Graphs self-complementary in $K_n - e$

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Graphs self-complementary in $K_n - e$ exist for those values of $n$ where self-complementary graphs do not exist. For these graphs, the structure of the complementing permutation is analysed and their diameter is determined. The definition is related to the notions of “self-complement index” and “self-complementary index” defined by other authors.

0. Introduction

A factorisation of a graph $H$ is a partition of its edges into disjoint classes. The graph with the same vertex-set as $H$ and with, as edges, the edges in one of the classes is a factor. A factorisation in which all the factors are isomorphic to each other is an isomorphic factorisation. Here we shall study isomorphic factorisations of $K_n - e$ into 2 factors.

Definition. A graph $G$ that is one of the factors in an isomorphic factorisation of $K_n - e$ into 2 factors is called self-complementary in $K_n - e$.

Since $K_n - e$ has $\frac{1}{2}n(n - 1) - 1$ edges, such a factorisation is only possible if this number is divisible by 2. Thus it is necessary that $n = 2$ or $3$ (mod 4). We can compare this with the fact that isomorphic factorisations of $K_n$ into 2 factors, i.e. self-complementary graphs, only exist if $n = 0$ or $1$ (mod 4). Graphs self-complementary in $K_n - e$ in a sense fill the gap where self-complementary graphs do not exist. In Section 1, we determine the structure of the ‘complementing permutation’ of such graphs and in Section 2 their diameter is determined. Other authors have defined the notions of “self-complement index” and “self-complementary index” and in Sections 3 and 4 these are related to the subject of the present paper.

1. The complementing permutation

It is known (see [3], [4], for example) that for a self-complementary (s.c.) graph, the complementing permutation $\tau$ of the vertices that maps $G$ onto its complement $\overline{G}$ is a product of disjoint cycles whose lengths are multiples of 4.
with the additional possibility of one fixed point. We shall investigate the corresponding notion in the present case.

Given a graph $G$ self-complementary in $K_n - e$, let the edges of $G$ be coloured red and the remaining edges of $K_n - e$ be coloured green. Since the 2 factors are isomorphic, there is a permutation $\tau$ of the vertices of $K_n - e$ that induces a mapping of the red edges onto the green edges. We shall consider $\tau$ as a permutation of the vertices of $K_n$ and denote by $\tau'$ the corresponding mapping that this induces on the set of edges of $K_n$. Thus $\tau'$ maps each red edge onto a green edge. However, the mapping $\tau'$ does not necessarily map each green edge onto a red edge. This would be so if $\tau'$ mapped $e$ onto itself, but it may be that $\tau'$ maps $e$ onto a red edge and some green edge onto $e$.

Such a $\tau$ (which for definiteness we shall always insist induces a mapping from red to green) will (as for S.C. graphs) be called a complementing permutation. It will be useful to consider the cycles of the induced mapping $\tau'$. A cycle of $\tau'$ that does not include $e$ must be of even length, consisting of edges alternately red and green. The cycle of $\tau'$ that does include $e$ has odd length, consisting of $e$ followed by red and green edges alternately; this length equals 1 when $\tau'$ maps $e$ onto itself.

Now let $e = (v_1, v_2)$, with $v_1$ and $v_2$ being the special vertices. We prove a number of results about the structure of a complementing permutation $\tau$.

**Lemma 1.** The cycles of $\tau$ that do not include the special vertices have lengths that are multiples of 4 with the possibility of one fixed point in $\tau$.

**Proof.** In fact the graph induced by the vertices in the cycles that do not include the special vertices is an s.c. graph, with the restriction of $\tau$ to these vertices as complementing permutation. This is because $\tau'$, restricted to edges between these vertices, maps red edges onto green and green edges onto red. So the result in the lemma is just the known result about complementing permutations for s.c. graphs. We need however to give a short proof of this to notice where it breaks down when the special vertices are involved.

If a cycle $(u_1, u_2, \ldots, u_L)$ does not involve the special vertices, the induced permutation $\tau'$ maps the edge $(u_1, u_2)$ to $(u_2, u_3)$, $(u_2, u_3)$ to $(u_3, u_4)$, and so on; and $(u_L, u_1)$ to $(u_1, u_2)$. This cycle of $\tau'$ must have even length, so $L$ is even. By considering now the cycle of $\tau'$ including the edge $(u_1, u_{(L/2)+1})$, we find that $L/2$ must be even. Thus $L$ is a multiple of 4.

Also there can be at most one fixed point, for if $\tau$ contains $(u_1)(u_2)$ then the edge $(u_1, u_2)$ forms a cycle of length 1 in $\tau'$ which cannot be.

**Lemma 2.** The only way in which the special vertices $v_1$ and $v_2$ can occur in different cycles of $\tau$ is as two fixed points $(v_1)(v_2)$ and then there can be no other fixed point in $\tau$.

**Proof.** Suppose that $v_1$ occurs in a cycle of length $L_1$ and $v_2$ occurs in a cycle of length $L_2$. The argument just given in Lemma 1 can be applied to each cycle to
show that each of $L_1$ and $L_2$ is, if not equal to 1, divisible by 4. However, consider the cycle of $\tau'$ including $e$. The number of edges in this cycle is the l.c.m. of $L_1$ and $L_2$ and is thus divisible by $L_1$ and $L_2$. Since this is to be odd, it must be that $L_1 = L_2 = 1$.

The argument in the final sentence of the proof of Lemma 1 shows that there can be no other fixed point. \[ \Box \]

**Lemma 3.** If $v_1$ and $v_2$ occur in the same cycle, then either (i) the cycle has length 3 or (ii) the cycle has length $4h + 2$ and $\tau^{2h+1}(v_1) = v_2$.

**Proof.** First suppose that $v_1$ and $v_2$ appear consecutively in a cycle of length $L$. Consideration of the cycle of $\tau'$ including $e$ shows that $L$ must be odd. If $L > 3$, one finds that there is another cycle of $\tau'$, not including $e$, that has odd length. Thus $L = 3$ is the only possibility.

Now suppose that $v_1$ and $v_2$ appear in a cycle of length $L$ but not consecutively. Then by considering the edge $(u_1, u_2)$ as in Lemma 1, we find that $L$ must be even. And the remaining argument of Lemma 1 follows, showing that $L$ is a multiple of 4, making all the corresponding cycles of $\tau'$ of even length and so precluding the existence of a cycle of $\tau'$ of odd length including $e$. But this happens, unless $u_1$ and $u_2$ appear as, say, $u_1$ and $u_{(L/2)+1}$. Then for the cycle of $\tau'$ involving $e$ to have odd length, $L/2$ must be odd. Thus $L = 4h + 2$ and $\tau^{2h+1}(v_1) = v_2$. (We note that $h = 0$ is included as a possibility in which case the cycle in question is just $(v_1 v_2)$.) \[ \Box \]

We can sum up these results as follows:

**Theorem.** Let $\tau$ be a complementing permutation for a graph self-complementary in $K_n - e$, where $e = (v_1, v_2)$. Then $n = 2$ or $3 \mod 4$. Moreover

(a) when $n = 2 \mod 4$, either

(i) $\tau$ has both special vertices $v_1, v_2$ as fixed points $(v_1)(v_2)$ and the other cycles have lengths that are multiples of 4, or

(ii) $\tau$ consists of a cycle of length $4h + 2$ including both special vertices, with $\tau^{2h+1}(v_1) = v_2$, and other cycles with lengths that are multiples of 4,

(b) when $n = 3 \mod 4$, either

(i) $\tau$ consists of a cycle of length 3 including both special vertices, and other cycles with lengths that are multiples of 4, or

(ii) $\tau$ consists of a cycle of length $4h + 2$ including both special vertices, with $\tau^{2h+1}(v_1) = v_2$, other cycles with lengths that are multiples of 4, and one fixed point.

Thus in the case $n = 6$, the possible forms for $\tau$ are:

$$(v_1 v_2)(\ldots), \quad (v_1 v_2)(\ldots), \quad (v_1 \ldots v_2 \ldots)$$

and in the case $n = 7$:

$$(v_1 v_2)(\ldots)(\cdot), \quad (v_1 v_2)(\ldots), \quad (v_1 \ldots v_2 \ldots)(\cdot).$$
The five graphs self-complementary in $K_n - e$ are shown in Fig. 1, with the special vertices marked in black.

2. Diameter

It is known (see [3]) that an s.c. graph has diameter 2 or 3. Now a graph s.c. in $K_n - e$ may not even be connected: For any value of $n = 2$ or 3 (mod 4), let $H$ be an s.c. graph with $n - 2$ vertices. Form $G$ by adding an isolated vertex $v_1$ and an additional vertex $v_2$, adjacent to every vertex of $H$. Then $G$ is s.c. in $K_n - e$ with $v_1$ and $v_2$ as special vertices. In fact, it is not difficult to see that if a graph s.c. in $K_n - e$ has an isolated vertex, then it must have this structure.

However, we can show that this is, in a sense, an exceptional case:

Theorem. Let $G$ be a graph s.c. in $K_n - e$ with no isolated vertex. Then the diameter of $G$ is 2 or 3.

Proof. We first recall the proof for s.c. graphs: Let $\tau$ be a complementing permutation for an s.c. graph, and let $x_1$ and $x_2$ ($\neq x_1$) be any two vertices. Then $x_1$ is either adjacent to $\tau^{-1}(x_1)$ or $\tau(x_1)$. Denote $x_1$ and whichever of these it is adjacent to by $y_1$ and $\tau(y_1)$ (so that $x_1$ may be $y_1$ or $\tau(y_1)$). Similarly denote $x_2$ and whichever of $\tau^{-1}(x_2)$ and $\tau(x_2)$ it is adjacent to by $y_2$ and $\tau(y_2)$. Then either $y_1$ is adjacent to $y_2$ or $\tau(y_1)$ is adjacent to $\tau(y_2)$. So these four vertices lie in a path (either in the order $\tau(y_1)$, $y_1$, $y_2$, $\tau(y_2)$ or in the order $y_1$, $\tau(y_1)$, $\tau(y_2)$, $y_2$) of length 3. Hence $d(x_1, x_2) \leq 3$.

(The special cases when, for example, $\tau(x_1) = x_1$ or when $\tau(x_1) = x_2$, are easily dealt with.)

Now this proof also shows that for two vertices $x_1$ and $x_2$ of a graph $G$ s.c. in $K_n - e$, the distance $d(x_1, x_2)$ is less than or equal to 3 except for the following cases where it breaks down:

(i) The statement \("x_1 is either adjacent to $\tau^{-1}(x_1)$ or $\tau(x_1)$\) breaks down if $x_1$ and $\tau(x_1)$ are the special vertices $v_1$, $v_2$.

(ii) The statement \("either $y_1$ is adjacent to $y_2$ or $\tau(y_1)$ is adjacent to $\tau(y_2)$\) breaks down if $\tau(y_1)$ and $\tau(y_2)$ are the special vertices.
We can first dismiss case (ii) because, when \( v_1 \) and \( v_2 \) are not fixed points or consecutive in \( \tau \) (and these cases will be dealt with below), this possibility cannot arise. For we then have \( v_1 \) and \( v_2 \) in the same cycle with \( \tau^{2k+1}(v_1) = v_2 \). Now suppose \( \tau(y_1) = v_1 \) and \( \tau(y_2) = v_2 \). Then \( \tau^{2k+1}(y_1) = y_2 \). However, in the proof we had \( y_1 \) adjacent to \( \tau(y_1) \) and \( y_2 \) adjacent to \( \tau(y_2) \), whereas \( y_1 \) adjacent to \( \tau(y_1) \) implies, by applying \( \tau^{2k+1} \), then \( y_2 \) is not adjacent to \( \tau(y_2) \).

We need to investigate \( d(v_1, v_2) \), \( d(v_1, x) \) and \( d(v_2, x) \), where \( x \) is a non-special vertex, in the following cases:

(a) when \( \tau \) contains the special vertices as fixed points \((v_1)(v_2)\)
(b) when \( \tau \) contains the cycle \((v_1v_2)\)
(c) when \( \tau \) contains the cycle \((v_2v_1)\).

Since \( n > 3 \), there are in \( \tau \) some cycles whose lengths are multiples of 4. In each of these, number the vertices in some consecutive way so that we may talk about the odd vertices and the even vertices in any particular cycle.

**Case (a).** For any cycle in \( \tau \) with length a multiple of 4, \( v_1 \) is either adjacent to the odd vertices and not adjacent to the even or vice versa. The same is true for \( v_2 \). Now in any cycle, each odd vertex \( x \) is adjacent to an even vertex (either \( \tau^{-1}(x) \) or \( \tau(x) \)) and each even vertex is adjacent to an odd vertex. So \( d(v_1, x) \leq 2 \) and \( d(v_2, x) \leq 2 \) (similarly) for all non-special vertices \( x \).

If there is a cycle such that \( v_1 \) and \( v_2 \) are both adjacent to the odd vertices or such that \( v_1 \) and \( v_2 \) are both adjacent to the even vertices, then \( d(v_1, v_2) = 2 \). If not, there is a cycle with \( v_1 \) adjacent to the odd vertices and \( v_2 \) adjacent to the even, so \( d(v_1, v_2) = 3 \).

**Case (b).** Now for any cycle if \( v_1 \) is adjacent to the odd vertices then \( v_2 \) is not adjacent to the even, and if \( v_1 \) is adjacent to the even vertices then \( v_2 \) is not adjacent to the odd. The set of cycles can thus be partitioned into three:

Let \( A_1 \) be the set of cycles all of whose vertices are adjacent to \( v_1 \) and not adjacent to \( v_2 \); let \( A_2 \) be the set of cycles all of whose vertices are adjacent to \( v_2 \) and not adjacent to \( v_1 \); and \( B \) be the set of cycles whose odd (or even) vertices are adjacent to both \( v_1 \) and \( v_2 \).

Clearly if \( x \) is a vertex in a cycle in \( A_1 \) then \( d(v_1, x) = 1 \); and if \( x \) is a vertex in a cycle in \( B \) then \( d(v_1, x) \leq 2 \). Also if \( B \) is not empty then \( d(v_1, v_2) = 2 \).

So we have to investigate (i) \( d(v_1, v_2) \) when \( B \) is empty, (ii) \( d(v_1, x) \) when \( x \) belongs to a cycle in \( A_2 \). (Identical results must be true for \( d(v_2, x) \).)

(i) If \( B \) is empty, then neither \( A_1 \) nor \( A_2 \) is empty otherwise \( v_1 \) or \( v_2 \) is isolated. Let \( \alpha_1 \) be a cycle in \( A_1 \) and \( \alpha_2 \) a cycle in \( A_2 \). There is certainly an edge between some vertex in \( \alpha_1 \) and some vertex of \( \alpha_2 \). So \( d(v_1, v_2) \leq 3 \).

(ii) Suppose that \( x \) belongs to a cycle in \( A_2 \). We know that \( A_1 \) and \( B \) are not both empty otherwise \( v_1 \) is isolated.

If \( A_1 \) is not empty, let \( \alpha_1 \) be a cycle in \( A_1 \). Either \( x \) is adjacent to some vertex in \( \alpha_1 \) or else \( \tau(x) \) and \( \tau^{-1}(x) \) both are. Since \( x \) is adjacent to either \( \tau(x) \) or \( \tau^{-1}(x) \), we have \( d(v_1, x) \leq 3 \).
If $B$ is not empty, then, as we have already seen, $d(v_1, v_2) = 2$. So $d(v_1, x) \leq 3$ since $x$ is adjacent to $v_2$.

Case (c). We suppose that $\tau$ contains a cycle $(v_1v_2w)$, where $v_2$ is adjacent to $w$. It is clear that if $x$ is a vertex in some other cycle then $d(v_2, x) \leq 2$ and $d(w, x) \leq 2$.

If $v_1$ is adjacent to $x$ then $d(v_1, x) = 1$. If $v_1$ is not adjacent to $x$, then by applying $\tau^{-3}$ and $\tau^3$, $v_1$ is adjacent to both $\tau^{-3}(x)$ and $\tau^3(x)$. But $x$ is either adjacent to $\tau^{-3}(x)$ or $\tau^3(x)$, so $d(v_1, x) \leq 2$.

Finally suppose that $v_1$ is adjacent to a certain vertex $x$ in one of the other cycles. Since $d(v_2, x) \leq 2$ we have $d(v_1, v_2) \leq 3$ and similarly $d(v_1, w) \leq 3$. □

3. Self-complement index

In [1], the self-complement index $s(G)$ of a graph $G$ was introduced to provide a measure of how close $G$ is to being self-complementary. The value $s(G)$ is defined to be the maximum number of vertices of an induced subgraph of $G$ whose complement is also an induced subgraph of $G$. For a graph with $n$ vertices, $s(G) = n$ if and only if $G$ is self-complementary.

**Theorem.** If $G$ is self-complementary in $K_n - e$, then $s(G) = n - 1$.

**Proof.** Let the edges of $G$ be coloured red and let $\tau$ be a complementary permutation that maps red edges onto green. Let $e = (v_1, v_2)$.

Let $H$ be the subgraph of $G$ induced by the set of vertices $V(K_n) - \{\tau^{-1}(v_1)\}$ and $K$ the subgraph induced by the set of vertices $V(K_n) - \{v_1\}$. Now $\tau$ restricted to the vertices of $H$ maps these onto the vertices of $K$. Moreover the induced permutation $\tau'$ restricted to the edges between vertices of $H$ maps red edges to green; it also maps green edges (and $e$ if it occurs) to red, since $e$ does not occur as an edge between vertices of $K$. Thus $K$ is isomorphic to $\overline{H}$. So the complement of the induced subgraph $H$ is also an induced subgraph of $G$ and hence $s(G) = n - 1$. □

We note that the converse of this theorem is not true. In the first case, graphs $G$ with $n$ vertices and $s(G) = n - 1$ exist for all values of $n$. Moreover, even when $n$ has a value for which graphs s.c. in $K_n - e$ exist, there are graphs $G$ with $s(G) = n - 1$ which are not s.c. in $K_n - e$. Take, for example, $n = 2 \pmod{4}$ and let $G$ consist of an s.c. graph with $n - 1$ vertices, together with an additional isolated vertex.

4. Self-complementary index

In a more recent paper [2], a different parameter is defined, unfortunately also denoted there by $s(G)$, to measure how close $G$ is to being self-complementary.
We shall denote this, the self-complementary index, by $t(G)$. It is related to the concept of the switched graph $S_H(G)$, where $H$ is a subset of $V(G)$. This is defined to be the graph with the same vertex set as $G$ and with the following edges:

$$
\{(u, v) \mid u, v \in H \text{ and } (u, v) \notin E(G)\}
\cup \{(u, v) \mid u \notin H \text{ and } (u, v) \in E(G)\}
\cup \{(u, v) \mid u \in H, v \notin H \text{ and } (u, v) \notin E(G)\}.
$$

Then $t(G)$ is the maximum number of vertices in a set $H$ such that $S_H(G)$ is isomorphic to $G$.

For a graph with $n$ vertices, $t(G) = n$ if and only if $G$ is self-complementary. It is never the case that $t(G) = n - 1$. The index is related to the subject of the present paper as follows:

**Theorem.** Let $G$ be a graph with $n$ vertices. Then $t(G) = n - 2$ if and only if either $G$ or $G - e$ is s.c. in $K_n - e$.

**Proof.** Suppose that $G$ is s.c. in $K_n - e$, where $e = (v_1, v_2)$. Let $H = V(G) - \{v_1, v_2\}$. Then it is clear that $S_H(G) \cong G$, so that $t(G) = n - 2$. The same is true if $G - e$ is s.c. in $K_n - e$.

Conversely, if $t(G) = n - 2$, let $H$ be a subset of $V(G)$ with $n - 2$ vertices such that $S_H(G) \cong G$. Let $v_1$ and $v_2$ be the two elements not in $H$ and let $e = (v_1, v_2)$. It is clear that if $e \notin E(G)$ then $G$ is s.c. in $K_n - e$ and if $e \in E(G)$ then $G - e$ is s.c. in $K_n - e$.  

**References**