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Safety estimation of structural systems via interval analysis

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Abstract Considering that the uncertain information has serious influences on the safety of structural systems and is always limited, it is reasonable that the uncertainties are generally described as interval sets. Based on the non-probabilistic set-theoretic theory, which is applied to measuring the safety of structural components and further combined with the branch-and-bound method for the probabilistic reliability analysis of structural systems, the non-probabilistic branch-and-bound method for determining the dominant failure modes of an uncertain structural system is given. Meanwhile, a new system safety measuring index obtained by the non-probabilistic set-theoretic model is investigated. Moreover, the compatibility between the classical probabilistic model as well as the proposed interval-set model will be discussed to verify the physical meaning of the safety measure in this paper. Some numerical examples are utilized to illustrate the validity and feasibility of the developed method.

1. Introduction

Structural safety or reliability analysis plays an important role in the analysis and design of structural systems. With the continuous development of technology, the complexity of the engineering structural systems increases gradually so that the anticipated influence of the uncertainty on them becomes more and more profound.[1] On the one hand, there exists multiple uncertain information, such as random, fuzzy, uncertain-but-bounded, etc. in the engineering structures[2]; on the other hand, the experimental data are usually scant.[3] Therefore, the applicable conditions of the probabilistic reliability model and fuzzy reliability model cannot always satisfy the requirements. The subjective assumptions about the probability density function and membership functions will lead to the infeasible solutions with large differences.[4] In practice, the statistical information on uncertainty may not be easily available whereas the bounds on the uncertain information can be obtained readily. Considering that, Ben-Haim[5] initially proposed the concept of the non-probabilistic and robust reliability of structures. Recently, the non-probabilistic reliability theory has been developed extensively. Elishakoff [6] proposed the concept of the non-probabilistic safety factor, which is defined as the ratio of the yield stress—in case it is a deterministic quantity—by the upper bound of stress. In the opposite case in which the stress is deterministic but the strength is a non-probabilistic variable, this safety factor equals the lower bound of the strength divided by the stress. In the general case in which both the stress and strength are interval variables, the non-probabilistic safety factor is defined as the ratio of the lower bound of the strength to the upper bound of the stress. Qiu et al.[7] extensively revis-

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ited the concept of "non-probabilistic reliability" in the critical light. They suggested a non-probabilistic convex reliability model by using the partial order relation of the superscribed hyper-rectangles or interval vectors. Guo et al.[8] quantified the uncertain structural parameters as interval variables and proposed another measure of the 'non-probabilistic reliability', which is taken as the shortest distance η from the origin to the failure surface. They pointed out that the structure must be safe when $\eta > 1$. The inequality $\eta > 1$ means that the set of the stress has no common points with the set of the strength, whereas the inequality $0 < \eta < 1$ implies their interference. The latter case will usually be the problem of our concern. In view of the latter case, Wang et al.[9] proposed a new non-probabilistic set-theoretic safety measure for structures, where based on the non-probabilistic set-theoretic stress–strength interference model, the ratio of the volume of the safe region to the total volume of the region associated with the variation of the basic interval variables is suggested as the measure of the non-probabilistic safety of the structural component. Nevertheless, the structural system reliability based on the non-probabilistic method has not been extensively studied. It has been recognized that a fully satisfactory estimation of the structural reliability must be grounded on a system approach. In some situations it is insufficient to estimate the reliability of the individual structural members of a structural system.[10]

In this paper, based on the non-probabilistic set-theoretic model of safety measure for structural components proposed in Ref. [9], the non-probabilistic set-theoretic branch-and-bound method is presented to determine the dominant failure modes and the safety measure of a structural system.

2. Interval finite element analysis

Consider the safety analysis of structures subject to external loads. Stress S and strength R are influenced by a great deal of factors, and thus they can be expressed as the following functions of variables, i.e.

$$\begin{cases} S = S(x_{S_1}, x_{S_2}, \cdots, x_{S_q}) \\ R = R(x_{R_1}, x_{R_2}, \cdots, x_{R_p}) \end{cases} \tag{1}$$

where the variables $x_{S_i} (i = 1, 2, \cdots, q)$ are related to the structural stress, such as force, moment, temperature, humidity, over-loading, etc., and the variables $x_{R_i} (i = 1, 2, \cdots, p)$ may be associated with the structural strength, such as surface roughness, material properties, scratch, and crack length, etc. However, these parameters are usually uncertain so that the structural stress and strength would have uncertainty too. It is assumed that the uncertain parameters vary within a given interval or hyper-rectangle, i.e.

$$\underline{\boldsymbol{x}} \leqslant \boldsymbol{x} \leqslant \bar{\boldsymbol{x}} \quad \text{or} \quad \underline{x}_i \leqslant x_i \leqslant \bar{x}_i \quad (i = 1, 2, \cdots, m) \tag{2}$$

where $\underline{\boldsymbol{x}} = (\underline{x}_i)$ and $\bar{\boldsymbol{x}} = (\bar{x}_i)$ are, respectively, the lower and upper bound vectors of uncertain parameter vector $\boldsymbol{x}$.

By use of the interval notations in interval mathematics,[11,12] the inequality (2) can be rewritten as

$$\boldsymbol{x} \in \boldsymbol{x}^{\mathrm{I}} \quad \text{or} \quad x_i \in x_i^{\mathrm{I}} \quad (i = 1, 2, \cdots, m) \tag{3}$$

where $\boldsymbol{x}^{\mathrm{I}} = [\underline{\boldsymbol{x}}, \bar{\boldsymbol{x}}]$, $x_i^{\mathrm{I}} = [\underline{x}_i, \bar{x}_i]$.

In order to determine the structural stress interval S^{I}, the static displacement response interval $\boldsymbol{u}^{\mathrm{I}}$ should be solved firstly. The static equilibrium equation of structures in the form of finite element method can be given as

$$\boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x}) \tag{4}$$

where $\boldsymbol{K}$ is the stiffness matrix, $\boldsymbol{f}$ the external load vector, and $\boldsymbol{u}$ the displacement vector. $\boldsymbol{K}$ and $\boldsymbol{f}$ are both the functions of uncertain parameter vector $\boldsymbol{x}$.

Eq. (3) can be regarded as the constraint condition. Hence, by solving the static displacement vector $\boldsymbol{u} = (\boldsymbol{u}_i)$, we mean to solve the family of equilibrium equations in which the structural parameters can vary inside the bounded sets. That is to say, the static displacement of the equilibrium equation with uncertain-but-bounded parameters should be a set, which is obtained by

$$\Gamma = \{\boldsymbol{u} : \boldsymbol{u} \in \mathbf{R}^n,\ \boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x}),\ \boldsymbol{x} \in \boldsymbol{x}^{\mathrm{I}}\} \tag{5}$$

The calculation of the solution set, in general, is extremely difficult, namely, the solution set Γ has a very complicated region and is not commonly convex. Taking this into account, one has to determine a closed convex interval set $[u_{i\min}, u_{i\max}]$ for each component of the static displacement vector, which is the smallest width one enclosing all possible values, satisfying $\boldsymbol{K}(\boldsymbol{x})\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x})$ when the structural parameters $\boldsymbol{x} = (x_i)$ vary within $\boldsymbol{x}^{\mathrm{I}} = [\underline{\boldsymbol{x}}, \bar{\boldsymbol{x}}]$. Therefore the static displacement problem (4) subject to inequality (2) or Eq. (3) can be transformed into the interval parameter linear equations

$$\boldsymbol{K}(\boldsymbol{x}^{\mathrm{I}})\boldsymbol{u} = \boldsymbol{f}(\boldsymbol{x}^{\mathrm{I}}) \tag{6}$$

or linear interval equations

$$\boldsymbol{K}^{\mathrm{I}}\boldsymbol{u} = \boldsymbol{f}^{\mathrm{I}} \tag{7}$$

In recent years, the two kinds of Eqs. (6) and (7) have been widely investigated by various approaches, including the approximate methods, such as interval matrix perturbation method, interval parameter perturbation method, etc., and the accurate methods, such as the vertex combination method of interval matrices, the vertex combination method of interval parameters and the optimization method, etc. However, the researches on the stress interval problem of structures with interval parameters were seldom done up to now.[13]

The relationship between the stress and the displacement reads

$$\boldsymbol{S} = \boldsymbol{D}\boldsymbol{B}\boldsymbol{u} \tag{8}$$

where $\boldsymbol{S}$ is the structural stress, $\boldsymbol{D}$ the elastic matrix and $\boldsymbol{B}$ the geometric matrix.

Taking the first-order Taylor's series expansion of the structural stress with uncertain parameter vector around the middle or nominal value $\boldsymbol{x}^{\mathrm{c}}$, the following expression can be easily obtained

$$\boldsymbol{S}(\boldsymbol{x}) = \boldsymbol{S}(\boldsymbol{x}^{\mathrm{c}}) + \sum_{i=1}^{m} \frac{\partial \boldsymbol{S}}{\partial x_i}\left(x_i - x_i^{\mathrm{c}}\right) \tag{9}$$

where

$$\frac{\partial \boldsymbol{S}}{\partial x_i} = \frac{\partial \boldsymbol{D}}{\partial x_i}\boldsymbol{B}\boldsymbol{u} + \boldsymbol{D}\boldsymbol{B}\frac{\partial \boldsymbol{u}}{\partial x_i} \tag{10}$$

and

$$\frac{\partial \boldsymbol{u}}{\partial x_i} = \boldsymbol{K}^{-1}\left(-\frac{\partial \boldsymbol{K}}{\partial x_i}\boldsymbol{u} + \frac{\partial \boldsymbol{f}}{\partial x_i}\right) \tag{11}$$

By virtue of the natural interval extension, the stress interval can be deduced by Eqs. (3) and (9)

$$\boldsymbol{S}^{\mathrm{I}}(\boldsymbol{x}) = \boldsymbol{S}(\boldsymbol{x}^{\mathrm{c}}) + \sum_{i=1}^{m} \frac{\partial \boldsymbol{S}}{\partial x_i} \Delta x_i^{\mathrm{I}} \tag{12}$$

where
$\Delta \boldsymbol{x}^{\mathrm{I}} = (\Delta x_i^{\mathrm{I}}),\ \Delta x_i^{\mathrm{I}} = [-\Delta x_i, \Delta x_i],$ and $\Delta x_i = (\bar{x}_i - \underline{x}_i)/2$.

In practical engineering, a rational strength interval R^{I} of structures can be selected to measure the scatter of material strength based on the manufacture accuracy and technological level.

3. Non-probabilistic set-theoretic model of structural safety measure

Once the stress interval S^{I} and the strength interval R^{I} are known, the non-probabilistic safety and failure measure of components would be computed based on the non-probabilistic set-theoretic model proposed in Ref. [9]. For the following convenience, the non-probabilistic set-theoretic model of structural safety measure will be introduced simply (see Ref. [9] for detail).

In the analysis and design of structures, the stress and strength have the same physical dimensions. Thus, their interval descriptions can be placed on a number of axes. During the design process, the strength is generally required to be larger than the actual stress, implying that the structure with respect to the median values must be safe, i.e. $R^{\mathrm{c}} > S^{\mathrm{c}}$. However, because of the scatter in the stress and strength, the intervals themselves may share the same numerical values to yield an intersection set as shown in Fig. 1 as the shaded region. This region can be called the interference region. Similar to the terminology in (probabilistic) reliability theory, Fig. 1 can be dubbed as non-probabilistic set-theoretic stress–strength interference model. We are interested in adopting a measure of safety (or reliability) that is interconnected with lower and upper bounds of the stress and the strength.

The limit state of the structure is expressed as the function of structural stress S and the structural strength R as follows:

$$M(R, S) = R - S \tag{13}$$

Given the specified values of S and R, it is possible to judge the structural state, in terms of it being in the state of safe or the state of failure. Basic variables space will be divided into two parts, namely the safe region and the failure region, by the failure plane or the limit state plane, i.e.

$$M(R, S) = R - S = 0 \tag{14}$$

The positive value of M indicates the safe region of basic variables, while the negative value of M represents the failure region.

When the interference between the stress interval and strength interval occurs as shown in Fig. 1, even though the median value S^{c} of the stress is smaller than the median value R^{c} of the strength, it cannot be ensured that the stress will take on values not in excess of the strength. Thus, the possibility that the stress is larger than the strength will be different from zero. This fact can be described as

$$\eta(M(R, S) < 0) > 0 \tag{15}$$

where $\eta(T)$ represents the possibility of the event T.

It is instructive to represent the stress and strength in a plane as shown in Fig. 2. The solid rectangle shows the region of variation of both stress and strength. It is crossed by the failure plane $R = S$. The safe region is again hatched, whereas the failure region is unshaded. The possibility that Eq. (15) holds or the possibility that the stress is larger than the strength will be referred to by us as the non-probabilistic set-theoretic failure measure, which can be defined as the ratio of the area of failure region to the total area of basic variables region, i.e.

$$F_M = \eta(M(R, S) < 0) = \frac{A_{\text{failure}}}{A_{\text{total}}} \tag{16}$$

The possibility that the stress is smaller than the strength is called here with us the non-probabilistic set-theoretic safety measure, which is naturally defined as the ratio of the area of the safe region to the total area of basic region of variation, i.e.

$$S_M = \eta(M(R, S) > 0) = \frac{A_{\text{safe}}}{A_{\text{total}}} \tag{17}$$

Obviously, the process for evaluation of A_{failure} will be easier than that of A_{safe}. So, we represent S_M as the complement to F_M,

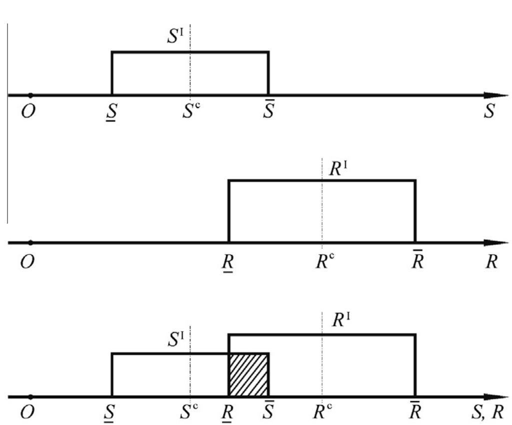

Fig. 1 Non-probabilistic set-theoretic stress–strength interference model.

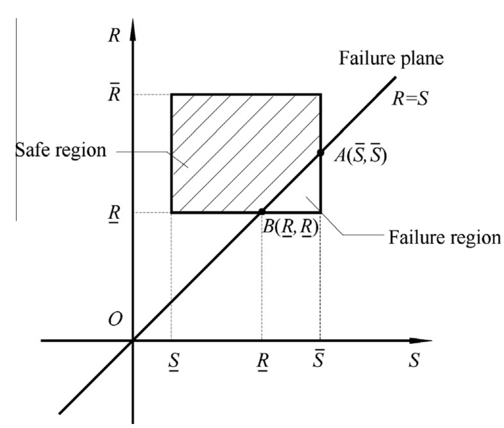

Fig. 2 Scheme for normalized space of variables under interference occurrence.

$$S_M = 1 - F_M = 1 - \frac{A_{\text{failure}}}{A_{\text{total}}} \tag{18}$$

The coincidence of Eq. (17) with the reliability obtained by probability theory in the case of uniform distribution of both the stress and strength has been demonstrated in Ref. [9].

When the interference between the stress and the strength does not take place, or else when the maximum value/upper bound of the stress is equal to or smaller than the minimum value/lower bound of the strength, the event that the stress is bigger than the strength is impossible. In other words, the possibility that the stress is larger than the strength must be zero, i.e.

$$F_M = \eta(M(R,S) < 0) = \frac{A_{\text{failure}}}{A_{\text{total}}} = 0 \tag{19}$$

The state when the upper bound $\overline{S}$ of the stress is equal to the lower bound $\underline{R}$ of the strength could be called "critical state" as shown in Fig. 3.

For the general nonlinear limit state function (shown in Fig. 4), the above concept of the non-probabilistic safety measure can still be applied, as a ratio of the appropriate areas.

Another case should be particularly concerned about, namely if the limit state plane is expressed as the function of multi-dimensional interval variables, the multi-dimensional region (actually a range of hyper-rectangle) enclosed by basic interval variables will be divided into safe region and failure region by hyper-surface. Under such circumstances, the non-probabilistic set-theoretic failure measure can be redefined as the ratio of the hyper-volume of failure region to the hyper-volume of basic interval variables region; meanwhile, the non-probabilistic set-theoretic safety measure can be redefined as the ratio of the hyper-volume of safe region to the hyper-volume of basic interval variables region.

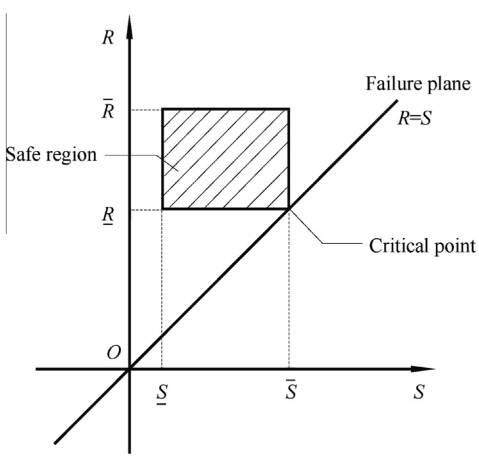

Fig. 3 Representation for critical state.

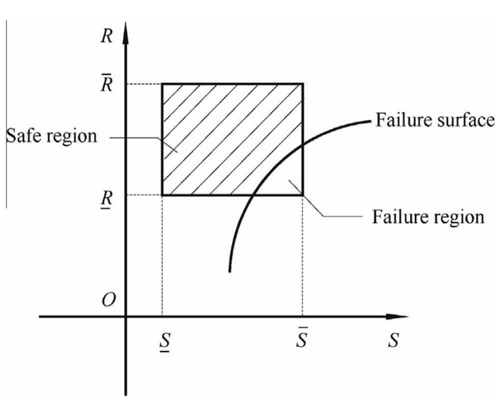

Fig. 4 Nonlinear limit state function.

4. Compatibility and transition between non-probabilistic set-theoretic model and probabilistic structural reliability

In order to illustrate the meaning of F_M/S_M in physics, the analysis of compatibility and transition between the possibility obtained by the non-probabilistic model and the traditional probability based on the probability theory will be demonstrated. In this section, a truncated normal distribution model is discussed as the expression of the probability density function to describe the random variables. When the value of parameters in the probability density function changes, the variability of random variables will either increase or decrease. With the increase of variability, the uncertain variable tends to the uniform distribution. If so, the two results of structural safety measure, derived from the probabilistic approach and the non-probabilistic set-theoretic modeling method, may show a good agreement. In other words, a nice transition from probabilistic to set-theoretic modeling may be implemented.

Considering that the truncated normal probability density function of two variables varying within a rectangle is used to describe the distribution for strength R and stress S. The expression of the above statements can be written as

$$f_{RS}(r,s) = \begin{cases} c\exp\left(-\frac{(r-R^{c})^2}{a^2} - \frac{(s-S^{c})^2}{b^2}\right), & \text{when } |r-R^{c}| \leqslant R^{r},\ |s-S^{c}| \leqslant S^{r} \\ 0, & \text{when } |r-R^{c}| > R^{r},\ |s-S^{c}| > S^{r} \end{cases} \tag{20}$$

where $f_{RS}(r,s)$ is the joint probability density function of R and S; R^{c} and S^{c} indicate the respective central values; R^{r} and S^{r} denote the bounds of uncertainty for the random variables R and S, respectively; a and b represent coefficients; the normalization constant c is further derived from

$$c = \left[4ab\,\mathrm{erf}\left(\frac{R^{r}}{a}, \frac{S^{r}}{b}\right)\right]^{-1} \tag{21}$$

where erf(x,y) is defined as

$$\mathrm{erf}(x,y) = \int_0^x \int_0^y \mathrm{e}^{-(\xi^2+\eta^2)}\mathrm{d}\eta\mathrm{d}\xi \tag{22}$$

The probability distribution is specified as truncated normal; however, we may not know precisely its exact shape. It is of utmost interest then to evaluate the safety as a function of a and b. It can be seen that if the ranges of R and S are available, then the probability density $f_{RS}(r,s)$ depends exclusively on a and b, namely, the variability of random variables increases with the increase of a and b; when a and b are large enough, the random variables tend to a uniform distribution within a range of rectangle, as shown in Fig. 5.

Monte Carlo simulation can be carried out to obtain possible values of random variables R and S when the parameters a, b, R^{r} and S^{r} are given (see Figs. 6–9 for detail). Consider the limit state function in Eq. (14) as the criterion to judge the structural safety/failure measure. It is easy to understand that

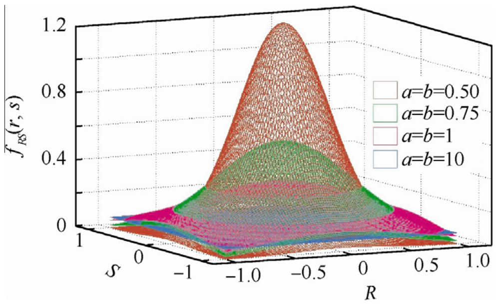

Fig. 5 Probability density function of random variables R and S with different a and b.

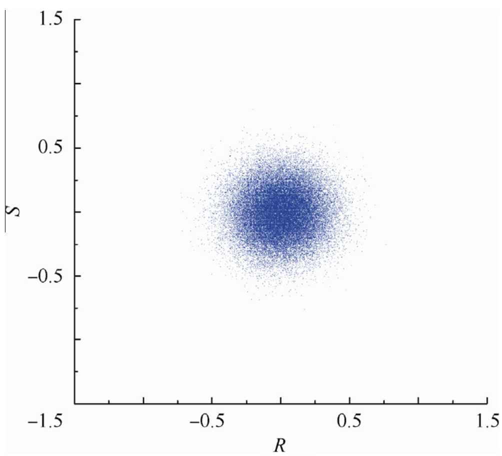

Fig. 6 Random samples obtained by Monte Carlo simulation when $a = b = 0.25$.

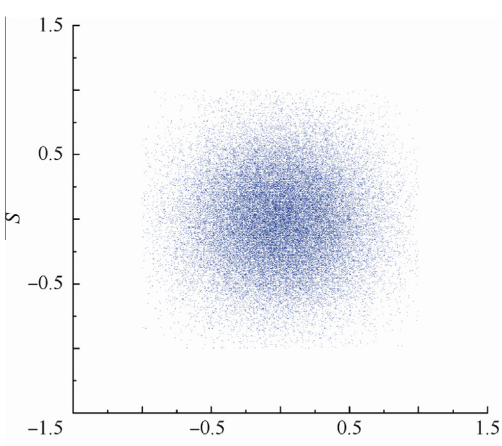

Fig. 7 Random samples obtained by Monte Carlo simulation when $a = b = 0.50$.

$$F_M^{\text{prob}} = \eta(R - S < 0) = \frac{k_f}{n} \tag{23}$$

and

$$S_M^{\text{prob}} = \eta(R - S > 0) = 1 - F_M^{\text{prob}} = 1 - \frac{k_f}{n} \tag{24}$$

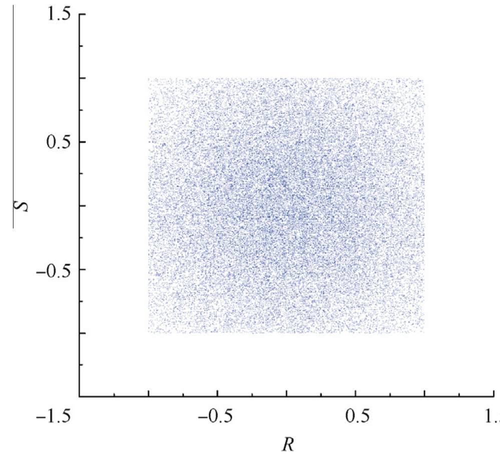

Fig. 8 Random samples obtained by Monte Carlo simulation when $a = b = 1$.

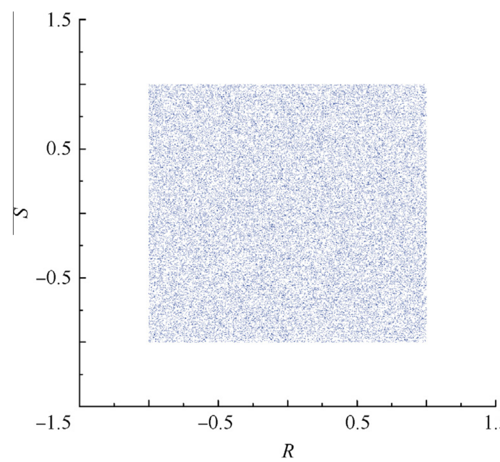

Fig. 9 Random samples obtained by Monte Carlo simulation when $a = b = 10$.

where n is the number of random samples created by Monte Carlo simulation, and k_f means the number of points, which may satisfy the criterion $R - S < 0$.

$$F_M^{\text{prob}} = F_M \text{ and } S_M^{\text{prob}} = S_M \tag{25}$$

Thus, with the increasing deviation of R and S, these two random variables tend to be uniformly distributed within a given rectangle, and hence the results of reliability based on probabilistic as well as set-theoretic modeling are anticipated to show a good consistency.[9]

As above mentioned, it is necessary to emphasize that the proposed possibility based on the non-probabilistic set-theoretic model and the classical reliability based on probabilistic theory has the same nature and physical meaning when they are used to measure the uncertain structural safety.

5. Non-probabilistic set-theoretic safety analysis of structural systems

There are so many failure modes in large structures with a high degree of redundancy that it is impossible to identify all of

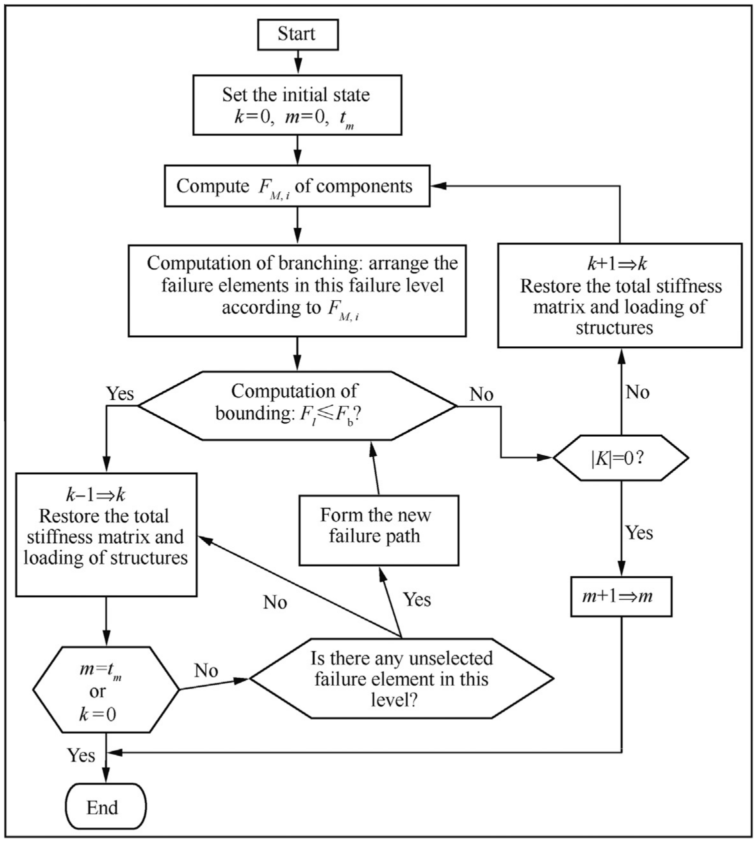

Fig. 10 Flow chart of the proposed non-probabilistic branch-and-bound method.

them a prior for estimating structural systems' reliability or possibility of failure. Different failure modes may have different possibilities of occurrence. By virtue of the idea of the branch-and-bound method[10] in the probabilistic reliability theory, the non-probabilistic set-theoretic branch-and-bound method of a structural system is built based on the non-probabilistic set-theoretic model of the safety measure for structural components in previous section. The difference and relation between the two kinds of branch-and-bound methods are: (1) both of them are proposed for analyzing the influences of uncertain factors on the structural safety; (2) they adopt two kinds of different descriptive forms of uncertainty, i.e., the bounded set for the non-probabilistic method and the random variables for the stochastic method.

In the non-probabilistic branch-and-bound method, branching is to choose the failure element in the failure path, while bounding is to remain the failure paths corresponding to the dominant failure modes and abandon the minor failure modes according to some certain criteria. The flowchart of the branch-and-bound method based on the non-probabilistic set-theoretic measure of structural system safety is given in Fig. 10. The procedures of the non-probabilistic branch-and-bound method for seeking the dominant failure paths are given as follows:

Step 1 Set the initial state: $k = 0$, $m = 0$, t_m.

Step 2 Operation of branching: compute the non-probabilistic set-theoretic failure measure $F_{M,i}$ of the candidate elements in this failure level and arrange them in order, then go to Step 3.

Step 3 Operation of bounding: judge whether this failure path should be retained according to the inequation $F_l \leqslant F_b$: (1) if this failure path is not retained (as ★ shown in Fig. 11), then go to Step 4; (2) if this failure path is retained (as ▲ shown in Fig. 11), then go to Step 5.

Step 4 Go back to the previous failure level along the original path, and let $k - 1 \Rightarrow k$. Restore the stiffness and loading states to the previous level structures, and go to Step 6.

Step 5 Check whether the structural system is failed: (1) if the structural system does not fail (as ▲ shown in Fig. 11), then let $k + 1 \Rightarrow k$, modify the stiffness and loading state of structures, and go to Step 2; (2)

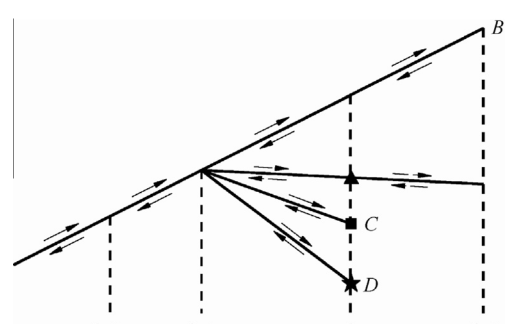

Fig. 11 Procedures of non-probabilistic branch-and-bound method.

if the structural system is failed so that the structural stiffness matrix becomes singular, i.e. $|K| = 0$, (as ■ shown in Fig. 11), then go to Step 8.

Step 6 Check whether the end criterion is satisfied: (1) if one of the end criteria is satisfied, then go to Step 9; (2) if one of the end criteria is not satisfied, then go to Step 7.

Step 7 Check whether there are still the candidate failure elements in this failure level: (1) if exits (as ★ shown in Fig. 11), then include the failure elements into the new failure paths, and go to Step 3; (2) if not, then go to Step 4.

Step 8 Form a new failure mode and let $m + 1 \Rightarrow m$, and go to Step 6.

Step 9 End.

Especially, in Step 6, two end criteria can be chosen: (1) $m = t_m$, i.e., the number of the retained failure modes reaches the pre-specified number; (2) $k = 0$, which implies that all optional failure paths have been searched.

In Fig. 10, k is the failure level, m the number of the retained failure modes, t_m the pre-specified number of the retained failure modes, $F_{M,i}$ the non-probabilistic set-theoretic failure measure for the ith component, F_b the bound of the failure measure of the retained failure modes, which will decide the total number of the remained failure modes, and F_l the failure measure corresponding to the lth failure mode, which can be computed as

$$F_l = \prod_{r_i} F_{M,r_i}, \quad r_i \in \{r_1, r_2, \cdots, r_q\} \tag{26}$$

where r_i is the failure element series in the lth failure mode.

Based on the obtained non-probabilistic set-theoretic failure measure F_l ($l = 1,2,\cdots,t_m$) corresponding to the t_m dominant failure modes of a structural system, the non-probabilistic set-theoretic failure measure F_s of a structural system can be computed as

$$F_s = 1 - \prod_{l=1}^{t_m}(1 - F_l) \tag{27}$$

6. Numerical examples

In this section, two numerical examples are performed to illustrate the validity of the presented non-probabilistic set-theoretic measure model of structural system safety.

6.1. A simple structural system

Consider the simple structural system shown in Fig. 12 loaded by a single concentrated load P. System failure is assumed to be the compressed failure in Element 1 or Element 2. The elastic moduli and cross-sectional areas of elements are 60 GPa and 0.015 m^2 for Element 1, and 80 GPa and 0.010 m^2 for Element 2. The external load and the compressive strength of these two materials are supposed to be uncertain and changing within the following intervals, respectively,

$$P^I = [P^c - \alpha P^c, P^c + \alpha P^c] \tag{28}$$

$$\sigma_{r1}^I = \left[\sigma_{r1}^c + \beta\sigma_{r1}^c, \sigma_{r1}^c - \beta\sigma_{r1}^c\right] \tag{29}$$

$$\sigma_{r2}^I = \left[\sigma_{r2}^c + \beta\sigma_{r2}^c, \sigma_{r2}^c - \beta\sigma_{r2}^c\right] \tag{30}$$

where their central values are $P^c = 1.0 \times 10^6$ N, $\sigma_{r1}^c = -55.0$ MPa, $\sigma_{r2}^c = -80.0$ MPa, and their uncertain coefficients are $\alpha = 10\%$ and $\beta = 10\%$.

It is obvious that there are two failure modes as shown in Fig. 13 for the two-bar plane truss.

According to the flowchart of the proposed non-probabilistic branch-and-bound method, the non-probabilistic set-theoretic failure measure for the components in the failure paths needs to be computed firstly.

The compressive stress of the two elements can be easily obtained as

$$\begin{cases} \sigma_{s1} = P/\sqrt{2}A_1 \\ \sigma_{s2} = P/\sqrt{2}A_2 \end{cases} \tag{31}$$

From Eq. (28), the intervals of stress σ_{s1} and σ_{s2} are expressed as

$$\sigma_{s1}^I = \frac{P^I}{\sqrt{2}A_1} = [\underline{\sigma}_{s1}, \bar{\sigma}_{s1}] = [42.43, 51.85] \tag{32}$$

and

$$\sigma_{s2}^I = \frac{P^I}{\sqrt{2}A_2} = [\underline{\sigma}_{s2}, \bar{\sigma}_{s2}] = [63.64, 77.78] \tag{33}$$

Similarly, the intervals of strength σ_{r1} and σ_{r2} are got from Eqs. (29) and (30)

$$\sigma_{r1}^I = [\underline{\sigma}_{r1}, \bar{\sigma}_{r1}] = [49.5, 60.5] \tag{34}$$

and

$$\sigma_{r2}^I = [\underline{\sigma}_{r2}, \bar{\sigma}_{r2}] = [72.0, 88.0] \tag{35}$$

By virtue of Eq. (16) and Eqs. (32)–(35), the non-probabilistic set-theoretic failure measure for Element 1 and Element 2 can be obtained as

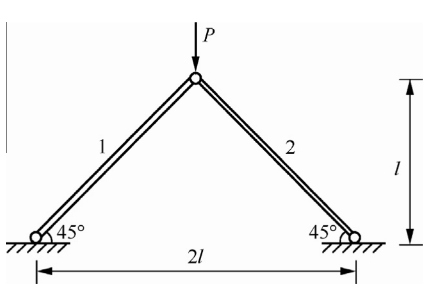

Fig. 12 Simple structural system.

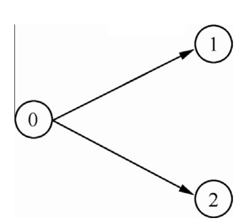

Fig. 13 Search tree of failure modes for simple structural system.

$$F_{M,1} = \frac{1/2(\bar{\sigma}_{s1} - \underline{\sigma}_{r1})^2}{(\bar{\sigma}_{r1} - \underline{\sigma}_{r1})(\bar{\sigma}_{s1} - \underline{\sigma}_{s1})} = 2.67\% \tag{36}$$

and

$$F_{M,2} = \frac{1/2(\bar{\sigma}_{s2} - \underline{\sigma}_{r2})^2}{(\bar{\sigma}_{r2} - \underline{\sigma}_{r2})(\bar{\sigma}_{s2} - \underline{\sigma}_{s2})} = 7.39\% \tag{37}$$

Further, the non-probabilistic failure measures corresponding to the two failure paths shown in Fig. 13 are, respectively,

$$F_1 = F_{M,1} = 2.67\% \text{ for the first failure path} : 0 \to 1 \tag{38}$$

and

$$F_2 = F_{M,2} = 7.39\% \text{ for the second failure path} : 0 \to 2 \tag{39}$$

Therefore, the non-probabilistic set-theoretic failure measure F_s of the structural system can be computed as

$$F_s = 1 - (1 - 2.67\%)(1 - 7.39\%) = 9.86\% \tag{40}$$

For the comparison with the probabilistic reliability method, the uncertain parameters are assumed to obey the truncated normal distribution within the given intervals for the probabilistic reliability analysis (for detail, see Section 4). By virtue of the numerical results obtained by two methods, the validity and feasibility of the new non-probabilistic structural safety measure developed in this paper will be demonstrated. In addition, the physical meaning of the proposed F_M/S_M may be more clearly embodied by the analysis of compatibility and transition.

In some cases that we can obtain statistical properties of these uncertain parameters from vast measurements or past experience, they should be treated as random variables. The normal distribution is a popular choice, but it may not be appropriate for realistic cases in which the uncertain parameters are measured to be limited in a certain range.

Under such circumstances, the truncated normal distribution model may take advantage. The univariate form can be applied as[14,15]

$$p(x) = \begin{cases} c_d \exp\left(-(x - x^c)^2/b_d^2\right) & |x - x^c| \leqslant \Delta x \\ 0 & |x - x^c| > \Delta x \end{cases} \tag{41}$$

where $p(x)$ is the probability density function of x; Δx is the uncertain bound for the random variable x; b_d is a parameter, and the normalization constant c_d can be derived from

$$c_d = [2b\mathrm{erf}(\Delta x/b_d)]^{-1} \tag{42}$$

where erf(·) is the error function and defined as

$$\mathrm{erf}(x) = \int_0^x \mathrm{e}^{-t^2}\mathrm{d}t \tag{43}$$

Fig. 14 show the probability density functions of the random variable at different parameters b_d and Δx. If Δx is given, then the probability density depends exclusively on b_d. The deviation of x increases with the growth of b_d, namely, a large b_d corresponds to a large deviation of x. When $b_d^2 \gg \Delta x^2$, x is nearly uniformly distributed, as shown by the case of b_d = 100.0 in Fig. 14.

For multi-dimensional uncertainty problem, the realization of x, denoted by $(x)_k (k = 1,2,\cdots)$, can be generated by

$$(x)_k = b\mathrm{erf}^{-1}\left[(2\delta_k - 1)\mathrm{erf}\left(\frac{\Delta x}{b_d}\right)\right] \tag{44}$$

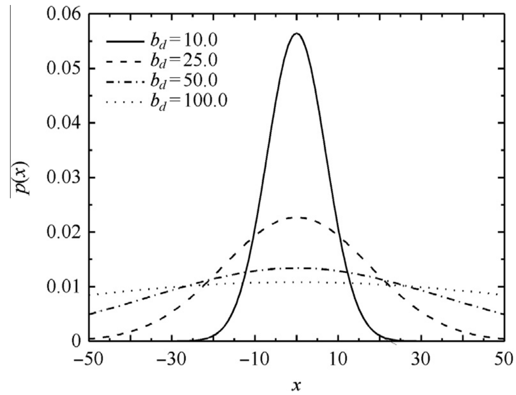

Fig. 14 Probability density function for a truncated normally distributed random variable (Δx = 50.0).

where $\delta_k (k = 1,2,\cdots)$ are independent random numbers uniformly distributed in [0,1].

With given parameters Δx and b_d in the probability density functions $p(x)$ of the uncertain parameters, Monte Carlo simulations can be carried out to obtain the reliability of structural system. Two cases with different deviations (b_d = 10.0 and b_d = 100.0) are investigated.

When b_d = 10.0, the failure probability for the failure paths 0 → 1 and 0 →2 can be computed as, respectively,

$$P_1 = 2.07\%, \quad P_2 = 5.28\% \tag{45}$$

So the failure probability of system is

$$P_f = 1 - (1 - 2.07\%)(1 - 5.28\%) = 7.24\% \tag{46}$$

When b_d = 100.0, the failure probability for the failure paths 0 → 1 and 0 → 2 can be respectively computed as

$$P_1 = 2.52\%, \quad P_2 = 7.26\% \tag{47}$$

So the failure probability of system is

$$P_f = 1 - (1 - 2.52\%)(1 - 7.26\%) = 9.60\% \tag{48}$$

It can be seen from the results of the two cases that the failure probability of system in the case of large deviation of uncertain parameters is closer to the non-probabilistic failure

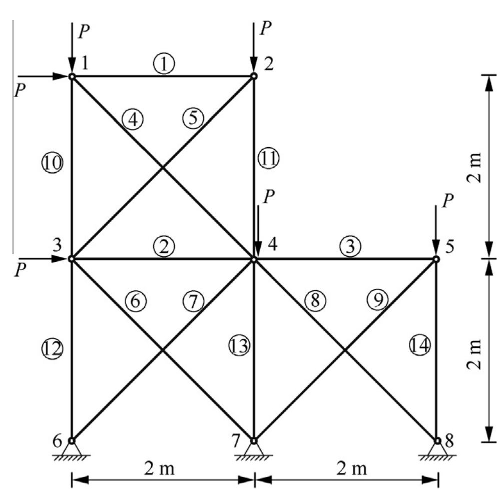

Fig. 15 14-bar plane truss structure.

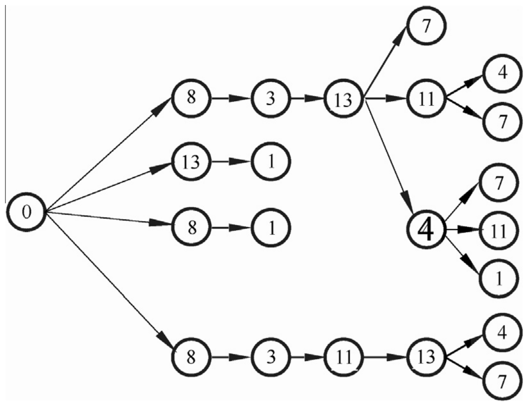

Fig. 16 Search tree of dominant failure modes for 14-bar plane truss structure.

measure than that in the case of small deviation. It also to some extent illustrates the rationality for the safety estimation deduced by interval analysis.

6.2. 14-bar truss structure

Consider a plane truss structure with 8 nodes and 14 elements as shown in Fig. 15. The elastic moduli and cross-sectional areas for all members are the same, which are 70 GPa and 0.004 m^2, respectively. The external loads, the tensile and compressive resistances of materials are supposed to be uncertain, and change within the following intervals, respectively,

$$P^{\mathrm{I}} = [P^{\mathrm{c}} - \alpha P^{\mathrm{c}}, P^{\mathrm{c}} + \alpha P^{\mathrm{c}}] \tag{49}$$

$$\sigma^{\mathrm{I}}_{\mathrm{t},i} = \left[\sigma^{\mathrm{c}}_{\mathrm{t},i} - \beta\sigma^{\mathrm{c}}_{\mathrm{t},i}, \sigma^{\mathrm{c}}_{\mathrm{t},i} + \beta\sigma^{\mathrm{c}}_{\mathrm{t},i}\right] \tag{50}$$

$$\sigma^{\mathrm{I}}_{\mathrm{c},i} = \left[\sigma^{\mathrm{c}}_{\mathrm{c},i} + \beta\sigma^{\mathrm{c}}_{\mathrm{c},i}, \sigma^{\mathrm{c}}_{\mathrm{c},i} - \beta\sigma^{\mathrm{c}}_{\mathrm{c},i}\right] \tag{51}$$

where their central values are $P^{\mathrm{c}} = 1.2 \times 10^5$ N, $\sigma^{\mathrm{c}}_{\mathrm{t}} = 60$ MPa, $\sigma^{\mathrm{c}}_{\mathrm{c}} = -60$ MPa, and their uncertain coefficients are $\alpha = 10\%$ and $\beta = 10\%$. For comparing the non-probabilistic set-theoretic safety method and the probabilistic reliability method, these uncertain parameters are assumed to obey truncated normal distributions within the given intervals, where only the case of $b_d = 10.0$ in Eq. (41) is considered.

Using the non-probabilistic and probabilistic branch-and-bound method, we can get the first ten dominant failure modes of the system. The search tree is shown in Fig. 16, which gives the failure paths of the dominant failure modes.

Table 1 shows the failure measure and failure probability of the first ten dominant failure modes obtained by the two methods. It can be seen from Table 1 that the two methods give the same failure paths while the failure measures of every paths obtained by the presented method are slightly larger than the failure probabilities obtained by the probabilistic method. Thus, the failure measure (16.88%) of structural system yielded by the former method is larger than the failure probability (14.51%) of structural system yielded by the latter method. It can be seen from this that at this time, the non-probabilistic set-theoretic safety theory is slightly more conservative than the probabilistic reliability theory.

7. Conclusions

In view of the limitations of high demand on the original data for the probabilistic reliability model and the fuzzy reliability model, the uncertain information in structural safety analysis is quantified as interval set in this paper. Intervals of displacement and stress of a structural system are computed by the interval finite element method.

Based on the non-probabilistic set-theoretic model, the new non-probabilistic set-theoretic branch-and-bound method is presented for determining the dominant failure modes. Moreover, the non-probabilistic set-theoretic safety measure of a structural system is computed and further compared with the classical probabilistic reliability. The study on compatibility and transition of two models (random variable model and bounded interval-set model) can not only show their inner relations well but also explicitly illustrate the essence of the safety measure in physics. The numerical examples demonstrate that the present non-probabilistic and probabilistic methods can identify the same dominant failure paths while the former method may be slightly more conservative than the latter method. That is to say, if the structural system is computed and judged as being reliable by the presented non-

Table 1 Failure measure and failure probability of the first ten dominant failure modes.

Sequence number of failure mode	Non-probabilistic method		Probabilistic method	
	Failure path	Failure measure (%)	Failure path	Failure probability (%)
1	8-3-13-7	3.35	8-3-13-7	2.87
2	8-3-13-11-4	3.35	8-3-13-11-4	2.87
3	8-3-13-11-7	3.35	8-3-13-11-7	2.87
4	8-3-13-4-7	3.35	8-3-13-4-7	2.87
5	8-3-13-4-11	3.35	8-3-13-4-11	2.87
6	8-3-13-4-1	0.42	8-3-13-4-1	0.33
7	13-1	0.27	13-1	0.20
8	8-1	0.26	8-1	0.20
9	8-3-11-13-4	0.25	8-3-11-13-4	0.19
10	8-3-11-13-7	0.25	8-3-11-13-7	0.19
Failure measure or probability of structural systems (%)	16.88		14.51	

probabilistic set-theoretic method (i.e. if $F_s > \alpha$, α is the specified safety measure level), then the structure system must be evaluated to be reliable by the probabilistic method.

It is necessary to emphasize that the proposed safety measure requires less information on the uncertainty than the probabilistic reliability model or fuzzy reliability model, where only the bounds on the magnitude of uncertain parameters are not sufficient. Therefore, in the absence of enough information on uncertainties, the presented non-probabilistic set-theoretic method can give a more feasible assessment for the structural system safety.

Of course, the purpose of the paper is not to replace the probabilistic reliability model by the presented non-probabilistic set-based reliability model, which is only an alternative or supplementary way to the structural reliability analysis. Since the description form of uncertainties depends on the type and the amount of uncertain information, and the type of the chosen reliability analysis model depends on the description form of uncertainties, which model will be selected absolutely depends on the type and the amount of the known uncertain information in practice.

Acknowledgments

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