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Tightness and distinguished Fréchet spaces ☆

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Abstract

Valdivia invented a nondistinguished Fréchet space whose weak bidual is quasi-Suslin but not K-analytic. We prove that Grothendieck/Köthe's original nondistinguished Fréchet space serves the same purpose. Indeed, a Fréchet space is distinguished if and only if its strong dual has countable tightness, a corollary to the fact that a (DF)-space is quasibarrelled if and only if its tightness is countable. This answers a Cascales/Kąkol/Saxon question and leads to a rich supply of (DF)-spaces whose weak duals are quasi-Suslin but not K-analytic, including the spaces $C_c(\kappa)$ for κ a cardinal of uncountable cofinality. Our level of generality rises above (DF)- or even dual metric spaces to Cascales/Orihuela's class $\mathfrak{G}$. The small cardinals $\mathfrak{b}$ and $\mathfrak{d}$ invite a novel analysis of the Grothendieck/Köthe example, and are useful throughout.

Keywords: Class $\mathfrak{G}$; Quasibarrelled; K-analytic; Quasi-Suslin; Compact-open

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1. Introduction

Let E be a Fréchet space with strong dual F. Recall that E is distinguished if and only if F is (quasi)barrelled. Since F is locally complete, it is barrelled when quasibarrelled, and the same is true of the Mackey space $(F, \mu(F, F'))$. Accordingly, we may restate Valdivia's [29, pp. 65–66, (23), (24)]: *The weak dual* $(F', \sigma(F', F))$ *of* F *is quasi-Suslin*; *it is K-analytic if and only if* $(F, \mu(F, F'))$ *is quasibarrelled*. Our Corollary 2 greatly extends the statement to the setting of locally convex spaces (lcs) F having a bornivorous $\mathfrak{G}$-representation, defined below. All dual metric spaces are included (Example 2).

In the extended setting we prove that "$(F, \mu(F, F'))$ is quasibarrelled" is equivalent to "$(F, \sigma(F, F'))$ has countable tightness" (Theorem 1(I)). We determine that the tightness of $(F, \sigma(F, F'))$ lies between the small cardinals $\mathfrak{b}$ and $\mathfrak{d}$, thus is uncountable, when F is the strong dual of the Grothendieck/Köthe (G/K) nondistinguished Fréchet space. Consequently, the weak bidual of the G/K space is quasi-Suslin but not K-analytic, an apparent novelty.

Cascales and Orihuela [5] introduced the class $\mathfrak{G}$ of those lcs E for which there is a family $\{A_\alpha\colon \alpha \in \mathbb{N}^\mathbb{N}\}$ of subsets of its topological dual E' (called a $\mathfrak{G}$-representation) such that:

(G1) $E' = \bigcup\{A_\alpha\colon \alpha \in \mathbb{N}^\mathbb{N}\}$;
(G2) $A_\alpha \subset A_\beta$ when $\alpha \leqslant \beta$;
(G3) in each A_α, sequences are equicontinuous,

where $\alpha \leqslant \beta$ means the inequality holds coordinatewise. We shall say that a $\mathfrak{G}$-representation is *closed* if every set A_α is $\sigma(E', E)$-closed; *bornivorous* if every $\beta(E', E)$-bounded set is contained in some A_α. The class $\mathfrak{G}$ is stable under the formation of subspaces, separated quotients, completions, countable direct sums and countable products, and contains many important spaces: all (*LM*)-spaces (hence metrizable and (*LF*)-spaces), dual metric spaces (hence (*DF*)-spaces), the space of distributions $D'(\Omega)$ and real analytic functions $A(\Omega)$ for open $\Omega \subset \mathbb{R}^\mathbb{N}$, etc.

The *tightness* $t(X)$ of a topological space X is the smallest cardinal κ such that for every set $A \subset X$ and every $x \in \bar{A}$ (the closure of A) there exists a set $B \subset A$ with $|B| \leqslant \kappa$ such that $x \in \overline{B}$. In [7,8,10–12] we find ideas related to countable tightness for quasibarrelled spaces in $\mathfrak{G}$.

Trivially, each metrizable lcs enjoys countable tightness; Kaplansky (cf. [17, §24,1.(6)]) proved that the corresponding weak topology does, as well. His conclusion is obvious for the distinguished Fréchet space $\mathbb{R}^\mathbb{N}$, whose metrizable and weak topologies coincide. On the other hand, the strong dual φ of $\mathbb{R}^\mathbb{N}$ is not metrizable, but still enjoys countable tightness in the original and weak topologies. Indeed, under any Hausdorff locally convex topology whatsoever, the $\aleph_0$-dimensional space φ is a countable union of separable metrizable subspaces, so all of its subsets are separable, which implies countable tightness.

Kaplansky and φ illustrate Proposition 4.7 and Theorem 4.8 of [7]:

(∗) *If an lcs* E *in class* $\mathfrak{G}$ *has countable tightness, the weak topology* $\sigma(E, E')$ *has countable tightness, too. Every quasibarrelled space in class* $\mathfrak{G}$ *has countable tightness.*

In particular, (∗) proves anew that the G/K space is nondistinguished, once we show (Example 4) that its strong dual has tightness $\mathfrak{d}$.

Conversely, $\mathbb{R}^\mathbb{N}$ and φ support Corollary 4, that a Fréchet space is distinguished if (and only if) its strong dual has countable tightness. More generally, Theorem 1(II) says that an lcs F with a bornivorous $\mathfrak{G}$-representation is quasibarrelled if and only if its tightness is countable.

The special case for F an arbitrary (DF)-space is the intermediate Corollary 3, which answers Question 3 in [7].

We further see in [7, Theorem 4.6], that:

(CKS1) *If E is in class $\mathfrak{G}$, then the following conditions are equivalent*:

- (a) $(E, \sigma(E, E'))$ *has countable tightness.*
- (b) *The weak dual* $(E', \sigma(E', E))$ *is K-analytic.*
- (c) $(E', \sigma(E', E))$ *is* realcompact (*i.e.,* $(E', \sigma(E', E))$ *is homeomorphic to a closed subspace of the product* $\mathbb{R}^I$ *for some* I).
- (d) $(E', \sigma(E', E))$ *is Lindelöf.*
- (e) $(E', \sigma(E', E))^n$ *is Lindelöf for every* $n \in \mathbb{N}$.

Note that (CKS1) applies to the space $C_p(X)$ only when it is metrizable, i.e., when X is countable, see [8, Corollary 2.8]. Nevertheless, the tightness of $C_p(X)$ is countable if and only if every finite product of X is Lindelöf [1, II.1.1]. The following result of [6, Corollary 1.4] is relevant: *A barrelled space E belongs to class $\mathfrak{G}$ if and only if its weak dual is K-analytic*.

In the very recent [8, Lemma 2], we find that:

(CKS2) *For a quasibarrelled space E, the following assertions are equivalent*:

- (a) *E belongs to class $\mathfrak{G}$.*
- (b) *The strong dual* $(E', \beta(E', E))$ *is a quasi-*(LB)*-space.*
- (c) *There exists a base* $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ *of neighborhoods of zero in E such that* $U_\alpha \subset U_\beta$ *for* $\beta \leqslant \alpha$ *in* $\mathbb{N}^{\mathbb{N}}$.

Let us say that $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$*-base* for an lcs E if it fulfills part (c) of (CKS2). We show in Section 4 that (CKS2) fails when quasibarrelled is omitted: while all (DF)-spaces belong to class $\mathfrak{G}$, some do not admit a $\mathfrak{G}$-base. Our examples use the concepts of the bounding cardinal $\mathfrak{b}$ and the dominating cardinal $\mathfrak{d}$. In particular, whether or not the (DF)-space $C_c(\omega_1)$ admits a $\mathfrak{G}$-base depends on the axiom system employed: $C_c(\omega_1)$ *has a* $\mathfrak{G}$-base *if and only if* $(\aleph_1 =)$ $\omega_1 = \mathfrak{b}$ (Theorem 3). In either case, the weak dual of $C_c(\omega_1)$ is quasi-Suslin but not K-analytic.

$\mathfrak{G}$-bases provide spaces $C_c(X)$ (different from what Talagrand presented in [26]) whose weak duals are not K-analytic but are covered by an ordered family of compact sets (Example 13). $\mathfrak{G}$-bases similarly benefit the G/K and Valdivia examples, for all strong duals F of Fréchet spaces have a $\mathfrak{G}$-base $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ (Example 1(E)), and the polars U_α° in F' are $\sigma(F', F)$-compact.

Recall that a Hausdorff topological space X is a *quasi-Suslin* (respectively *K-analytic*) space if there exists a map $T : \mathbb{N}^{\mathbb{N}} \to 2^X$ (respectively a map $T : \mathbb{N}^{\mathbb{N}} \to 2^X$ such that $T(\alpha)$ is compact for each $\alpha \in \mathbb{N}^{\mathbb{N}}$) such that

(K1) $\bigcup\{T(\alpha)\colon \alpha \in \mathbb{N}^{\mathbb{N}}\} = X$ and

(K2) if $\alpha^{(n)}$ is a sequence in $\mathbb{N}^{\mathbb{N}}$ which converges to α in $\mathbb{N}^{\mathbb{N}}$ and $x_n \in T(\alpha^{(n)})$ for all $n \in \mathbb{N}$, then the sequence $(x_n)_n$ has an adherence point in X belonging to $T(\alpha)$; see [29] (and [4] for more references).

Rogers [22] proved that K-analytic spaces and K-Suslin spaces (in the sense of [29]) coincide in the category of completely regular Hausdorff spaces. For every K-analytic space X there exists an ordered family $\{T(\alpha)\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact subsets of X covering X [27], but, as Talagrand has shown [26], there are topological spaces not K-analytic, but covered by an ordered family $\{T(\alpha)\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets. Again, $\mathfrak{G}$-bases give us many additional examples of this type.

2. Basic results

Let us develop some basic examples and interrelations among $\mathfrak{G}$-bases, $\mathfrak{G}$-representations that are closed and/or bornivorous, (ℓ^∞-)quasibarrelled and Mackey spaces, tightness and character.

Proposition 1. *If $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base for an lcs E, then E is in class $\mathfrak{G}$; in fact, the set of polars $\{U_\alpha^\circ\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a closed $\mathfrak{G}$-representation. It is bornivorous if and only if E is quasibarrelled.*

Proof. Transparently, the set of polars is a closed $\mathfrak{G}$-representation. E is quasibarrelled iff each bounded set in the strong dual is equicontinuous iff each such set is contained in some U_α° iff the set of polars is bornivorous. □

Proposition 2. *If $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a bornivorous $\mathfrak{G}$-representation for E, then E is ℓ^∞-quasibarrelled, $\{A_\alpha^{\circ\circ}\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a closed and bornivorous $\mathfrak{G}$-representation, and the strong dual of E is a quasi-(LB)-space; i.e., is covered by an ordered family $\{B_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of Banach disks.*

Proof. By (G3), each A_α is $\beta(E', E)$-bounded, hence so is each bipolar $A_\alpha^{\circ\circ}$. Since $\{A_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is bornivorous, each $A_\alpha^{\circ\circ}$ is contained in some A_β, making sequences in $A_\alpha^{\circ\circ}$ equicontinuous; also, every $\beta(E', E)$-bounded sequence is contained in some A_α, and is equicontinuous. Therefore $\{A_\alpha^{\circ\circ}\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a closed and bornivorous $\mathfrak{G}$-representation, and E is ℓ^∞-quasibarrelled. Finally, since sequences in $A_\alpha^{\circ\circ}$ are equicontinuous, $B_\alpha := A_\alpha^{\circ\circ}$ is sequentially complete and a Banach disk in the strong dual [21, 3.2.5]. □

By an *(LM)-space* we mean the inductive limit of an increasing sequence of metrizable lcs. Every (*LM*)-space is quasibarrelled.

Example 1. The following spaces admit a $\mathfrak{G}$-base:

(A) *Every quasibarrelled space in* $\mathfrak{G}$ [8, Lemma 2].
(B) *The strong dual of every locally complete quasi-*(*LB*)*-space.* The proof is the one we gave for Corollary 2.3 of [8].
(C) *The space $D'(\Omega)$ of distributions and the real analytic function space $A(\Omega)$.* Indeed, for open sets $\Omega \subset \mathbb{R}^n$, the space of test functions $D(\Omega)$ is a complete Montel (*LF*)-space, and therefore a locally complete quasi-(*LB*)-space. Its strong dual $D'(\Omega)$ falls into category (B). Similarly for $A(\Omega)$ (see [2,18]).
(D) *Every* (*LM*)*-space E.* This is an immediate consequence of (A), if we accept that E is in class $\mathfrak{G}$. Here is a simple constructive proof. Let (E_n) be an increasing sequence of metrizable spaces whose union is the inductive limit space E. For every $j \in \mathbb{N}$ let $(U_n^j)_n$ be a decreasing basis of neighborhoods of zero in E_j and, for every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$, set $U_\alpha := (\bigcup_k U_{n_k}^k)^{\circ\circ}$ to obtain a $\mathfrak{G}$-base $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$.

(E) *Every strong dual F of a metrizable lcs E.* Indeed, let $(V_n)_n$ be a decreasing basis of balanced neighborhoods of zero for E, and for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ set $U_\alpha = (\bigcap_k n_k V_k)^\circ$ to obtain a $\mathfrak{G}$-base $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ for F.

(F) *The strong bidual E'' of every quasibarrelled E in $\mathfrak{G}$.* By (A), E has a $\mathfrak{G}$-base; the set of polars in E'' of the polars in E' is a $\mathfrak{G}$-base for the strong bidual. Or, one could argue via (A) and Propositions 1, 2 that the strong dual of E is a locally complete quasi-(*LB*)-space, and then use (B).

Recall that an lcs is *dual metric* if it has a fundamental sequence of bounded sets and is ℓ^∞-*quasibarrelled*, i.e., every strongly bounded sequence in the dual is equicontinuous.

Example 2. The following spaces admit a bornivorous $\mathfrak{G}$-representation:

(A′) *Every quasibarrelled space in $\mathfrak{G}$.* Example 1(A) and Proposition 1 apply.

(B′) *The spaces $D'(\Omega)$ and $A(\Omega)$.* By (C) and Proposition 1, these spaces are in $\mathfrak{G}$, and strong duals of Montel spaces are (quasi)barrelled, so that (A′) applies.

(C′) *Every (LM)-space.* This follows from (D) and Proposition 1.

(D′) *Every dual metric space E; hence every (DF)-space.* Indeed, let (B_n) be a fundamental sequence of bounded sets of E. For every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ let $A_\alpha := \bigcap_k n_k (B_k)^\circ$. Conditions (G1)–(G3) are easily checked. Also, every bounded set in the strong dual is contained in some A_α.

The following three statements may or may not hold for an lcs E in class $\mathfrak{G}$:

(α) E has a $\mathfrak{G}$-base.

(β) E has a bornivorous $\mathfrak{G}$-representation.

(γ) E has countable tightness.

We know from (A), (A′), ($*$) that

$$E \text{ is quasibarrelled} \quad \Rightarrow \quad (\alpha) \wedge (\beta) \wedge (\gamma).$$

In the next section, Theorem 1(II), we prove the converse. Better yet, omitting (α), we show that

$$E \text{ is quasibarrelled} \quad \Leftrightarrow \quad (\beta) \wedge (\gamma),$$

so that $(\beta) \wedge (\gamma) \Rightarrow (\alpha)$. Only ($\alpha$) is omissible. Indeed, $(\alpha) \wedge (\beta) \nRightarrow (\gamma)$, as Example 4 shows, and Example 3 shows that $(\alpha) \wedge (\gamma) \nRightarrow (\beta)$, even when E is Mackey. In this light, (β) is a most felicitous setting for Theorem 1(I), (II).

Example 3 requires some preparation. Recall that if (E, τ) is an lcs with dual E' and M is an $\aleph_0$-dimensional subspace of the algebraic dual E^* such that $M \cap E' = \{0\}$, then the coarsest locally convex topology η on E finer than both τ and $\sigma(E, E' + M)$, denoted as $\sup\{\tau, \sigma(E, E' + M)\}$, is said to be a *countable enlargement* (CE) of (E, τ), or of τ. Basic 0-neighborhoods are sets of the form $U \cap A^\circ$, where U is a basic τ-neighborhood of 0 and A is a finite subset of M.

Proposition 3. *Let $\eta = \sup\{\tau, \sigma(E, E' + M)\}$ be a CE of (E, τ). If (E, τ) is in class $\mathfrak{G}$, has a $\mathfrak{G}$-base, or has countable tightness, so is or has (E, η), respectively.*

Proof. Let (B_n) be an increasing sequence of finite sets whose union is a Hamel basis for M.

(a) If $\{A_\alpha\colon \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-representation for (E, τ), then

$$\left\{A_\alpha + kB_k^{\circ\circ}\colon \alpha \in \mathbb{N}^\mathbb{N} \text{ and } k \text{ is the first coordinate of } \alpha\right\}$$

is a $\mathfrak{G}$-representation for (E, η). Note that any sequence from $A_\alpha + kB_k^{\circ\circ}$ is equicontinuous on a dense finite-codimensional subspace of (E, τ), and thus is the sum of two sequences, one equicontinuous on (E, τ) and the other on $(E, \sigma(E, E' + M))$.

(b) If $\{U_\alpha\colon \alpha \in \mathbb{N}^\mathbb{N}\}$ is a $\mathfrak{G}$-base for (E, τ), we easily see that

$$\left\{U_\alpha \cap k^{-1}B_k^{\circ}\colon \alpha \in \mathbb{N}^\mathbb{N} \text{ and } k \text{ is the first coordinate of } \alpha\right\}$$

is a $\mathfrak{G}$-base for (E, η).

(c) Let us suppose (E, τ) has countable tightness, and let A be a subset of E whose η-closure contains 0. For each $n \in \mathbb{N}$, the set $n^{-1}B_n^\circ$ is an η-neighborhood of 0, and thus 0 is in the η-closure, hence in the τ-closure of $n^{-1}B_n^\circ \cap A$. Countable tightness ensures a countable subset C_n of $n^{-1}B_n^\circ \cap A$ whose τ-closure contains 0. For V any η-neighborhood of 0, there exist a τ-neighborhood U of 0 and a positive integer n such that $U \cap n^{-1}B_n^\circ \subset V$. Now C_n is a subset of $n^{-1}B_n^\circ$ and meets every τ-neighborhood of 0; thus C_n meets V. Therefore, $C := \bigcup_n C_n$ is a countable subset of A whose η-closure contains 0; i.e., (E, η) has countable tightness. □

Lemma 1. *Let $\eta = \sup\{\tau, \sigma(E, E' + M)\}$ be a CE of (E, τ). If τ is quasibarrelled and η is ℓ^∞-quasibarrelled, then η is also quasibarrelled.*

Proof. Suppose $B \subset E' + M$ is $\beta(E' + M, E)$-bounded. If $(f_n + h_n)_n$ is a sequence in B, where $(f_n)_n \subset E'$ and $(h_n)_n \subset M$, then by hypothesis $(f_n + h_n)_n$ is η-equicontinuous and in the polar of $U \cap A^\circ$ for some τ-neighborhood U of 0 and some finite $A \subset M$. Now any point g in $(U \cap A^\circ)^\circ$ is relatively τ-continuous when restricted to the dense finite-codimensional subspace $E \cap A^\perp$, and is the sum $f + h$ for some $f \in E'$ and some h in the span of A. Thus $(f_n + h_n)_n \subset E' + N$, where N is a finite-dimensional subspace of M, so that $(h_n)_n$ has a finite-dimensional span. Since $(f_n + h_n)_n$ was an arbitrarily chosen sequence in B, it follow that $B \subset E' + L$ for some finite-dimensional subspace L of M. But, by Theorem 2.6 of [28], this property characterizes quasibarrelledness of the CE η. □

Every barrelled space E with $E' \neq E^*$ admits a CE which is not barrelled. On the other hand, every CE of a metrizable space is still metrizable, hence still quasibarrelled. Nonetheless, there are many quasibarrelled spaces which do admit a CE that is not quasibarrelled. Example 4.5 of [28] is given explicitly as such an example, and can be adapted to every quasibarrelled (DF)-space E in which each member of a fundamental sequence of bounded sets has uncountable-codimensional span. All such E have a $\mathfrak{G}$-base (D′, A) and have countable tightness $(*)$, and admit CEs that are not quasibarrelled.

Example 3. There are many Mackey spaces with countable tightness and $\mathfrak{G}$-bases which have no bornivorous $\mathfrak{G}$-representations. In fact, any nonquasibarrelled CE (E, η) of a quasibarrelled space (E, τ) in $\mathfrak{G}$ serves as an example.

Proof. Our Theorem 2.4 in [28] says every CE of a quasibarrelled space is Mackey. Since (E, τ) has countable tightness $(*)$ and a $\mathfrak{G}$-base (A), so does (E, η) (Proposition 3). Since (E, η) is not quasibarrelled, it is not ℓ^∞-quasibarrelled (Lemma 1), and has no bornivorous $\mathfrak{G}$-representation (Proposition 2). □

Proposition 4. *If an ℓ^∞-quasibarrelled space (E,τ) has countable tightness, then it is Mackey.*

Proof. Suppose τ is not the Mackey topology $\mu := \mu(E, E')$. Then there exists a μ-closed set A that is not τ-closed. Select $x \in \bar{A}^\tau \setminus A$, and let p be a μ-continuous seminorm such that $p(x-y) \geqslant 1$ for each $y \in A$. Let $\{x_n\}_n$ be an arbitrary countable subset of A. The Hahn–Banach theorem yields $\{f_n\}_n \subset E^*$ such that, for each $n \in \mathbb{N}$,

$$f_n(x - x_n) = 1 \quad \text{and} \quad \left|f_n(z)\right| \leqslant p(z) \quad \text{for all } z \in E.$$

The inequality implies that $\{f_n\}_n$ is μ-equicontinuous and hence $\beta(E', E)$-bounded. By hypothesis, then, it is τ-equicontinuous, and its polar U is a τ-neighborhood of the origin. Therefore $x + \frac{1}{2}U$ is a τ-neighborhood of x which misses $\{x_n\}_n$, violating countable tightness. □

The *character* of an lcs E, denoted $\chi(E)$, is the smallest cardinality for a base of neighborhoods of 0. An lcs is metrizable, for example, if and only if its character is countable. The next result was likely known to Kaplansky. The first inequality is obvious, the second follows from [7, Theorem 4.2(i)]. One may begin an elementary proof with the fact that if U is a basic neighborhood of 0 in E, then U° is $\sigma(E', E)$-compact, so the n-fold product $(U^\circ)^n$ is also compact, etc.

Proposition 5 *(*(Kaplansky)*). For any lcs E,*

$$t(E), t\big(E, \sigma(E, E')\big) \leqslant \chi(E).$$

The paper [23] introduced to the study of lcs the cardinals $\mathfrak{b}$ and $\mathfrak{d}$, whose very definitions suggest an affinity with class $\mathfrak{G}$. Given $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ with $\alpha = (a_k)_k$ and $\beta = (b_k)_k$ we write $\alpha \leqslant^* \beta$ to mean that $a_k \leqslant b_k$ for almost all (i.e., all but finitely many) $k \in \mathbb{N}$. Thus $\alpha \leqslant \beta$ implies $\alpha \leqslant^* \beta$, but not conversely. It is easily seen that every countable set in $(\mathbb{N}^\mathbb{N}, \leqslant^*)$ has an upper bound, and this is not true for $(\mathbb{N}^\mathbb{N}, \leqslant)$. The bounding cardinal $\mathfrak{b}$ and the dominating cardinal $\mathfrak{d}$ are defined as the least cardinality for unbounded, respectively, cofinal subsets of the quasi-ordered space $(\mathbb{N}^\mathbb{N}, \leqslant^*)$. It is clear that in any ZFC-consistent system, one has $\aleph_1 \leqslant \mathfrak{b} \leqslant \mathfrak{d} \leqslant \mathfrak{c}$. The Continuum Hypothesis (CH) requires all four of these cardinals to coincide. Yet it is ZFC-consistent to assume that any of the three inequalities is strict. Scales exist, i.e., well ordered cofinal subsets of $(\mathbb{N}^\mathbb{N}, \leqslant^*)$ exist, if and only if $\mathfrak{b} = \mathfrak{d}$ (cf. [23, Remark, p. 144]). One may consult [9] for a fuller discussion of these fundamental ideas.

Let us define an equivalence relation $=^*$ on $(\mathbb{N}^\mathbb{N}, \leqslant^*)$ so that $\alpha =^* \beta$ if and only if $a_k = b_k$ for almost all $k \in \mathbb{N}$. Thus $\alpha =^* \beta$ if and only if $\alpha \leqslant^* \beta$ and $\beta \leqslant^* \alpha$. Let $\hat{\alpha}$ denote the equivalence class represented by α, and observe that each $\hat{\alpha}$ is countable.

Every metrizable lcs admits a $\mathfrak{G}$-base with each U_α determined by the first coordinate of α. Nonmetrizable lcs with $\mathfrak{G}$-bases have precisely limited characters.

Proposition 6. *The character $\chi(E)$ of a nonmetrizable lcs E having a $\mathfrak{G}$-base must satisfy*

$$\mathfrak{b} \leqslant \chi(E) \leqslant \mathfrak{d}.$$

Proof. Let $\{U_\alpha\colon \alpha \in \mathbb{N}^\mathbb{N}\}$ be a $\mathfrak{G}$-base for E. By definition of $\mathfrak{d}$, there exists a cofinal set D in $(\mathbb{N}^\mathbb{N}, \leqslant^*)$ with $|D| = \mathfrak{d}$. The set

$$\hat{D} := \bigcup\{\hat{\alpha}\colon \alpha \in D\}$$

satisfies $|\hat{D}| = \aleph_0 \cdot \mathfrak{d} = \mathfrak{d}$ and is cofinal in $(\mathbb{N}^{\mathbb{N}}, \leqslant)$ so that $\{U_\beta\colon \beta \in \hat{D}\}$ is a base of neighborhoods of zero of cardinality $\mathfrak{d}$. Therefore $\chi(E) \leqslant \mathfrak{d}$. Since the cardinals are well ordered, there exists a subset A of $\mathbb{N}^{\mathbb{N}}$ with $|A| = \chi(E)$ such that $\{U_\alpha\colon \alpha \in A\}$ is a base of neighborhoods of zero in E. Let us see that $|A| \geqslant \mathfrak{b}$. Suppose, to the contrary, that $|A| < \mathfrak{b}$. Then by the definition of $\mathfrak{b}$ there is some $\beta \in \mathbb{N}^{\mathbb{N}}$ such that $\alpha \leqslant^* \beta$ for every $\alpha \in A$. Hence, for every $\alpha \in A$ there exists $\gamma \in \hat{\beta}$ such that $\alpha \leqslant \gamma$. It follows that $\{U_\gamma\colon \gamma \in \hat{\beta}\}$ is a countable base of neighborhoods of zero, contradicting nonmetrizability of E. $\square$

Corollary 1. *If an lcs space E admits a $\mathfrak{G}$-base, then*

$$t(E), t\big(\big(E, \sigma(E, E')\big)\big) \leqslant \mathfrak{d}.$$

The use of $\mathfrak{b}$ and $\mathfrak{d}$ is optimal; i.e., in the previous two results, $\mathfrak{b}$ cannot be replaced by any larger cardinal, and $\mathfrak{d}$ cannot be replaced by any smaller cardinal. The proof is Proposition 9 and Example 12. Also, the strong dual of any nonnormable metrizable lcs has a $\mathfrak{G}$-base (Example 1(E)), and we proved [23, Corollary 2] that its character is always exactly $\mathfrak{d}$.

Note, too, that the converse of Proposition 6 fails. As a product of complete metrizable spaces, $\mathbb{R}^X$ is always a Baire space, hence b-Baire-like. In class $\mathfrak{G}$, the latter is equivalent to metrizable [8, Theorem 2.2]. For X uncountable, $\mathbb{R}^X$ is nonmetrizable, thus not in $\mathfrak{G}$, thus without a $\mathfrak{G}$-base, and $\chi(\mathbb{R}^X) = |X|$; we take $\mathfrak{b} \leqslant |X| \leqslant \mathfrak{d}$ to contradict the converse of Proposition 6.

3. Distinguished Fréchet spaces, K-analytic duals and tightness

Banach spaces are distinguished, and their strong duals certainly have countable tightness. We saw in the introduction that the strong dual of the distinguished $\mathbb{R}^{\mathbb{N}}$ has countable tightness. Let us fathom the tightness of the strong dual E' of the original nondistinguished Fréchet space E, and also of its weak topology $\sigma(E', E'')$.

Probably the first and most famous example of a nondistinguished Fréchet space is attributed to Grothendieck and Köthe [17, bottom of p. 436]. We follow Schaefer [25, p. 193], who describes it as the vector space E of all numerical double sequences $x = (x_{ij})$ such that for each $n \in \mathbb{N}$, $p_n(x) = \sum_{i,j} |a_{ij}^{(n)} x_{ij}| < \infty$, where $a_{ij}^{(n)} = j$ for $i \leqslant n$ and all j, $a_{i,j}^{(n)} = 1$ for $i > n$ and all j. The semi-norms p_n $(n \in \mathbb{N})$ generate a locally convex topology under which E is a Fréchet space. The dual E' is identified with the space of double sequences $u = (u_{ij})$ such that $|u_{ij}| \leqslant c a_{ij}^{(n)}$ for all i, j and suitable $c > 0$, $n \in \mathbb{N}$. (The duality is given by $\langle x, u\rangle = \sum_{i,j} x_{ij} u_{ij}$; absolute convergence makes the order of summation irrelevant.) Schaefer goes on to outline the standard proof that E is nondistinguished. We take a different route.

Example 4. The tightness $t(E')$ of the strong dual E' of the Grothendieck/Köthe space E is $\mathfrak{d}$, the dominating cardinal.

Proof. For each $f : \mathbb{N} \to \mathbb{N}$, i.e., for $f \in \mathbb{N}^{\mathbb{N}}$, define the double sequence $v^f := (v_{ij}^f) \in E'$ so that, for all $i, j \in \mathbb{N}$,

$$v_{ij}^f = \begin{cases} 0 & \text{if } j \leqslant f(i), \\ 1 & \text{if } j > f(i). \end{cases}$$

Thus if i is held fixed, the single sequence $(v_{ij}^f)_j$ consists of zeros for the first $f(i)$ coordinates, and ones thereafter. We set $A = \{v^f\colon f \in \mathbb{N}^{\mathbb{N}}\}$ and complete our demonstration in three steps.

I. *The origin* 0 *is in the* $\beta(E', E)$*-closure of* A. Let B be an arbitrary bounded set in E, choose $g \in \mathbb{N}^{\mathbb{N}}$ such that $g(i)$ is an upper bound for $p_i(B)$ for each $i \in \mathbb{N}$, and set $f(i) := 2^i \cdot g(i)$, thus determining $v^f =: v \in A$. Let $x := (x_{ij})$ be an arbitrary member of B. Then

$$\begin{aligned}\left|\langle x, v\rangle\right| &\leqslant \sum_{i,j}\left|x_{ij} v^f_{ij}\right| = \sum_{i\geqslant 1}\sum_{j>f(i)} |x_{ij}| \leqslant \sum_{i\geqslant 1}\sum_{j>f(i)} \frac{j}{f(i)}|x_{ij}| \\ &\leqslant \sum_{i\geqslant 1}\frac{1}{f(i)}\sum_{j\geqslant 1} j|x_{ij}| \leqslant \sum_{i\geqslant 1}\frac{1}{f(i)} p_i(x) \leqslant \sum_{i\geqslant 1}\frac{1}{2^i} = 1,\end{aligned}$$

proving $v \in A \cap B^\circ$. Thus A meets every $\beta(E', E)$-neighborhood of 0, and claim (I) follows.

II. $t(E') \geqslant \mathfrak{d}$. It suffices to show that 0 is not in the closure of any subset of A having fewer than $\mathfrak{d}$ elements. Let $C := \{v^f\colon f \in \mathcal{F}\}$, where $\mathcal{F}$ is a subset of $\mathbb{N}^{\mathbb{N}}$ with $|\mathcal{F}| < \mathfrak{d}$. By definition of $\mathfrak{d}$, the set $\mathcal{F}$ is not cofinal in $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$, so there exists $h \in \mathbb{N}^{\mathbb{N}}$ such that

$$h \nleqslant^* f \quad \text{for every } f \in \mathcal{F}.$$

For each $r \in \mathbb{N}$ define $x^r = (x^r_{ij})_{ij} \in E$ by writing

$$x^r_{ij} = \begin{cases} 2 & \text{if } (i,j) = (r, h(r)), \\ 0 & \text{if } (i,j) \neq (r, h(r)). \end{cases}$$

Note that $D := \{x^r\colon r \in \mathbb{N}\}$ is bounded in E since, for a given n, we have $p_n(x^r) = 2$ for all $r > n$, implying that $p_n(D)$ is a finite, hence bounded set. Let f be an arbitrary member of $\mathcal{F}$. Because $h \nleqslant^* f$, there exists some $r \in \mathbb{N}$ with $h(r) > f(r)$. Therefore,

$$\left\langle x^r, v^f\right\rangle = \sum_{i,j} x^r_{ij} v^f_{ij} = 2 \cdot v^f_{r,h(r)} = 2.$$

We conclude that $v^f \notin D^\circ$ and, in fact, D° is a neighborhood of 0 in E' which misses C entirely.

III. $t(E') \leqslant \mathfrak{d}$. This is a consequence of Example 1(E) and Corollary 1. □

Example 5. If E is the above Grothendieck/Köthe space and E' its strong dual, then the tightness of $(E', \sigma(E', E''))$ is between $\mathfrak{b}$ and $\mathfrak{d}$.

Proof. I. $t(E', \sigma(E', E'')) \leqslant \mathfrak{d}$. Again, Example 1(E) and Corollary 1 apply.

II. $t(E', \sigma(E', E'')) \geqslant \mathfrak{b}$. Define A as in the proof of the previous example. Since its $\beta(E', E)$-closure contains 0, so does its closure in the coarser topology $\sigma(E', E'')$. It suffices to prove that 0 is not in the $\sigma(E', E'')$-closure of any subset $C := \{v^f\colon f \in \mathcal{F}\}$ of A with $|\mathcal{F}| < \mathfrak{b}$. By definition of $\mathfrak{b}$, the set $\mathcal{F}$ is bounded in $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$, yielding $g \in \mathbb{N}^{\mathbb{N}}$ such that $f \leqslant^* g$ for every $f \in \mathcal{F}$. Thus for each $f \in \mathcal{F}$ there exists $m(f) \in \mathbb{N}$ such that $f(n) \leqslant g(n)$ for all $n > m(f)$.

If we define $h \in \mathbb{N}^{\mathbb{N}}$ so that each $h(n) = g(n) + 1$, then, for every $f \in \mathcal{F}$,

$$h(n) > f(n) \quad \text{for all } n > m(f).$$

Just as before, we define x^r in terms of this h and note that $D := \{x^r\colon r \in \mathbb{N}\}$ is bounded in E, so that its polar D° is a neighborhood of 0 in E'. Thus D, viewed canonically as a subset of E'', is equicontinuous on E'. The Alaoglu–Bourbaki theorem provides $z \in E''$ such that $x^r \in z + V$ for infinitely many $r \in \mathbb{N}$ whenever V is a $\sigma(E'', E')$-neighborhood of 0. For an arbitrary $f \in \mathcal{F}$, set $V = \{u \in E'': |u(v^f)| < 1\}$ and choose $r > m(f)$ such that $x^r - z \in V$. Then

$$2 - \left|z\left(v^f\right)\right| \leqslant \left|2 - z\left(v^f\right)\right| = \left|\left\langle x^r, v^f\right\rangle - z\left(v^f\right)\right| = \left|\left(x^r - z\right)\left(v^f\right)\right| < 1,$$

which implies that $|z(v^f)| > 1$. We conclude that $\{z\}^\circ$ is a $\sigma(E', E'')$-neighborhood of 0 which excludes each $v^f \in C$. □

Distinct from a bornivorous $\mathfrak{G}$-representation, an increasing sequence $(D_n)_n$ of sets in an lcs E is *bornivorous* if each bounded set in E is absorbed by some D_n. A general theorem now emerges.

Theorem 1. *Suppose (E, τ) has a bornivorous $\mathfrak{G}$-representation $\{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}$.*

I. *The following three assertions are equivalent*:
   (i) *The space $(E, \sigma(E, E'))$ has countable tightness.*
   (ii) *The space $(E, \mu(E, E'))$ is quasibarrelled.*
   (iii) *The space $(E, \mu(E, E'))$ has countable tightness.*

II. *The next three assertions are equivalent*:
   (iv) *(E, τ) has countable tightness.*
   (v) *(E, τ) is quasibarrelled.*
   (vi) *There exists a family of absolutely convex closed subsets*

$$\mathcal{F} := \{D_{n_1,n_2,\ldots,n_k}: k, n_1, n_2, \ldots, n_k \in \mathbb{N}\}$$

   of E such that
   (a) *$D_{n_1,n_2,\ldots,n_k} \subset D_{m_1,m_2,\ldots,m_k}$, if $m_i \leqslant n_i$ for $i = 1, 2, \ldots, k$;*
   (b) *For every $\alpha = (n_k) \in \mathbb{N}^{\mathbb{N}}$ we have $D_{n_1} \subset D_{n_1,n_2} \subset \cdots \subset D_{n_1,n_2,\ldots,n_k}$ and the sequence is bornivorous*;
   (c) *If $W_\alpha := \bigcup_k D_{n_1,n_2,\ldots,n_k}$ for each $\alpha = (n_k)_k \in \mathbb{N}^{\mathbb{N}}$, then the family $\{W_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a basis of neighborhoods of zero in (E, τ).*

Proof. I. (i) ⇒ (ii): (CKS1) implies $(E', \sigma(E', E))$ is realcompact and thus so is the closed subset $A_\alpha^{\circ\circ}$. The latter is also countably compact due to equicontinuity of sequences. Hence $A_\alpha^{\circ\circ}$ is $\sigma(E', E)$-compact, and $A_\alpha^{\circ\circ\circ} = A_\alpha^\circ$ is a Mackey neighborhood of 0. Each $\beta(E', E)$-bounded set is contained in some A_α, thus is $\mu(E, E')$-equicontinuous; i.e., the Mackey topology is quasibarrelled.

(ii) ⇒ (iii): Trivially, $\{A_\alpha: \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-representation for $(E, \mu(E, E'))$, as well, and (iii) follows from $(*)$ under the assumption of (ii).

(iii) ⇒ (i): Again, $(*)$ applies. This concludes the proof of I.

II. (iv) ⇒ (v): Propositions 2 and 4 combine with (iv) to yield $\tau = \mu(E, E')$. Thus (iv) ensures that $\tau = \mu(E, E')$ has countable tightness. By I, then, $\tau = \mu(E, E')$ is quasibarrelled.

(v) ⇒ (vi): This is part of Lemma 2 in [8].

(vi) ⇒ (iv): Let A be any subset of (E, τ) with 0 in its closure. Thus each W_α meets A, and some of the sets $D_{n_1,n_2,\ldots,n_k}$ do, as well. Choose some

$$x_{n_1,n_2,\ldots,n_k} \in A \cap D_{n_1,n_2,\ldots,n_k}$$

whenever the intersection is nonempty, and let C be the totality of such points $x_{n_1,n_2,\dots,n_k}$. It is clear that C is a countable subset of A and that each W_α meets C, so that 0 is in the closure of C; i.e., (E,τ) has countable tightness. □

We need the following proposition, derived from [4, Proposition 1].

Proposition 7 *(Cascales). If an lcs E has a closed $\mathfrak{G}$-representation, then $(E',\sigma(E',E))$ is quasi-Suslin. In particular, the weak dual of a dual metric space is quasi-Suslin.*

We now generalize Valdivia's [29, pp. 65–66, (23), (24)], from strong duals of Fréchet spaces to all lcs having a bornivorous $\mathfrak{G}$-representation (see Example 2(D′)).

Corollary 2. *Let E be an lcs with a bornivorous $\mathfrak{G}$-representation. The weak dual $(E',\sigma(E',E))$ is always quasi-Suslin, but is K-analytic if and only if $(E,\mu(E,E'))$ is quasibarrelled.*

Proof. By Propositions 2 and 7, the weak dual is quasi-Suslin. For the rest, apply Theorem 1(i), (ii) and (CKS1)(a), (b). □

Part II yields the (positive) answer to Question 3 of [7]:

Corollary 3. *A (DF)-space, more generally, a dual metric space has countable tightness if and only if it is quasibarrelled.*

Corollary 4. *A Fréchet space E is distinguished if and only if its strong dual has countable tightness.*

Proof. A Fréchet space is distinguished if and only if its strong dual, a (DF)-space, is quasibarrelled [17, §29,4.(3)]. The previous corollary applies. □

Corollary 4 and Example 4 provide a novel proof for the following.

Corollary 5 *(Grothendieck, Köthe). The Grothendieck/Köthe space is, indeed, a nondistinguished Fréchet space.*

If we combine (CKS1), Examples 2 and 5, and Proposition 7, we obtain new information about the G/K example.

Corollary 6. *The weak bidual of the Grothendieck/Köthe space is a quasi-Suslin space that is not K-analytic.*

A separable lcs need not have countable tightness. Indeed, the product space $\mathbb{R}^{\mathbb{R}}$ is barrelled (Baire, even) and separable. Yet $t(\mathbb{R}^{\mathbb{R}}) = \chi(\mathbb{R}^{\mathbb{R}}) = \mathfrak{c}$, for the origin is in the closure of the set A of elements having zeros in finitely many coordinates and ones elsewhere, and no subset of A with fewer than $\mathfrak{c}$ members has this property. The story is different for dual metric spaces.

Corollary 7. *Every separable dual metric space E has countable tightness.*

Proof. E is quasibarrelled by [21, 8.2.20]. □

By $C_c(X)$ and $C_p(X)$ we denote the space $C(X)$ of real-valued continuous maps on a completely regular Hausdorff topological space X endowed with the compact-open and pointwise topology, respectively.

For dual metric Mackey spaces, all the conditions (i)–(vi) of Theorem 1, together with (a)–(e) of (CKS1) are equivalent. If "dual metric" is omitted, this is certainly not the case: Every $C_p(X)$ space is Mackey, indeed, is quasibarrelled [3, Proposition 1.4], but some have countable tightness (e.g., when $C_c(X)$ is separable and metrizable), and some have uncountable tightness (e.g., when X has the discrete topology and $|X| = \mathfrak{c}$, so that $C_p(X)$ is just the space $\mathbb{R}^{\mathbb{R}}$). We note in [11, Example 2], that if X is an uncountable K-analytic space, then the weak dual of $C_p(X)$ is not K-analytic. If X is countable, then $C_p(X)$ is metrizable, and its weak dual *is* K-analytic. We note that when X is infinite, $C_p(X)$ never has a fundamental sequence of bounded sets, and therefore cannot be dual metric. In fact, this argument holds for all dense quasibarrelled subspaces of $\mathbb{R}^X$.

On the other hand,

Example 6. *Many separable quasibarrelled spaces with countable tightness are not in class* $\mathfrak{G}$. We hinted at such spaces already: Let locally compact X be uncountable, separable, metrizable and a countable union of compact sets, so that $C_c(X)$ is metrizable and separable by a theorem of Warner [30]. (For example, one could take $X = \mathbb{R}$, or the closed interval [0, 1].) Every subset of $C_c(X)$ is separable, and thus so is every subset of $C_p(X)$. Hence $C_p(X)$ is not only separable, it also has countable tightness (also, see [1, II.1.1]). As noted, $C_p(X)$ is always quasibarrelled. Finally, $C_p(X)$ cannot be in $\mathfrak{G}$, for this only happens when X is countable [8, Corollary 2.8].

4. (*DF*)-spaces with and without $\mathfrak{G}$-bases

By (CKS2), a quasibarrelled space is in $\mathfrak{G}$ if and only if it has a $\mathfrak{G}$-base. The strong dual of a nondistinguished Fréchet space is not quasibarrelled, and still has a $\mathfrak{G}$-base (Example 1(E)). However, the quasibarrelled assumption cannot be omitted at the more general level of (*DF*)-spaces: all the spaces of this section are (*DF*)-spaces, hence in $\mathfrak{G}$, and most do not have a $\mathfrak{G}$-base. None is quasibarrelled. Exactly one has a $\mathfrak{G}$-base when we assume that $\aleph_1 = \mathfrak{b} = \mathfrak{d}$; none has a $\mathfrak{G}$-base if we assume that $\aleph_1 \neq \mathfrak{b}$.

The yet more general dual metric space F satisfies precisely one of these two statements:

(i) the weak dual of F is K-analytic, or
(ii) the weak dual of F is quasi-Suslin but not K-analytic (see Proposition 7).

Valdivia's may be the first space F proven to satisfy (ii), and in the previous section we found a second example by taking F to be the strong dual of the G/K space. The next section exhibits many F that satisfy (ii); some of these admit $\mathfrak{G}$-bases, most do not. The spaces F of the current section are such that $(F, \mu(F, F'))$ is a Banach space, so that F satisfies (i) by Corollary 2. For both (i) and (ii), then, the existence of a $\mathfrak{G}$-base turns out to be neither necessary nor sufficient.

We reprise from [24] an idea found useful in [7]. Fix p with $1 \leqslant p < \infty$, let Λ be an uncountable indexing set, and let G denote $\ell^p(\Lambda)$ endowed with the locally convex topology having as a base of neighborhoods of zero all sets of the form

$$[n; T] := n^{-1} D + G_{\Lambda \setminus T},$$

where $n \in \mathbb{N}$; D is the unit ball in the Banach space $\ell^p(\Lambda)$; T is a countable subset of Λ; and for each $S \subset \Lambda$,

$$G_S := \{u \in G\colon\ u(x) = 0 \text{ for all } x \notin S\}.$$

Note that, for each countable T, the subspaces G_T and $G_{\Lambda\setminus T}$ are topologically complementary in G, and G_T inherits the same Banach topology from G as it does from the Banach space $\ell^p(\Lambda)$, and the dual G' of G is the same as that of $\ell^p(\Lambda)$.

The next result shows, among other things, that separability cannot be omitted in Corollary 7.

Proposition 8. *G is a sequentially complete (DF)-space with*

$$t(G) = \aleph_1, \qquad t(\sigma(G, G')) = \aleph_0, \quad \text{and} \quad \chi(G) \geqslant |\Lambda|.$$

Proof. Since every sequence in G is contained in a Banach subspace of G, sequential completeness is clear. Since G' coincides with the dual of the Banach space $\ell^p(\Lambda)$, Kaplansky's theorem ensures that $t((G, \sigma(G, G'))) = \aleph_0$. Also, $(nD)_n$ is a fundamental sequence of bounded sets. Next we see that G is $\aleph_0$-quasibarrelled: Let $(U_n)_n$ be a sequence of balanced convex closed neighborhoods of zero whose intersection U is bornivorous in G. For each $n \in \mathbb{N}$ there exists a countable $T_n \subset \Lambda$ and $k_n \in \mathbb{N}$ such that $[k_n; T_n] \subset U_n$. Therefore, each $G_{\Lambda\setminus T_n} \subset U_n$, and $G_{\Lambda\setminus T} \subset U$, where $T := \bigcup_n T_n$. The direct summand G_T, a Banach space, is $\aleph_0$-quasibarrelled, and $U \cap G_T$ is a relative neighborhood of zero. Since U meets both summands G_T and $G_{\Lambda\setminus T}$ in relative neighborhoods of zero, it is a neighborhood of zero in G.

Because G is not the Banach space $\ell^p(\Lambda)$, it is not Mackey, therefore is not quasibarrelled. By Corollary 3, $t(G) \geqslant \aleph_1$. For a more direct proof, let A be the set of canonical unit vectors in $\ell^p(\Lambda)$. The closure of A in G contains 0, and the same is never true for a countable subset of A, implying $t(G) \geqslant \aleph_1$.

To reverse the inequality, let A be an arbitrary set whose closure contains the origin in G. We must find $B \subset A$ whose closure contains the origin, and such that $|B| \leqslant \aleph_1$. The set of all countable ordinals is denoted ω_1 and has cardinality $\aleph_1$. We define

$$X_0 = \{\alpha \in \omega_1\colon\ \alpha \text{ has no immediate predecessor}\},$$

and for $n = 1, 2, \dots$ set

$$X_n = \{\alpha + n\colon\ \alpha \in X_0\}.$$

Note that $X_0, X_1, X_2, \dots$ partition ω_1 into cofinal sets. Define

$$X := \bigcup_{n\geqslant 1} X_n = \omega_1 \setminus X_0.$$

We induct on the well-ordered set X to obtain, for each $\alpha \in X$, countable sets $T_\alpha, S_\alpha \subset \Lambda$ and $x_\alpha \in A$ such that

$$T_\alpha = \bigcup\{T_\beta \cup S_\beta\colon\ \beta \in X \text{ and } \beta < \alpha\} \quad \text{and} \quad x_\alpha \in [n; T_\alpha] \cap G_{S_\alpha},$$

where n is the unique positive integer such that $\alpha \in X_n$. The induction rests on the fact that if $\alpha \in X$ is given and T_β, S_β and x_β have been suitably chosen for all β in X less than α, then, as the above-defined countable union of countable sets, T_α is a countable set, and therefore by hypothesis on A there exists $x_\alpha \in A \cap [n; T_\alpha]$. Moreover, since each point in G is in G_S for some countable $S \subset \Lambda$, there exists a suitable choice for S_α.

Now $B := \{x_\alpha\colon \alpha \in X\}$ satisfies $|B| \leqslant |\omega_1| = \aleph_1$, and we will show that the closure of B contains the origin. Given $n \geqslant 1$ and a countable $T \subset \Lambda$, the countable $T \cap (\bigcup_{\alpha\in X} T_\alpha)$ is contained in $\bigcup_{\alpha\in Y} T_\alpha$ for some countable $Y \subset X$, and since X_n is cofinal, we may choose $\gamma \in X_n$ with $\gamma > \sup Y$. We now have

$$T \cap S_\gamma \subset T \cap \left(\bigcup_{\alpha\in X} T_\alpha\right) \subset \bigcup_{\alpha\in Y} T_\alpha \subset T_\gamma.$$

Therefore

$$[n; T_\gamma] \subset [n; T \cap S_\gamma],$$

and we have

$$x_\gamma \in [n; T_\gamma] \cap G_{S_\gamma} \subset [n; T \cap S_\gamma] \cap G_{S_\gamma} = [n; T] \cap G_{S_\gamma} \subset [n; T].$$

Hence B meets every neighborhood of the origin, and $t(G) = \aleph_1$.

Finally, we show that $\chi(G) \geqslant |\Lambda|$. If $\mathcal{U}$ is a collection of neighborhoods of the origin in G with $|\mathcal{U}| < |\Lambda|$, then for each $U \in \mathcal{U}$ we may choose a countable set $T_U \subset \Lambda$ with $G_{\Lambda\setminus T_U} \subset U$, and since $|\bigcup\{T_U\colon U \in \mathcal{U}\}| \leqslant \aleph_0 \cdot |\mathcal{U}| < |\Lambda|$, we have $S := \Lambda \setminus \bigcup\{T_U\colon U \in \mathcal{U}\} \neq \emptyset$. This, in turn, implies that

$$\{0\} \neq G_S \subset \bigcap \mathcal{U}.$$

That is to say, $\bigcap \mathcal{U}$ contains a nonzero subspace, and thus $\mathcal{U}$ is not a base of neighborhoods of the origin for the Hausdorff space G. $\square$

Although G admits a bornivorous (and closed) $\mathfrak{G}$-representation for every uncountable Λ, we shall find that G admits a $\mathfrak{G}$-base only when $|\Lambda|$ is severely restricted under an axiomatic assumption milder than CH.

Example 7. If we assume that $\aleph_1 = \mathfrak{b} = |\Lambda|$, then G has a $\mathfrak{G}$-base.

Proof. By definition of $\mathfrak{b}$ there is a one-to-one map ϕ from the set $[0, \mathfrak{b})$ of ordinals less than $\mathfrak{b}$ onto a set A unbounded in $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$, and by hypothesis there is a one-to-one map ψ from Λ onto $[0, \mathfrak{b})$. For arbitrary $\sigma = (a_1, a_2, a_3, \ldots) \in \mathbb{N}^{\mathbb{N}}$, let $\beta(\sigma)$ be the first member of $[0, \mathfrak{b})$ such that $\phi(\beta(\sigma)) \nleqslant^* (a_2, a_3, \ldots)$, and define the corresponding neighborhood U_σ of the origin by writing

$$U_\sigma = \bigl[a_1; \psi^{-1}\bigl([0, \beta(\sigma)]\bigr)\bigr].$$

Note that $\beta(\sigma) < \mathfrak{b} = \aleph_1$ implies the set $[0, \beta(\sigma)]$ is countable, hence $\psi^{-1}([0, \beta(\sigma)])$ is a countable subset of Λ, and U_σ is, in truth, a well-defined neighborhood of the origin in G. Obviously, $\sigma \leqslant \tau \Rightarrow \beta(\sigma) \leqslant \beta(\tau)$, and after comparison of the first coordinates we conclude that $U_\tau \subset U_\sigma$. In fact, $\{U_\sigma\colon \sigma \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base: Given $n \in \mathbb{N}$ and countable $T \subset \Lambda$, set

$$\alpha = \sup \psi(T)$$

and note that $\alpha < \mathfrak{b}$, since $\mathfrak{b}$ has uncountable cofinality and $\psi(T)$ is countable. Since $\phi([0, \alpha])$ has fewer than $\mathfrak{b}$ elements, it is bounded in $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$ by some $(a_2, a_3, \ldots)$. Putting $a_1 = n$ and $\sigma = (a_1, a_2, a_3, \ldots)$ thus yields $\beta(\sigma) \notin [0, \alpha]$, so that $[0, a] \subset [0, \beta(\sigma)]$. It follows that

$$T \subset \psi^{-1}\bigl([0, \alpha]\bigr) \subset \psi^{-1}\bigl(\bigl[0, \beta(\sigma)\bigr]\bigr)$$

and, finally, that $U_\sigma \subset [n; T]$. $\square$

Example 8. If we assume that $\aleph_1 < \mathfrak{b}$, then G does not have a $\mathfrak{G}$-base.

Proof. If G has a $\mathfrak{G}$-base, then so does the subspace G_{Λ_1}, where Λ_1 is a subset of Λ of size $\aleph_1$. But the character of G_{Λ_1} is clearly $\aleph_1$. This contradicts Proposition 6 and the assumption that $\aleph_1 < \mathfrak{b}$. □

From the two previous examples, we have

Theorem 2. *For $|\Lambda| = \aleph_1$, the space G has a $\mathfrak{G}$-base if and only if we make the assumption that $\aleph_1 = \mathfrak{b}$.*

Propositions 6, 8 give us

Example 9. If $|\Lambda| > \mathfrak{d}$, then G does not have a $\mathfrak{G}$-base.

It is now obvious that the converse to Example 7 holds under the assumption that scales exist, equivalently, that $\mathfrak{b} = \mathfrak{d}$.

5. Dual metric spaces $C_c(X)$

Recall that an lcs E is a *df-space* if it contains a fundamental sequence of bounded sets and is c_0-quasibarrelled, i.e., every null sequence in the strong dual is equicontinuous. It is known (e.g., see [13, Corollary 3.3]) that a $C_c(X)$ space is a dual metric space if and only if it is a *df*-space, and we will use the two terms interchangeably in the $C_c(X)$ context. However, the paper [16] answers a 30-year-old question by showing that there does exist a $C_c(X)$ space that is a *df*-space and not a (*DF*)-space. For the moment, we will consider spaces $C_c(\kappa)$, where κ is an infinite cardinal, among which there is no distinction between *df*- and (*DF*)-spaces. The space $C_c(\omega_1)$ was studied by Morris and Wulbert in 1967 [20]; $\omega_1 = \aleph_1$ is the first uncountable ordinal (cardinal).

We take the view that the cardinal κ is also an ordinal, and is the set $[0, \kappa)$ of all ordinals of cardinality less than κ endowed with its usual interval topology, so that $C_c(\kappa)$ is well defined. For each ordinal α, the closed interval $[0, \alpha]$ is compact via a simple transfinite induction. However, for κ an infinite cardinal, $\kappa = [0, \kappa)$ is not compact, but has a fundamental system of compact sets consisting of the sets $[0, \alpha]$ as the ordinal α ranges over a cofinal subset A of $[0, \kappa)$. Thus the (compact-open) topology for $C_c(\kappa)$ has a base of neighborhoods of zero consisting of sets of the form

$$U_{n,\alpha} := \{ f \in C(\kappa):\ |f(\gamma)| \leqslant n^{-1} \text{ for all } \gamma \in [0, \alpha] \},$$

where $n \in \mathbb{N}$ and $\alpha \in A$. Since

$$|\{U_{n,\alpha}:\ n \in \mathbb{N} \text{ and } \alpha \in A\}| = \aleph_0 \cdot |A| = |A|,$$

it is easy to see that character $\chi(C_c(\kappa))$ equals cofinality $\mathrm{cf}(\kappa)$. As to tightness, the collection C of characteristic functions of the open intervals (α, κ) is a subset of $C_c(\kappa)$ whose closure contains 0. If B is any subset of C of size less than $\mathrm{cf}(\kappa)$, then the corresponding collection of left endpoints has supremum $\beta < \kappa$, so that all members of B are identically one on the open interval (β, κ). Choose $\gamma \in (\beta, \kappa)$. The evaluation functional δ_γ is in the dual $C_c(\kappa)'$ and bounds B away from 0, so that, even in the weak topology $\sigma(C_c(\kappa), C_c(\kappa)')$, zero is not in the closure of B. It follows that the tightness of both the original and weak topologies for $C_c(\kappa)$ is at least $\mathrm{cf}(\kappa)$, and by Proposition 5 is at most $\mathrm{cf}(\kappa)$. We have proved

Proposition 9. $t(C_c(\kappa)) = t(\sigma(C_c(\kappa), C_c(\kappa)')) = \chi(C_c(\kappa)) = \mathrm{cf}(\kappa)$.

Thus with spaces $C_c(\kappa)$ the tightness for the original and weak topologies is always the same, unlike the spaces G of the last section. Much similarity remains between these two types of spaces, nevertheless.

Proposition 10. *$C_c(\kappa)$ is a (DF)-space when $\mathrm{cf}(\kappa)$ is uncountable.*

Proof. Warner [30] proved that $C_c(X)$ is a (DF)-space if and only if every countable union of compact sets is relatively compact in X. A countable union of compact sets in $[0,\kappa)$ is contained in a countable union of closed intervals, and their right endpoints have supremum $\beta < \kappa$ when $\mathrm{cf}(\kappa)$ is uncountable. Therefore the countable union is contained in the compact interval $[0,\beta]$, and is relatively compact. □

It is a simple exercise to see that $C_c(\kappa)$ is always sequentially complete, and thus is a Fréchet space when $\mathrm{cf}(\kappa) = \aleph_0$.

If we combine the previous two propositions, Example 2(D′), Theorem 1(I) and Corollary 2, we find a large class of lcs that are quasi-Suslin but not K-analytic.

Example 10. If $\mathrm{cf}(\kappa)$ is uncountable, then the weak dual of $C_c(\kappa)$ is quasi-Suslin but not K-analytic.

Example 11. If $\aleph_0 < \mathrm{cf}(\kappa) < \mathfrak{b}$, or if $\mathfrak{d} < \mathrm{cf}(\kappa)$, then the (DF)-space $C_c(\kappa)$ does not admit a $\mathfrak{G}$-base. Indeed, this is immediate from Propositions 6, 9.

None of the spaces G of the last section admits a $\mathfrak{G}$-base under the ZFC-consistent assumption that $\aleph_1 \neq \mathfrak{b}$. Not so with the spaces $C_c(\kappa)$; some of them admit $\mathfrak{G}$-bases regardless of the (ZFC-consistent) model in which we work.

Example 12. $C_c(\mathfrak{b})$ and $C_c(\mathfrak{d})$ admit $\mathfrak{G}$-bases. If $\mathrm{cf}(\kappa) = \aleph_0, \mathfrak{b}, \mathfrak{d}$, then $C_c(\kappa)$ admits a $\mathfrak{G}$-base.

Proof. For the $C_c(\mathfrak{b})$ case, choose $A \subset \mathbb{N}^{\mathbb{N}}$ with $|A| = \mathfrak{b}$ such that A is unbounded in $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$. Let ϕ be a one-to-one mapping from $[0,\mathfrak{b})$ onto A. For arbitrary $\sigma = (a_1, a_2, a_3, \ldots) \in \mathbb{N}^{\mathbb{N}}$, we define the corresponding neighborhood U_σ of the origin by writing

$$U_\sigma = U_{a_1,\beta},$$

where β is the first member of $[0,\mathfrak{b})$ such that $\phi(\beta) \nleqslant^* (a_2, a_3, \ldots)$. Obviously, $\sigma \leqslant \tau \Rightarrow U_\tau \subset U_\sigma$. In fact, $\{U_\sigma\colon \sigma \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base: Given $n \in \mathbb{N}$ and ordinal $\alpha < \mathfrak{b}$, some $(a_2, a_3, \ldots)$ bounds the set $\phi([0,\alpha])$ in $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$, by definition of $\mathfrak{b}$. Putting $a_1 = n$ and $\sigma = (a_1, a_2, a_3, \ldots)$ produces a corresponding $\beta \notin [0,\alpha]$, so that $[0,\alpha] \subset [0,\beta]$ and $U_\sigma = U_{a_1,\beta} \subset U_{n,\alpha}$.

One repeats the construction for $C_c(\mathfrak{d})$ with $\mathfrak{d}$ replacing $\mathfrak{b}$, with A cofinal in $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$, and with β the first member of $[0,\mathfrak{d})$ such that $(a_2, a_3, \ldots) \leqslant^* \phi(\beta)$. From the definition of $\mathfrak{d}$ one concludes that the result is a $\mathfrak{G}$-base.

When $\mathrm{cf}(\kappa) = \aleph_0$, the Fréchet space $C_c(\kappa)$ has a $\mathfrak{G}$-base by Example 1(D). If $\mathrm{cf}(\kappa)$ is $\mathfrak{b}$ or $\mathfrak{d}$, let ϕ be a one-to-one map from a cofinal subset M of $[0, \kappa)$ onto a subset A of $\mathbb{N}^{\mathbb{N}}$ such that $|M| = \mathfrak{b}$ or $\mathfrak{d}$ and A is unbounded or cofinal in $(\mathbb{N}^{\mathbb{N}}, \leqslant^*)$, respectively. We proceed just as before to construct a $\mathfrak{G}$-base for $C_c(\kappa)$. □

This example in conjunction with Proposition 9 affirms the optimality of $\mathfrak{b}$ and $\mathfrak{d}$ in Proposition 6 and its corollary.

Example 13. If $\mathrm{cf}(\kappa)$ is $\mathfrak{b}$ or $\mathfrak{d}$, then the weak dual of $C_c(\kappa)$ is not K-analytic, but is covered by an ordered family $\{B_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of compact sets.

Proof. Use previous examples and take $B_\alpha = U_\alpha^\circ$, where $\{U_\alpha\colon \alpha \in \mathbb{N}^{\mathbb{N}}\}$ is a $\mathfrak{G}$-base for $C_c(\kappa)$. □

Combining our work, we obtain

Theorem 3. *The Morris/Wulbert space $C_c(\omega_1)$ admits a $\mathfrak{G}$-base if and only if we assume that $\aleph_1 = \mathfrak{b}$.*

Some notation will aid the application of Theorem 1(I) to *df*-spaces $C_c(X)$.

If $[X, 1] := \{f \in C(X)\colon |f(x)| \leqslant 1 \text{ for all } x \in X\}$ is absorbing in $C(X)$, i.e., if X is pseudocompact, we let $C_u(X)$ denote $C(X)$ endowed with the uniform norm topology having unit ball $[X, 1]$. We need the following lemma; lacking a reference, we give a proof.

Lemma 2. *If X is pseudocompact and $C_c(X)' = C_u(X)'$, then X is compact.*

Proof. Assume that X is noncompact. Let η be the Banach uniform topology generated by $[X, 1]$, and let βX be the Čech compactification of X. Fix $x \in \beta X \setminus X$. For every $f \in C_c(X)$ let f^β be its continuous extension to the space βX. Define the evaluation map e by writing

$$e(f) := f^\beta(x)$$

for every $f \in C_c(X)$. Clearly, $e \in C_u(X)'$ with norm 1. Each compact $K \subset X$ routinely admits $f \in C(X)$ with $f(K) = \{0\}$ and $e(f) = 1$, so that e is not continuous in the compact-open topology, a contradiction to hypothesis. □

We remind ourselves that [$C_c(X)$ is a *df*-space] $\Leftrightarrow$ [$C_c(X)$ is a dual metric space] $\Rightarrow$ [$C_c(X)$ has a fundamental sequence of bounded sets] $\Leftrightarrow$ [$[X, 1]$ is a bornivore in $C_c(X)$]. The last equivalence, due to Warner [30], is proved more simply in [14].

Theorem 4. *The following assertions are equivalent for a df-space $C_c(X)$:*

(0) *X is compact.*
(1) *$C_c(X)$ coincides with the Banach space $C_u(X)$.*
(2) *$C_c(X)$ has countable tightness.*
(3) *The weak topology of $C_c(X)$ has countable tightness.*
(4) *The Mackey topology $\mu(C_c(X), C_c(X)')$ has countable tightness.*

Recall that condition (3) is equivalent to any condition from (CKS1).

Proof. (0) $\Rightarrow$ (1) $\Rightarrow$ (2) is obvious, (2) $\Rightarrow$ (3) follows from ($*$), and (3) $\Leftrightarrow$ (4) from Theorem 1. Note also that (2) $\Rightarrow$ (4) follows from Proposition 4.

There remains to show that (4) $\Rightarrow$ (0): By Theorem 1(I), condition (4) implies that the topology $\mu(E, E')$ is quasibarrelled, and therefore the bornivorous barrel $[X, 1]$ is a $\mu(E, E')$-neighborhood of zero. Hence $C_c(X)' = C_u(X)'$, which implies (by the lemma) that X is compact. $\square$

Recall that an lcs is *docile* if every infinite-dimensional subspace contains an infinite-dimensional bounded set [12]. Increasingly, we find properties of $C_c(X)'$ that characterize properties of X. For instance, X is Warner bounded if and only if the *strong dual* of $C_c(X)$ is docile [14], and is pseudocompact if and only if the *weak dual* is docile [13]. Again, $C_c(X)$ is a *df*-space if and only if the *strong dual* is a Banach space, if and only if the *weak dual* is docile and locally complete [13].

Corollary 8. *A completely regular Hausdorff space X is compact if and only if $(C_c(X)', \sigma(C_c(X)', C_c(X)))$ is docile, locally complete and realcompact.*

Proof. If X is compact, then $C_c(X)$ is a Banach space; the weak dual of every Banach space is docile and locally complete and, by ($*$) and (CKS1), is realcompact.

Conversely, the weak dual hypotheses ensure, via the main theorem of [13], that $C_c(X)$ is a *df*-space, equivalently, a dual metric space, and hence in $\mathfrak{G}$. The realcompact hypothesis and (CKS1) now prove that the weak topology of $C_c(X)$ has countable tightness, so by Theorem 4, X is compact. $\square$

Whether or not $C_c(X)$ is a Banach space is therefore entirely determined by its weak dual, whereas in the larger class of lcs, a Banach space and a nonBanach space may share the same weak dual. In considering realcompactness, note that $\mathbb{R}^I$ itself is docile if and only if $|I| < \mathfrak{b}$, according to Example 4.1 of [15]. The corollary remains valid when realcompact is replaced by any of the conditions (b)–(e) of (CKS1).

The corollary is interesting in connection with the spaces $C_c(\kappa)$ considered earlier in this section. $X = [0, \kappa] \setminus \{\kappa\}$ is just one point short of being compact. If κ has countable cofinality, then $[0, \kappa)$ is a countable union of compact sets and $C(\kappa)$ is a nonnormable Fréchet space. Therefore its weak dual is locally complete [21], and by ($*$) and (CKS1), is realcompact. Since $X = [0, \kappa)$ is not compact, we conclude from the corollary that the weak dual of $C_c(\kappa)$ is not docile. This is not surprising, for an easy consequence of [23, Theorem 8], is the fact that every barrelled metrizable nonnormable space has a nondocile weak dual. The analysis via Corollary 8 of the uncountable cofinality case gives an alternate proof of Example 10:

Proof. Since κ has uncountable cofinality, every countable union of compact sets is relatively compact in $[0, \kappa)$, and therefore, by a theorem of Warner, $C_c(\kappa)$ is a (DF)-space, hence a *df*-space. By the main theorem of [13], the weak dual of $C_c(\kappa)$ is docile and locally complete. Since $[0, \kappa)$ is not compact, Corollary 8 ensures that the weak dual is not realcompact, thus not K-analytic. (In general, K-analytic implies realcompact.) $\square$

One may wish to know precisely when $C_c(X)$ has countable tightness; the same problem for $C_p(X)$ was solved by McCoy in [19, Theorem 2], see also [1, Theorem II. 1.1]. McCoy and Ntantu solved the problem for $C_c(X)$ by defining an open cover Σ of X to be *compact-open* if every compact subset of X is contained in some member of Σ, and then proving that $C_c(X)$ *has countable tightness if and only if every compact-open cover of* X *has a compact-open countable subcover.*

Theorem 4 allows us to extend Example 10 to spaces $C_c(X)$ where X is not necessarily a cardinal κ.

Corollary 9. *Let $C_c(X)$ be a df-space. Its weak dual is quasi-Suslin*; *it is K-analytic if and only if X is compact.*

Proof. The first part follows from Example 2(D′) and Proposition 7. Theorem 4 and (CKS1) prove the second part. □

In [13] and [16] we show that there exists X for which $C_c(X)$ is a *df*-space and not a (*DF*)-space, more than answering the Buchwalter–Schmets [3] question from 1973. Thus the following corollary has a nonvacuous hypothesis and provides a class of spaces having the desired Valdivia-like conclusion, and yet having no overlap with the spaces covered in Example 10, since the latter are all (*DF*)-spaces (Proposition 10). The only new part of the proof is the observation that Banach spaces are (*DF*)-spaces.

Corollary 10. *Let $C_c(X)$ be any df-space which is not a* (*DF*)*-space. The weak dual is quasi-Suslin but not K-analytic.*

We conclude the paper with two open problems.

Problem 1. Does there exists an lcs which admits a closed $\mathfrak{G}$-representation but no bornivorous $\mathfrak{G}$-representation?

Problem 2. Does every lcs in $\mathfrak{G}$ have a quasi-Suslin weak dual? (See Proposition 7.)

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