}I\oplus\mathbf{R}K}(z)=(z-KzK)/2=vI+xK,$$
$$P_{\mathbf{R}I\oplus\mathbf{R}L}(z)=(z-LzL)/2=vI+yL,$$
$$P_{\mathbf{R}J\oplus\mathbf{R}K}(z)=(z+LzL)/2=wJ+xK,$$
$$P_{\mathbf{R}J\oplus\mathbf{R}L}(z)=(z+KzK)/2=wJ+yL,$$
$$P_{\mathbf{R}K\oplus\mathbf{R}L}(z)=(z+JzJ)/2=xK+yL,$$
$$P_{\mathbf{R}J\oplus\mathbf{R}K\oplus\mathbf{R}L}(z)=(3z+JzJ+KzK+LzL)/4=wJ+xK+yL$$

for $\xi=\mathbf{R}J\oplus\mathbf{R}K\oplus\mathbf{R}L$, etc. Particular cases of special unitary transformations also correspond to $w=0$ or $x=0$ or $y=0$ for $M\neq 0$. To each closed curve γ in $\mathbf{H}$ and each quaternions a and b with $ab \neq 0$ there corresponds a closed curve $a\gamma b$ in $\mathbf{H}$, for example, for $a = J$ and $b = K$ there is the identity $JzK = vL - wK - xJ + yI$ for each $z = vI + wJ + xK + yL$ in $\mathbf{H}$.

Instead of the Riemann two dimensional surface of the complex logarithm function we get the four dimensional manifold W, that is, a subset of $Y^{\aleph_0} := \prod_{i\in\mathbf{Z}} Y_i$, where $Y_i = Y$ for each i, such that each Y is a copy of $\mathbf{H}$ embedded into $\mathbf{H}\times\mathbf{R}^3$ and cut by a three dimensional submanifold Q and with diffeomorphic bending of a neighbourhood of Q such that two three dimensional edges Q_1 and Q_2 of Y diffeomorphic to Q do not intersect outside zero, $Q_1 \cap Q_2 = \{0\}$, that is, the boundary ∂Q is also cut everywhere outside zero. We have $\partial Q = \partial Q^w \cup \partial Q^x \cup \partial Q^y$, where

$$\partial Q^w := \left\{\theta(w,x,y)\colon\ w = 0,\ (w,x,y) \in \bigcup_{k=1}^{4} S_k\right\},$$

$$\partial Q^x := \left\{\theta(w,x,y)\colon\ x = 0,\ (w,x,y) \in \bigcup_{k=1}^{4} S_k\right\} \quad \text{and}$$

$$\partial Q^y := \left\{\theta(w,x,y)\colon\ y = 0,\ (w,x,y) \in \bigcup_{k=1}^{4} S_k\right\}.$$

This means, that $\partial Q^w = \{z = xK + yL\colon\ (x,y) \in \mathbf{R}^2,\ x$ and y are not simultaneously positive$\}$. Similarly for ∂Q^x and ∂Q^y with $z = wJ + yL$ and $z = wJ + xK$ instead of $z = xK + yL$ respectively. To exclude rotations in each subspace $vI + s(aK + bL)$ isomorphic with $\mathbf{R}^2$ and embedded into $\mathbf{R}I + \partial Q^w$ and similarly for $vI + s(aJ + bL)$ and $vI + s(aJ + bK)$ we have cut ∂Q, where $v, s \in \mathbf{R}$ are variables and a, b are two real constants such that $ab \leqslant 0$, $a^2 + b^2 > 0$. Then in $\mathbf{H}\times\mathbf{R}^3$ two copies Y_i, and Y_{i+1} are glued by the equivalence relation of $Q_{2,i}$ with $Q_{1,i}$ via the segments $\{s_{l,i}(wJ + xK + yL)\colon\ s_{l,i} \in \overline{\mathbf{R}}_+\}$ such that $s_{1,i+1} = s_{2,i}$ for each $s_{l,i} \in \overline{\mathbf{R}}_+$ and each given real $(w,x,y) \in P$ with $w^2 + x^2 + y^2 = 1$. This defines the four dimensional manifold W embedded into $\mathbf{H}\times\mathbf{R}^3$ and $\mathrm{Ln}\colon \mathbf{H}\setminus\{0\} \to W$ is the univalued function, that is, $\mathrm{Ln}(z)$ is a singleton in W for each $z \in \mathbf{H}\setminus\{0\}$.

3.8. Theorem. *The function* Ln *is quaternion holomorphic on any domain* U *in* $\mathbf{H}$ *obtained by a quaternion holomorphic diffeomorphism of* $\mathbf{H}\setminus Q$ *onto* U. *Each path* γ *in* $\mathbf{H}$ *such that* $\gamma(s) = r\exp(2\pi sn(wJ + xK + yL))$ *with* $s \in [0,1]$, $n \in \overline{\mathbf{R}}_+$, $w^2 + x^2 + y^2 = 1$ *is closed in* $\mathbf{H}$ *if and only if* $n \in \mathbf{N}$, *where* $r > 0$. *In this case*

$$\int_\gamma z^{-1}\,dz = \int_\gamma d(\mathrm{Ln}\,z) = 2\pi n(wJ + xK + yL). \tag{3.8}$$

Proof. If U and V are two open subsets in $\mathbf{H}$ and $g: V \to U$ is a quaternion holomorphic diffeomorphism of V onto U and f is a quaternion holomorphic function on V, then $f\circ g^{-1}$ is quaternion holomorphic function on U, since

$$(f\circ g^{-1})'(z).h = \big(f'(\zeta)\big)\big|_{\zeta=g^{-1}(z)}.\big(g^{-1}(z)\big)'.h$$

for each $z \in U$ and each $h \in \mathbf{H}$ and

$$\partial\big(f \circ g^{-1}(z)\big)/\partial\tilde{z} = \big(\partial f(\zeta)/\partial\tilde{\zeta}\big)\big|_{\zeta=g^{-1}(z)}.\big(\partial\tilde{g}^{-1}(z)/\partial\tilde{z}\big) + \big(\partial f(\zeta)/\partial\zeta\big)\big|_{\zeta=g^{-1}(z)}.\big(\partial g^{-1}(z)/\partial\tilde{z}\big) = 0.$$

Since exp is the diffeomorphism from $\mathbf{H}\setminus Q$ onto $\mathbf{H}\setminus\exp(Q)$, we have that Ln is quaternion holomorphic on $\mathbf{H}\setminus Q$ and on each of its quaternion holomorphic images after choosing a definite branch of the multivalued function $\mathrm{Ln}(z)$ (see formula (3.7)).

A path γ is defined for each $s \in \mathbf{R}$ not only for $s \in [0,1]$ due to the existence of exp. In view of formula (3.2) a path γ is closed (that is, $\gamma(s_0) = \gamma(s_0+1)$ for each $s_0 \in \mathbf{R}$) if and only if $\cos(2\pi n) = \cos(0) = 1$ and $\sin(2\pi n) = 0$, that is, $n \in \mathbf{N}$.

From the definition of the line integral we get the equality:

$$\int_\gamma d(\mathrm{Ln}\,z) = \int_0^1 (\mathrm{Ln}\,z)'.\gamma'(s)\,ds.$$

Considering integral sums by partitions P of $[0,1]$ and taking the limit by the family of all P we get, that $\int_\gamma d(\mathrm{Ln}\,z) = \mathrm{Arg}(\gamma(1)) - \mathrm{Arg}(\gamma(0))$ for a chosen branch of the function $\mathrm{Arg}(z)$ (see formula (3.7)). Therefore, $\int_\gamma d(\mathrm{Ln}\,z) = 2\pi n(wJ + xK + yL)$.

Since z commutes with itself, we have: $\exp(z)'.z = \exp(z)z$. Therefore,

$$\exp\big(\mathrm{Ln}(z)\big)'.I = \big(\partial\exp(\eta)/\partial\eta\big)\big|_{\eta=\mathrm{Ln}(z)}.\big(\mathrm{Ln}(z)\big)'.I = \exp\big(\mathrm{Ln}(z)\big)\big(\mathrm{Ln}(z)\big)'.I,$$

consequently, $(\mathrm{Ln}(z))'.I = \exp(-\mathrm{Ln}(z)) = z^{-1}$ and inevitably

$$\lim_P I(z^{-1},\gamma;P) = \lim_P \sum_l \hat{z}_l^{-1}\Delta z_l = \lim_P \Delta\,\mathrm{Ln}(z_l) = \int_\gamma d\,\mathrm{Ln}(z),$$

hence $\int_\gamma z^{-1}\,dz = \int_\gamma d\,\mathrm{Ln}(z)$. That is, $\int_\gamma d\,\mathrm{Ln}(z)$ can be considered as the definition of $\int_\gamma z^{-1}\,dz$.

3.9. Theorem. *Let f be a continuous quaternion holomorphic function on an open domain U in $\mathbf{H}$. If $(\gamma + z_0)$ and ψ are presented as piecewise unions of paths $\gamma_j + z_0$ and ψ_j with respect to parameter $s \in [a_j, b_j]$ and $s \in [c_j, d_j]$ respectively with $a_j < b_j$ and $c_j < d_j$ for each $j = 1,\dots,n$ and $\bigcup_j [a_j,b_j] = \bigcup_j [c_j,d_j] = [0,1]$ homotopic relative to $U_j\setminus\{z_0\}$, where $U_j\setminus\{z_0\}$ is a 3-connected open domain in $\mathbf{H}$ such that $\pi_{1,t}(U_j\setminus\{z_0\})$ and $\pi_{2,u}(U_j\setminus\{z_0\})$ are simply connected in $\mathbf{C}$ for each t and $u \in \mathbf{C}$ for which there exists $z \in \mathbf{H}$ for each j. If $(\gamma + z_0)$ and ψ are closed rectifiable paths in U such that $\gamma(s) = r\exp(2\pi s(wJ + xK + yL))$ with $s \in [0,1]$ and $w^2 + x^2 + y^2 = 1$ and $z_0 \notin \psi$. Then*

$$f(z) = (2\pi)^{-1}\bigg(\int_\psi f(\zeta)(\zeta - z)^{-1}\,d\zeta\bigg)(wJ + xK + yL)^{-1} \tag{3.9}$$

for each $z \in U$ such that $|z - z_0| < \inf_{\zeta\in\psi([0,1])}|\zeta - z_0|$.

Proof. Join γ and ψ by a rectifiable path ω such that $z_0 \notin \omega$, which is going in one direction and the opposite direction, denoted ω^-, such that $\omega_j \cup \psi_j \cup \gamma_j \cup \omega_{j+1}$ is homotopic to a point relative to $U_j \setminus \{z_0\}$ for suitable ω_j and ω_{j+1}, where ω_j joins $\gamma(a_j)$ with $\psi(c_j)$ and ω_{j+1} joins $\psi(d_j)$ with $\gamma(b_j)$ such that z and $z_0 \notin \omega_j$ for each j. Then $\int_{\omega_j} f(\zeta)(\zeta - z)^{-1}\,d\zeta = -\int_{\omega_j^-} f(\zeta)(\zeta - z)^{-1}\,d\zeta$ for each j. In view of Theorem 2.15 there is the equality $-\int_{\gamma^- + z} f(\zeta)(\zeta - z)^{-1}\,d\zeta = \int_\psi f(\zeta)(\zeta - z)^{-1}\,d\zeta$. Since $\gamma + z$ is a circle around z its radius $r > 0$ can be chosen so small, that $f(\zeta) = f(z) + \alpha(\zeta, z)$, where α is a continuous function on U^2 such that $\lim_{\zeta \to z} \alpha(\zeta, z) = 0$, then

$$\int_{\gamma+z} f(\zeta)(\zeta - z)^{-1}\,d\zeta = \int_{\gamma+z} f(z)(\zeta - z)^{-1}\,d\zeta + \delta(r)$$
$$= 2\pi f(z)(wJ + xK + yL) + \delta(r),$$

where

$$\left|\delta(r)\right| \leqslant \left|\int_\gamma \alpha(\zeta, z)(\zeta - z)^{-1}\,d\zeta\right| \leqslant 2\pi \sup_{\zeta \in \gamma}\left|\alpha(\zeta, z)\right| C_1 \exp(C_2 r^6),$$

where C_1 and C_2 are positive constants (see inequality (2.12)), hence there exists $\lim_{r \to 0,\, r>0} \delta(r) = 0$. Taking the limit while $r > 0$ tends to zero yields the conclusion of this theorem.

3.9.1. Corollary. *Let f, U, ψ, z and z_0 be as in Theorem* 3.9*, then*

$$\left|f(z)\right| \leqslant \sup_{(\zeta \in \psi,\, h \in \mathbf{H},\, |h| \leqslant 1)} \left|\hat{h}(\zeta).h\right|.$$

3.10. Theorem. *Let f be a continuous function on an open subset U of* **H**. *If f is quaternion integral holomorphic on U, then f is quaternion locally z-analytic on U.*

Proof. Let $z_0 \in U$ be a marked point and let Γ denotes the family of all rectifiable paths $\gamma : [0, 1] \to U$ such that $\gamma(0) = z_0$, then $U_0 = \{\gamma(1)\colon\ \gamma \in \Gamma\}$ is a connected component of z_0 in U. Therefore, $g = \{\gamma(1), \int_\gamma f(z)\,dz\}$ is the function with the domain U_0. Let X be a compact metric space and F be a function continuous on $U \times X$ with values in $\mathbf{H}$ and for each $p \in X$ let $f_p(z) := F(z, p)$ be quaternion holomorphic on U by $z \in U$. Define G on $U^2 \times X$ by $G(z, w, p) := [F(z, p) - F(w, p)](w - z)^{-1},\ w \neq z$. Then $G(z, z, p) = (\partial f_p(z)/\partial z).I$. It can be seen with the help of formula (3.9) that G is continuous on $U^2 \times X$, since

$$G(b, c, q) - G(a, a, p) = \int_\gamma \left[\left(\partial f_q(z)/\partial z\right) - \left(\partial f_p(a)/\partial a\right)\right].dz.(c - b)^{-1},$$

where γ is a rectifiable curve such that $\gamma(0) = b$, $\gamma(1) = c$. Moreover, G is uniformly continuous on $V^2 \times X$ for each compact canonical closed subset V in $\mathbf{H}$ such that $V \subset U$. As in §2.15 it can be proved, that $F(z) := \int_\gamma f(z)\,dz$, for each rectifiable γ in U, depends only on initial and final points. This integral is finite, since $\gamma([0, 1])$ is contained in a compact canonical closed subset $W \subset U$ on which f is bounded. Therefore,

$(\partial\int_{z_0}^{z} f(\zeta)\,d\zeta/\partial z).h = \hat{f}(z).h$ for each $z \in U$ and $h \in \mathbf{H}$, $(\partial\int_{z_0}^{z} f(\zeta)\,d\zeta/\partial\tilde{z}) = 0$ for each $z \in U$ and $h \in \mathbf{H}$, where z_0 is a marked point in U such that z and z_0 are in one connected component of U. In particular, $\hat{f}(z).I = f(z)$ for each $z \in U$. Here $\hat{f}$ is correctly defined for each $f \in C^{1,0}(U,\mathbf{H})$ by continuity of the differentiable integral functional on $C^0(U,\mathbf{H})$. In particular, $\hat{f}(z).I = f(z)$ for each $z \in U$. For a given $z \in \mathbf{U}$ choose a neighbourhood W satisfying the conditions of Theorem 3.9. Then there exists a rectifiable path $\psi \subset W$ such that $F(z)$ is presented by formula (3.9). The latter integral is infinite differentiable by z such that

$$\left(\partial^k F(z)/\partial z^k\right) = \frac{k!}{2\pi}\left(\int_{\psi} f(\zeta)(\zeta - z)^{-k-1}\,d\zeta\right)(w_0 J + x_0 K + y_0 L)^{-1}, \tag{3.10}$$

where w_0, x_0 and $y_0 \in \mathbf{R}$ are fixed and $w_0^2 + x_0^2 + y_0^2 = 1$. In particular, we may choose a ball $W = B(a,R,\mathbf{H}) := \{\xi \in \mathbf{H}\colon |\xi - a| \leqslant R\} \subset U$ for a sufficiently small $R > 0$ and $\psi = \gamma + a$, where $\gamma(s) = r\exp(2\pi s(w_0 J + x_0 K + y_0 L))$ with $s \in [0,1]$, $0 < r < R$. If we prove that $F(z)$ is quaternion locally z-analytic, then evidently its z-derivative $f(z)$ will also be quaternion locally z-analytic. Consider $z \in B(a,r',\mathbf{H})$ with $0 < r' < r$, then $|z-a| < |\zeta - a|$ for each $\zeta \in \psi$ and $(\zeta - a - (z-a))^{-1} = (1 - (\zeta - a)^{-1}(z-a))^{-1}(\zeta - a)^{-1} = \sum_{k=0}^{\infty}((\zeta - a)^{-1}(z-a))^k(\zeta - a)^{-1}$, where $0 \notin \psi$. Therefore,

$$F(z) = (2\pi)^{-1}\sum_{k=0}^{\infty}\phi_k(z), \tag{3.11}$$

where

$$\phi_k(z) := \left(\int_{\psi} f(\zeta)\left((\zeta - a)^{-1}(z-a)\right)^k(\zeta - a)^{-1}\,d\zeta\right)(w_0 J + x_0 K + y_0 L)^{-1}.$$

Thus $|\phi_k(z)| \leqslant \sup_{\zeta\in\psi}|f(\zeta)|(r'/r)^{-k}$ for each $z \in B(a,r',\mathbf{H})$ and series (3.11) converges uniformly on $B(a,r',\mathbf{H})$. Each function $\phi_k(z)$ is evidently quaternion locally z-analytic on $B(a,r',\mathbf{H})$, hence $F(z)$ is such too. Since for each $a \in U$ there is an $r' > 0$, for which the foregoing holds, it follows that $f(z)$ is the quaternion locally z-analytic function.

3.11. Note. Theorems 2.11, 2.15, 2.16, 3.10 and Corollary 2.13 establish the equivalence of notions of quaternion holomorphic, quaternion integral holomorphic and quaternion locally z-analytic classes of functions on domains satisfying definite conditions. Before, the notion of quaternion holomorphicity was defined relative to a right superdifferentiation, similarly it can be defined relative to a left superdifferentiation. Quaternion local z-analyticity shows, that a function is quaternion holomorphic relative to a right superdifferentiation if and only if it is quaternion holomorphic relative to a left superdifferentiation.

In particular, if $f \in {}_lC^{\omega}(U,\mathbf{H})$, then evidently $F(z) := \int_{z_0}^{z} f(\zeta)\,d\zeta$ and $(\partial f(\zeta)/\partial\zeta).I$ belong to ${}_lC^{\omega}(U,\mathbf{H})$, where z and $z_0 \in U_0$, $\zeta \in U$, U_0 is a connected component of U open in $\mathbf{H}$, since $(b_n\hat{\zeta}^n).\Delta\zeta = b_n(\partial\zeta^{n+1}/\partial\zeta).\Delta\zeta$ for each $\zeta \in \mathbf{H}$, $\Delta\zeta \in \mathbf{H}$, $n \in \mathbf{N}$, $b_n \in \mathbf{H}$.

3.11.1. Definitions. Let U be an open subset in $\mathbf{H}$ and $f \in C^0(U,\mathbf{H})$, then we say that f possesses a primitive $g \in C^1(U,\mathbf{H})$ if $g'(z).I = f(z)$ for each $z \in U$. A region U in $\mathbf{H}$

is said to be quaternion holomorphically simply connected if every function quaternion holomorphic on it possesses a primitive.

From §3.10 we get.

3.11.2. Theorem. *If $f \in C^\omega(U, \mathbf{H})$, where U is 3-connected; $\pi_{1,t}(U)$ and $\pi_{2,u}(U)$ are simply connected in $\mathbf{C}$ for each t and $u \in \mathbf{C}$ for which there exists $z \in U$, U is an open subset in $\mathbf{H}$, then there exists $g \in C^\omega(U, \mathbf{H})$ such that $g'(z).I = f(z)$ for each $z \in U$.*

3.11.3. Theorem. *Let U and V be quaternion holomorphically simply connected regions in $\mathbf{H}$ with $U \cap V \neq \emptyset$ connected. Then $U \cup V$ is quaternion holomorphically simply connected.*

3.12. Corollary. *Let U be an open subset in $\mathbf{H}^n$, then the family of all quaternion holomorphic functions $f : U \to \mathbf{H}$ has a structure of an $\mathbf{H}$-algebra.*

Proof. If $f_1(z) = \alpha g(z)\beta + \gamma h(z)\delta$ or $f_2(z) = g(z)h(z)$ for each $z \in U$, where α, β, γ and $\delta \in \mathbf{H}$ are constants, g and h are quaternion holomorphic functions on U, then F_1 and F_2 are Frechét differentiate on U by (v, w, x, y) (see §2.1 and §2.2) and $D_{\tilde{z}} f_1(z) = \alpha(D_{\tilde{z}} g)\beta + \gamma(D_{\tilde{z}} h)\delta = 0$ and $D_{\tilde{z}} f_2(z) = (D_{\tilde{z}} g)h + g(D_{\tilde{z}} h) = 0$, hence f_1 and f_2 are also quaternion holomorphic on U.

3.13. Proposition. *For each complex holomorphic function f in a neighbourhood $B(t_0, r, \mathbf{C})$ of a point $t_0 \in \mathbf{C}$ there exists a quaternion z-analytic function g on a neighbourhood $B(a, r, \mathbf{H})$ of $a \in \mathbf{H}$ such that $a_{1,1} = t_0$ (or $a_{1,2} = t_0$) and $g_{1,1}(t, u_0) = f(t)$ (or $g_{1,2}(u_0, t) = f(t)$ respectively) on $B(t_0, r, \mathbf{C})$, where $B(x, r, X) := \{y \in X\colon\ \rho_X(x, y) \leqslant r\}$ is the ball in a space X with a metric ρ, $r > 0$, $u_0 = a_{1,2}$ (or $u_0 = a_{1,1}$ correspondingly).*

Proof. Write conditions (2.2) for a right superlinearly superdifferentiable function in the complex form. This yields:

$$\partial f_{1,1}/\partial \bar{t} = 0, \quad \partial f_{1,2}/\partial t = 0, \quad \partial f_{1,1}/\partial u = 0, \quad \partial f_{1,2}/\partial \bar{u} = 0. \tag{3.12}$$

There are also skew conditions:

$$\partial(g_{1,1} + ih_{1,2})/\partial(w + ix) = 0, \quad \partial(g_{1,2} + ih_{1,1})/\partial(w + ix) = 0. \tag{3.13}$$

Other conditions derive from these. For example, for the pair of variables (v, y) using the matrix L we get

$$\partial g_{1,1}/\partial v = \partial h_{1,2}/\partial y, \quad \partial h_{1,2}/\partial v = -\partial g_{1,1}/\partial y, \tag{3.14}$$

$$\partial g_{1,2}/\partial v = \partial h_{1,1}/\partial y, \quad \partial h_{1,1}/\partial v = -\partial g_{1,2}/\partial y, \tag{3.15}$$

which in complex form is the following:

$$\partial(g_{1,1} + ih_{1,2})/\partial(v - iy) = 0, \quad \partial(g_{1,2} + ih_{1,1})/\partial(v - iy) = 0. \tag{3.16}$$

Eqs. (3.13) and (3.16) are equivalent to:

$$\partial(f_{1,1} + f_{1,2})/\partial(w + ix) = 0, \quad \partial(\bar{f}_{1,1} - \bar{f}_{1,2})/\partial(w + ix) = 0, \tag{3.13$'$}$$

$$\partial(f_{1,1}+f_{1,2})/\partial(v-iy)=0, \quad \partial(\bar{f}_{1,1}-\bar{f}_{1,2})/\partial(v-iy)=0, \tag{3.16'}$$

that is, there are two functions p and q holomorphic in complex variables $w-ix$ and $v+iy$ such that $f_{1,1}(z)=p(w-ix,v+iy)+q(w-ix,v+iy)$ and $f_{1,2}(z)=p(w-ix,v+iy)-q(w-ix,v+iy)$.

Consider first an extension in the class of quaternion holomorphic functions with a right superdifferential not necessarily right superlinear on the superalgebra $\mathbf{H}^n$. Since f is holomorphic in $B(t_0,r,\mathbf{C})$, it has a decomposition $f(t)=\sum_{n=0}^{\infty} f_n(t-t_0)^n$, where $f_n\in\mathbf{C}$. Consider its extension in $B\big(\big(\begin{smallmatrix} t_0 & 0\\ 0 & \bar{t}_0\end{smallmatrix}\big),r,\mathbf{H}\big)$ such that

$$f(z)=\sum_{n=0}^{\infty}\begin{pmatrix} f_n & 0\\ 0 & \bar{f}_n\end{pmatrix}\left(z-\begin{pmatrix} t_0 & 0\\ 0 & \bar{t}_0\end{pmatrix}\right)^n.$$

Evidently this series converges for each $z\in B\big(\big(\begin{smallmatrix} t_0 & 0\\ 0 & \bar{t}_0\end{smallmatrix}\big),r,\mathbf{H}\big)$ and this extension of f is quaternion holomorphic, since $\big(\begin{smallmatrix} f_n & 0\\ 0 & \bar{f}_n\end{smallmatrix}\big)\in\mathbf{H}$ for each n and $\big(\begin{smallmatrix} t_0 & 0\\ 0 & \bar{t}_0\end{smallmatrix}\big)\in\mathbf{H}$, that is, $\partial f/\partial\tilde{z}=0$. If $z=\big(\begin{smallmatrix} t & u\\ -\bar{u} & \bar{t}\end{smallmatrix}\big)$ and $u=0$, then

$$f(z)=\sum_{n=0}^{\infty}\begin{pmatrix} f(t) & 0\\ 0 & \bar{f}(t)\end{pmatrix}.$$

Another type of a solution is:

$$f(z)=\sum_{n=0}^{\infty}\begin{pmatrix} f_n & 0\\ 0 & \bar{f}_n\end{pmatrix}\left((z-JzJ)/2-\begin{pmatrix} t_0 & 0\\ 0 & \bar{t}_0\end{pmatrix}\right)^n,$$

since $\big(\begin{smallmatrix} t & 0\\ 0 & \bar{t}\end{smallmatrix}\big)=(z-JzJ)/2$ for each $z=\big(\begin{smallmatrix} t & u\\ -\bar{u} & \bar{t}\end{smallmatrix}\big)$.

Consider now more narrow class of quaternion holomorphic functions with a right superlinear superdifferential on the superalgebra $\mathbf{H}^n$. There is another way to construct f on $B\big(\big(\begin{smallmatrix} t_0 & 0\\ 0 & \bar{t}_0\end{smallmatrix}\big),r,\mathbf{H}\big)$, because due to Theorems 2.15 and 3.10 a quaternion holomorphic function in interior of this ball is quaternion z-analytic in it. The construction of $f_{1,1}(t,u)$ satisfying the conditions above and such that $f_{1,1}(t,0)=f(t)$ in $B(t_0,r,\mathbf{C})$ then comes down to finding $f_{1,2}(t,u)$ with partial differential skew conditions arising from those for $f_{1,1}$ and specific conditions on $f_{1,2}$ such that $f_{1,1}$ is holomorphic in t and antiholomorphic in u, $f_{1,2}$ is holomorphic in u and antiholomorphic in t (where antiholomorphic means holomorphic in the complex conjugate variable $\bar{u}$ or $\bar{t}$ respectively).

The second type of extension can be obtained from the first by applying right multiplication by $\widetilde{K}$ on the right, that is,

$$f(z)\mapsto f(z)\widetilde{K}=\begin{pmatrix} f_{1,2} & -f_{1,1}\\ \bar{f}_{1,1} & \bar{f}_{1,2}\end{pmatrix}.$$

3.14. Proposition. *If f is a quaternion holomorphic function on an open subset U in $\mathbf{H}$, where $f'(z)\neq 0$ and $f'(z)$ is right superlinear, then it is a conformal mapping in each point $z\in U$, that is preserving angles between differentiable curves.*

Proof. Let $z \in U$, then f is differentiable at z and there exists $\lambda = f'(z) \in \mathbf{H}$. Each quaternion $h = \begin{pmatrix} h_t & h_u \\ -\bar{h}_u & \bar{h}_t \end{pmatrix} \in \mathbf{H}$ can be considered as vector (h_t, h_u) in $\mathbf{C}^2$. Consider a scalar product in $\mathbf{C}^2$: $(h,k) := h_t\bar{k}_t + h_u\bar{k}_u$. On the other hand, if $\lambda \neq 0$, then $\lambda = |\lambda|\zeta$, where $|\zeta| = 1$. Rows and columns of the 2×2 complex matrix ζ are orthonormal, hence it is unitary and $(\zeta h, \zeta k) = (h,k)$ for each h and k in $\mathbf{C}^2$ or for the corresponding quaternions in $\mathbf{H}$. Therefore, for each vectors $h \neq 0$ and $k \neq 0$ in $\mathbf{H}$

$$(\lambda h, \lambda k)/\big(|\lambda h||\lambda k|\big) = (h,k)/\big(|h||k|\big). \tag{3.17}$$

If ψ and $\phi : (-1,1) \to U$ are two differentiable curves crossing in a point $z \in U$, then we have two vectors $\psi'(0) =: h$ and $\phi'(0) = k$, where $\psi(0) = \phi(0) = z$. Then $f(\psi(s))' = f'(z)|_{z=\psi(s)}.\psi'(s)$. From formula (3.17) it follows, that f preserves the angle α between curves ψ and ϕ, where $\cos(\alpha) = \mathrm{Re}(\psi'(0), \phi'(0))/(|\psi'(0)||\phi'(0)|)$ for $\psi'(0) \neq 0$ and $\phi'(0) \neq 0$.

3.15. Theorem. *Let f be a quaternion holomorphic function on an open subset U in $\mathbf{H}$ such that $\sup_{z\in U,\, h\in B(0,1,\mathbf{H})} |[f(z)(\zeta - z)^{-2}]\hat{.}h| \leqslant C/|\zeta - z|^2$ for each $\zeta \in \mathbf{H}\setminus \mathrm{cl}(U)$. Then $|f'(z)| \leqslant C/d(z)$ for each $z \in U$, where $d(z) := \inf_{\zeta\in\mathbf{H}\setminus U} |\zeta - z|$; $|f(\xi) - f(z)|/|\xi - z| \leqslant 2C/r$ for each ξ and $z \in B(a, r/2, \mathbf{H}) \subset \mathrm{Int}(B(a,r,\mathbf{H})) \subset U$, where $r > 0$. In particular, if f is a quaternion holomorphic function with bounded $[f(z)(\zeta - z)^{-2}]\hat{.}h|\zeta - z|^2$ on $\mathbf{H}^2 \times B(0,1,\mathbf{H})$ with $|\zeta| \geqslant 2|z|$, that is,*

$$\sup_{\zeta,z\in\mathbf{H},\, |\zeta|\geqslant 2|z|,\, h\in B(0,1,\mathbf{H})} \big|\big[f(z)(\zeta - z)^{-2}\big]\hat{.}h\big||\zeta - z|^2 < \infty,$$

then f is constant.

Proof. In view of Theorem 3.9 there exists a rectifiable path γ in U such that

$$\big(\partial^k f(z)/\partial z^k\big) = k!(2\pi)^{-1}\bigg(\int_{\gamma+z_0} f(\zeta)(\zeta - z)^{-k-1}\, d\zeta\bigg)(wJ + xK + yL)^{-1}, \tag{3.18}$$

where $\gamma(s) = r' \exp(2\pi s(wJ + xK + yL))$ with $s \in [0,1]$, $0 < r'$. Therefore, $|f'(z)| \leqslant C/d(z)$. Since $\int_\zeta^z df(z) = f(z) - f(\zeta)$, then

$$\big|f(\xi) - f(z)\big|/|\xi - z| \leqslant \sup_{z\in B(a,r/2,\mathbf{H})} \big[C/d(z)\big] \leqslant 2C/r,$$

where $r' < r/2$, ξ and $z \in B(a, r/2, \mathbf{H}) \subset \mathrm{Int}(B(a,r,\mathbf{H})) \subset U$. Taking r tending to infinity, if f is quaternion holomorphic with bounded $[f(z)(\zeta - z)^{-2}]\hat{.}h|\zeta - z|^2$ on $\mathbf{H}^2 \times B(0,1,\mathbf{H})$ for $|\zeta| \geqslant 2|z|$, then $f'(z) = 0$ for each $z \in \mathbf{H}$, since f is locally z-analytic and

$$\sup_{\zeta,z\in U,\, |\zeta|\geqslant 2|z|,\, h\in B(0,1,\mathbf{H})} \big|\big[f(z)(\zeta - z)^{-2}\big]\hat{.}h\big||\zeta - z|^2 < \infty$$

is bounded, hence f is constant on $\mathbf{H}$.

3.16. Remark. Theorems 3.9, 3.10 and 3.15 are the quaternion analogs of the Cauchy, Morera and Liouville theorems correspondingly. Evidently, Theorem 3.15 is also true for right superlinear $\hat{f}(z)$ on $\mathbf{H}$ for each $z \in U$ and with bounded $\hat{f}(z).h$ on $U \times B(0,1,\mathbf{H})$ instead of $[f(z)((\zeta - z)^{-2}]\hat{}.h|\zeta - z|^2$. In particular, if f is quaternion holomorphic on $\mathbf{H}$ and $\hat{f}(z)$ is right superlinear on $\mathbf{H}$ for each $z \in \mathbf{H}$ and $\hat{f}(z).h$ is bounded on $U \times B(0,1,\mathbf{H})$, then F is constant.

3.17. Theorem. *Let $P(z)$ be a polynomial on $\mathbf{H}$ such that $P(z) = z^{n+1}\sum_{\eta(k)=0}^{n}(A_k, z^k)$, where $A_k = (a_{1,k}, \dots, a_{s,k})$, $a_{j,l} \in \mathbf{H}$, $k = (k_1, \dots, k_s)$, $0 \leqslant k_j \in \mathbf{Z}$, $\eta(k) = k_1 + \cdots + k_s$, $0 \leqslant s = s(k) \in \mathbf{Z}$, $s(k) \leqslant \eta(k) + 1$, $(A_k, z^k) := a_{1,k}z^{k_1} \dots a_{s,k}z^{k_s}$, $z^0 := 1$. Then $P(z)$ has a root in $\mathbf{H}$.*

Proof. Consider a polynomial $Q(z) := z^{n+1} + \sum_{\eta(k)=0}^{n}(z^k, \tilde{A}_k)$, where $(z^k, \tilde{A}_k) := z^{k_1}\tilde{a}_{1,k} \dots z^{k_s}\tilde{a}_{s,k}$. Then PQ is a quaternion holomorphic function on $\mathbf{H}$. Suppose that $P(z) \neq 0$ for each $z \in H$. Consider a rectifiable path γ_R in $\mathbf{H}$ such that $\gamma_R([0,1]) \cap \mathbf{H} = [-R, R]$ and outside $[-R, R]$: $\gamma_R(s) = R\exp(2\pi s M)$, where M is a unit vector in $\mathbf{H}_i$. Since $\lim_{|z|\to\infty} P(z)z^{-n-1} = 1$, then due to Theorem 2.11

$$\lim_{R\to\infty}\int_{\gamma R}(PQ)^{-1}(z)\,dz = \int_{-R}^{R}(PQ)^{-1}(v)\,dv = \int_{-R}^{R}\left|P(v)\right|^{-2}dv \geqslant 0.$$

The latter integral is equal to zero if and only if $|P(v)|^{-2} = 0$ for each $v \in \mathbf{R}$. This contradicts our supposition, hence there exists a root $z_0 \in \mathbf{H}$, that is, $P(z_0) = 0$.

3.17.1. Note. Consider, for example, the polynomial $P(z) = (z-a)^2 + J(z-a)K$ on $\mathbf{H}$, then there does not exist $\lim_{z\to a,\, z\neq a} f(z)(z-a)^{-1}$ and there also does not exist $\lim_{z\to a,\, z\neq a}(z-a)^{-1}f(z)$, though $f(a) = 0$. This makes an obstacle for a quaternion analog of the Gauss theorem about zeros of a derivative of a complex polynomial. Even in a particular case, when a polynomial has a decomposition $f(z) = (z-a_1)\dots(z-a_m)$, where $a_1, \dots, a_m \in \mathbf{H}$, then

$$\begin{aligned} &f(z)^{-1}f'(z).h \\ &\quad = \sum_j (z-a_m)^{-1}\dots(z-a_{j+1})^{-1}(z-a_j)^{-1}h(z-a_{j+1})\dots(z-a_m), \end{aligned}$$

consequently,

$$\left(f(z)^{-1}f'(z).I\right)\tilde{} = \sum_j \lambda_j(z-a_j)\lambda_j^{-1}/|z-a_j|^2,$$

where $\lambda_j = [(z-a_{j+1})\dots(z-a_m)]\tilde{}$. Hence $z\sum_j c_j|z-a_j|^{-2} = \sum_j \lambda_j a_j \lambda_j^{-1}$, where $\lambda_j z =: zc_j\lambda_j$, $c_j \in \mathbf{H}$, $|c_j| = 1$, z is a root of $f'(z).I$. In the case of pairwise commuting $a_1, \dots, a_m$ this formula simplifies, but in general $a_1, \dots, a_m$ need not be commuting.

3.17.2. Remark. The noncommutative geometry in terms of a scheme theory for associative algebras depends heavily on sheaf theory own a Zariski topology, even a noncommutative version thereof [33]. In this theory, instead of starting from a noncommutative algebra and dealing with its geometry as being "virtual" we now can consider concretely defined geometrical objects, but defined over a noncommutative field **H**. The noncommutative **H**-algebras that appear are the rings of locally analytic functions in an open set U for the real topology, i.e. $C^{\omega}_{(z,\tilde{z})}(U,\mathbf{H})$. This obviously leads to the possibility to define presheaves and sheaves on the objects embedded into $\mathbf{H}^n$ and endowed with the induced real topology; it also applies to **H**-analytic objects like the four dimensional manifold W constructed before Theorem 3.8, and indeed to any quaternion version of a general manifold, that is a "manifold" with a local $\mathbf{H}^n$-structure generalizing in the obvious way the local $\mathbf{R}^m$-structure. In later work we aim to study the quaternion version of sheaf cohomology and Cartan Theorems A and B, as well as noncommutative Stein manifolds, i.e. the quaternion version of holomorphy domains.

3.18. Theorem. *Let f be a quaternion holomorphic function on an open subset U in* **H**. *Suppose that $\varepsilon > 0$ and K is a compact subset of U. Then there exists a function $g(z) = P_\infty(z) + \sum_{k=1}^{\nu} P_k[(z-a_k)^{-1}]$, $z \in \mathbf{H}\setminus\{a_1,\dots,a_\nu\}$, $\nu \in \mathbf{N}$, where P_∞ and P_j are polynomials, $a_j \in \mathrm{Fr}(U)$, $\mathrm{Fr}(U)$ denotes a topological boundary of U in* **H**, *such that $|f(z) - g(z)| < \varepsilon$ for each $z \in \mathsf{K}$.*

Proof is analogous to the proof of Runge's theorem (see [18]) due to Theorem 3.9 and considering four dimensional cubes $S_{j,k} = {}_1S_j \times {}_2S_k$ with ribs of length n^{-1} in **H** instead of two dimensional cubes in **C** and putting $S := \bigcup_{j,k} S_{j,k}$ such that $\mathsf{K} \subset \mathrm{Int}(S)$, where $n \in \mathbf{N}$ tends to infinity, ${}_1S_j$ and ${}_2S_k$ are two dimensional cubes in $\mathbf{C}_1$ and $\mathbf{C}_2$ which are two copies of **C** embedded orthogonally in **H** as $\mathbf{R}I \oplus \mathbf{R}J$ and $\mathbf{R}K \oplus \mathbf{R}L$ correspondingly. Since f is quaternion holomorphic and **K** is compact, we may apply formula (3.9) to each $B_{j,k} = \gamma$ such that $\gamma = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} \\ -\bar{\gamma}_{1,2} & \bar{\gamma}_{1,1} \end{pmatrix}$, $\gamma_{1,1} = \partial\, {}_1S_j$, $\gamma_{1,2} = \partial\, {}_2S_k$, it can be seen, that f can be approximated uniformly on K by a sum of the form $\sum_{k=1}^{\mu}(a_{1,k}(\zeta_k - z)^{-1}a_{2,k})$, where $a_{j,k} \in \mathbf{H}$, $\zeta_k \in \mathrm{Fr}(S)$. For a given $n \in \mathbf{N}$ if $b \in \mathrm{Fr}(S)$, then there exists $a \in \mathrm{Fr}(U) \cup \partial B(0,r,\mathbf{H})$ such that $|b-a| \leqslant n^{-1}$. If $z \in \mathsf{K}$ and $|z-a| \geqslant n^{-1}$, then the series $(z-b)^{-1} = (\sum_{k=0}^{\infty}[(z-a)^{-1}(b-a)]^k)(z-a)^{-1}$ converges uniformly on K and it is clear that f can be approximated uniformly on K by a function of the indicated form (see also §3.17).

3.19. Note and Definitions. Consider a one-point (Alexandroff) compactification $\widehat{\mathbf{H}}$ of the locally compact topological space **H**. It is homeomorphic to a unit four dimensional sphere S^4 in the Euclidean space $\mathbf{R}^5$. If ζ is a point in S^4 different from $(1,0,0,0,0)$, then the straight line containing $(1,0,0,0,0)$ and ζ crosses π_S in a finite point z, where π_S is the four dimensional plane orthogonal to the vector $(1,0,0,0,0)$ and tangent to S^4 at the south pole $(-1,0,0,0,0)$. This defines the bijective continuous mapping from $S^4\setminus\{(1,0,0,0,0)\}$ onto π_S such that $(1,0,0,0,0)$ corresponds to the point of infinity. Therefore each function on a subset U of **H** as a topological space can be considered on the homeomorphic subset V in S^4. If U is a locally compact subset of **H** and $\lim_{z\in U,\,|z|\to\infty} f(z)$ exists, then f has an extension on $\widehat{U}$.

Let $z_0 \in \widehat{\mathbf{H}}$ be a marked point. If a function f is defined and quaternion holomorphic on $V\setminus\{z_0\}$, where V is a neighbourhood of z_0, then z_0 is called a point of an isolated singularity of f.

Suppose that f is a quaternion holomorphic function in $B(a,0,r,\mathbf{H})\setminus\{a\}$ for some $r>0$. Then we say that f has an isolated singularity at a. Let $B(\infty,r,\mathbf{H}):=\{z\in\widehat{\mathbf{H}}$ such that $r^{-1}<|z|\leqslant\infty\}$. The we say that f has an isolated singularity at ∞ if it is quaternion holomorphic in some $B(\infty,r,\mathbf{H})$.

Let $f:U\to\mathbf{H}$ be a function, where U is a neighbourhood of $z\in\widehat{\mathbf{H}}$. Then f is said to be meromorphic at z if f has an isolated singularity at z. If U is an open subset in $\widehat{\mathbf{H}}$, then f is called meromorphic in U if f is meromorphic at each point $z\in U$. If U is a domain of f and f is meromorphic in U, then f is called meromorphic on U. Denote by $\mathbf{M}(U)$ the set of all meromorphic functions on U. Let f be meromorphic on a region U in $\widehat{\mathbf{H}}$. A point $c\in\bigcap_{V\subset U,V}$ is compact $\mathrm{cl}(f(U\setminus V))$ is called a cluster value of f.

3.20. Proposition. *Let f be a quaternion holomorphic function with a right $\mathbf{H}$-superlinear superdifferential on an open connected subset $U\subset\widehat{\mathbf{H}}$ and suppose that there exists a sequence of points $z_n\in U$ having a cluster point $z\in U$ such that $f(z_n)=0$ for each $n\in\mathbf{N}$, then $f=0$ everywhere on U.*

Proof follows from the local z-analyticity of f and the fact $f^{(k)}(z)=0$ for each $0\leqslant k\in\mathbf{Z}$ (see Theorems 2.11 and 3.10), when $f'(z)$ is right $\mathbf{H}$-superlinear on U, since

$$f^{(k)}(z)=\lim_{n+m\to\infty}\big(f^{(k-1)}(\zeta_n)-f^{(k-1)}(\zeta_m)\big)(\zeta_n-\zeta_m)^{-1},$$

where ζ_n is a subsequence of $\{z_n:n\}$ of pairwise distinct points converging to z. Therefore, f is equal to zero on a neighbourhood of z. The maximal subset of U on which f is equal to zero is open in U. On the other hand it is closed, since f is continuous, hence f is equal to zero on U, since U is connected.

3.21. Note. Without the condition of right $\mathbf{H}$-superlinearity of $f'(z)$ on U Proposition 3.20 is not true in general, since $f_1(z):=azb$ and $f_2(z):=abz$ coincide on $\mathbf{R}I$, but not on any neighbourhood of zero, when a and b are noncommuting fixed quaternions, $z\in\mathbf{H}$.

Consider a function $f(z)=z^{-1}az$ on $\mathbf{H}\setminus\{0\}$, where $0\neq z\in\mathbf{H}$ and $a\in\mathbf{H}$. If $a\neq vI$, then there exists $0\neq h\in\mathbf{H}$ such that $h^{-1}ah=:b\neq a$. For $z=sh$ there exists $\lim_{z=sh,\,s\neq0,\,s\to0}f(z)=b$, for $z=sI$ $\lim_{z=sI,\,s\neq0,\,s\to0}f(z)=a$. Therefore, if $a\neq vI$ for some $v\in\mathbf{R}$, then there does not exist a limit of $f(z)$ for z tending to zero. This makes clear, that for the quaternion field it is important to consider an analog of a Laurent series of a function quaternion holomorphic on $U\setminus\{0\}$ not only in terms az^k, but also in $a_1z^{k_1}\dots a_nz^{k_n}$, where k_j are integers, $z^0:=1$.

3.22. Theorem. *Let $\mathbf{A}$ denote the family of all functions f such that f is quaternion holomorphic on $U:=\mathrm{Int}(B(a,r,R,\mathbf{H}))$, where a is a marked point in $\mathbf{H}$, $0\leqslant r<R<\infty$ are fixed. Let $\mathbf{S}$ denote a subset of $\mathbf{Z}^{\mathbf{N}}$ such that for each $k\in S$ there exists $m(k):=\max\{j:\ k_j\neq0,\ k_i=0$ for each $i>j\}\in\mathbf{N}$ and let $\mathbf{B}$ be a family of finite sequences $b=(b_1,\dots,b_n)$ such that $b_j\in\mathbf{H}$ for each $j=1,\dots,n$, $n\in\mathbf{N}$. Then there exists a bijective correspondence between $\mathbf{A}$ and $q\in\mathbf{B}^S$ such that*

$$\lim_{m+\eta\to\infty}\ \sup_{z\in B(a,r_1,R_1,\mathbf{H})}\ \sum_{k,\ m(k)=m,\ \eta(k)=\eta}\big|(b_k,z^k)\big|=0 \tag{3.19}$$

for each r_1 and R_1 such that $r < r_1 < R_1 < R$, where $\eta(k) := k_1 + \cdots + k_{m(k)}$, $q(k) =:$ $b_k = (b_{k,1}, \ldots, b_{k,m(k)})$, $(b_k, z^k) = b_{k,1}z^{k_1} \ldots b_{k,m(k)}z^{k_{m(k)}}$ for each $k \in \mathbf{S}$, that is, $f \in \mathbf{A}$ can be presented by a convergent series

$$f(z) = \sum_{b \in q} (b_k, z^k). \tag{3.20}$$

Proof. If condition (3.19) is satisfied, then the series (3.20) converges on $B(a, r', R', \mathbf{H})$ for each r' and R such that $r < r' < R < R$, since r_1 and R_1 are arbitrary such that $r < r_1 < R_1 < R$ and $\sum_{n=0}^{\infty} p^n$ converges for each $|p| < 1$. In particular taking $r_1 < r' < R' < R_1$ for $p = R'/R_1$ or $p = r_1/r'$.

Therefore, from (3.19) and (3.20) it follows, that f presented by the series (3.20) is quaternion holomorphic on U.

Vice versa let f be in $\mathbf{A}$. In view of Theorems 2.11 and 3.9 there are two rectifiable closed paths γ_1 and γ_2 such that $\gamma_2(s) = a + r' \exp(2\pi s M_2)$ and $\gamma_1(s) = a + R' \exp(2\pi s M_1)$, where $s \in [0, 1]$, M_1 and $M_2 \in \mathbf{H}$ with $|M_1| = 1$ and $|M_2| = 1$, where $r < r' < R' < R$, because as in §3.9 U can be presented as a finite union of regions U_j each of which satisfies the conditions of Theorem 2.11. Using a finite number of rectifiable paths w_j (joining γ_1 and γ_2 within U_j) going twice in one and the opposite directions leads to the conclusion that for each $z \in \mathrm{Int}(B(a, r', R', \mathbf{H}))$ the function $f(z)$ is presented by the integral formula:

$$f(z) = (2\pi)^{-1}\left\{\left(\int_{\gamma_1} f(\zeta)(\zeta - z)^{-1}\, d\zeta\right) M_1^{-1} - \left(\int_{\gamma_2} f(\zeta)(\zeta - z)^{-1}\, d\zeta\right) M_2^{-1}\right\}. \tag{3.21}$$

On γ_1 we have the inequality: $|(\zeta - a)^{-1}(z - a)| < 1$, on γ_2 another inequality holds: $|(\zeta - a)(z - a)^{-1}| < 1$. Therefore, for γ_1 the series

$$(\zeta - z)^{-1} = \left(\sum_{k=0}^{\infty} \bigl((\zeta - a)^{-1}(z - a)\bigr)^k\right)(\zeta - a)^{-1}$$

converges uniformly by $\zeta \in B(a, R_2 + \varepsilon, R_1, \mathbf{H})$ and $z \in B(a, r_2, R_2, \mathbf{H})$, while for γ_2 the series

$$(\zeta - z)^{-1} = -(z - a)^{-1}\left(\sum_{k=0}^{\infty} \bigl((\zeta - a)(z - a)^{-1}\bigr)^k\right)$$

converges uniformly by $\zeta \in B(a, r_1, r_2, -\varepsilon, \mathbf{H})$ and $z \in B(a, r_2, R_2, \mathbf{H})$ for each $r' < r_2 < R_2 < R'$ and each $0 < \varepsilon < \min(r_2 - r_1, R_1 - R_2)$. Consequently,

$$f(z) = \sum_{k=0}^{\infty} \bigl(\phi_k(z) + \psi_k(z)\bigr), \tag{3.22}$$

where

$$\phi_k(z) := (2\pi)^{-1}\left\{\left(\int_{\gamma_1} f(\zeta)\big((\zeta-a)^{-1}(z-a)\big)^k(\zeta-a)^{-1}\,d\zeta\right)M_1^{-1}\right\},$$

$$\psi_k(z) := (2\pi)^{-1}\left\{\left(\int_{\gamma_2} f(\zeta)(z-a)^{-1}\big((\zeta-a)(z-a)^{-1}\big)^k\,d\zeta\right)M_2^{-1}\right\},$$

and where $\phi_k(z)$ and $\psi_k(z)$ are quaternion holomorphic functions, hence f has decomposition (3.20) in U, since due to §§2.15 and 3.9 there exists $\delta > 0$ such that integrals for ϕ_k and ψ_k by γ_1 and γ_2 are the same for each $r' \in (r_1, r_1+\delta)$, $R' \in (R_1-\delta, R_1)$. Using the definition of the quaternion line integral we get (3.20) converging on U. Varying $z \in U$ by $|z|$ and $\mathrm{Arg}(z)$ we get that (3.20) converges absolutely on U, consequently, (3.19) is satisfied.

3.23. Notes and Definitions. Let γ be a closed curve in $\mathbf{H}$. There are natural projections from $\mathbf{H}$ on complex planes: $\pi_1(z) = (v, w)$, $\pi_2(z) = (v, x)$, $\pi_3(z) = (v, y)$, where $z = vI + wJ + xK + yL$ with real v, w, x and y. Therefore, $\pi_l(\gamma) =: \gamma_l$ are curves in complex planes $\mathbf{C}_1$ isomorphic to $\mathbf{R}I \oplus \mathbf{R}J$, $\mathbf{C}_2$ isomorphic to $\mathbf{R}I \oplus \mathbf{R}K$ and $\mathbf{C}_3$ isomorphic to $\mathbf{R}I \oplus \mathbf{R}L$, where $l = 1, 2, 3$ respectively. A curve γ in $\mathbf{H}$ is closed (a loop, in another words) if and only if γ_l are closed for each $l = 1, 2, 3$, that is, $\gamma(0) = \gamma(1)$ and $\gamma_l(0) = \gamma_l(1)$ correspondingly. For each point $a \in \mathbf{H}$ we have its projections $a_l := \pi_l(a)$. In each complex plane there is the standard complex notion of a topological index $\mathrm{In}(a_l, \gamma_l)$ of a curve γ_l at a_l. Therefore, there exists a vector $\mathrm{In}(a, \gamma) := \{\mathrm{In}(a_1, \gamma_1), \mathrm{In}(a_2, \gamma_2), \mathrm{In}(a_3, \gamma_3)\}$ which we call the topological index of γ at a point $a \in \mathbf{H}$. This topological index is invariant relative to homotopies satisfying conditions of Theorem 3.9. Consider now a standard closed curve $\gamma(s) = a + r\exp(2\pi snM)$, where $M \in \mathbf{H}_i$ with $|M| = 1$, $n \in \mathbf{Z}$, $r > 0$, $s \in [0, 1]$. Then $\hat{\mathrm{In}}(a, \gamma) := (2\pi)^{-1}(\int_\gamma d\,\mathrm{Ln}(z-a)) = nM$ is called the quaternion index of γ at a point a. It is also invariant relative to homotopies satisfying the conditions of Theorem 3.9. Moreover, $\hat{\mathrm{In}}(h_1 a h_2, h_1\gamma h_2) = \hat{\mathrm{In}}(a, \gamma)$ for each h_1 and $h_2 \in \mathbf{H}\setminus\{0\}$ such that $h_1 M h_2 = M$. For $M = wJ + xK + yL$ there is the equality $\hat{\mathrm{In}}(a, \gamma) = \mathrm{In}(a_1, \gamma_1)wJ + \mathrm{In}(a_2, \gamma_2)xK + \mathrm{In}(a_3, \gamma_3)yL$ (adopting the corresponding convention for signs of indexes in each $\mathbf{C}_j$ and the convention of positive directions of going along curves). In view of the properties of Ln for each curve ψ in $\mathbf{H}$ there exists $\int_\gamma d\,\mathrm{Ln}(z-a) = 2\pi qM$ for some $q \in \mathbf{R}$ and $M \in \mathbf{H}_i$ with $|M| = 1$. For a closed curve ψ up to a composition of homotopies each of which is charaterized by homotopies in $\mathbf{C}_l$ for $l = 1, 2, 3$ there exists a standard γ with a generator M for which $\hat{\mathrm{In}}(a, \gamma) = qM$, where $q \in \mathbf{Z}$. Therefore, we can take as a definiton $\hat{\mathrm{In}}(a, \psi) = \hat{\mathrm{In}}(a, \gamma)$. Define also the residue of a meromorphic function with an isolated singularity at a point $a \in \mathbf{H}$ as $\mathrm{res}(a, f) := (\int_\gamma f(z)\,dz)(2\pi M)^{-1}$, where $\gamma(s) = a + r\exp(2\pi sM) \subset V$, $r > 0$, $|M| = 1$, $M \in \mathbf{H}_i$, $s \in [0, 1]$, f is quaternion holomorphic on $V\setminus\{a\}$.

If f has an isolated singularity at $a \in \widehat{\mathbf{H}}$, then coefficients b_k of its Laurent series (see §3.22) are independent of $r > 0$. The common series is called the a-Laurent series. If $a = \infty$, then $g(z) := f(z^{-1})$ has a 0-Laurent series c_k such that $c_{-k} = b_k$. Let $\beta := \sup_{b_k \neq 0} \eta(k)$, where $\eta(k) = k_1 + \cdots + k_m$, $m = m(k)$ for $a = \infty$; $\beta = \inf_{b_k \neq 0} \eta(k)$ for $a \neq \infty$. We say that f has a removable singularity, pole, essential singularity at ∞ according as $\beta \leqslant 0$, $0 < \beta < \infty$, $\beta = +\infty$. In the second case β is called the order of the pole at ∞. For a finite a the corresponding cases are: $\beta \geqslant 0$, $-\infty < \beta < 0$, $\beta = -\infty$. If f has a pole at a, then $|\beta|$ is called the order of the pole at a.

A value of a function $\partial_f(a) := \inf\{\eta(k)\colon\ b_k \neq 0\}$ is called a divisor of f at $a \neq \infty$, $\partial_f(a) := \inf\{-\eta(k)\colon\ b_k \neq 0\}$ for $a = \infty$. Then $\partial_{f+g}(a) \geqslant \min\{\partial_f(a), \partial_g(a)\}$ for each $a \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$ and $\partial_{fg}(a) = \partial_f(a) + \partial_g(a)$. For a function f meromorphic on an open subset U in $\widehat{\mathbf{H}}$ the function $\partial_f(p)$ by the variable $p \in U$ is called the divisor of f.

3.24. Theorem. *Let U be an open region in $\widehat{\mathbf{H}}$ with n distinct marked points $p_1, \dots, p_n$, and let f be a quaternion holomorphic function on $U \setminus \{p_1, \dots, p_n\} =: U_0$ and ψ be a rectifiable closed curve lying in U_0 such that U_0 satisfies the conditions of Theorem* 3.9 *for each $z_0 \in \{p_1, \dots, p_n\}$. Then*

$$\int_{\psi} f(z)\,dz = 2\pi \sum_{j=1}^{n} \hat{\mathrm{I}}\mathrm{n}(p_j, \psi)\,\mathrm{res}(p_j, f).$$

Proof. For each p_j consider the principal part T_j of a Laurent series for f in a neighbourhood of p_j, that is, $T_j(z) = \sum_{k,\eta(k)<0} (b_k, (z - p_j)^k)$, where $\eta(k) = k_1 + \cdots + k_n$ for $k = (k_1, \dots, k_n)$ (see Theorem 3.22). Therefore, $h(z) := f(z) - \sum_j T_j(z)$ is a function having a quaternion holomorphic extension on U. In view of Theorem 3.9 for a quaternion holomorphic function g in a neighbourhood V of a point p and a rectifiable closed curve ζ we have

$$\hat{\mathrm{I}}\mathrm{n}(p, \zeta) g(p) = (2\pi)^{-1} \left(\int_{\zeta} g(z)(z - p)^{-1}\,dz \right)$$

(see §3.23). We may consider small ζ_j around each p_j with $\hat{\mathrm{I}}\mathrm{n}(p_j, \zeta_j) = \hat{\mathrm{I}}\mathrm{n}(p_j, \gamma)$ for each $j = 1, \dots, n$. Then $\int_{\zeta_j} f(z)\,dz = \int_{\zeta_j} T_j(z)\,dz$ for each j. Representing U_0 as a finite union of open regions U_j and joining ζ_j with γ by paths w_j going in one and the opposite direction as in Theorem 3.9 we get

$$\int_{\gamma} f(z)\,dz + \sum_j \int_{\zeta_j^-} f(z)\,dz = 0,$$

consequently,

$$\int_{\gamma} f(z)\,dz = \sum_j \int_{\zeta_j} f(z)\,dz = \sum_j 2\pi \hat{\mathrm{I}}\mathrm{n}(p_j, \gamma)\,\mathrm{res}(p_j, f),$$

where $\hat{\mathrm{I}}\mathrm{n}(p_j, \gamma)$ and $\mathrm{res}(p_j, f)$ are invariant relative to homotopies satisfying conditions of Theorem 3.9.

3.25. Corollary. *Let f and T be the same as in* §3.24*, then* $\mathrm{res}(p_j, f) = \mathrm{res}(p_j, T_j) = \mathrm{res}(p_j, \sum_{k,\eta(k)=-1} (b_k, (z - p_j)^k))$*, in particular,* $\mathrm{res}(p_j, b(z - p_j)^{-1}) = b$.

3.26. Corollary. *Let U be an open region in $\widehat{\mathbf{H}}$ with n distinct points $p_1, \dots, p_n$, let also f be a quaternion holomorphic function on $U \setminus \{p_1, \dots, p_n\} =: U_0$, $p_n = \infty$, and*

U_0 satisfies conditions of Theorem 3.9 *with at least one ψ, γ and each $z_0 \in \{p_1, \dots, p_n\}$. Then $\sum_{p_j \in U} \operatorname{res}(p_j, f) = 0$.*

Proof. If γ is a closed curve encompassing $p_1, \dots, p_{n-1}$, then $\gamma^-(s) := \gamma(1-s)$, where $s \in [0,1]$, encompasses $p_n = \infty$ with positive going by γ^- relative to p_n. Since $\int_\gamma f(z)\,dz + \int_{\gamma^-} f(z)\,dz = 0$, we get the statement of this corollary from Theorem 3.24.

3.27. Definitions. Let f be a holomorphic function on a neighbourhood V of a point $z \in \mathbf{H}$. Then the infimum: $n(z; f) := \inf\{k\colon\ k \in \mathbf{N},\ f^{(k)}(z) \neq 0\}$ is called a multiplicity of f at z. Let f be a holomorphic function on an open subset U in $\widehat{\mathbf{H}}$. Suppose $w \in \widehat{\mathbf{H}}$, then the valence $\nu_f(w)$ of f at w is by the definition $\nu_f(w) := \infty$, when the set $\{z\colon\ f(z) = w\}$ is infinite, and otherwise $\nu_f(w) := \sum_{z, f(z)=w} \eta(z; f)$.

3.27.1. Theorem. *Let f be a meromorphic function on a region $U \subset \widehat{\mathbf{H}}$. If $b \in \widehat{\mathbf{H}}$ and $\nu_f(b) < \infty$, then b is not a cluster value of f and the set $\{z\colon\ \nu_f(z) = \nu_f(b)\}$ is a neighbourhood of b. If $U \neq \widehat{\mathbf{H}}$ or f is not constant, then the converse statement holds. Nevertheless, it is false, when $f =$ const on $\widehat{\mathbf{H}}$.*

3.27.2. Theorem. *Let U be a proper open subset of $\widehat{\mathbf{H}}$, let also f and g be two continuous functions from $\overline{U} := \operatorname{cl}(U)$ into $\widehat{\mathbf{H}}$ such that on a topological boundary $\operatorname{Fr}(U)$ of U they satisfy the inequality $|f(z)| < |g(z)|$ for each $z \in \operatorname{Fr}(U)$. Suppose f and g are meromorphic functions in U and h be a unique continuous map from $\overline{U}$ into $\widehat{\mathbf{H}}$ such that $h|_E = f|_E + g|_E$, where $E := \{z\colon\ f(z) \neq \infty,\ g(z) \neq \infty\}$. Then $\nu_{g|U}(0) - \nu_{g|U}(\infty) = \nu_{h|U}(0) - \nu_{h|U}(\infty)$.*

Proofs of these two theorems are analogous to that of Theorems VI.4.1, 4.2 [18].

3.28. Theorem. *Let U be an open subset in $\mathbf{H}^n$, then there exists a representation of the $\mathbf{R}$-linear space $C^\omega_{z,\tilde{z}}(U, \mathbf{H})$ of locally $(z, \tilde{z})$-analytic functions on U such that it is isomorphic to the $\mathbf{R}$-linear space $C^\omega_z(U, \mathbf{H})$ of quaternion holomorphic functions on U.*

Proof. Evidently, the proof can be reduced to the case $n = 1$ by induction considering local $(z, \tilde{z})$-series decompositions by $({}^n z, {}^n \tilde{z})$ with coefficients being convergent series of $({}^1 z, {}^1 \tilde{z}, \dots, {}^{n-1} z, {}^{n-1} \tilde{z})$. For each $z \in \mathbf{H}$ there are identities: $JzJ = -vI - wJ + xK + yL$, $KzK = -vI + wJ - xK + yL$, $LzL = -vI + wJ + xK - yL$, where $z = vI + wJ + xK + yL$ with v, w, x and $y \in \mathbf{R}$. Hence

$$P_{\mathbf{R}I}(z) = vI = (z - JzJ - KzK - LzL)/4,$$

$$P_{\mathbf{R}J}(z) = wJ = (z - JzJ + KzK + LzL)/4,$$

$$P_{\mathbf{R}K}(z) = xK = (z + JzJ - KzK + LzL)/4,$$

$$P_{\mathbf{R}L}(z) = yL = (z + JzJ + KzK - LzL)/4$$

are projection operators on $\mathbf{R}I, \mathbf{R}J, \mathbf{R}K$ and $\mathbf{R}L$ respectively, where I, J, K and L are orthogonal vectors relative to the scalar product in $\mathbf{C}^4$, $\mathbf{H} \ni z \mapsto (t, u, -\bar{u}, \bar{t}) \in \mathbf{C}^4$. Therefore,

$$\tilde{z} = vI - wJ - xK - yL = -(z + JzJ + KzK + LzL)/2$$

and

$$d\tilde{z} = (dv)I - (dw)J - (dx)K - (dy)L \\ = -\big(dz + J(dz)J + K(dz)K + L(dz)L\big)/2.$$

Consequently, each polynomial in $(z, \tilde{z})$ is also a polynomial in z only, moreover, each polynomial locally $(z, \tilde{z})$ analytic function on U is polynomial locally z-analytic on U. Then if a series by $(z, \tilde{z})$ converges in a ball $B(z_0, r, \mathbf{H}^n)$, then its series in the z-representation converges in a ball $B(z_0, r/2, \mathbf{H}^n)$. Then $\int_\gamma \tilde{z}\,dz = -(\int_\gamma z\,dz + \int_\gamma JzJ\,dz + \int_\gamma KzK\,dz + \int_\gamma LzL\,dz)/2$ and $\int_\gamma \tilde{z}\,dz = 0$ for a closed rectifiable curve γ in $\mathbf{H}$ in such representation. This is not contradictory, because from $f_1|_\gamma = f_2|_\gamma$ it does not follow $\hat{f}_1|_\gamma = \hat{f}_2|_\gamma$, since $\hat{f}(z)$ is defined by values of a function f on an open neighbourhood of a point $z \in \mathbf{H}$, where f, f_1 and $f_2 \in C^0(U, \mathbf{H})$. Therefore, $\int_\gamma d\,\mathrm{Ln}\,z$ is quite different in general from $\int_\gamma \tilde{z}\,dz$ (see §§2.5 and 3.8). Considering basic polynomials of any polynomial basis in $C^\omega_{z,\tilde{z}}(U, \mathbf{H})$ we get (due to infinite dimensionality of this space) a polynomial base of $C^\omega_z(U, \mathbf{H})$. This establishes the $\mathbf{R}$-linear isomorphism between these two spaces. Moreover, in such representation of the space $C^\omega_{z,\tilde{z}}(U, \mathbf{H})$ we can put $D_{\tilde{z}} = 0$, yielding for differential forms $\partial_{\tilde{z}} = 0$, this leads to differential calculus and integration with respect to D_z and dz only.

3.29. Notes. The latter paragraph also shows that for ${}_lC^\omega_{z,\tilde{z}}(U, \mathbf{H})$ and for ${}_rC^\omega_{z,\tilde{z}}(U, \mathbf{H})$ operators D_z and $D_{\tilde{z}}$ are different and neither D_z nor $D_{\tilde{z}}$ may be omitted from the differential calculus, since automorphisms $z \mapsto azb$ of $\mathbf{H}$ with given quaternions a and b such that $ab \neq 0$ do not leave ${}_lC^\omega_{z,\tilde{z}}(U, \mathbf{H})$ and ${}_rC^\omega_{z,\tilde{z}}(U, \mathbf{H})$ invariant.

Apart from the complex polynomial case in the quaternion case a polynomial may have infinite family of roots, for example, $P(z) = z^2 + zJzJ + zKzK + zLzL - 1$ has a 3-dimensional over $\mathbf{R}$ manifold of roots $P(z) = 0$, since $P(z) = -2|z|^2 - 1$.

Theorem 3.27.2 is the quaternion analog of the Rouché theorem.

The function $f(z) := \cos(z\tilde{z}) := [\exp(Jz\tilde{z}) + \exp(-Jz\tilde{z})]/2$ is bounded on $\mathbf{H}$, but neither the operator $\hat{f}$ is right superlinear, nor the operator $[f(z)(\zeta - z)]^{-2}]\hat{}.h|\zeta - z|^2$ is bounded on $\mathbf{H}^2 \times B(0, 1, \mathbf{H})$ for $|\zeta| \geqslant 2|z|$, then $f(z) = \cos(z\tilde{z})$ can be written in the corresponding representation as a quaternion locally z-analytic function, since $\sum_{n=1}^\infty (2R)^n/n!$ converges for each $0 \leqslant R < \infty$. This shows that in the last part of Theorem 3.15 its conditions cannot be replaced by boundedness of a quaternion holomorphic function f. An interesting analog of the Liouville theorem for real harmonic functions was investigated in [5]. Possibly the particular case of the quaternion analog of the Liouville theorem for right superlinear $\hat{f}$ may be deduced from [5] with the help of Eq. (2.4) of Corollary 2.4 above.

There are other ways to define superdifferentiations of algebras of quaternion functions:

(1) factorize an algebra of quaternion locally $(z,\tilde{z})$-analytic functions $f: U \to \mathbf{H}$ by all relations of the form $[\sum_j S_{j,1}zS_{j,2} - \tilde{z}]$, where $S_{j,k} \in \mathbf{H}$ are fixed and $\sum_j S_{j,1}zS_{j,2} = \tilde{z}$ for each $z \in \mathbf{H}$;

(2) use as a starting point superlinearly superdifferentiable functions $f: U \to \mathbf{H}$ and then prolong a superdifferentiation on products of such functions with milder conditions on a superdifferential, but they lead to the same result. This approach can be generalized for general Clifford algebras over $\mathbf{R}$, but some results will be weaker or take another form, than in the case of $\mathbf{H}^n$.

3.30. Theorem (Argument principle). *Let f be a quaternion holomorphic function on an open region U satisfying conditions of* §3.9 *and let γ be a closed curve contained in U, then* $\hat{\mathrm{I}}\mathrm{n}(0; f \circ \gamma) = \sum_{\partial_f(a) \neq 0} \hat{\mathrm{I}}\mathrm{n}(a;\gamma)\partial_f(a)$.

Proof. There is the equality $\hat{\mathrm{I}}\mathrm{n}(0; f \circ \gamma) = \int_{\zeta \in \gamma} d\,\mathrm{Ln}(f(\zeta)) = \int_0^1 d\,\mathrm{Ln}(f \circ \gamma(s)) = \int_\gamma f^{-1}(\zeta)\,df(\zeta)$. Let $\partial_f(a) = n \in \mathbf{N}$, then

$$f^{-1}(a)f'(a).S = \sum_{l,k;\ n_1+\cdots+n_k=\partial_f(a),\ 0 \leqslant n_j \in \mathbf{Z},\ j=1,\ldots,k} (z-a)^{n_1} g_{S,l,k,1;n_1,\ldots,n_k}(z) \times (z-a)^{n_2} g_{S,l,k,2;n_1,\ldots,n_k}(z) \ldots (z-a)^{n_k} g_{S,l,k,k;n_1,\ldots,n_k}(z),$$

where $g_{S,l,p,k;n_1,\ldots,n_k}(z)$ are quaternion holomorphic functions of z on U such that $g_{S,l,p,k;n_1,\ldots,n_k}(a) \neq 0$, $S \in \{I,J,K,L\}$, where $l = 1,\ldots,m$, $1 \leqslant m \leqslant 4^{\partial_f(a)}$ (see §§2.8, 3.7, 3.22, 3.28), since each term $\xi(z)(v-v_0)^{n_1}(w-w_0)^{n_2}(x-x_0)^{n_3}(y-y_0)^{n_4}$ with $n_1+\cdots+n_4 \geqslant \partial_f(a)$, $n_j \geqslant 0$, has such decomposition, where $\xi(z)$ is a quaternion holomorphic function on a neighbourhood of a such that $\xi(a) \neq 0$. Suppose ψ is a closed curve such that $\hat{\mathrm{I}}\mathrm{n}(p,\psi) = 2\pi n M$, $|M| = 1$, $M \in \mathbf{H}_i$, $0 \neq n \in \mathbf{Z}$. Then we can define a curve $\psi^{1/n} =: \omega$ as a closed curve for which $\hat{\mathrm{I}}\mathrm{n}(p,\omega) = 2\pi M$ and $\omega([0,1]) \subset \psi([0,1])$. Then we call $\omega^n = \psi$. That is, $\hat{\mathrm{I}}\mathrm{n}(p,\psi^{1/n}) = \hat{I}(p,\psi)/n$. The latter formula allows an interpretation also when $\hat{\mathrm{I}}\mathrm{n}(p,\psi)/n$ is equal to $2\pi q M$, where $0 \neq q \in \mathbf{Q}$. That is, a curve $\psi^{1/n}$ can be defined for each $0 \neq n \in \mathbf{Z}$. This means that γ can be presented as union of curves ω_j for each of which there exists $n_j \in \mathbf{N}$ such that $\omega_j^{n_j}$ is a closed curve. Using Theorem 3.9 for each $a \in U$ with $\partial_f(a) \neq 0$, also using the series given above we can find a finite family of ω_j for which one of the terms in the series is not less, than any other term. We may also use small homotopic deformations of ω_j satisfying the conditions of Theorem 3.9 such that in the series one of the terms is greater than any other for almost all points on ω_j. Such deformation is permitted, since otherwise two terms would coincide on an open subset of U, that is impossible. Considering such series, formulas (2.6), (2.7) and using Theorem 3.27.2 we get the statement of this theorem.

3.31. Theorem. *If f has an essential singularity at a, then* $\mathrm{cl}(f(V)) = \widehat{\mathbf{H}}$ *for each* $V \subset \mathrm{dom}(f)$, $V = U \setminus \{a\}$, *where U is a neighbourhood of a.*

Proof. Suppose that the statement of this theorem is false, then there would exist $r > 0$ and $m > 0$ and a quaternion $A \in \mathbf{H}$ such that f is z-analytic in $B(a,0,r,\mathbf{H}) \setminus \{a\}$ and $|f(z) - A| \geqslant m$ for each z such that $0 < |z-a| < r$. If $\infty \notin \mathrm{cl}(f(V))$, then there exists

$R > 0$ such that $A \notin \mathrm{cl}(f(V))$ for each $|A| > R$. Therefore, the function $[f(z) - A]^{-1}$ is quaternion holomorphic in $B(a, 0, r, \mathbf{H})\setminus\{a\}$. Hence $[f(z) - A]^{-1} = \sum_k (p_k, (z-a)^k)$, where in this sum $k = (k_1, \dots, k_{m(k)})$ with $k_j \geqslant 0$ for each $j = 1, \dots, m(k) \in \mathbf{N}$, p_k are finite sequences of coefficients for $[f(z) - A]^{-1}$ as in §3.22. If $D_z^n([f(z) - A]^{-1})|_{z=a} = 0$ for each $n \geqslant 0$, then $[f(z) - A]^{-1} = 0$ in a neighbourhood of a. Therefore, $[f(z) - A]^{-1} = \sum_{n_1+\cdots+n_l=n} g_1 z^{n_1} \dots g_l z^{n_l}$ for some n such that $0 \leqslant n \in \mathbf{N}$, $n_j \geqslant 0$ for each $j = 1, \dots, l \in \mathbf{N}$, each g_j is a quaternion holomorphic function (of z). Consequently, taking inverses of both sides $[f(z) - A]$ and $(\sum_{n_1+\cdots+n_l=n} g_1 z^{n_1} \dots g_l z^{n_l})^{-1}$ and comparing their expansion series we see that finite sequences b_k of expansion coefficients for f have the property $b_k = 0$ for each $\eta(k) < -n$. This contradicts the hypothesis and proves the theorem.

3.32. Definition. Let a and b be two points in $\mathbf{H}$ and θ be a stereographic mapping of the unit four dimensional real sphere S^4 on $\widehat{\mathbf{H}}$. Then $\chi(a, b) := |\phi(a) - \phi(b)|_{\mathbf{R}^5}$ is called the chordal metric, where $\phi := \theta^{-1} : \widehat{\mathbf{H}} \to \mathbf{S}^4$, S^4 is embedded in $\mathbf{R}^5$ and $|*|_{\mathbf{R}^5}$ is the Euclidean distance in $\mathbf{R}^5$.

3.32.1. Theorem. *Let U be an open region in $\widehat{\mathbf{H}}$, $\{f_n\colon n \in \mathbf{N}\}$ be a sequence of functions meromorphic on U tending uniformly in U to f relative to the chordal metric. Then either f is the constant ∞ or else f is meromorphic on U.*

3.32.2. Theorem. *Let $\{f_k : k \in \mathbf{N}\}$ be a sequence of meromorphic functions on an open subset U in $\widehat{\mathbf{H}}$, which tends uniformly in the sence of the chordal metric in U to f, $f \neq \mathrm{const}$. If $f(a) = b$ and $r > 0$ are such that $B(a, r, \mathbf{H}) \subset U$ and $f(z) \neq b$ for each $z \in B(a, r, \mathbf{H})\setminus\{a\}$, then there exists $m \in \mathbf{N}$ such that the value of the valence of $f_k|_{B(a,r,\mathbf{H})}$ at b is $n(b; f) = n(a; f)$ for each $k \geqslant m$.*

3.32.3. Note. The proofs of these theorems are formally similar to the proofs of VI.4.3 and 4.4 [18]. Theorem 3.32.2 is the quaternion analog of the Hurwitz theorem. There are also the following quaternion analogs of the Mittag-Leffler and Weierstrass theorems. Their proofs are similar to those for Theorems VIII. 1.1 and 1.2 respectively. Nevertheless the second part of the Weierstrass theorem is not true in general because of noncommutativity of $\mathbf{H}$, that is, a function $h \in \mathbf{M}(U)$ with $\partial_h = \partial$ is not necessarily representable as $h = fg$, where g is quaternion holomorphic on U and f is another marked function $f \in \mathbf{M}(U)$ such that $\partial_f = \partial$. In the proofs ordered products of more elementary polynomial functions and in particular linear terms $(z - b_k)$ have to be considered as in §3.28, using Theorems 3.17 and 3.22. Theorem 3.33.2 is not true in general without condition of right superlinearity (or left superlinearity) of the superdifferential, for example, the function $f(z) = xK$ serves as a counterexample, where $z = vI + wJ + xK + yL$, v, w, x and $y \in \mathbf{R}$, $z \in \mathbf{H}$.

3.33. Theorem. *Let U be a nonempty proper open subset of $\widehat{\mathbf{H}}$, let $A \subset U$ not containing any cluster point in U. Let there be a function $g_b \in \mathbf{M}(\widehat{\mathbf{H}})$ for each $b \in A$ having a pole at b and no other. Then there exists $f \in \mathbf{M}(U)$ quaternion holomorphic on $U\setminus B$ and having the same principal part at b as g_b. If f is such a function, then each other such function is the function $f + g$, where g is quaternion holomorphic on U.*

3.33.1. Theorem. *Let U be a proper nonempty open subset of $\widehat{\mathbf{H}}$. Let $\partial : U \to \mathbf{Z}$ be a function such that $\{\partial(z) \neq 0\}$ does not have a cluster point in U. Then there exists $f \in \mathbf{M}(U)$ such that $\partial_f = \partial$.*

3.33.2. Theorem. *Let U be an open region in $\mathbf{H}$ and f be a function quaternion holomorphic on U with a right superlinear superdifferential on U. Suppose f is not constant and $B(a,r,\mathbf{H}) \subset U$, where $0 < r < \infty$. Then $f(B(a,r,\mathbf{H}))$ is a neighbourhood of $f(a)$ in $\mathbf{H}$.*

3.34. Remarks. For calculating expansion coefficients b_k of a function f quaternion holomorphic on $U \setminus \{z_0\}$, where U is open in $\mathbf{H}$, it is possible to use the residues $\operatorname{res}[(f(z)(z-z_0)^l)^{(n)}.(S_{j_1},\dots,S_{j_n})]$, where $S_j \in \{I,J,K,L\}$, $0 \leqslant l \in \mathbf{Z}$, $0 \leqslant n \in \mathbf{Z}$. But the system of equations for each $b_k = (b_{k,1},\dots,b_{k,m})$ is nonlinear in general. The calculation of a residue of a term (b_k, z^k) along the closed curve $\gamma(s) = r\exp(2\pi s M)$ (or ψ homotopic to it and satisfying conditions of Theorem 3.9) with $|M| = 1$, $M \in \mathbf{H}_i$, $s \in [0,1]$, reduces to a calculation of a $\mathbf{R}$-linear combination of integrals of the form $\int_0^1 \exp(2\pi s n_1 M_1)\dots\exp(2\pi s n_l M_l)\,ds A$, where $n_1,\dots,n_l \in \mathbf{Z}$, $n_1+\dots+n_l = 0$, $M_j := \widetilde{S}_j M S_j$, $S_j \in \{I,J,K,L\}$ for each $j = 1,\dots,l$, $A \in \{J,K,L\}$. The case of $f \in {}_lC^\omega(U,\mathbf{H})$ is trivial due to Corollary 3.25.

For several quaternion variables a multiple quaternion line integral

$$\mathbf{I} := \int_{\gamma_n}\biggl(\dots\biggl(\int_{\gamma_1} f({}^1z,\dots,{}^nz)\,d\,{}^1z\biggr)\dots\biggr)\,d\,{}^nz$$

may be naturally considered for rectifiable curves $\gamma_1,\dots,\gamma_n$ in $\mathbf{H}$. If $\gamma_j = r_j\exp(2\pi s_j M_j)$ with $0 < r_j < \infty$, $s_j \in [0,1]$ and $[M_k, M_j] = 0$ commute for each $k, j = 1,\dots,n$, then this integral $\mathbf{I}$ does not depend on the order of integration for $f \in C^0(U,\mathbf{H})$, where U is an open subset in $\widehat{\mathbf{H}}^n$ and $\gamma_j \subset {}^jU$ for each j, $U = {}^1U \times \dots \times {}^nU$, ${}^jU$ is an open subset in $\mathbf{H}$. Therefore, there is the natural generalization of Theorem 3.9 for several quaternion variables:

$$(2\pi)^n f(z_0) = \biggl(\int_{\psi_n}\biggl(\dots\biggl(\int_{\psi_1} f({}^1\zeta,\dots,{}^n\zeta)({}^1\zeta - {}^1z_0)^{-1}\,d\,{}^1\zeta\biggr)M_1^{-1}\dots\biggr) \times ({}^n\zeta - {}^nz_0)^{-1}\,d\,{}^n\zeta\biggr)M_n^{-1} \tag{3.9'}$$

for the corresponding $U = {}^1U \times \dots \times {}^nU$, where ψ_j and ${}^jU$ satisfy conditions of Theorem 3.9 for each j and f is a continuous quaternion holomorphic function on U.

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