Computers and Mathematics with Applications 62 (2011) 225–234

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

A new homotopy perturbation method for solving systems of partial differential equations

Jafar Biazar[a,*], Mostafa Eslami[b,c]

[a] Department of Mathematics, Faculty of Sciences, Islamic Azad University, Rasht Branch, Guilan, Iran
[b] Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Guilan, Rasht, Iran
[c] Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

ARTICLE INFO

Article history:
Received 16 February 2011
Received in revised form 27 April 2011
Accepted 27 April 2011

Keywords:
System of partial differential equations
New homotopy perturbation method

ABSTRACT

In this paper, a new homotopy perturbation method (NHPM) is introduced for obtaining solutions of systems of non-linear partial differential equations. Theoretical considerations are discussed. To illustrate the capability and reliability of the method three examples are provided. Comparison of the results of applying NHPM with those of applying HPM reveal the effectiveness and convenience of the new technique.

1. Introduction

In this work, we study systems of partial differential equations (PDEs). Such systems arise in many areas of mathematics, engineering and physical sciences. These equations are often too complicated to be solved exactly and even if an exact solution is obtained, the required calculations may be too complicated. Very recently, many powerful methods have been presented, such as the Adomian decomposition method [1,2], the variational iteration method [3,4], the homotopy perturbation method (HPM) [5–21], and the differential transform method [22–24].

The general form of a system of PDEs can be considered as the following:

$$\frac{\partial u_j}{\partial t} + N_j(x_1, \dots, x_{n-1}, t, u_1, \dots, u_n) = g_j(x_1, x_2, \dots, x_{n-1}, t), \quad j = 1, \dots, n, \tag{1}$$

with the following initial conditions:

$$u_j(x_1, x_2, \dots, x_{n-1}, t_0) = f_j(x_1, x_2, \dots, x_{n-1}), \quad j = 1, \dots, n$$

where $N_1, \dots, N_n$ are non-linear operators, which usually depend on the functions u_i and their derivatives, and $g_1, g_2, \dots, g_n$ are inhomogeneous terms.

This paper is arranged as follows. In Section 2, the new modification of HPM, called NHPM, for solving systems of partial differential equations is presented. The efficiency of the new method is verified by the numerical results for three sample examples in Section 3. Comparisons between this method and HPM are illustrated in this section. The conclusions appear in Section 4.

* Corresponding author.
E-mail addresses: jafar.biazar@gmail.com (J. Biazar), eslami_mostafa@yahoo.com, m.eslami@guilan.ac.ir (M. Eslami).

0898-1221/$ – see front matter © 2011 Elsevier Ltd. All rights reserved.
doi:10.1016/j.camwa.2011.04.070

2. The basic idea of NHPM

For solving system (1), by using NHPM, we construct the following homotopies:

$$(1-p)\left(\frac{\partial U_j}{\partial t}-u_{j,0}\right)+p\left(\frac{\partial U_j}{\partial t}+N_j(x_1,\dots,x_{n-1},t,U_1,\dots,U_n)-g_j\right)=0,\quad j=1,\dots,n \tag{2}$$

or

$$\frac{\partial U_j}{\partial t}=u_{j_0}-p\left(u_{j_0}+N_j(x_1,x_2,\dots,x_{n-1},t,U_1,\dots,U_n)-g_j\right),\quad j=1,\dots,n. \tag{3}$$

Applying the inverse operator, $L^{-1}=\int_{t_0}^{t}(.)\mathrm{d}t$, to both sides of Eq. (3), we obtain

$$\begin{aligned}U_j(x_1,x_2,\dots,x_{n-1},t) &= U_j(x_1,x_1,\dots,x_{n-1},t_0)+\int_{t_0}^{t}u_{j_0}\mathrm{d}t\\ &\quad -p\int_{t_0}^{t}\left(u_{j_0}+N_j(x_1,x_2,\dots,x_{n-1},t,U_1,\dots,U_n)-g_j\right)\mathrm{d}t,\quad j=1,\dots,n\end{aligned} \tag{4}$$

where

$$U_j(x_1,x_2,\dots,x_{n-1},t_0)=u_j(x_1,x_2,\dots,x_{n-1},t_0),\quad j=1,\dots,n.$$

Let us present the solution of the system (4) as the following:

$$U_j=U_{j0}+pU_{j1}+p^2U_{j2}+\cdots,\quad j=1,\dots,n \tag{5}$$

where $U_{ij},\ i=1,\dots,n, j=0,\dots,n$, are functions which should be determined. Suppose that the initial approximations of the solutions of Eqs. (1) are in the following form:

$$u_{i,0}(x_1,x_2,\dots,x_{n-1},t)=\sum_{j=0}^{\infty}a_{i,j}(x_1,x_2,\dots,x_{n-1})P_j(t),\quad i=1,\dots,n \tag{6}$$

where $a_{i,j}(x_1,x_2,\dots,x_{n-1}),\ i=1,\dots,n, j=0,\dots,n$, are unknown coefficients and $P_0(t),P_1(t),P_2(t),\dots$ are specific functions.

Substituting (5) and (6) into (4) and equating the coefficients of p with the same powers leads to

$$\begin{aligned}
&p^0:\ U_{i,0}(x_1,x_2,\dots,x_{n-1},t)=f_i(x_1,x_2,\dots,x_{n-1})+\sum_{j=0}^{\infty}a_{ij}\int_{t_0}^{t}P_j(t)\mathrm{d}t,\\
&p^1:\ U_{i,1}(x_1,x_2,\dots,x_{n-1},t)=-\sum_{j=0}^{\infty}a_{ij}\int_{t_0}^{t}P_j(t)\mathrm{d}t-\int_{t_0}^{t}\left(N_i(x_1,x_2,\dots,x_{n-1},t,U_{1,0},\dots,U_{n,0})-g_i\right)\mathrm{d}t,\\
&p^2:\ U_{i,2}(x_1,x_2,\dots,x_{n-1},t)=-\int_{t_0}^{t}\left(N_i(x_1,x_2,\dots,x_{n-1},t,U_{1,0},\dots,U_{n,0},U_{1,1},\dots,U_{n,1})\right)\mathrm{d}t,\\
&\vdots\\
&p^j:\ U_{i,j}(x_1,x_2,\dots,x_{n-1},t)=-\int_{t_0}^{t}\left(N_i(x_1,x_2,\dots,x_{n-1},t,U_{1,0},\dots,U_{n,0},\dots,U_{1,j-1},\dots,U_{n,j-1})\right)\mathrm{d}t,\\
&\vdots
\end{aligned} \tag{7}$$

Now if we solve these equations in such a way that $U_{i,1}(x_1x_2,\dots,x_{n-1},t)=0$, then Eqs. (7) yield

$$U_{i,2}(x_1x_2,\dots,x_{n-1},t)=U_{i,3}(x_1x_2,\dots,x_{n-1},t)=\cdots=0.$$

Therefore the exact solution may be obtained as the following:

$$u_i(x_1,x_2,\dots,x_{n-1},t)=U_{i,0}(x_1,x_2,\dots,x_{n-1},t)=f_i(x_1,x_2,\dots,x_{n-1})+\sum_{j=0}^{\infty}a_{i,j}\int_{t_0}^{t}P_j(t)\mathrm{d}t,\quad i=1,\dots,n. \tag{8}$$

It is worth mentioning that if $g_i(x_1, x_2, \ldots, x_{n-1}, t)$, and $u_{i,0}(x_1, x_2, \ldots, x_{n-1}, t)$, are analytic around $t = t_0$, then their Taylor series can be defined as

$$u_{i,0}(x_1, x_2, \ldots, x_{n-1}, t) = \sum_{j=0}^{\infty} a_{i,j}(x_1, x_2, \ldots, x_{n-1})(t - t_0)^n,$$
$$g_i(x_1, x_2, \ldots, x_{n-1}, t) = \sum_{j=0}^{\infty} a^*_{i,j}(x_1, x_2, \ldots, x_{n-1})(t - t_0)^n, \quad (9)$$

which can be used in Eqs. (7), where $a_{i,j}(x_1, x_2, \ldots, x_{n-1}),\ i = 1, \ldots, n,\ j = 0, \ldots, n$, are unknown coefficients which must be computed, and $a^*_{i,j}(x_1, x_2, \ldots, x_{n-1}),\ i = 1, \ldots, n,\ j = 0, \ldots, n$, are known ones.

To show the capability of the method, NHPM has been applied to some examples in the next section.

3. Numerical results

To demonstrate the effectiveness of the method three examples of systems of non-linear partial differential equations are presented.

Example 1. Consider the following system of three-dimensional partial differential equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - v\dfrac{\partial u}{\partial x} - \dfrac{\partial v}{\partial t}\dfrac{\partial u}{\partial y} = 1 - x + y + t, \\ \dfrac{\partial v}{\partial t} - u\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial t}\dfrac{\partial v}{\partial y} = 1 - x - y - t, \end{cases} \quad (10)$$

with initial conditions

$$u(x, y, 0) = x + y - 1,$$
$$v(x, y, 0) = x - y + 1.$$

The exact solutions are

$$u(x, y, t) = x + y + t - 1,$$
$$v(x, y, t) = x - y - t + 1.$$

The HPM and NHPM methods are used to approximate the solutions.

First we apply the HPM method.

According to the homotopy perturbation method, we have

$$\begin{cases} U_0 = x + y - 1, \\ V_0 = x - y + 1, \end{cases}$$

$$\begin{cases} U_1 = 2t + \dfrac{1}{2}t^2, \\ V_1 = -\dfrac{1}{2}t^2, \end{cases}$$

$$\begin{cases} U_2 = -\dfrac{1}{6}t^3 - \dfrac{1}{2}t^2, \\ V_2 = \dfrac{1}{2}t^2 + \dfrac{1}{6}t^3 - 2t, \end{cases}$$

$$\begin{cases} U_3 = \dfrac{1}{3}t^3 + \dfrac{1}{24}t^4 - \dfrac{1}{2}t^2 - 2t, \\ V_3 = -\dfrac{1}{24}t^4 + \dfrac{1}{2}t^2, \end{cases}$$

$$\vdots$$

Therefore, the solution will be as follows:

$$\begin{cases} u = -1 + x + y + 2t - \dfrac{1}{5040}t^7 + \dfrac{1}{40}t^5 - \dfrac{1}{2}t^3 + \cdots, \\ v = 1 + x - y + 2t + \dfrac{1}{5040}t^7 - \dfrac{1}{40}t^5 + \dfrac{1}{2}t^3 + \cdots. \end{cases}$$

To solve Eq. (10) by using NHPM, we construct the following homotopies:

$$\begin{aligned}\frac{\partial U}{\partial t}(x,y,t)&=u_0(x,y,t)-p\left(u_0(x,y,t)-V\frac{\partial U}{\partial x}-\frac{\partial V}{\partial t}\frac{\partial U}{\partial y}-1+x-y-t\right),\\ \frac{\partial V}{\partial t}(x,y,t)&=v_0(x,y,t)-p\left(v_0(x,y,t)-U\frac{\partial V}{\partial x}-\frac{\partial U}{\partial t}\frac{\partial V}{\partial y}-1+x+y+t\right).\end{aligned}\tag{11}$$

Applying the inverse operator, $L^{-1}=\int_0^t(.)\mathrm{d}t$, to both sides of these equations, we obtain

$$\begin{aligned}U(x,y,t)&=U(x,y,0)+\int_0^t u_0(x,y,t)\mathrm{d}t-p\int_0^t\left(u_0(x,y,t)-V\frac{\partial U}{\partial x}-\frac{\partial V}{\partial t}\frac{\partial U}{\partial y}-1+x-y-t\right)\mathrm{d}t,\\ V(x,y,t)&=V(x,y,0)+\int_0^t v_0(x,y,t)\mathrm{d}t-p\int_0^t\left(v_0(x,y,t)-U\frac{\partial V}{\partial x}-\frac{\partial U}{\partial t}\frac{\partial V}{\partial y}-1+x+y+t\right)\mathrm{d}t.\end{aligned}\tag{12}$$

Suppose that the solutions of system (12) are as assumed in (5); substituting Eqs. (5) into Eqs. (12), collecting the same powers of p, and equating each coefficient of p to zero yields

$$p^0:\ \begin{cases}U_0(x,y,t)=U(x,y,0)+\displaystyle\int_0^t u_0(x,y,t)\mathrm{d}t,\\ V_0(x,y,t)=V(x,y,0)+\displaystyle\int_0^t v_0(x,y,t)\mathrm{d}t,\end{cases}$$

$$p^1:\ \begin{cases}U_1(x,y,t)=\displaystyle\int_0^t\left(-u_0(x,y,t)+V_0\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial t}\frac{\partial U_0}{\partial y}+1-x+y+t\right)\mathrm{d}t,\\ V_1(x,y,t)=\displaystyle\int_0^t\left(-v_0(x,y,t)+U_0\frac{\partial V_0}{\partial x}+\frac{\partial U_0}{\partial t}\frac{\partial V_0}{\partial y}+1-x-y-t\right)\mathrm{d}t,\end{cases}$$

$$p^2:\ \begin{cases}U_2(x,y,t)=\displaystyle\int_0^t\left(V_0\frac{\partial U_1}{\partial x}+V_1\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial t}\frac{\partial U_1}{\partial y}+\frac{\partial V_1}{\partial t}\frac{\partial U_0}{\partial y}\right)\mathrm{d}t,\\ V_2(x,y,t)=\displaystyle\int_0^t\left(U_0\frac{\partial V_1}{\partial x}+U_1\frac{\partial V_0}{\partial x}+\frac{\partial U_0}{\partial t}\frac{\partial V_1}{\partial y}+\frac{\partial U_1}{\partial t}\frac{\partial V_0}{\partial y}\right)\mathrm{d}t,\end{cases}$$

$$\vdots$$

$$p^j:\ \begin{cases}U_{j+1}(x,y,t)=\displaystyle\int_0^t\sum_{k=0}^{j}\left(V_k\frac{\partial U_{j-k}}{\partial x}+\frac{\partial V_k}{\partial t}\frac{\partial U_{j-k}}{\partial y}\right)\mathrm{d}t,\\ V_{j+1}(x,y,t)=\displaystyle\int_0^t\sum_{k=0}^{j}\left(U_k\frac{\partial V_{j-k}}{\partial x}+\frac{\partial U_k}{\partial t}\frac{\partial V_{j-k}}{\partial y}\right)\mathrm{d}t,\end{cases}$$

$$\vdots$$

Assume $\begin{cases}u_0(x,y,t)=\sum_{n=0}^{\infty}a_n(x,y)P_n(t),\ P_k(t)=t^k,\ U(x,y,0)=u(x,y,0),\\ v_0(x,y,t)=\sum_{n=0}^{\infty}b_n(x,y)P_n(t),\ P_k(t)=t^k,\ V(x,y,0)=v(x,y,0).\end{cases}$

Solving the above equations for $U_1(x,y,t)$, $V_1(x,y,t)$ leads to the result

$$\begin{aligned}U_1(x,y,t)=&\ (-a_0(x,y)+b_0(x,y)+2)t\\ &+\left(-\frac{1}{2}a_1(x,y)+\frac{1}{2}b_0(x,y)+\frac{1}{2}(x-y+1)a_{0x}(x,y)+\frac{1}{2}b_1(x,y)+\frac{1}{2}a_{0,y}(x,y)b_0(x,y)+\frac{1}{2}\right)t^2\\ &+\begin{pmatrix}-\frac{1}{3}a_2(x,y)+\frac{1}{6}b_1(x,y)+\frac{1}{6}(x-y+1)a_{1x}(x,y)+\frac{1}{3}a_{0x}(x,y)b_0(x,y)\\ +\frac{1}{3}b_2(x,y)+\frac{1}{3}a_{0y}(x,y)b_1(x,y)+\frac{1}{6}a_{1y}(x,y)b_0(x,y)\end{pmatrix}t^3\\ &+\begin{pmatrix}-\frac{1}{4}a_3(x,y)+\frac{1}{12}b_2(x,y)+\frac{1}{12}(x-y+1)a_{2x}(x,y)+\frac{1}{8}a_{0x}(x,y)b_1(x,y)+\frac{1}{8}a_{1x}(x,y)b_0(x,y)\\ +\frac{1}{4}b_3(x,y)+\frac{1}{4}a_{0y}(x,y)b_2(x,y)+\frac{1}{8}a_{1y}(x,y)b_1(x,y)+\frac{1}{12}a_{2y}(x,y)b_0(x,y)\end{pmatrix}t^4\\ &+\cdots,\end{aligned}$$

$$
\begin{aligned}
V_1(x,y,t) &= (-b_0(x,y)-a_0(x,y))t \\
&+ \left(-\frac{1}{2}b_1(x,y)+\frac{1}{2}a_0(x,y)+\frac{1}{2}(x+y-1)b_{0x}(x,y)-\frac{1}{2}a_1(x,y)+\frac{1}{2}a_0(x,y)b_{0y}(x,y)-\frac{1}{2}\right)t^2 \\
&+ \begin{pmatrix} -\frac{1}{3}b_2(x,y)+\frac{1}{6}a_1(x,y)+\frac{1}{6}(x+y-1)b_{1x}(x,y)+\frac{1}{3}a_0(x,y)b_{0x}(x,y) \\ -\frac{1}{3}a_2(x,y)+\frac{1}{6}a_0(x,y)b_{1y}(x,y)+\frac{1}{3}a_1(x,y)b_{0y}(x,y) \end{pmatrix} t^3 \\
&+ \begin{pmatrix} -\frac{1}{4}b_3(x,y)+\frac{1}{12}a_2(x,y)+\frac{1}{12}(x+y-1)b_{2x}(x,y)+\frac{1}{8}a_1(x,y)b_{0x}(x,y)+\frac{1}{8}a_0(x,y)b_{1x}(x,y) \\ -\frac{1}{4}a_2(x,y)+\frac{1}{4}a_2(x,y)b_{0y}(x,y)+\frac{1}{8}a_1(x,y)b_{1y}(x,y)+\frac{1}{12}a_0(x,y)b_{2y}(x,y) \end{pmatrix} t^4 \\
&+ \cdots.
\end{aligned}
$$

By the vanishing of $U_1(x,y,t)$, $V_1(x,y,t)$, the coefficients $a_n(x,y)$, $b_n(x,y)$ $(n=1,2,3,\ldots)$ are determined as

$$
\begin{aligned}
&a_0(x,y)=1, \qquad a_1(x,y)=a_2(x,y)=a_3(x,y)=a_4(x,y)=\cdots 0, \\
&b_0(x,y)=-1, \qquad b_1(x,y)=b_2(x,y)=b_3(x,y)=b_4(x,y)=\cdots 0.
\end{aligned}
$$

Therefore we obtain the solution of Eq. (14) as

$$
\begin{aligned}
u(x,y,t) &= U_0(x,y,t) = x+y-1+a_0(x,y)t+\frac{1}{2}a_1(x,y)t^2+\frac{1}{3}a_2(x,y)t^3+\frac{1}{4}a_3(x,y)t^4+\cdots = x+y+t-1, \\
v(x,y,t) &= V_0(x,y,t) = x-y+1+b_0(x,y)t+\frac{1}{2}b_1(x,y)t^2+\frac{1}{3}b_2(x,y)t^3+\frac{1}{4}b_3(x,y)t^4+\cdots = x-y-t+1,
\end{aligned}
$$

which is an exact solution.

Example 2. Consider the following system of two non-linear equations:

$$
\begin{cases}
\dfrac{\partial u}{\partial x}-v\dfrac{\partial u}{\partial t}+u\dfrac{\partial v}{\partial t}=-1+e^x\sin t, \\
\dfrac{\partial v}{\partial x}+\dfrac{\partial u}{\partial t}\dfrac{\partial v}{\partial x}+\dfrac{\partial v}{\partial t}\dfrac{\partial u}{\partial x}=-1-e^{-x}\cos t,
\end{cases} \tag{13}
$$

with boundary conditions

$$
\begin{aligned}
u(0,t) &= \sin t, \\
v(0,t) &= \cos t.
\end{aligned}
$$

The exact solutions of this system are

$$
\begin{aligned}
u(x,t) &= e^x\sin t, \\
v(x,t) &= e^{-x}\cos t.
\end{aligned}
$$

This system will be solved by using HPM and NHPM.

First we apply the HPM method.

According to the homotopy perturbation method, we have

$$
\begin{cases} U_0=\sin t, \\ V_0=\cos t, \end{cases}
$$

$$
\begin{cases} U_1=e^x\sin t-\sin t, \\ V_1=-x+e^{-x}\cos t-\cos t, \end{cases}
$$

$$
\begin{cases} U_2=e^x-e^{-x}-2x-\frac{1}{2}x^2\cos(t), \\ V_2=-e^{-x}\cos^2 t+e^x\sin^2 t+2\cos^2 t+x\cos t-1, \end{cases}
$$

$$\vdots$$

Therefore, the solution will be as follows:

$$
\begin{cases} u=e^x\sin t+e^x-e^{-x}-2x-\frac{1}{2}x^2\cos(t)+\cdots, \\ v=-e^{-x}\cos^2 t+e^x\sin^2 t+e^{-x}\cos t+2\cos^2 t+x\cos t-x-1+\ldots. \end{cases}
$$

To solve system (12) by using NHPM, we construct the following homotopy:

$$\begin{aligned}
\frac{\partial U}{\partial x}(x,t) &= u_0(x,t) - p\left(u_0(x,t) - V\frac{\partial U}{\partial t} + U\frac{\partial V}{\partial t} + 1 - e^x \sin t\right),\\
\frac{\partial V}{\partial x}(x,t) &= v_0(x,t) - p\left(v_0(x,t) + \frac{\partial U}{\partial t}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t}\frac{\partial U}{\partial x} + 1 + e^{-x}\cos t\right).
\end{aligned} \tag{14}$$

Applying the inverse operator, $L^{-1} = \int_0^x (.)\mathrm{d}x$, to both sides of the above equations, we obtain

$$\begin{aligned}
U(x,t) &= U(0,t) + \int_0^x u_0(x,t)\mathrm{d}x - p\int_0^x \left(u_0(x,t) - V\frac{\partial U}{\partial t} + U\frac{\partial V}{\partial t} + 1 - e^x \sin t\right)\mathrm{d}x,\\
V(x,t) &= V(0,t) + \int_0^x v_0(x,t)\mathrm{d}x - p\int_0^x \left(v_0(x,t) + \frac{\partial U}{\partial t}\frac{\partial V}{\partial x} + \frac{\partial V}{\partial t}\frac{\partial U}{\partial x} + 1 + e^{-x}\cos t\right)\mathrm{d}x.
\end{aligned} \tag{15}$$

Suppose that the solutions of system (15) have the form (5); substituting Eqs. (5) into Eqs. (15), collecting terms with the same powers of p, and equating each coefficient of p to zero results in

$$p^0: \begin{cases} U_0(x,t) = U(0,t) + \int_0^x u_0(x,t)\mathrm{d}x,\\ V_0(x,t) = V(0,t) + \int_0^x v_0(x,t)\mathrm{d}x, \end{cases}$$

$$p^1: \begin{cases} U_1(x,t) = \int_0^x \left(-u_0(x,t) + V_0\frac{\partial U_0}{\partial t} - U_0\frac{\partial V_0}{\partial t} - 1 + e^x\sin t\right)\mathrm{d}x,\\ V_1(x,t) = \int_0^x \left(-v_0(x,t) - \frac{\partial U_0}{\partial t}\frac{\partial V_0}{\partial x} - \frac{\partial V_0}{\partial t}\frac{\partial U_0}{\partial x} - 1 - e^{-x}\cos t\right)\mathrm{d}x, \end{cases}$$

$$p^2: \begin{cases} U_2(x,t) = \int_0^x \left(V_0\frac{\partial U_1}{\partial t} + V_1\frac{\partial U_0}{\partial t} - U_0\frac{\partial V_1}{\partial t} - U_1\frac{\partial V_1}{\partial t}\right)\mathrm{d}x,\\ V_2(x,t) = \int_0^x \left(-\frac{\partial U_0}{\partial t}\frac{\partial V_1}{\partial x} - \frac{\partial U_1}{\partial t}\frac{\partial V_0}{\partial x} - \frac{\partial V_0}{\partial t}\frac{\partial U_1}{\partial x} - \frac{\partial V_1}{\partial t}\frac{\partial U_0}{\partial x}\right)\mathrm{d}x, \end{cases}$$

$$\vdots$$

$$p^j: \begin{cases} U_{j+1}(x,t) = \int_0^x \sum_{k=0}^{j} V_k\left(\frac{\partial U_{j-k}}{\partial t} - U_k\frac{\partial V_{j-k}}{\partial t}\right)\mathrm{d}x,\\ V_{j+1}(x,t) = \int_0^x \sum_{k=0}^{j}\left(-\frac{\partial U_k}{\partial t}\frac{\partial V_{j-k}}{\partial x} - \frac{\partial V_k}{\partial t}\frac{\partial U_{j-k}}{\partial y}\right)\mathrm{d}x, \end{cases}$$

$$\vdots$$

Assuming that $\begin{cases} u_0(x,t) = \sum_{n=0}^{\infty} a_n(t)x^k, & U(0,t) = u(0,t),\\ v_0(x,t) = \sum_{n=0}^{\infty} b_n(t)x^k, & V(0,t) = v(0,t), \end{cases}$ and solving equations $U_1(x,t)$, $V_1(x,t)$ leads to the following results:

$$\begin{aligned}
U_1(x,t) &= (-a_0(t) + \sin t)\,x + \left(-\frac{1}{2}a_1(t) + \frac{a_0'(t)\cos t}{2} + \frac{b_0(t)\cos t}{2} - \frac{b_0'(t)\sin t}{2} + \frac{a_0(t)\sin t}{2} + \frac{\sin t}{2}\right)x^2\\
&\quad + \left(-\frac{1}{3}a_2(t) + \frac{a_1'(t)\cos t}{6} + \frac{b_1(t)\cos t}{6} + \frac{b_0(t)a_0'(t)}{3} - \frac{b_1'(t)\sin t}{6} + \frac{a_1(t)\sin t}{6} - \frac{b_0'(t)a_0(t)}{3} + \frac{\sin t}{6}\right)x^3\\
&\quad + \begin{pmatrix} -\frac{1}{4}a_3(t) + \frac{a_2'(t)\cos t}{12} + \frac{b_2(t)\cos t}{12} + \frac{b_1(t)a_0'(t)}{8} + \frac{b_0(t)a_1'(t)}{8}\\ -\frac{b_2'(t)\sin t}{12} + \frac{a_2(t)\sin t}{12} - \frac{b_0'(t)a_1(t)}{8} - \frac{b_1'(t)a_0(t)}{8} + \frac{\sin t}{24} \end{pmatrix} x^4 + \cdots,\\
V_1(x,t) &= (-b_0(t) - b_0(t)\cos t + a_0(t)\sin t - \cos t - 1)\,x\\
&\quad + \left(-\frac{1}{2}b_1(t) - \frac{b_1(t)\cos t}{2} - \frac{b_0(t)a_0'(t)}{2} + \frac{a_1(t)\sin t}{2} - \frac{b_0'(t)a_0(t)}{2} + \frac{\cos t}{2}\right)x^2
\end{aligned}$$

$$+\left(-\frac{1}{3}b_2(t)-\frac{b_2(t)\cos t}{3}-\frac{b_0(t)a_1'(t)}{6}-\frac{b_1(t)a_0'(t)}{3}+\frac{a_2(t)\sin t}{3}-\frac{b_1'(t)a_0(t)}{6}-\frac{b_0'(t)a_1(t)}{3}-\frac{\cos t}{6}\right)x^3$$
$$+\begin{pmatrix}-\frac{1}{4}b_3(t)-\frac{b_3(t)\cos t}{4}-\frac{b_0(t)a_2'(t)}{12}-\frac{b_1(t)a_1'(t)}{8}-\frac{b_2(t)a_0'(t)}{4}\\+\frac{a_3(t)\sin t}{4}-\frac{b_0'(t)a_2(t)}{4}-\frac{b_1'(t)a_1(t)}{8}-\frac{b_2'(t)a_0(t)}{12}+\frac{\cos t}{24}\end{pmatrix}x^4+\cdots.$$

By the vanishing of $U_1(x,t)$, $V_1(x,t)$, the coefficients $a_n(t)$, $b_n(t)$ $(n=1,2,3,\ldots)$ are obtained as follows:

$$a_0(t)=\sin t,\qquad a_1(t)=\sin t,\qquad a_2(t)=\frac{1}{2}\sin t,\qquad a_3(t)=\frac{1}{6}\sin t,\qquad a_4(t)=\frac{1}{24}\sin t,\ldots$$
$$b_0(t)=-\cos t,\qquad b_1(t)=\cos t,\qquad b_2(t)=-\frac{1}{2}\cos t,\qquad b_3(t)=\frac{1}{6}\cos t,\qquad b_4(t)=-\frac{1}{24}\cos t,\ldots.$$

This implies that

$$u(x,t)=U_0(x,t)=\sin t+a_0(t)x+\frac{1}{2}a_1(t)x^2+\frac{1}{3}a_2(t)x^3+\frac{1}{4}a_3(t)x^4+\cdots=e^x\sin t,$$
$$v(x,t)=V_0(x,t)=\cos t+b_0(t)x+\frac{1}{2}b_1(t)x^2+\frac{1}{3}b_2(t)x^3+\frac{1}{4}b_3(t)x^4+\cdots=e^{-x}\cos t.$$

In this example the exact solutions are gained.

Example 3. Consider the following non-linear system of inhomogeneous partial differential equations:

$$\begin{cases}\frac{\partial u}{\partial t}-\frac{\partial w}{\partial x}\frac{\partial v}{\partial t}-\frac{1}{2}\frac{\partial w}{\partial t}\frac{\partial^2 u}{\partial x^2}=-4xt,\\ \frac{\partial v}{\partial t}-\frac{\partial w}{\partial t}\frac{\partial^2 u}{\partial x^2}=6t,\\ \frac{\partial w}{\partial t}-\frac{\partial^2 u}{\partial x^2}-\frac{\partial v}{\partial x}\frac{\partial w}{\partial t}=4xt-2t-2,\end{cases}\tag{16}$$

subject to the initial condition

$$u(x,0)=x^2+1,$$
$$v(x,0)=x^2-1,$$
$$w(x,0)=x^2-1.$$

The exact solutions are

$$u(x,t)=x^2-t^2+1,$$
$$v(x,t)=x^2+t^2-1,$$
$$w(x,t)=x^2-t^2-1.$$

Using the homotopy perturbation method leads to

$$\begin{cases}U_0=x^2+1,\\ V_0=x^2-1,\\ W_0=x^2-1,\end{cases}$$
$$\begin{cases}U_1=-2xt^2,\\ V_1=3t^2,\\ W_1=2xt^2-t^2,\end{cases}$$
$$\begin{cases}U_2=\frac{1}{2}(16x-2)t^2,\\ V_2=\frac{1}{2}(8x-4)t^2,\\ W_2=x(4x-2)t^2,\end{cases}$$
$$\begin{cases}U_3=3t^4+(x(8x-4)+x(4x-2))t^2,\\ V_3=2x(4x-2)t^2,\\ W_3=2x^2(4x-2)t^2,\end{cases}$$
$$\vdots$$

Then, the approximate solutions, in series form, are

$$\begin{cases} u \approx 1 + x^2 - t^2, \\ v \approx x^2 - 1 + t^2, \\ w \approx x^2 - 1 - t^2. \end{cases}$$

For solving this system by using NHPM, we consider the following homotopy:

$$\begin{aligned} \frac{\partial U}{\partial t}(x,t) &= u_0(x,t) - p\left(u_0(x,t) - \frac{\partial W}{\partial x}\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial W}{\partial t}\frac{\partial^2 U}{\partial x^2} + 4xt\right), \\ \frac{\partial V}{\partial t}(x,t) &= v_0(x,t) - p\left(v_0(x,t) - \frac{\partial W}{\partial t}\frac{\partial^2 U}{\partial x^2} - 6t\right), \\ \frac{\partial W}{\partial t}(x,t) &= w_0(x,t) - p\left(w_0(x,t) - \frac{\partial^2 U}{\partial x^2} - \frac{\partial V}{\partial x}\frac{\partial W}{\partial t} - 4xt + 2t + 2\right). \end{aligned} \tag{17}$$

Applying the inverse operator, $L^{-1} = \int_0^t (.)\mathrm{d}t$, to both sides of the system (17) leads to

$$\begin{aligned} U(x,t) &= U(x,0) + \int_0^t u_0(x,t)\mathrm{d}t - p\int_0^t \left(u_0(x,t) - \frac{\partial W}{\partial x}\frac{\partial V}{\partial t} - \frac{1}{2}\frac{\partial W}{\partial t}\frac{\partial^2 U}{\partial x^2} + 4xt\right)\mathrm{d}t, \\ V(x,t) &= V(x,0) + \int_0^t v_0(x,t)\mathrm{d}t - p\int_0^t \left(v_0(x,t) - \frac{\partial W}{\partial t}\frac{\partial^2 U}{\partial x^2} - 6t\right)\mathrm{d}t, \\ W(x,t) &= W(x,0) + \int_0^t w_0(x,t)\mathrm{d}t - p\int_0^t \left(w_0(x,t) - \frac{\partial^2 U}{\partial x^2} - \frac{\partial V}{\partial x}\frac{\partial W}{\partial t} - 4xt + 2t + 2\right)\mathrm{d}t. \end{aligned} \tag{18}$$

Suppose that the solutions of system (18) are in the form (5); substituting Eqs. (5) into Eqs. (18), collecting the terms with the same powers of p, and equating each coefficient of p to zero results in

$$p^0: \begin{cases} U_0(x,t) = U(x,0) + \int_0^t u_0(x,t)\mathrm{d}t, \\ V_0(x,t) = V(x,0) + \int_0^t v_0(x,t)\mathrm{d}t, \\ W_0(x,t) = W(x,0) + \int_0^t w_0(x,t)\mathrm{d}t, \end{cases}$$

$$p^1: \begin{cases} U_1(x,t) = \int_0^t \left(-u_0(x,t) + \frac{\partial W_0}{\partial x}\frac{\partial V_0}{\partial t} + \frac{1}{2}\frac{\partial W_0}{\partial t}\frac{\partial^2 U_0}{\partial x^2} - 4xt\right)\mathrm{d}t, \\ V_1(x,t) = \int_0^t \left(-v_0(x,t) + \frac{\partial W_0}{\partial t}\frac{\partial^2 U_0}{\partial x^2} + 6t\right)\mathrm{d}t, \\ W_1(x,t) = \int_0^t \left(-w_0(x,t) + \frac{\partial^2 U_0}{\partial x^2} + \frac{\partial V_0}{\partial x}\frac{\partial W_0}{\partial t} + 4xt - 2t - 2\right)\mathrm{d}t, \end{cases}$$

$$p^2: \begin{cases} U_2(x,t) = \int_0^t \left(\frac{\partial W_1}{\partial x}\frac{\partial V_0}{\partial t} + \frac{\partial W_0}{\partial x}\frac{\partial V_1}{\partial t} + \frac{1}{2}\frac{\partial W_0}{\partial t}\frac{\partial^2 U_1}{\partial x^2} + \frac{1}{2}\frac{\partial W_1}{\partial t}\frac{\partial^2 U_0}{\partial x^2}\right)\mathrm{d}t, \\ V_2(x,t) = \int_0^t \left(\frac{\partial W_0}{\partial x}\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial W_1}{\partial x}\frac{\partial^2 U_0}{\partial x^2}\right)\mathrm{d}t, \\ W_2(x,t) = \int_0^t \left(\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial V_0}{\partial x}\frac{\partial W_1}{\partial t} + \frac{\partial V_1}{\partial x}\frac{\partial W_0}{\partial t}\right)\mathrm{d}t \end{cases}$$

$$\vdots$$

$$p^j: \begin{cases} U_{j+1}(x,t) = \int_0^t \left(\sum_{k=0}^{j} \left(\frac{\partial W_k}{\partial x} \frac{\partial V_{j-k}}{\partial} + \frac{1}{2} \frac{\partial W_k}{\partial t} \frac{\partial^2 U_{j-k}}{\partial x^2} \right) \right) \mathrm{d}t, \\ V_{j+1}(x,t) = \int_0^t \left(\sum_{k=0}^{j} \frac{\partial W_k}{\partial x} \frac{\partial^2 U_{j-k}}{\partial x^2} \right) \mathrm{d}t, \\ W_{j+1}(x,t) = \int_0^t \left(\frac{\partial^2 U_j}{\partial x^2} + \sum_{k=0}^{j} \frac{\partial V_k}{\partial x} \frac{\partial W_{j-k}}{\partial t} \right) \mathrm{d}t \end{cases}$$

$$\vdots$$

Assume that

$$\begin{cases} u_0(x,t) = \sum_{n=0}^{\infty} a_n(x) t^k, \qquad U(x,0) = u(x,0), \\ v_0(x,t) = \sum_{n=0}^{\infty} b_n(x) t^k, \qquad V(x,0) = v(x,0), \\ w_0(x,t) = \sum_{n=0}^{\infty} c_n(x) t^k, \qquad W(x,0) = w(x,0). \end{cases}$$

If we set the Taylor series of $U_1(x,t)$, $V_1(x,t)$, and $W_1(x,t)$ at $t = 0$ equal to zero, then

$$\begin{aligned} U_1(x,y) \;=\; & (-a_0(x) + 2xb_0(x) + c_0(x))\, t + \left(-\frac{1}{2} a_1(x) + xb_1(x) + \frac{1}{2} b_0(x) c_0'(x) + \frac{1}{2} c_1(x) + \frac{1}{4} a_0''(x) c_0(x) - 2x \right) t^2 \\ & + \left(-\frac{1}{3} a_2(x) + \frac{2}{3} xb_2(x) + \frac{1}{3} b_1(x) c_0'(x) + \frac{1}{6} b_0(x) c_1'(x) + \frac{1}{3} c_2(x) + \frac{1}{12} a_1''(x) c_0(x) + \frac{1}{6} a_0''(x) c_1(x) \right) t^3 \\ & + \begin{pmatrix} -\frac{1}{4} a_3(x) + \frac{1}{2} xb_3(x) + \frac{1}{4} b_2(x) c_0'(x) + \frac{1}{8} b_1(x) c_1'(x) + \frac{1}{12} b_0(x) c_2'(x) \\ +\frac{1}{4} c_3(x) + \frac{1}{24} a_2''(x) c_0(x) + \frac{1}{16} a_1''(x) c_1(x) + \frac{1}{8} a_0''(x) c_2(x) \end{pmatrix} t^4 + \cdots = 0, \\ V_1(x,y) \;=\; & (-b_0(x) + 2c_0(x))\, t + \left(-\frac{1}{2} b_1(x) + c_1(x) + \frac{1}{2} c_0(x) a_0''(x) + 3 \right) t^2 \\ & + \left(-\frac{1}{3} b_2(x) + \frac{2}{3} c_2(x) + \frac{1}{6} c_0(x) a_1''(x) + \frac{1}{3} c_1(x) a_0''(x) \right) t^3 \\ & + \left(-\frac{1}{4} b_3(x) + \frac{1}{2} c_3(x) + \frac{1}{12} c_0(x) a_2''(x) + \frac{1}{8} c_1(x) a_1''(x) + \frac{1}{4} c_2(x) a_0''(x) \right) t^4 + \cdots = 0, \\ W_1(x,y) \;=\; & (-c_0(x) + 2xc_0(x))\, t + \left(-\frac{1}{2} c_1(x) + \frac{1}{2} a_0''(x) + xc_1(x) + b_0'(x) c_0(x) + 2x - 1 \right) t^2 \\ & + \left(-\frac{1}{3} c_3(x) + \frac{1}{6} a_1''(x) + \frac{2}{3} xc_2(x) + \frac{1}{3} b_0'(x) c_1(x) + \frac{1}{6} b_1'(x) c_0(x) \right) t^3 \\ & + \left(-\frac{1}{4} c_3(x) + \frac{1}{12} a_2''(x) + \frac{1}{2} xc_3(x) + \frac{1}{4} b_0'(x) c_2(x) + \frac{1}{8} b_1'(x) c_1(x) + \frac{1}{12} b_2'(x) c_0(x) \right) t^4 + \cdots = 0. \end{aligned}$$

It follows easily that

$$\begin{aligned} & a_0(x) = 0, \quad a_1(x) = -2, \quad a_2(x) = 0, \quad a_3(x) = 0, \quad a_4(x) = 0, \ldots \\ & b_0(x) = 0, \quad b_1(x) = 2, \quad b_2(x) = 0, \quad b_3(x) = 0, \quad b_4(x) = 0, \ldots \\ & c_0(x) = 0, \quad c_1(x) = -2, \quad c_2(x) = 0, \quad c_3(x) = 0, \quad c_4(x) = 0, \ldots. \end{aligned}$$

Therefore, the exact solutions of the system of partial differential equations can be expressed as follows:

$$u(x,t) = U_0(x,t) = x^2 + 1 + a_0(x)t + \frac{1}{2} a_1(x) t^2 + \frac{1}{3} a_2(x) t^3 + \frac{1}{4} a_3(x) t^4 + \cdots = x^2 - t^2 + 1,$$

$$v(x,t) = V_0(x,t) = x^2 - 1 + b_0(x)t + \frac{1}{2} b_1(x) t^2 + \frac{1}{3} b_2(x) t^3 + \frac{1}{4} b_3(x) t^4 + \cdots = x^2 + t^2 - 1,$$

$$w(x,t) = W_0(x,t) = x^2 - 1 + c_0(x)t + \frac{1}{2} c_1(x) t^2 + \frac{1}{3} c_2(x) t^3 + \frac{1}{4} c_3(x) t^4 + \cdots = x^2 - t^2 - 1.$$

4. Conclusions

In this article, a new modification of HPM, called NHPM, has been introduced for solving systems of non-linear partial differential equations. This method has been applied to three examples successfully, and exact solutions of the equations are achieved, where traditional HPM leads to an approximate solution. Numerical results reveal that NHPM is a powerful tool for solving linear and non-linear initial and boundary value problems. The basic idea described in this paper is strong enough to be employed to solve other functional equations. The convergence of the method is under study by our research group. The computations associated with the examples were performed using Maple 13.

References

[1] Y. Cherruault, G. Adomian, Decomposition methods: a new proof of convergence, Mathematical and Computer Modelling 18 (1993) 103–106.
[2] A.M. Wazwaz, The decomposition method applied to systems of partial differential equations and to the reaction–diffusion Brusselator model, Applied Mathematics and Computation 110 (2000) 251–264.
[3] J.H. He, Variational iteration method a kind of non-linear analytical technique: some examples, International Journal of Non-Linear Mechanics 34 (1999) 699–708.
[4] S. Momani, S. Abuasad, Application of He's variational iteration method to Helmholtz equation, Chaos, Solitons and Fractals 27 (5) (2005) 1119–1123.
[5] J.H. He, Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering 178 (1999) 257–262.
[6] J.H. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems, International Journal of Non-Linear Mechanics 35 (1) (2000) 37–43.
[7] J. Biazar, M. Eslami, H. Ghazvini, Homotopy perturbation method for systems of partial differential equations, International Journal of Nonlinear Sciences and Numerical Simulation 8 (3) (2007) 411–416.
[8] J. Biazar, H. Ghazvini, M. Eslami, He's homotopy perturbation method for systems of integro-differential equations, Chaos, Solitons and Fractals 39 (2009) 1253–1258.
[9] Z. Odibat, S. Momani, Modified homotopy perturbation method: application to quadratic Riccati differential equation of fractional order, Chaos, Solitons and Fractals 36 (1) (2008) 167–174.
[10] D.D. Ganji, The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Physics Letters A 355 (2006) 337–341.
[11] L. Cveticanin, Homotopy–perturbation method for pure nonlinear differential equation, Chaos, Solitons and Fractals 30 (2006) 1221–1230.
[12] A. Beléndez, T. Beléndez, A. Márquez, C. Neipp, Application of He's homotopy perturbation method to conservative truly nonlinear oscillators, Chaos, Solitons and Fractals 37 (3) (2008) 770–780.
[13] S. Abbasbandy, Numerical solutions of the integral equations: homotopy perturbation method and Adomian's decomposition method, Applied Mathematics and Computation 173 (2006) 493–500.
[14] M. Dehghan, F. Shakeri, Use of He's homotopy perturbation method for solving a partial differential equation arising in modeling of flow in porous media, Journal of Porous Media 11 (2008) 765–778.
[15] A. Golbabai, M. Javidi, Application of homotopy perturbation method for solving eighth-order boundary value problems, Applied Mathematics and Computation 191 (1) (2007) 334–346.
[16] M.A. Rana, A.M. Siddiqui, Q.K. Ghori, Application of He's homotopy perturbation method to Sumudu transform, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 185–190.
[17] T. Ozis, A. Yildirim, Traveling wave solution of Korteweg–de Vries equation using He's homotopy perturbation method, International Journal of Nonlinear Sciences and Numerical Simulation 8 (2) (2007) 239–242.
[18] A. Yildirim, Application of He's homotopy perturbation method for solving the Cauchy reaction–diffusion problem, Computers and Mathematics with Applications 57 (4) (2009) 612–618.
[19] Muhammad Aslam Noor, Syed Tauseef Mohyud-Din, Homotopy perturbation method for solving sixth-order boundary value problems, Computers and Mathematics with Applications 55 (2008) 2953–2972.
[20] M.A. Jafari, A. Aminataei, Application of homotopy perturbation method in the solution of Fokker–Planck equation, Physica Scripta 80 (5) (2009) Article ID 055001.
[21] M.S.H. Chowdhury, I. Hashim, Application of multistage homotopy-perturbation method for the solutions of the Chen system, Nonlinear Analysis. Real World Applications 10 (2009) 381–391.
[22] C.K. Chen, Solving partial differential equations by two dimensional differential transform, Applied Mathematics and Computation 106 (1999) 171–179.
[23] J. Biazar, M. Eslami, Analytic solution for Telegraph equation by differential transform method, Physics Letters. A 374 (2010) 2904–2906.
[24] F. Ayaz, On the two-dimensional differential transform method, Applied Mathematics and Computation 143 (2003) 361–374.