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On transitive algebras containing a standard finite von Neumann subalgebra

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Abstract

Let $\mathcal{M}$ be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $\mathfrak{A}$ be a transitive algebra containing $\mathcal{M}'$. In this paper we prove that if $\mathfrak{A}$ is 2-fold transitive, then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$. This implies that if a transitive algebra containing a standard finite von Neumann algebra (in the sense of [U. Haagerup, The standard form of von Neumann algebras, Math. Scand. 37 (1975) 271–283]) is 2-fold transitive, then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$. Non-selfadjoint algebras related to free products of finite von Neumann algebras, e.g., $\mathcal{L}\mathbb{F}_n$ and $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$, are studied. Brown measures of certain operators in $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$ are explicitly computed.

Keywords: Transitive algebras; n-Fold transitive; Operator ranges; Standard finite von Neumann algebras; Free products; Brown measures; Hyperinvariant subspaces

0. Introduction

Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. A subalgebra $\mathfrak{A}$ of $\mathcal{B}(\mathcal{H})$ is *transitive* if it contains the identity operator and has no invariant (closed) subspace other than the two trivial ones. The *transitive algebra problem* asks: if $\mathfrak{A}$ is a transitive algebra on $\mathcal{H}$, is $\mathfrak{A}$ strongly dense in $\mathcal{B}(\mathcal{H})$? This problem was implicitly contained in a question of R. Kadison [18] concerning algebras whose invariant subspaces have invariant

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complementary subspaces, and it was W. Arveson [1] who first explicitly stated the problem, coined the term “transitive algebra,” and began an in-depth study of the problem. Note that an affirmative answer to the transitive algebra problem would give rise to an affirmative answer to (hyper)invariant subspace problem. The (hyper)invariant subspace problem asks if an algebra generated by (the commutant of) a single bounded operator on $\mathcal{H}$ can be transitive.

On the other hand, one's intuition expects that there exists a transitive algebra which is not strongly dense in $\mathcal{B}(\mathcal{H})$. It is of interest, then, to know how one might strengthen the hypothesis so as to get a provable result. The first partial solutions of the transitive algebra problem were given by Arveson [1]. Arveson proved that if $\mathfrak{A}$ is a transitive algebra containing a MASA (maximal abelian von Neumann subalgebra) of $\mathcal{B}(\mathcal{H})$, then the strong closure of $\mathfrak{A}$ is $\mathcal{B}(\mathcal{H})$. For various generalizations of Arveson's results, we refer to [5,15,16]. Inspired by the invariant subspace problem affiliated with a von Neumann algebra, we consider the following construction: Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space and $\mathcal{M}'$ be the commutant of $\mathcal{M}$. Suppose $\{T_\alpha\} \subseteq \mathcal{M}$ has no non-trivial common invariant subspace relative to $\mathcal{M}$, i.e., if $E \in \mathcal{M}$ satisfies $ET_\alpha E = T_\alpha E$ for all T_α then $E = 0$ or I. It is easy to see that the algebra, $\mathfrak{A}$, generated by $\{T_\alpha\}$ and $\mathcal{M}'$ is a transitive algebra. Now we may ask that if $\mathfrak{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$ when we choose $\mathcal{M}$ and $\{T_\alpha\}$ suitably. To make the question non-trivial, we first choose the “size” of $\mathcal{M}$ suitably large.

Recall that a von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$ is said to be *standard* if there exists a conjugate unitary operator $J : \mathcal{H} \to \mathcal{H}$, such that the mapping $X \to JX^*J$ is a *-anti-isomorphism from $\mathcal{M}$ onto $\mathcal{M}'$. Haagerup [11] proved that every von Neumann algebra is *-isomorphic to a standard von Neumann algebra on a Hilbert space. If a von Neumann algebra $\mathcal{M}$ is standard on a Hilbert space $\mathcal{H}$, then $\mathcal{M}'$ is also standard on $\mathcal{H}$. We may ask the following question: if a von Neumann algebra $\mathcal{M}$ is standard on a Hilbert space $\mathcal{H}$ and $\mathfrak{A}$ is a transitive algebra on $\mathcal{H}$ which contains $\mathcal{M}'$, is $\mathfrak{A}$ strongly dense in $\mathcal{B}(\mathcal{H})$? By [21, Theorem 8.26], if $\mathcal{M}$ is a type I_∞ factor, this question is equivalent to the transitive algebra question. So it is more interesting to restrict one's attention to the case where $\mathcal{M}$ is a finite von Neumann algebra. Notably, if an abelian von Neumann algebra $\mathcal{M}$ is standard on a Hilbert space $\mathcal{H}$, then $\mathcal{M}$ is a MASA of $\mathcal{B}(\mathcal{H})$. By [1], any transitive algebra which contains $\mathcal{M}' = \mathcal{M}$ is strongly dense in $\mathcal{B}(\mathcal{H})$ and thus the answer to above question is affirmative. Suppose $\mathcal{M}$ is a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $\mathfrak{A}$ is a transitive algebra which contains $\mathcal{M}'$. In Section 3 of this paper, we prove the following result. If $\mathfrak{A}$ is 2-fold transitive (i.e., for any linearly independent vectors ξ, η in $\mathcal{H}$, the closure of $\{(T\xi, T\eta)\colon T \in \mathfrak{A}\}$ is $\mathcal{H} \oplus \mathcal{H}$), then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$. As a corollary, this implies that if $\mathcal{M}$ is a standard finite von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\mathfrak{A}$ is a 2-fold transitive algebra containing $\mathcal{M}$, then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$. This partly answers a question of Arveson [2] (also see [21, 10.5]), which asks for a transitive algebra $\mathfrak{A}$ whether 2-fold transitivity implies that the strong closure of $\mathfrak{A}$ is $\mathcal{B}(\mathcal{H})$. The proof of our result relies on a new characterization of n-fold transitivity and operator theory techniques.

As applications, we study some non-selfadjoint algebras related to (reduced) free products of finite von Neumann algebras. In Section 4, we prove the following result. Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra with a faithful normal trace τ. Suppose $\mathcal{N} \subseteq \mathcal{M}$ is a von Neumann subalgebra and $Z \in \mathcal{M}$ satisfies the following conditions:

1. $Z \neq 0$ and $\tau(Z) = 0$;
2. $Z, Z((\mathcal{N}' \cap \mathcal{M}) \ominus \mathbb{C}I), ((\mathcal{N}' \cap \mathcal{M}) \ominus \mathbb{C}I)Z$ are mutually orthogonal in $L^2(\mathcal{M}, \tau)$.

Let $\mathfrak{A} \subseteq \mathcal{B}(L^2(\mathcal{M}, \tau))$ be the algebra generated by $Z, \mathcal{N}$ and $\mathcal{M}'$ (relative to $\mathcal{B}(L^2(\mathcal{M}, \tau))$). Then $\mathfrak{A}$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. As a corollary, we prove the following. Let $(\mathcal{M}_1, \tau_1), (\mathcal{M}_2, \tau_2)$ be finite von Neumann algebras and $\mathcal{M} = (\mathcal{M}_1, \tau_1) * (\mathcal{M}_2, \tau_2)$ be the reduced free product von Neumann algebra and τ be the induced faithful normal trace on $\mathcal{M}$. Suppose $\mathcal{N} \subseteq \mathcal{M}_1$ is a diffuse von Neumann subalgebra and $Z \in \mathcal{M}_2$ is not a scalar. Let $\mathfrak{A} \subseteq \mathcal{B}(L^2(\mathcal{M}, \tau))$ be the algebra generated by $Z, \mathcal{N}$ and $\mathcal{M}'$. Then $\mathfrak{A}$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. For example, let $\mathcal{M} = \mathcal{L}\mathbb{F}_n$ $(2 \leqslant n \leqslant \infty)$ be the type II_1 factor associated with the left regular representation λ of the free group $\mathbb{F}_n$ on n generators and a, b be two generators of $\mathbb{F}_n$. Let k be a positive integer and $Z \in \{\lambda(a), \lambda(a)^*\}''$ be a non-scalar operator. We show that the algebra $\mathfrak{A}$ on $L^2(\mathcal{M}, \tau)$ generated by $Z, \lambda(b)^k$, and $\mathcal{M}'$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. The proof of above results relies on techniques of type II_1 factors. Freeness also plays a key role.

In Section 5, we consider some interesting (but by no means trivial) non-selfadjoint algebras related to $\mathcal{M} = (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$ (reduced free product von Neumann algebra with respect to the normalized trace on $M_2(\mathbb{C})$). Let $(E_{ij})_{i,j=1,2}$ and $(F_{ij})_{i,j=1,2}$ be the matrix units in $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $1 * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$, respectively. Consider the following subalgebras of $\mathcal{B}(L^2(\mathcal{M}, \tau))$:

1. $\mathfrak{A}_1$, the algebra generated by E_{11}, E_{12}, F_{11} and $\mathcal{M}'$.
2. $\mathfrak{A}_2$, the algebra generated by E_{11}, E_{12}, F_{12} and $\mathcal{M}'$.
3. $\mathfrak{A}_3$, the algebra generated by E_{11}, F_{12} and $\mathcal{M}'$.
4. $\mathfrak{A}_4$, the algebra generated by E_{12}, F_{12} and $\mathcal{M}'$.

In this paper, we prove that $\mathfrak{A}_1$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. We then prove that $\mathfrak{A}_3$ and $\mathfrak{A}_4$ are not transitive algebras. Indeed, for any $0 \leqslant r \leqslant 1$, there are invariant subspaces E_r and F_r of $\mathfrak{A}_3$ and $\mathfrak{A}_4$, respectively, such that $\tau(E_r) = \tau(F_r) = r$. The main idea to prove that $\mathfrak{A}_3$ and $\mathfrak{A}_4$ are not transitive is as following. Note that

$$\mathfrak{A}_3 \subseteq \left\{(E_{11} - E_{22} + F_{12})^2\right\}' \quad \text{and} \quad \mathfrak{A}_4 \subseteq \left\{(E_{12} + F_{12})^2\right\}'.$$

Based on the results of Brown measures of R-diagonal operators computed by Haagerup and Larson [13], we explicitly compute the Brown measures of $E_{11} - E_{22} + F_{12}$ and $E_{12} + F_{12}$ by using techniques of free probability theory and operator theory. Then we apply Theorem 7.1 of [14] on the existence of hyperinvariant subspaces of operators whose Brown measures are not concentrated on a single point. The question that $\mathfrak{A}_2$ is transitive or not remains open!

Remarkable progress on the (hyper)invariant subspace problem relative to a factor of type II_1 has been made during past ten years (see for example [6,7,12,14,22]). The fact that 2-fold transitivity of a transitive algebra containing a standard type II_1 factor implies that the strong closure of the transitive algebra is $\mathcal{B}(\mathcal{H})$ can be viewed as a support for the existence of non-trivial (hyper)invariant subspaces of operators relative to a factor of type II_1.

Besides the Introduction and Sections 3–5, there are two more sections in this paper. In Section 1, we provide some characterizations of n-fold transitivity. To prove our main result (Theorem 3.1), some auxiliary lemmas are proved in Section 2.

For the general theory of operator theory and invariant subspaces, we refer to [21]. For the general theory of von Neumann algebras, we refer to [19]. For the general theory of free probability theory, we refer to [24].

1. On *n*-fold transitivity

We begin this section by establishing some notation and terminology. Let $\mathcal{H}$ be a Hilbert space. A *linear manifold* in $\mathcal{H}$ is a subset of $\mathcal{H}$ which is closed under vector addition and under multiplication by complex numbers. A *subspace* of $\mathcal{H}$ is a linear manifold which is closed in the norm topology; the trivial subspaces are $\{0\}$ and $\mathcal{H}$. If $\mathcal{D}$ is a linear manifold in $\mathcal{H}$, then $[\mathcal{D}]$ denotes the norm closure of $\mathcal{D}$.

If $T \in \mathcal{B}(\mathcal{H})$, the collection of all subspaces of $\mathcal{H}$ invariant under T is denoted by $\operatorname{Lat} T$; if $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$, then $\operatorname{Lat}\mathfrak{A} = \bigcap_{T\in\mathfrak{A}} \operatorname{Lat} T$. A subspace $\mathcal{K}$ is *hyperinvariant* for T if $\mathcal{K} \in \operatorname{Lat}\{T\}'$, i.e., $\mathcal{K} \in \operatorname{Lat} S$ for every S commutes with T. Let $P \in \mathcal{B}(\mathcal{H})$ be a projection, i.e., $P = P^* = P^2$. We say $P \in \operatorname{Lat} T$ if $P\mathcal{H} \in \operatorname{Lat} T$. $P \in \operatorname{Lat} T$ if and only if $PTP = TP$.

If $\mathcal{H}$ is a Hilbert space and n is a positive integer, then $\mathcal{H}^{(n)}$ denotes the direct sum of n copies of $\mathcal{H}$, i.e., the Hilbert space $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. If T is an operator on $\mathcal{H}$, then $T^{(n)}$ denotes the direct sum of n copies of T (regarded as an operator on $\mathcal{H}^{(n)}$). However, we will use I_n instead of $I^{(n)}$ to denote the identity operator on $\mathcal{H}^{(n)}$. If $\mathfrak{A}$ is a set of operators on $\mathcal{H}$, then $\mathfrak{A}^{(n)} = \{T^{(n)}\colon T \in \mathfrak{A}\}$. We will identify $\mathcal{B}(\mathcal{H}^{(n)})$ with $M_n(\mathbb{C}) \bar{\otimes} \mathcal{B}(\mathcal{H})$ by writing $T \in \mathcal{B}(\mathcal{H}^{(n)})$ as matrix form $(T_{ij})_{n\times n}$. With this identification, $\mathfrak{A}^{(n)} = \mathbb{C}I_n \bar{\otimes} \mathfrak{A}$.

Let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be a transitive algebra and $n \in \mathbb{N}$. Recall that $\mathfrak{A}$ is said to be *n-fold transitive* if for any linearly independent vectors $\xi_1, \xi_2, \ldots, \xi_n$ in $\mathcal{H}$, $[(S\xi_1, S\xi_2, \ldots, S\xi_n)\colon S \in \mathfrak{A}] = \mathcal{H}^{(n)}$. Note that if $\mathfrak{A}$ is n-fold transitive, then it is also m-fold transitive for each $m < n$. We consider $n = 2$, first. The following lemma is well known (cf. [1]). For the sake of completeness, we provide the proof.

Lemma 1.1. *Let $\mathfrak{A}$ be a transitive algebra on a Hilbert space $\mathcal{H}$. For any $\xi, \eta \in \mathcal{H}$, $\xi, \eta \neq 0$, either $[(T\xi, T\eta)\colon T \in \mathfrak{A}] = \mathcal{H}^{(2)}$ or there is a closed, densely defined operator S such that $ST = TS$ for every $T \in \mathfrak{A}$ and $\mathcal{G}(S)$, the graph of S, equals to $[(T\xi, T\eta)\colon T \in \mathfrak{A}]$. If $\mathcal{G}(S) = [(T\xi, T\eta)\colon T \in \mathfrak{A}]$, then S^{-1} (as a mapping) exists and $\mathcal{G}(S^{-1}) = [(T\eta, T\xi)\colon T \in \mathfrak{A}]$.*

Proof. We can assume that ξ, η are linearly independent. Suppose $\mathcal{G} = [(T\xi, T\eta)\colon T \in \mathfrak{A}] \neq \mathcal{H}^{(2)}$. If there is $z \neq 0$ such that $(0, z) \in \mathcal{G}$, then the closure of $\{(0, Tz)\colon T \in \mathfrak{A}\}$ is $0 \oplus \mathcal{H}$ since $\mathfrak{A}$ is transitive. Thus $0 \oplus \mathcal{H} \subseteq \mathcal{G}$ and $\mathcal{G} = \mathcal{H}^{(2)}$. It is a contradiction. So $(0, z) \in \mathcal{G}$ implies that $z = 0$. Define $ST\xi = T\eta$. Then S is well defined and $\mathcal{G}(S) = \mathcal{G}$. So S is a closed, densely defined operator. By symmetry of ξ and η, $S^{-1}T\eta = T\xi$ is a closed, densely defined operator such that $\mathcal{G}(S^{-1}) = [(T\eta, T\xi)\colon T \in \mathfrak{A}]$. $\square$

In the following proposition we summarize some characterizations of 2-fold transitivity. "$1 \Leftrightarrow 2$" is proved by Arveson in [1]. The authors cannot find the equivalence of 1, 3, 4 in the literature (for example [1,21]). For the sake of completeness, we provide the proof.

Proposition 1.2. *Let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be a transitive algebra. Then the following conditions are equivalent*:

1. *$\mathfrak{A}$ is 2-fold transitive.*
2. *If S is a closed, densely defined operator such that for any $T \in \mathfrak{A}$, $TS = ST$, i.e., for any $\xi \in \mathcal{D}(S)$, $TS\xi = ST\xi$, then $S = \lambda I$ for some scalar λ.*
3. *For any ξ, η and $\zeta \in \mathcal{H}$, $\xi \neq 0$, there exists a sequence T_n in $\mathfrak{A}$ such that $T_n\xi$ converges to ζ and $\sup_n \|T_n\eta\| < \infty$.*
4. *$\operatorname{Lat}\mathfrak{A}^{(2)} = \operatorname{Lat}\mathcal{B}(\mathcal{H})^{(2)}$, i.e., for any projection $P \in M_2(\mathbb{C}) \bar{\otimes} \mathcal{B}(\mathcal{H})$, if $P \in \operatorname{Lat}(\mathbb{C}I_2 \bar{\otimes} \mathfrak{A})$, then $P \in M_2(\mathbb{C}) \bar{\otimes} \mathbb{C}I$.*

Proof. "1 ⇒ 2." Firstly, we prove that for any $\xi \in \mathcal{D}(S)$, $\eta = S\xi$ is linearly dependent on ξ. Otherwise, assume that η, ξ are linearly independent. Since $\mathfrak{A}$ is 2-fold transitive, $\mathcal{H}^{(2)} = [(T\xi, T\eta)\colon T \in \mathfrak{A}] = [(T\xi, ST\xi)\colon T \in \mathfrak{A}] \subseteq \mathcal{G}(S)$. It is a contradiction. Suppose for ξ_1, ξ_2 in $\mathcal{D}(S)$, $S\xi_1 = \lambda_1\xi_1$, $S\xi_2 = \lambda_2\xi_2$ and $S(\xi_1 + \xi_2) = \lambda(\xi_1 + \xi_2)$. Then $\lambda_1 = \lambda_2 = \lambda$. This implies that $S = \lambda I$.

"2 ⇒ 1." Suppose for two linearly independent vectors ξ, η in $\mathcal{H}$, $\{(T\xi, T\eta)\colon T \in \mathfrak{A}\} \neq \mathcal{H}^{(2)}$. By Lemma 1.1, $\{(T\xi, T\eta)\colon T \in \mathfrak{A}\}$ is the graph of a closed, densely defined operator S, i.e., $ST\xi = T\eta = TS\xi$. By assumption, $S = \lambda I$. So $\eta = \lambda\xi$. It is a contradiction.

"1 ⇒ 3" is obvious. "3 ⇒ 1." Let ξ, η be linearly independent vectors. Suppose that $[(T\xi, T\eta)\colon T \in \mathfrak{A}] \neq \mathcal{H}^{(2)}$, by Lemma 1.1, there is a closed, densely defined operator S such that $\mathcal{G}(S) = [(T\xi, T\eta)\colon T \in \mathfrak{A}] = [(T\xi, ST\xi)\colon T \in \mathfrak{A}]$. For any $\zeta \in \mathcal{H}$, by assumption of 3, there exist a sequence T_n in $\mathfrak{A}$ such that $T_n\xi$ converges to ζ and $\sup_n \|T_n\eta\| < \infty$. We can assume that $T_n\eta$ weakly converges to z. By Mazur's theorem, there is a sequence S_n such that $S_n\eta$ is the convex combination of $T_n\eta$ and $S_n\eta$ strongly converges to z. Note that $S_n\xi$ strongly converges to ζ. Since $SS_n\xi = S_n\eta$ and S is a closed, densely defined operator, $S\zeta = z$. This implies that $\zeta \in \mathcal{D}(S)$. Since $\zeta \in \mathcal{H}$ is arbitrary, $\mathcal{D}(S) = \mathcal{H}$. By closed graph theorem, S is a bounded operator. By Lemma 1.1 and symmetry of ξ and η, S^{-1} is a bounded operator on $\mathcal{H}$. Thus S is a bounded, invertible operator with inverse S^{-1} also a bounded operator. For any $\lambda \in \mathbb{C}$, consider vectors $\xi, \lambda\xi + \eta$. Similar arguments show that $S + \lambda I$ is a bounded, invertible operator with inverse $(S + \lambda I)^{-1}$ also a bounded operator. So $\sigma(S) = \emptyset$. It is a contradiction.

"4 ⇒ 1." Otherwise, there exist linearly independent vectors ξ, η in $\mathcal{H}$ such that $\mathcal{G} = [(T\xi, T\eta)\colon T \in \mathfrak{A}] \neq \mathcal{H}^{(2)}$. Let P be the projection from $\mathcal{H}^{(2)}$ onto subspace $\mathcal{G}$. Then $0 < P < I$ and $P \in \operatorname{Lat}\mathfrak{A}^{(2)}$. By assumption, $P \in M_2(\mathbb{C}) \bar{\otimes} 1$. Since $P \neq I$, $\operatorname{rank} P = 1$. Therefore, there exists a unitary matrix $U \in M_2(\mathbb{C}) \bar{\otimes} \mathbb{C}I$ such that $Q = UPU^* = 0 \oplus I$. Let

$$\begin{pmatrix} \zeta \\ \omega \end{pmatrix} = U \begin{pmatrix} \xi \\ \eta \end{pmatrix}.$$

Then

$$Q\begin{pmatrix} \zeta \\ \omega \end{pmatrix} = UPU^*\begin{pmatrix} \zeta \\ \omega \end{pmatrix} = UP\begin{pmatrix} \xi \\ \eta \end{pmatrix} = U\begin{pmatrix} \xi \\ \eta \end{pmatrix} = \begin{pmatrix} \zeta \\ \omega \end{pmatrix},$$

which implies that $\zeta = 0$. Since ξ, η are linearly independent and U is a unitary operator, ζ, ω are linearly independent. In particular, $\zeta \neq 0$. It is a contradiction.

"1 ⇒ 4." Suppose $\mathfrak{A}$ is 2-fold transitive. Let $I_2 \neq P = (P_{ij}) \in M_2(\mathbb{C}) \bar{\otimes} \mathcal{B}(\mathcal{H})$ such that $P \in \operatorname{Lat}\mathfrak{A}^{(2)}$. It is easy to see

$$P\mathcal{H}^{(2)} = \bigvee\left[\mathfrak{A}^{(2)}\begin{pmatrix} \xi \\ \eta \end{pmatrix} : \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in P\mathcal{H}^{(2)}\right].$$

We only need to prove that if P is the projection onto space $\left[\mathfrak{A}^{(2)}\left(\begin{smallmatrix} \xi \\ \eta \end{smallmatrix}\right)\right]$ for some $\xi, \eta \in \mathcal{H}$, then $P \in M_2(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. If ξ, η are linearly independent, by assumption 1, $P = I_2$. If ξ, η are linearly dependent, then $\eta = \lambda\xi$ for some $\lambda \in \mathbb{C}$.

$$\left[\mathfrak{A}^{(2)}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right] = \left[\begin{pmatrix} T\xi \\ \lambda T\xi \end{pmatrix} : T \in \mathfrak{A}\right] = \left[\begin{pmatrix} \zeta \\ \lambda\zeta \end{pmatrix} : \zeta \in \mathcal{H}\right]$$

is the closure of the range of the operator

$$\begin{pmatrix} 1 & 0 \\ \lambda & 0 \end{pmatrix} \in M_2(\mathbb{C}) \bar{\otimes} \mathbb{C}I.$$

Therefore, $P \in M_2(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. □

By using induction on n and similar idea of proof of Proposition 1.2, we can prove the following proposition.

Proposition 1.3. *Let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be a transitive algebra and $n \in \mathbb{N}$. Then $\mathfrak{A}$ is n-fold transitive if and only if $\operatorname{Lat}\mathfrak{A}^{(n)} = \operatorname{Lat}\mathcal{B}(\mathcal{H})^{(n)}$, i.e., for any projection $P \in \operatorname{Lat}(\mathbb{C}I_n \bar{\otimes} \mathcal{L})$, $P \in M_n(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. Furthermore, the strong closure of $\mathfrak{A}$ is $\mathcal{B}(\mathcal{H})$ if and only if for any $n \in \mathbb{N}$, $\operatorname{Lat}\mathfrak{A}^{(n)} = \operatorname{Lat}\mathcal{B}(\mathcal{H})^{(n)}$.*

Compare with Arveson's characterizations of n-fold transitivity by graph transformations (see [1]), the characterization of n-fold transitivity given by Proposition 1.3 is more "computable."

2. From $(n-1)$-fold transitivity to n-fold transitivity

Suppose $\mathfrak{A}$ is a transitive algebra on $\mathcal{H}$ and $\mathfrak{A}$ is $(n-1)$-fold $(n \geqslant 2)$ transitive. By Proposition 1.3, $\mathfrak{A}$ is n-fold transitive if and only if for any projection $P \in \operatorname{Lat}(\mathbb{C}I_n \bar{\otimes} \mathfrak{A})$, $P \in M_n(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. It is interesting to know under what conditions a projection $P \in \operatorname{Lat}(\mathbb{C}I_n \bar{\otimes} \mathfrak{A})$ implies that $P \in M_n(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. In the following, we will provide certain conditions along this line.

Recall that for each bounded operator $T \in \mathcal{B}(\mathcal{H})$, we can associate two closed subspaces of $\mathcal{H}$, the *null* space $\{\xi \in \mathcal{H}\colon\ T\xi = 0\}$ and the *range* space, $[T(\mathcal{H})]$, which is the closure of the range $T(\mathcal{H}) = \{T\xi\colon\ \xi \in \mathcal{H}\}$. The corresponding projections are called the *null projection*, denoted by $N(T)$, and the *range projection*, $R(T)$, respectively.

Lemma 2.1. *Let $P = \left(\begin{smallmatrix} T_1 & S \\ S^* & T_2 \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ be a projection and $Z = \left(\begin{smallmatrix} X & 0 \\ 0 & Y \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. If $P \in \operatorname{Lat} Z$, then $R(T_1)$ and $N(I - T_1)$ are in $\operatorname{Lat} X$.*

Proof. Since P is a projection, we have

$$T_1(I - T_1) = SS^*, \tag{1}$$

which implies that $R(S) \subseteq R(T_1)$. If $P \in \operatorname{Lat} Z$, then we have equation

$$T_1 X T_1 + S Y S^* = X T_1. \tag{2}$$

Therefore, $X(R(T_1)) \subseteq R(T_1)$, which implies that $R(T_1) \in \operatorname{Lat} X$.

$\forall \xi \in N(I - T_1)$, by Eq. (1), $SS^*\xi = 0$, which implies that $S^*\xi = 0$. By Eq. (2), $T_1 X\xi = X\xi$, which implies that $X\xi \in N(I - T_1)$. So $N(I - T_1)$ is in $\operatorname{Lat} X$. □

Corollary 2.2. *Let $\mathfrak{A}$ be a transitive algebra. Given a projection $P=(P_{ij})_{n\times n}\in M_n(\mathbb{C})\bar{\otimes}\mathcal{B}(\mathcal{H})$, if $P\in \operatorname{Lat}\mathfrak{A}^{(n)}$, then $P_{ii}=0$ or $P_{ii}=I$ or $P_{ii}-P_{ii}^2$ is a one-to-one self-adjoint operator with dense range for $1\leqslant i\leqslant n$, i.e., $N(P_{ii}-P_{ii}^2)=0$ and $R(P_{ii}-P_{ii}^2)=I$.*

Proof. For $1\leqslant i\leqslant n$, if $0<P_{ii}<I$, then $R(P_{ii})\neq 0$ and $N(I-P_{ii})\neq I$. Since $\mathfrak{A}$ is a transitive algebra, by Lemma 2.1, $R(P_{ii})=I$ and $N(I-P_{ii})=0$. This implies that both P_{ii} and $I-P_{ii}$ are one-to-one operators. Therefore $P_{ii}(I-P_{ii})$ is a one-to-one self-adjoint operator. So $N(P_{ii}-P_{ii}^2)=0$ and $R(P_{ii}-P_{ii}^2)=I$. □

Lemma 2.3. *Let $\mathfrak{A}$ be a transitive algebra and $n\geqslant 2$ be a positive integer. Suppose $\mathfrak{A}$ is $(n-1)$-fold transitive and $P=(P_{ij})_{n\times n}\in M_n(\mathbb{C})\bar{\otimes}\mathcal{B}(\mathcal{H})$ is a projection in $\operatorname{Lat}\mathfrak{A}^{(n)}$. Write $P=\left(\begin{smallmatrix}T_1 & S\\ S^* & T_2\end{smallmatrix}\right)\in\mathcal{B}(\mathcal{H}^{(m)}\bigoplus\mathcal{H}^{(n-m)})$, where $1\leqslant m\leqslant n-1$. If $R(T_1)\neq I_m$ or $N(I_m-T_1)\neq 0_m$, then $P\in M_n(\mathbb{C})\bar{\otimes}\mathbb{C}I$.*

Proof. By Lemma 2.1, $R(T_1), N(I-T_1)\in\operatorname{Lat}\mathfrak{A}^{(m)}$. By assumption and Proposition 1.3, there exist projections $Q_1,Q_2\in M_m(\mathbb{C})\bar{\otimes}\mathbb{C}I$ such that $Q_1=R(T_1)$ and $Q_2=N(I_m-T_1)$.

If $R(T_1)\neq I_m$, rank $Q_1\leqslant m-1$. Since $Q_1\in M_m(\mathbb{C})\bar{\otimes}\mathbb{C}I$, there exists a unitary operator $U_1\in M_m(\mathbb{C})\bar{\otimes}\mathbb{C}I$ such that

$$U_1Q_1U_1^*=0_k\oplus I_{m-k},$$

where $1\leqslant k\leqslant m$. Let $W_1=U_1\oplus I_{n-m}$. Note that $W_1(I_n\otimes Z)W_1^*=(I_n\otimes Z)$, for any $Z\in\mathfrak{A}$. Therefore, $W_1PW_1^*\in\operatorname{Lat}\mathfrak{A}^{(n)}$. Since $P\in M_n(\mathbb{C})\bar{\otimes}\mathbb{C}I$ if and only if $W_1PW_1^*\in M_n(\mathbb{C})\bar{\otimes}\mathbb{C}I$, we can assume $R(T_1)=0_k\oplus I_{m-k}$. This implies that $P_{11}=0$ and therefore $P_{1i}=P_{i1}=0$ for all $1\leqslant i\leqslant n$. So

$$P=\begin{pmatrix}0 & 0\\ 0 & P_1\end{pmatrix}\in\operatorname{Lat}\mathfrak{A}^{(n)}$$

and therefore $P_1\in\operatorname{Lat}\mathfrak{A}^{(n-1)}$. By assumption, $P_1\in M_{n-1}(\mathbb{C})\bar{\otimes}\mathbb{C}I$ and therefore $P\in M_n(\mathbb{C})\bar{\otimes}\mathbb{C}I$.

If $N(I_m-T_1)\neq 0_m$, rank $Q_2\geqslant 1$. Since $Q_2\in M_m(\mathbb{C})\bar{\otimes}\mathbb{C}I$, there exists a unitary operator $U_2\in M_m(\mathbb{C})\bar{\otimes}\mathbb{C}I$ such that

$$U_2Q_2U_2^*=I_{k'}\oplus 0_{m-k'},$$

where $k'\geqslant 1$. Let $W_2=U_2\oplus I_{n-m}$. Note that $W_2(I_n\otimes Z)W_2^*=(I_n\otimes Z)$, for any $Z\in\mathfrak{A}$. Therefore, $W_2PW_2^*\in\operatorname{Lat}\mathfrak{A}^{(n)}$. Since $P\in M_n(\mathbb{C})\bar{\otimes}\mathbb{C}I$ if and only if $W_2PW_2^*\in M_n(\mathbb{C})\bar{\otimes}\mathbb{C}I$, we can assume $N(I_{(m)}-T_1)=I_{k'}\bigoplus 0_{m-k'}$. This implies that $P_{11}=I$ and therefore $P_{1i}=P_{i1}=0$ for all $1\leqslant i\leqslant n$. So

$$P=\begin{pmatrix}I & 0\\ 0 & P_2\end{pmatrix}\in\operatorname{Lat}\mathfrak{A}^{(n)}$$

and therefore $P_2\in\operatorname{Lat}\mathfrak{A}^{(n-1)}$. By assumption, $P_2\in M_{n-1}(\mathbb{C})\bar{\otimes}1$ and therefore $P\in M_n(\mathbb{C})\bar{\otimes}\mathbb{C}I$. □

Lemma 2.4. *Let $\mathfrak{A}$ be a transitive algebra and $n \geqslant 3$ be a positive integer. Suppose $\mathfrak{A}$ is $(n-1)$-fold transitive and $P = (P_{ij})_{n\times n} \in M_n(\mathbb{C}) \bar{\otimes} \mathcal{B}(\mathcal{H})$ is a projection in $\operatorname{Lat}\mathfrak{A}^{(n)}$. Write $P = \left(\begin{smallmatrix} T_1 & S \\ S^* & T_2 \end{smallmatrix}\right) \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}^{(n-1)})$. If the range of T_1 is $\mathcal{H}$, then $P \in M_n(\mathbb{C}) \bar{\otimes} \mathbb{C}I$.*

Proof. Assume $P \notin M_n(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. By Corollary 2.2 and Lemma 2.3, both $T_1 - T_1^2$ and $T_2 - T_2^2$ are one-to-one operators with dense ranges. By assumption, T_1 is an invertible operator and therefore $T_1^{-1} \in \mathcal{B}(\mathcal{H})$. Since P is a projection, we have $T_1 - T_1^2 = SS^*$ and $T_2 - T_2^2 = S^*S$. Let $S = U|S|$ be the polar decomposition. Then $\mathcal{H} \oplus 0^{(n-1)} = R(T_1 - T_1^2) = R(S) = UU^*$ and $0 \oplus \mathcal{H}^{(n-1)} = R(T_2 - T_2^2) = R(S^*) = U^*U$. Hence, $U = (U_1, \ldots, U_{n-1})$ is a unitary operator from $0 \oplus \mathcal{H}^{(n-1)}$ onto $\mathcal{H} \oplus 0^{(n-1)}$ such that $R(T_1 - T_1^2) = UU^*$ and $R(T_2 - T_2^2) = U^*U$. Let $H = \sqrt{T_1 - T_1^2}$. It is easy to see

$$
\begin{aligned}
P &= \begin{pmatrix} T_1 & HU_1 & \cdots & HU_{n-1} \\ U_1^*H & U_1^*(I-T_1)U_1 & \cdots & U_1^*(I-T_1)U_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ U_{n-1}^*H & U_{n-1}^*(I-T_1)U_2 & \cdots & U_{n-1}^*(I-T_1)U_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} I & & & \\ & U_1^* & & \\ & & \ddots & \\ & & & U_{n-1}^* \end{pmatrix} \begin{pmatrix} T_1 & H & \cdots & H \\ H & I-T_1 & \cdots & I-T_1 \\ \vdots & \vdots & \ddots & \vdots \\ H & I-T_1 & \cdots & I-T_1 \end{pmatrix} \begin{pmatrix} I & & & \\ & U_1 & & \\ & & \ddots & \\ & & & U_{n-1} \end{pmatrix}.
\end{aligned}
$$

For every $0 \neq \xi \in \mathcal{H}$, we have

$$
\begin{pmatrix} T_1 & H \\ H & I-T_1 \end{pmatrix} \begin{pmatrix} -T_1^{-1}HU_1\xi \\ U_1\xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
$$

Therefore,

$$
\begin{pmatrix} T_1 & HU_1 \\ U_1^*H & U_1^*(I-T_1)U_1 \end{pmatrix} \begin{pmatrix} -U_1^*T_1^{-1}HU_1\xi \\ \xi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},
$$

which implies that $N\left(\begin{smallmatrix} T_1 & HU_1 \\ U_1^*H & U_1^*(I-T_1)U_1 \end{smallmatrix}\right) \neq 0_2$ and therefore $R\left(\begin{smallmatrix} T_1 & HU_1 \\ U_1^*H & U_1^*(I-T_1)U_1 \end{smallmatrix}\right) \neq I_2$. By Lemma 2.3, $P \in M_n(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. It is a contradiction. □

As a corollary of Proposition 1.3, Lemmas 2.3 and 2.4, we obtain Foias' result [8, Proposition 5].

Corollary 2.5 *(Foias' Theorem). Let $\mathfrak{A}$ be a transitive algebra on a Hilbert space $\mathcal{H}$. If $\mathfrak{A}$ has no non-trivial invariant operator ranges, then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$.*

3. On transitive algebras containing a standard finite von Neumann subalgebra

Theorem 3.1. *Let $\mathcal{M}$ be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $\mathfrak{A}$ be a transitive algebra which contains $\mathcal{M}'$. If $\mathfrak{A}$ is 2-fold transitive, then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$.*

Proof. We need to prove that $\mathfrak{A}$ is n-fold transitive for $n \in \mathbb{N}$. We use induction on n. Suppose $\mathfrak{A}$ is n-fold transitive for $n \geqslant 2$ but $\mathfrak{A}$ is not $(n+1)$-fold transitive. By Proposition 1.3, let $P = (P_{ij}) \in M_{n+1}(\mathbb{C}) \bar{\otimes} \mathcal{B}(\mathcal{H})$ be a projection such that $P \in \operatorname{Lat} \mathfrak{A}^{(n+1)}$ but $P \notin M_{n+1}(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. Note that $\mathcal{M}' \subseteq \mathfrak{A}$, $P \in \operatorname{Lat} \mathcal{M}'^{(n+1)}$. Since $\mathcal{M}'$ is a von Neumann algebra, $P \in M_{n+1}(\mathbb{C}) \bar{\otimes} \mathcal{M}$. Write $P = \begin{pmatrix} T_1 & S \\ S^* & T_2 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H}^{(n)})$. By Corollary 2.2 and Lemma 2.3, $R(T_1 - T_1^2) = \mathcal{H} \oplus 0^{(n)}$ and $R(T_2 - T_2^2) = 0 \oplus \mathcal{H}^{(n)}$. Since P is a projection, we have $T_1(I - T_1) = SS^*$ and $T_2(I - T_2) = S^*S$. By polar decomposition, there is a unitary operator $U = (U_2, \ldots, U_{n+1})$ from $0 \oplus \mathcal{H}^{(n)}$ onto $\mathcal{H} \oplus 0^{(n)}$ such that

$$UU^* = I \oplus 0_n, \tag{3}$$

$$U^*U = 0 \oplus I_n. \tag{4}$$

By Eqs. (3) and (4), $I \oplus 0_n$ and $0 \oplus I_n$ are equivalent in $M_{n+1}(\mathbb{C}) \bar{\otimes} \mathcal{M}$. Since $M_{n+1}(\mathbb{C}) \bar{\otimes} \mathcal{M}$ is a finite von Neumann algebra, it is a contradiction. □

Corollary 3.2. *Let $\mathcal{M}$ be a standard finite von Neumann algebra on a Hilbert space $\mathcal{H}$ and $\mathfrak{A}$ be a transitive algebra which contains $\mathcal{M}$. If $\mathfrak{A}$ is 2-fold transitive, then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$.*

Corollary 3.3 *(Arveson's Theorem). Let $\mathfrak{A} \subseteq \mathcal{B}(\mathcal{H})$ be a transitive algebra which contains a MASA of $\mathcal{B}(\mathcal{H})$. Then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(\mathcal{H})$.*

Proof. By Theorem 3.1, we need to prove that $\mathfrak{A}$ is 2-fold transitive. Assume $\mathfrak{A}$ is not 2-fold transitive. By Proposition 1.2, there is a non-scalar, closed densely defined operator S such that $TS = ST$ for every $T \in \mathfrak{A}$. Since $\mathfrak{A}$ contains a MASA $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, S is affiliated with $\mathcal{A}$. So S is a (unbounded) normal operator. Since S is not a scalar, there is a non-trivial spectral projection E of S. By Fuglede's theorem [9], $ET = TE$ for all $T \in \mathfrak{A}$. Since $\mathfrak{A}$ is a transitive algebra, it is a contradiction. □

Corollary 3.4. *Let $\mathcal{M}$ be a type II_1 factor with a faithful normal trace τ. If $\mathfrak{A}$ is a transitive subalgebra of $\mathcal{B}(L^2(\mathcal{M}, \tau))$ which contains $\mathcal{M}'$ and a MASA of $\mathcal{M}$, then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.*

Proof. Suppose $\mathcal{A}$ is a MASA in $\mathcal{M}$ and $\mathfrak{A}$ is a transitive algebra which contains $\mathcal{A}$ and $\mathcal{M}'$. By Theorem 3.1, we need to prove that $\mathfrak{A}$ is 2-fold transitive. Let $\mathcal{L}$ be the von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{M}'$. Then $\mathcal{L}' = \mathcal{A}' \cap \mathcal{M} = \mathcal{A}$. Let S be a closed, densely defined operator such that $ST = TS$ for every $T \in \mathfrak{A}$. Then S is affiliated with $\mathcal{A}$. Similar arguments as the proof of Corollary 3.3 show that $S = \lambda I$ for some λ. By Proposition 1.2, $\mathfrak{A}$ is 2-fold transitive. □

Remark. Let $\mathcal{M}$ be a type II_1 factor and $\mathcal{A}$ be a MASA in $\mathcal{M}$. Let $\mathcal{L}$ be the von Neumann subalgebra of $\mathcal{B}(L^2(\mathcal{M}, \tau))$ generalized by $\mathcal{A}$ and $\mathcal{M}'$. Does $\mathcal{L}$ contains a MASA of $\mathcal{B}(L^2(\mathcal{M}, \tau))$? It seems that $\mathcal{M} = \mathcal{L}\mathbb{F}_2$ and $\mathcal{A}$ be the MASA generated by one generator is a counterexample.

Combine Corollary 3.2 and Proposition 1.2, we have the following corollary.

Corollary 3.5. *Let $\mathcal{M}$ be a type II_1 factor with a faithful normal trace τ. Then the following conditions are equivalent:*

1. *For every non-scalar, closed, densely defined operator S affiliated with $\mathcal{M}$, $\{S\}' \cap \mathcal{B}(L^2(\mathcal{M},\tau))$ has a non-trivial invariant subspace (affiliated with $\mathcal{M}$).*
2. *Every transitive algebra on $L^2(\mathcal{M},\tau)$ containing $\mathcal{M}$ is strongly dense in $\mathcal{B}(L^2(\mathcal{M},\tau))$.*

Lemma 3.6. *Let $\mathcal{M}$ be a finite von Neumann algebra and $T \in \mathcal{M}$ be a normal operator, i.e., $TT^* = T^*T$. If $E \in \operatorname{Lat} T \cap \mathcal{M}$, then $ET = TE$. In particular, $E \in \operatorname{Lat} T^*$.*

Proof. Let $E \in \operatorname{Lat} T$. With respect to $E, I - E$, we can write $T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$. Now we have the following:

$$TT^* = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix} = \begin{pmatrix} AA^* + BB^* & BC^* \\ CB^* & CC^* \end{pmatrix},$$
$$T^*T = \begin{pmatrix} A^* & 0 \\ B^* & C^* \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} A^*A & A^*B \\ B^*A & B^*B + C^*C \end{pmatrix}.$$

By $TT^* = T^*T$, $AA^* + BB^* = A^*A$. Let τ be the unique center-valued trace on $\mathcal{M}$. Apply trace τ_E induced by τ on $E\mathcal{M}E$, we get $\tau_E(BB^*) = 0$. This implies that $B = 0$ and $ET = TE$. □

Corollary 3.7. *Let $\mathcal{M}$ be a finite von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and $T \in \mathcal{M}$ be a normal operator. If $\mathfrak{A}$ is an algebra (not necessarily transitive) containing $\mathcal{M}'$ and $T \in \mathfrak{A}$, then T^* is in the strong closure of $\mathfrak{A}$.*

Proof. For $n \in \mathbb{N}$, let $E \in \operatorname{Lat} \mathfrak{A}^{(n)}$. Then $E \in \operatorname{Lat} \mathcal{M}'^{(n)}$ and $E \in \operatorname{Lat} T^{(n)}$. Hence $E \in \operatorname{Lat} T^{(n)} \cap (M_n(\mathbb{C}) \bar{\otimes} \mathcal{M})$. Since $T^{(n)}$ is a normal operator in $M_n(\mathbb{C}) \bar{\otimes} \mathcal{M}$ and $M_n(\mathbb{C}) \bar{\otimes} \mathcal{M}$ is finite, $E \in \operatorname{Lat} T^{*(n)}$ by Lemma 3.6. By [21, Theorem 7.1], T^* is in the strong closure of $\mathfrak{A}$. □

The following question asked by Radjavi and Rosenthal [21] remains open: if $\mathfrak{A}$ is a transitive algebra generated by normal operators, is $\mathfrak{A}$ strongly dense in $\mathcal{B}(\mathcal{H})$? But we have the following result.

Corollary 3.8. *Let $\mathcal{M}$ be a type II_1 factor with a faithful normal trace τ. If $\mathfrak{A}$ is a transitive subalgebra of $\mathcal{B}(L^2(\mathcal{M},\tau))$ generated by normal operators in $\mathcal{M}$ and by $\mathcal{M}'$, then $\mathfrak{A}$ is strongly dense in $\mathcal{B}(L^2(\mathcal{M},\tau))$.*

Proof. Suppose $\mathfrak{A}$ is generated by $\{T_\alpha\} \subseteq \mathcal{M}$ and $\mathcal{M}'$, where $T_\alpha T_\alpha^* = T_\alpha^* T_\alpha$. By Corollary 3.7, the strong closure of $\mathfrak{A}$ contains the von Neumann algebra $\mathcal{L}$ generated by $\{T_\alpha\} \subseteq \mathcal{M}$ and $\mathcal{M}'$. Since $\mathcal{L}$ is transitive, $\mathcal{L} = \mathcal{B}(L^2(\mathcal{M},\tau))$. □

4. Non-selfadjoint algebras related to free products of finite von Neumann algebras

The main results of this section are Theorem 4.1 and Corollary 4.2. Examples (Example 4.9) related to free group factors are given at the end of this section.

Theorem 4.1. *Let $(\mathcal{M},\tau)$ be a finite von Neumann algebra with a faithful normal trace τ. Suppose $\mathcal{N} \subseteq \mathcal{M}$ is a von Neumann subalgebra and $Z \in \mathcal{M}$ satisfies the following conditions:*

1. *$Z \neq 0$ and $\tau(Z) = 0$.*
2. *$Z, Z((\mathcal{N}' \cap \mathcal{M}) \ominus \mathbb{C}1), ((\mathcal{N}' \cap \mathcal{M}) \ominus \mathbb{C}1)Z$ are mutually orthogonal in $L^2(\mathcal{M}, \tau)$.*

Let $\mathfrak{A} \subseteq \mathcal{B}(L^2(\mathcal{M}, \tau))$ be the algebra generated by $Z, \mathcal{N}$ and $\mathcal{M}'$. Then $\mathfrak{A}$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.

Corollary 4.2. *Let $(\mathcal{M}_1, \tau_1), (\mathcal{M}_2, \tau_2)$ be finite von Neumann algebras and $\mathcal{M} = (\mathcal{M}_1, \tau_1) * (\mathcal{M}_2, \tau_2)$ be the reduced free product von Neumann algebra and τ be the induced faithful normal trace on $\mathcal{M}$. Suppose $\mathcal{N} \subseteq \mathcal{M}_1$ is a diffuse von Neumann subalgebra and $Z \in \mathcal{M}_2$ is not a scalar. In $\mathcal{B}(L^2(\mathcal{M}, \tau))$, let $\mathfrak{A}$ be the algebra generated by Z, $\mathcal{N}$ and $\mathcal{M}'$. Then $\mathfrak{A}$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.*

To prove Theorem 4.1, we need to prove the following auxiliary lemmas.

Lemma 4.3. *Let $(\mathcal{M}, \tau)$ be a finite von Neumann algebra with a faithful normal trace τ and $\mathcal{N} \subseteq \mathcal{M}$ be a von Neumann subalgebra. Suppose $Z \in \mathcal{M}$ such that $\tau(Z) = 0$ and $Z, Z((\mathcal{N}' \cap \mathcal{M}) \ominus \mathbb{C}1), ((\mathcal{N}' \cap \mathcal{M}) \ominus \mathbb{C}1)Z$ are mutually orthogonal in $L^2(\mathcal{M}, \tau)$, i.e., $\tau(ZXZ^*) = \tau(Z^*XZ) = \tau(Z^*XZY) = 0$ for any $X, Y \in \mathcal{N}' \cap \mathcal{M}$ and $\tau(X) = \tau(Y) = 0$. Then $\forall A, B, C, D \in \mathcal{N}' \cap \mathcal{M}$, we have*

$$\tau\big((AZB)^*(CZD)\big) = \tau\big(A^*C\big)\tau\big(B^*D\big)\tau\big(Z^*Z\big). \tag{5}$$

Proof. Write $A^*C = X + \tau(A^*C)1$ and $D^*B = Y + \tau(D^*B)1$, where $X, Y \in \mathcal{N}' \cap \mathcal{M}$ and $\tau(X) = \tau(Y) = 0$. Then

$$\begin{aligned}
&\tau\big((AZB)^*(CZD)\big) \\
&\quad = \tau\big(B^*Z^*A^*CZD\big) \\
&\quad = \tau\big(Z^*\big(A^*C\big)Z\big(DB^*\big)\big) \\
&\quad = \tau\big(Z^*\big(X + \tau\big(A^*C\big)1\big)Z\big(Y + \tau\big(DB^*\big)1\big)\big) \\
&\quad = \tau\big(Z^*XZY\big) + \tau\big(Z^*XZ\big)\tau\big(DB^*\big) + \tau\big(A^*C\big)\tau\big(ZYZ^*\big) + \tau\big(A^*C\big)\tau\big(Z^*Z\big)\tau\big(DB^*\big) \\
&\quad = \tau\big(A^*C\big)\tau\big(Z^*Z\big)\tau\big(B^*D\big). \qquad \square
\end{aligned}$$

Corollary 4.4. *With same assumption of Lemma 4.3, $\forall A, B \in \mathcal{N}' \cap \mathcal{M}$, $AZB = 0$ implies that $A = 0$ or $B = 0$ or $Z = 0$.*

Corollary 4.5. *With same assumption of Lemma 4.3 and assume $Z \neq 0$. Then $\forall A, B, C, D \in \mathcal{N}' \cap \mathcal{M}$, $\tau((AZB)^*(CZD)) = 0$ if and only if $\tau(A^*C) = 0$ or $\tau(B^*D) = 0$.*

Lemma 4.6. *With same assumption of Lemma 4.3 and assume $Z \neq 0$. For any positive integer n, let $P = (P_{ij})_{2\times 2} \in M_2(\mathbb{C}) \mathbin{\bar{\otimes}} (\mathcal{N}' \cap \mathcal{M})$ be a projection. If $P \in \mathrm{Lat}(I_2 \mathbin{\bar{\otimes}} Z)$, then $P \in M_2(\mathbb{C}) \mathbin{\bar{\otimes}} \mathbb{C}I$.*

Proof. $P(I_2 \mathbin{\bar{\otimes}} Z)P = (I_2 \mathbin{\bar{\otimes}} Z)P$ implies the following equation

$$P_{11}ZP_{11} + P_{12}ZP_{21} = ZP_{11}. \tag{6}$$

For P_{11}, P_{12}, by the Gram–Schmidt's orthogonal process (with respect to τ), we have

$$P_{11} = x_1, \tag{7}$$

$$P_{12} = \alpha_{12}x_1 + x_2, \tag{8}$$

where $x_1, x_2 \in \mathcal{N}' \cap \mathcal{M}$ are orthogonal in $L^2(\mathcal{M}, \tau)$, i.e., $\tau(x_2^* x_1) = 0$ and $\alpha_{12} \in \mathbb{C}$. By taking $*$ of Eqs. (7), (8), we have

$$P_{11} = x_1^*, \tag{9}$$

$$P_{21} = \bar{\alpha}_{12}x_1^* + x_2^*, \tag{10}$$

where $x_1^*, x_2^* \in \mathcal{N}' \cap \mathcal{M}$ are orthogonal in $L^2(\mathcal{M}, \tau)$. Plug Eqs. (7)–(10) in Eq. (6), we have

$$\bigl(1 + |\alpha_{12}|^2\bigr)x_1 Z x_1^* + x_2 Z x_2^* + \alpha_{12}x_1 Z x_2^* + \bar{\alpha}_{12}x_2 Z x_1^* = Z x_1^*. \tag{11}$$

By Corollary 4.5, $x_2 Z x_2^*$ is perpendicular with $x_1 Z x_1^*$, $x_1 Z x_2^*$, $x_2 Z x_1^*$ and $Z x_1^*$. Therefore, $x_2 Z x_2^* = 0$. By Corollary 4.4, we have $x_2 = 0$. So Eq. (11) implies that

$$\bigl(1 + |\alpha_{12}|^2\bigr)x_1 Z x_1^* = Z x_1^*.$$

By Corollary 4.4, either $x_1 = 0 \in \mathbb{C}$ or $x_1 = \frac{1}{1+|\alpha_{12}|^2} \in \mathbb{C}$. Therefore $P_{11} = x_1 \in \mathbb{C}$, $P_{12} = \alpha_{12}x_1$ and $P_{21} = P_{12}^* \in \mathbb{C}$. By symmetry, we have $P_{22} \in \mathbb{C}$, which implies that $P \in M_2(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. □

Proof of Theorem 4.1. Let $\mathcal{L}$ be the von Neumann algebra generated by $\mathcal{N}$ and $\mathcal{M}'$ and $P \in \mathcal{B}(\mathcal{H})$ be a projection such that $P \in \operatorname{Lat}\mathfrak{A}$. Then $P \in \mathcal{N}' \cap \mathcal{M}$. Since $P \in \operatorname{Lat} Z$, we have $(I - P)ZP = 0$. By assumptions of Theorem 4.1 and Corollary 4.4, $P = I$ or $P = 0$, which implies that $\mathfrak{A}$ is a transitive algebra. If $Q \in M_2(\mathbb{C}) \bar{\otimes} \mathcal{B}(L^2(\mathcal{M}, \tau))$ is a projection such that $Q \in \operatorname{Lat}(I_2 \bar{\otimes} \mathfrak{A})$, then $Q \in \operatorname{Lat}(I_2 \bar{\otimes} \mathcal{L})$. Since $\mathcal{L}$ is a von Neumann algebra, $Q \in (I_2 \bar{\otimes} \mathcal{L})' = M_2(\mathbb{C}) \bar{\otimes} \mathcal{L}' = M_2(\mathbb{C}) \bar{\otimes} (\mathcal{N}' \cap \mathcal{M})$. Note $Q \in \operatorname{Lat}(I_2 \bar{\otimes} Z)$. By Lemma 4.6, $Q \in M_2(\mathbb{C}) \bar{\otimes} \mathbb{C}I$. By Proposition 1.2, $\mathfrak{A}$ is 2-fold transitive. By Theorem 3.1, $\mathfrak{A}$ is strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. □

The following lemma is proved by Popa in [20].

Lemma 4.7. *Let $\mathcal{B}$ be a diffuse von Neumann subalgebra of a finite von Neumann algebra $\mathcal{M}$ (with a faithful normal trace τ) and U be a unitary operator in $\mathcal{M}$. If $U\mathcal{B}U^*$ and $\mathcal{B}$ are orthogonal with respect to τ, then U is orthogonal to $\{V \in \mathcal{M}\colon V\mathcal{B}V^* = \mathcal{B}, VV^* = V^*V = I\}$, the set of normalizers of $\mathcal{B}$ in $\mathcal{M}$. In particular, U is orthogonal to $\mathcal{B}' \cap \mathcal{M}$.*

The following result is well known. For the sake of completeness, we include a proof in the following. Our proof follows Popa's idea in [20].

Lemma 4.8. *Let $(\mathcal{M}_1, \tau_1)$, $(\mathcal{M}_2, \tau_2)$ be finite von Neumann algebras and $\mathcal{M} = (\mathcal{M}_1, \tau_1) * (\mathcal{M}_2, \tau_2)$ be the reduced free product von Neumann algebra and τ be the induced faithful normal trace on $\mathcal{M}$. Suppose $\mathcal{N}$ is a diffuse von Neumann subalgebra of $\mathcal{M}_1$, then $\mathcal{N}' \cap \mathcal{M} = \mathcal{N}' \cap \mathcal{M}_1$.*

Proof. We only need to prove $\mathcal{N}' \cap \mathcal{M} \subseteq \mathcal{N}' \cap \mathcal{M}_1$. Let $\dot{\mathcal{M}}_1, \dot{\mathcal{M}}_2$ be the set $\{T \in \mathcal{M}_1\colon \tau_1(T) = 0\}$ and $\{T \in \mathcal{M}_2\colon \tau_2(T) = 0\}$, respectively. Note that $L^2(\mathcal{M}, \tau)$ is the closure of $\mathbb{C}I \oplus \dot{\mathcal{M}}_1 \oplus \dot{\mathcal{M}}_2 \oplus \dot{\mathcal{M}}_1 \otimes \dot{\mathcal{M}}_2 \oplus \dot{\mathcal{M}}_2 \otimes \dot{\mathcal{M}}_1 \oplus \cdots$. To prove $\mathcal{N}' \cap \mathcal{M} \subseteq \mathcal{N}' \cap \mathcal{M}_1$, we only need to prove $\dot{\mathcal{M}}_2, \dot{\mathcal{M}}_1 \otimes \dot{\mathcal{M}}_2, \dot{\mathcal{M}}_2 \otimes \dot{\mathcal{M}}_1, \ldots$ are orthogonal to $\mathcal{N}' \cap \mathcal{M}_1$. Note that $\dot{\mathcal{M}}_1$ and $\dot{\mathcal{M}}_2$ are the closure (with respect to strong operator topology) of linear span of $\{U \in \mathcal{M}_1\colon \tau_1(U) = 0, UU^* = U^*U = I\}$ and $\{V \in \mathcal{M}_2\colon \tau_2(V) = 0,\ VV^* = V^*V = I\}$, respectively. We need to prove the following words: $U_1V_1 \ldots U_nV_n$, $U_1V_1 \ldots U_n$ $(n \geqslant 2)$, $V_1U_1V_2U_2 \ldots$, are orthogonal to $\mathcal{N}' \cap \mathcal{M}_1$, where $U_1, U_2, \ldots$ and $V_1, V_2, \ldots$ are unitary operators in $\dot{\mathcal{M}}_1$ and $\dot{\mathcal{M}}_2$, respectively. Note that $U_1V_1 \ldots U_nV_n\dot{\mathcal{M}}_1V_n^*U_n^* \ldots V_1^*U_1^*$ and $\mathcal{M}_1$ are orthogonal in $L^2(\mathcal{M}, \tau)$. Von Neumann algebra $U_1V_1 \ldots U_nV_n\mathcal{N}V_n^*U_n^* \ldots V_1^*U_1^*$ and $\mathcal{N}$ are orthogonal with respect to τ. By Lemma 4.7, $U_1V_1 \ldots U_nV_n$ is orthogonal to $\mathcal{N}' \cap \mathcal{M}$. Similarly, we can prove other cases. □

Proof of Corollary 4.2. We can assume $Z \neq 0$ and $\tau(Z) = 0$ (otherwise, consider $Z - \tau(Z)$). By Lemma 4.8, $\mathcal{N}' \cap \mathcal{M} = \mathcal{N}' \cap \mathcal{M}_1$ and therefore Z is free with $\mathcal{N}' \cap \mathcal{M}$. So $Z, Z((\mathcal{N}' \cap \mathcal{M}) \ominus \mathbb{C}1), ((\mathcal{N}' \cap \mathcal{M}) \ominus \mathbb{C}1)Z$ are mutually orthogonal in $L^2(\mathcal{M}, \tau)$. By Theorem 4.1, $\mathfrak{A}$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. □

Example 4.9. Let $\mathcal{M} = \mathcal{L}\mathbb{F}_n$ $(2 \leqslant n \leqslant \infty)$ be the type II_1 factor associated with the left regular representation λ of the free group $\mathbb{F}_n$ on n generators and a, b be two generators of $\mathbb{F}_n$. Let k be a positive integer and $Z \in \{\lambda(a), \lambda(a)^*\}''$ be a non-scalar operator. By Corollaries 3.7 and 4.2, the algebra $\mathfrak{A}$ on $L^2(\mathcal{M}, \tau)$ generated by $Z, \lambda(b)^k$, and $\mathcal{M}'$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.

5. Non-selfadjoint algebras related to $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$

Let $\mathcal{M} = (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$ be the reduced free product von Neumann algebra with respect to the normalized trace on $M_2(\mathbb{C})$. Then $\mathcal{M}$ is a type II_1 factor with a faithful normal trace τ. Let $(E_{ij})_{i,j=1,2}$ and $(F_{ij})_{i,j=1,2}$ be the matrix units in $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $1 * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$, respectively. Consider the following subalgebras of $\mathcal{B}(L^2(\mathcal{M}, \tau))$:

1. $\mathfrak{A}_1$, the algebra generated by E_{11}, E_{12}, F_{11} and $\mathcal{M}'$.
2. $\mathfrak{A}_2$, the algebra generated by E_{11}, E_{12}, F_{12} and $\mathcal{M}'$.
3. $\mathfrak{A}_3$, the algebra generated by E_{11}, F_{12} and $\mathcal{M}'$.
4. $\mathfrak{A}_4$, the algebra generated by E_{12}, F_{12} and $\mathcal{M}'$.

We have the following questions: for $1 \leqslant i \leqslant 4$, is $\mathfrak{A}_i$ a transitive algebra? If $\mathfrak{A}_i$ is a transitive algebra, is $\mathfrak{A}_i$ strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$? In this section we will prove the following results.

1. $\mathfrak{A}_1$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.
2. $\mathfrak{A}_3, \mathfrak{A}_4$ are not transitive. The invariant subspaces of $\mathfrak{A}_3$ and $\mathfrak{A}_4$ are abundant.

The question on $\mathfrak{A}_2$ remains open!

To prove our main results, we need to introduce the following operators. In $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1$, let

$$W_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad W_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad W_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad W_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have

1. $W_1^2 = W_2^2 = W_3^2 = W_0$.
2. $W_1W_2 = -W_2W_1 = -W_3$, $W_1W_3 = -W_3W_1 = -W_2$, $W_2W_3 = -W_3W_2 = -W_1$.
3. W_0, W_1, W_2, W_3 form an orthonormal base of $L^2((M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1, \tau)$.

Similarly, in $1 * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$, let

$$V_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad V_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad V_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad V_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have similar equations and $\mathcal{M}$ is the von Neumann algebra generated by $\{W_0, W_1, W_2, W_3\} * \{V_0, V_1, V_2, V_3\}$.

5.1. The case of $\mathfrak{A}_1$

Let $W = W_1$, $V = V_1$ and $U = WV$. Then U is a Haar unitary operator in $\mathcal{M}$. Let $\mathcal{A}$ be the von Neumann algebra generated by U and U^*. The following lemma is an easy exercise.

Lemma 5.1. *$\mathcal{A}$ is a MASA of $\mathcal{M}$.*

Theorem 5.2. *Let $(\mathcal{M}, \tau)$ and $\mathcal{A}$ be as above. For $Z \in (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1$, let $\mathfrak{A}$ be the algebra generated by $Z, \mathcal{A}$ and $\mathcal{M}'$ in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. Then $\mathfrak{A}$ is a transitive algebra if and only if $Z \notin \{W\}''$. In this case, $\mathfrak{A}$ is strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.*

Corollary 5.3. *$\mathfrak{A}_1$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.*

We need the following lemmas.

Lemma 5.4. *With the same assumption of Theorem 5.2 and assume $Z = \lambda W$ for some $\lambda \in \mathbb{C}$. Then $\mathfrak{A}$ is not transitive.*

Proof. We can assume $Z = W$. Then $\mathfrak{A}$ is a von Neumann algebra. Note

$$\big(U + U^*\big)W = (WV + VW)W = WVW + V = WVW + W^2V = W\big(U + U^*\big).$$

So $U + U^* \in \mathfrak{A}'$, which implies that $\mathfrak{A}$ is not transitive. □

Lemma 5.5. *With the same assumption of Theorem 5.2 and assume $Z \neq 0$ and $\tau(Z) = \tau(Z^*W) = 0$. Then $\mathfrak{A}$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$.*

Proof. Note $\tau(Z^*V) = 0$ and $W^* = W$. It is easy to verify

$$\tau\big(Z(WV)^nZ^*\big) = \tau\big(Z^*(WV)^nZ\big) = \tau\big((WV)^nZ(WV)^mZ^*\big) = 0$$

for all $m, n \in \mathbf{Z} \setminus \{0\}$. Since $\mathcal{A} \ominus \mathbb{C}I$ is generated by $(WV)^n$ for all $n \in \mathbf{Z}$ and $n \neq 0$, Z, $Z(\mathcal{A} \ominus \mathbb{C}1)$, $(\mathcal{A} \ominus \mathbb{C}1)Z$ are mutually orthogonal in $L^2(\mathcal{M}, \tau)$. Since $\mathcal{A}$ is a MASA of $\mathcal{M}$, $\mathcal{A}' \cap \mathcal{M} = \mathcal{A}$. By Theorem 4.1, $\mathfrak{A}$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. □

Lemma 5.6. *With the same assumption of Theorem* 5.2 *and assume* $Z = W + X$, $X \neq 0$ *and* $\tau(X) = \tau(X^*W) = 0$. *Then* $\mathfrak{A}$ *is transitive.*

Proof. Let $P \in \mathcal{B}(L^2(\mathcal{M}, \tau))$ be a projection such that $P \in \operatorname{Lat}\mathfrak{A}$. Since $\mathfrak{A}$ contains $\mathcal{A}$ and $\mathcal{M}'$, $P \in \{\mathcal{A}, \mathcal{M}'\}' = \mathcal{A}' \cap \mathcal{M} = \mathcal{A}$. By identifying $\mathcal{M}$ as a subset of $L^2(\mathcal{M}, \tau)$, we can write $P = \sum_{n=-\infty}^{\infty} \lambda_n U^n$, where $\lambda_n = \bar{\lambda}_{-n}$. Since $P \in \operatorname{Lat} Z$, we have $PZP = ZP$. Thus, we have the following equation:

$$\sum_{m,n\in\mathbf{Z}} \lambda_m \lambda_n U^m W U^n + \sum_{m,n\in\mathbf{Z}} \lambda_m \lambda_n U^m X U^n = \sum_{n\in\mathbf{Z}} \lambda_n W U^n + \sum_{n\in\mathbf{Z}} \lambda_n X U^n.$$

It is easy to verify that $U^{-n} X U^n$, $n \in \mathbf{Z} \setminus \{0\}$ are mutually orthogonal in $L^2(\mathcal{M}, \tau)$ and perpendicular to $U^l W U^m, U^r X U^s, W U^m, X U^m$ for all $l, m, r, s \in \mathbf{Z}$ and $r \neq -s$. Therefore, for $n \neq 0$, the coefficients of $U^{-n} X U^n$ are zero, i.e., $|\lambda_n|^2 = 0$, which implies that $\lambda_n = 0, \forall n \in \mathbf{Z} \setminus \{0\}$. So $P = \lambda_0 I$. Since $P = P^* = P^2$, $P = 0$ or I. This implies that $\mathfrak{A}$ is transitive. $\square$

Proof of Theorem 5.2. If $Z = \lambda I$, then $\mathfrak{A}$ is the von Neumann algebra generated by $\mathcal{A}$ and $\mathcal{M}'$. So $\mathfrak{A}$ is not transitive. If $Z \in \{W\}''$ and $Z \neq \lambda I$, then $\mathfrak{A}$ is the algebra generated by $W, \mathfrak{A}$ and $\mathcal{M}'$. By Lemma 5.4, $\mathfrak{A}$ is not transitive. Assume that $Z \notin \{W\}''$. We can assume that $Z \neq 0$ and $\tau(Z) = 0$. Write $Z = \alpha W + X$, where $X \neq 0$ and $\tau(X) = \tau(X^*W) = 0$. If $\alpha = 0$, by Lemma 5.5, $\mathfrak{A}$ is a transitive algebra and strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. If $\alpha \neq 0$, we can assume $\alpha = 1$. By Lemma 5.6, $\mathfrak{A}$ is transitive. By Lemma 5.1 and Corollary 3.4, $\mathfrak{A}$ is strongly dense in $\mathcal{B}(L^2(\mathcal{M}, \tau))$. $\square$

5.2. Brown measures of $W_1 + F_{12}$ *and* $E_{12} + F_{12}$

The following observations are crucial to prove that $\mathfrak{A}_3$ and $\mathfrak{A}_4$ are not transitive: $\mathfrak{A}_3 \subseteq \{(W_1 + F_{12})^2\}'$ and $\mathfrak{A}_4 \subseteq \{(E_{12} + F_{12})^2\}'$. So we only need to prove that the support of Brown measures of $(W_1 + F_{12})^2$ and $(E_{12} + F_{12})^2$ are not single point and then conclude the existence of non-trivial hyperinvariant subspaces of $(W_1 + F_{12})^2$ and $(E_{12} + F_{12})^2$ by [14].

5.2.1. Brown measures of R*-diagonal operators*

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal trace τ. The Fuglede–Kadison determinant [10] of $T \in \mathcal{M}$ is defined by

$$\Delta(T) = \exp\bigl[\tau\bigl(\ln\bigl(T^*T\bigr)^{\frac{1}{2}}\bigr)\bigr]$$

is a generalization of a determinant of a finite matrix.

The Brown measure [4] μ_T of the element T is a Schwartz distribution on the complex plane defined by

$$\mu_T = \frac{1}{2\pi}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) \ln \Delta\bigl[T - (x + iy)I\bigr].$$

If $\mathcal{M} = M_n(\mathbb{C})$ and $\tau = \frac{1}{n}\operatorname{Tr}$ is the normalized trace on $M_n(\mathbb{C})$, then μ_T is the normalized counting measure $\frac{1}{n}(\delta_{\lambda_1} + \delta_{\lambda_2} + \cdots + \delta_{\lambda_n})$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of T repeated according to root multiplicity. If T is normal, μ_T is the trace τ composed with the spectral measure for T. From the definition, Brown measure μ_T only depends on the joint distribution of T and T^*.

The Brown measure has the following properties (see [4]): μ_T is the unique compactly supported measure on $\mathbb{C}$ such that $\ln \Delta[T-(x+iy)I] = \int_{\mathbb{C}} \ln|z-\lambda|\, d\mu_T(z)$ for all $\lambda \in \mathbb{C}$. The support of μ_T is contained in $\sigma(T)$, the spectra of T. $\mu_{AB} = \mu_{BA}$ for arbitrary A, B in $\mathcal{M}$, and if $f(z)$ is analytic in a neighborhood of $\sigma(A)$, $\mu_{f(T)} = (\mu_T)_f$. If $E \in \mathcal{M}$ is a projection such that $E \in \operatorname{Lat} T$, then with respect to $E, I-E$ we can write

$$T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix},$$

where $A = ETE$ and $C = (I-E)T(I-E)$ are elements of II_1 factors $\mathcal{M}_1 = E\mathcal{M}E$ and $\mathcal{M}_2 = (I-E)\mathcal{M}(I-E)$, respectively. Let μ_A and μ_C be the Brown measure of A and C computed relative to $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively, then $\mu_T = \alpha\mu_A + (1-\alpha)\mu_C$, where $\alpha = \tau(E)$.

R-diagonal operators are introduced by Nica and Speicher in [17]. Recall that an operator T in a non-commutative probability space is an R-diagonal operator if the R-transform $R_{\mu(T,T^*)}$ of the joint distribution $\mu(T, T^*)$ of T, T^* is of the form

$$R_{\mu(T,T^*)}(z_1, z_2) = \sum_{n=1}^{\infty} \alpha_n (z_1 z_2)^n + \sum_{n=1}^{\infty} \alpha_n (z_2 z_1)^n.$$

Nica and Speicher [17] proved that T is an R-diagonal operator if and only if T has same *-distribution as product UH, where U and H are *-free random variables in some tracial non-commutative C*-probability space, U is a Haar unitary operator and H is positive. In [13], Haagerup and Larsen explicitly computed the Brown measures of *R-diagonal* operators in a finite von Neumann algebra.

Theorem 5.7. *(See [14, Theorem 4.4].) Let U, H be *-free random variables in $(\mathcal{M}, \tau)$, with U a Haar unitary operator and H a positive operator such that the distribution μ_H of H is not a Dirac measure. Then the Brown measure μ_{UH} of UH can be computed as the following.*

1. *μ_{UH} is rotation invariant and its support is the annulus with inner radius $\|H^{-1}\|_2^{-1}$ and outer radius $\|H\|_2$.*
2. *$\mu_{UH}(\{0\}) = \mu_H(\{0\})$ and for $t \in]\mu_H(\{0\}), 1]$,*

$$\mu_{UH}\big(\mathbf{B}\big(0, \big(\mathcal{S}_{\mu_{H^2}}(t-1)\big)^{-1/2}\big)\big) = t,$$

   where $\mathcal{S}_{\mu_{H^2}}$ is the $\mathcal{S}$-transform of H^2 and $\mathbf{B}(0,r)$ is the closed disc with center 0 and radius r.
3. *μ_{UH} is the only rotation invariant symmetric probability measure satisfying 2.*

Based on Theorem 5.7, the Brown measure of the sum of a random variable with an arbitrary distribution and a free R-diagonal element, e.g. $U_n + U_\infty$, where U_n and U_∞ are the generators of $\mathbb{Z}_n$ and $\mathbb{Z}$, respectively, in the free product $\mathbb{Z}_n * \mathbb{Z}$, is computed by Biane and Lehner in [3].

5.2.2. Brown measure of $W_1 F_{12}$

Lemma 5.8. *$T = W_1 F_{12}$ is an R-diagonal operator with Brown measure μ_T satisfying*:

1. *μ_T is rotation invariant and the support of μ_T is $\mathbf{B}(0, \frac{1}{\sqrt{2}})$.*
2. *$d\mu_T(z) = \frac{1}{2}\delta_0 + \frac{1}{2\pi}\frac{1}{(1-r^2)^2}\,dr\,d\theta$ for $z = re^{i\theta}$ and $0 < r \leqslant \frac{1}{\sqrt{2}}$.*

Proof. Note that $F_{12} = V_2^* F_{22}$. Since W_1 and V_2 are free unitary operator such that $\tau(W_1) = \tau(V_2) = 0$, simple computations show that $\widetilde{U} = W_1 V_2^*$ is a Haar unitary operator. To prove T is an R-diagonal operator, we only need to check that $\widetilde{U}$ is *-free with F_{22}. Note that F_{22} is in the algebra generated by V_1, we only need to prove that $\widetilde{U}$ is *-free with V_1. Note that $V_1^2 = I$ and $\tau(V_1) = 0$. We need to check $\tau(\widetilde{U}^{n_1} V_1 \widetilde{U}^{n_2} V_1 \widetilde{U}^{n_3} \cdots V_1 \widetilde{U}^{n_k}) = 0$, where $n_1, n_2, \ldots, n_k \in \mathbb{Z}$ and $n_2, \ldots, n_{k-1}$ are not equal 0. Consider the word $\widetilde{U}^l V_1 \widetilde{U}^m$, $l, m \neq 0$. We have the following four cases:

1. If $l, m > 0$, then $\widetilde{U}^l V_1 \widetilde{U}^m = W_1 V_2^* \ldots W_1 (V_2^* V_1) W_1 V_2^* \ldots W_1 V_2^*$.
2. If $l > 0, m < 0$, then $\widetilde{U}^l V_1 \widetilde{U}^m = W_1 V_2^* \ldots W_1 (V_2^* V_1 V_2^*) W_1 V_2^* \ldots W_1$.
3. If $l < 0, m > 0$, then $\widetilde{U}^l V_1 \widetilde{U}^m = V_2^* W_1 \ldots W_1 V_1 W_1 V_2^* \ldots W_1 V_2^*$.
4. If $l < 0, m < 0$, then $\widetilde{U}^l V_1 \widetilde{U}^m = V_2^* W_1 \ldots W_1 (V_1 V_2^*) W_1 V_2^* \ldots W_1$.

Note that $\tau(V_2^* V_1) = \tau(V_2^* V_1 V_2^*) = \tau(V_1 V_2^*) = 0$. $\widetilde{U}^{n_1} V_1 \widetilde{U}^{n_2} V_1 \widetilde{U}^{n_3} \ldots V_1 \widetilde{U}^{n_k}$ is an alternating product of centered elements from $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $1 * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$. Thus $\tau(\widetilde{U}^{n_1} V_1 \widetilde{U}^{n_2} V_1 \widetilde{U}^{n_3} \ldots V_1 \widetilde{U}^{n_k}) = 0$. This proves that T is an R-diagonal operator.

Now we apply Theorem 5.7 to compute the Brown measure of T. Note that $H = F_{22}$ and $\|F_{22}\|_2 = (\tau(F_{22}^2))^{1/2} = \frac{1}{\sqrt{2}}$. Since the kernel of F_{22} is non-trivial, $\|F_{22}^{-1}\|_2^{-1} = 0$. By 1 of Theorem 5.7, we obtain 1 of Lemma 5.8. By 2 of Theorem 5.7, $\mu_T(0) = \mu_H(0) = \frac{1}{2}$. To compute the density function of Brown measure of T, we first compute the $\mathcal{S}$-transform of $H^2 = H = F_{22}$. Simple computation shows that $\mathcal{S}_{\mu_{H^2}}(\omega) = \frac{2(\omega+1)}{2\omega+1}$. By 2 of Theorem 5.7,

$$t = \mu_T\big(\mathbf{B}\big(0, \big(\mathcal{S}_{\mu_{H^2}}(t-1)\big)^{-1/2}\big)\big) = \mu_T\left(\mathbf{B}\left(0, \sqrt{1 - \frac{1}{2t}}\right)\right) \quad \text{for } t \in \left]\frac{1}{2}, 1\right].$$

Let $r = \sqrt{1 - \frac{1}{2t}}$. Then $t = \frac{1}{2(1-r^2)}$ and $\mu_T(\mathbf{B}(0, r)) = \frac{1}{2(1-r^2)}$ for $0 < r < \frac{1}{\sqrt{2}}$. This implies 2 of Lemma 5.8. $\square$

The following lemma is useful to compute the Brown measures of $E_{12} + F_{12}$ and $W_1 + F_{12}$.

Lemma 5.9. *Let $\mathcal{N} = 1 * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$. Then $W_1 \mathcal{N} W_1$ is free with $\mathcal{N}$.*

Proof. Consider an alternating product of elements of $W_1 \mathcal{N} W_1$ and $\mathcal{N}$: $A_1(W_1 B_1 W_1) \cdot A_2(W_2 B_2 W_2) A_3 \ldots (W_1 B_{n-1} W_1) A_n$, where $A_2, \ldots, A_{n-1}, B_1, \ldots, B_{n-1}$ are centered elements in $\mathcal{N}$ and A_1, A_n are either centered elements or 1. Then it an alternating product of elements of $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $\mathcal{N} = 1 * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$. Thus the trace is 0. $\square$

Since the Brown measure of T only depends on the joint distribution of T and T^*, the Brown measures of $E_{12} + F_{12}$ and $W_1 + F_{12}$ are same as the Brown measures of $W_1 F_{12} W_1 + F_{12}$ and $W_1 V_1 W_1 + F_{12}$, respectively.

With respect to matrix units of $\mathcal{N}$, write $W_1 = \left(\begin{smallmatrix} A & B^* \\ B & C \end{smallmatrix}\right)$. Since $W_1^2 = I$, we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} = \begin{pmatrix} A^2 + B^*B & AB^* + B^*C \\ BA + CB & C^2 + BB^* \end{pmatrix}.$$

Hence,

$$-CB = BA. \tag{12}$$

5.2.3. Brown measure of $E_{12} + F_{12}$

Since $\mu_{(E_{12}+F_{12})} = \mu_{(W_1 F_{12} W_1 + F_{12})}$, we need to compute the Brown measure of $W_1 F_{12} W_1 + F_{12}$. Note that

$$\begin{aligned}
(W_1 F_{12} W_1 + F_{12})^2 &= W_1 F_{12} W_1 F_{12} + F_{12} W_1 F_{12} W_1 \\
&= \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \\
&\quad + \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B^* \\ B & C \end{pmatrix} \\
&= \begin{pmatrix} B^2 & AB + BC \\ 0 & B^2 \end{pmatrix}.
\end{aligned}$$

So the Brown measure of $(W_1 F_{12} W_1 + F_{12})^2$ is same as the Brown measure of B^2 and the Brown measure of $W_1 F_{12} W_1 + F_{12}$ is same as the Brown measure of B (because Brown measures of $W_1 F_{12} W_1 + F_{12}$ and B are both rotation invariant). Now we only need to compute the Brown measure of B. Note that

$$W_1 F_{12} = \begin{pmatrix} 0 & A \\ 0 & B \end{pmatrix}.$$

Thus $\mu_{W_1 F_{12}} = \frac{1}{2}\delta_0 + \frac{1}{2}\mu_B$. By Lemma 5.7, we conclude that μ_B is rotation invariant and the support of μ_B is $\mathbf{B}(0, \frac{1}{\sqrt{2}})$ and $d\mu_B(z) = \frac{1}{\pi}\frac{1}{(1-r^2)^2}\,dr\,d\theta$ for $z = re^{i\theta}$ and $0 \leqslant r \leqslant \frac{1}{\sqrt{2}}$. Summarize above, we have the following proposition.

Proposition 5.10. *Let $\mu_{(E_{12}+F_{12})}$ be the Brown measure of $E_{12} + F_{12}$. Then we have the following*:

1. *$\mu_{(E_{12}+F_{12})}$ is rotation invariant and the support of $\mu_{(E_{12}+F_{12})}$ is $\mathbf{B}(0, \frac{1}{\sqrt{2}})$.*
2. *$d\mu_{(E_{12}+F_{12})}(z) = \frac{1}{\pi}\frac{1}{(1-r^2)^2}\,dr\,d\theta$ for $z = re^{i\theta}$ and $0 \leqslant r \leqslant \frac{1}{\sqrt{2}}$.*

Corollary 5.11. *Let $\mu_{(E_{12}+F_{12})^2}$ be the Brown measure of $(E_{12} + F_{12})^2$. Then we have the following*:

1. $\mu_{(E_{12}+F_{12})^2}$ *is rotation invariant and the support of* $\mu_{(E_{12}+F_{12})^2}$ *is* $\mathbf{B}(0, \frac{1}{2})$.
2. $d\mu_{(E_{12}+F_{12})^2}(z) = \frac{1}{4\pi}\frac{1}{r(1-r)^2}\,dr\,d\theta$ *for* $z = re^{i\theta}$ *and* $0 \leqslant r \leqslant \frac{1}{2}$.

5.2.4. Brown measure of $W_1 + F_{12}$

Since $\mu_{(W_1+F_{12})} = \mu_{(W_1V_1W_1+F_{12})}$, we need to compute the Brown measure of $W_1V_1W_1 + F_{12}$. We compute the Brown measures of $(W_1V_1W_1 + F_{12})^2$ first. Note that

$$\begin{aligned}(W_1V_1W_1 + F_{12})^2 &= I_2 + W_1V_1W_1F_{12} + F_{12}W_1V_1W_1\\ &= I_2 + \begin{pmatrix} A & B^* \\ B & C \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B^* \\ B & C \end{pmatrix}\begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\\ &\quad + \begin{pmatrix} 0 & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A & B^* \\ B & C \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} A & B^* \\ B & C \end{pmatrix}\\ &= \begin{pmatrix} I + 2BA & A^2 - B^*B + BB^* - C^2 \\ 0 & I + 2BA \end{pmatrix},\end{aligned}$$

the last equation follows by (12). So the Brown measure of $(W_1F_{12}W_1 + F_{12})^2$ is same as the Brown measure of $1 + 2BA$. Now we only need to compute the Brown measure of $2BA$. Note that

$$W_1V_1W_1F_{12} = \begin{pmatrix} 0 & A^2 - B^*B \\ 0 & 2BA \end{pmatrix}.$$

Since $W_1V_1W_1$ is *-free with F_{12}, $\mu_{W_1F_{12}} = \mu_{W_1V_1W_1F_{12}} = \frac{1}{2}\delta_0 + \frac{1}{2}\mu_{2BA}$. By Lemma 5.7, we conclude that μ_{2BA} is rotation invariant and the support of μ_{2BA} is $\mathbf{B}(0, \frac{1}{\sqrt{2}})$ and $d\mu_{2BA}(z) = \frac{1}{\pi}\frac{1}{(1-r^2)^2}\,dr\,d\theta$ for $z = re^{i\theta}$ and $0 \leqslant r \leqslant \frac{1}{\sqrt{2}}$. Summarize above, we have the following proposition.

Proposition 5.12. *Let* $\mu_{(W_1+F_{12})^2}$ *be the Brown measure of* $(W_1 + F_{12})^2$. *Then we have the following*:

1. $\mu_{(W_1+F_{12})^2}$ *is rotation invariant with respect to* 1 *and the support of* $\mu_{(W_1+F_{12})^2}$ *is* $\mathbf{B}(1, \frac{1}{\sqrt{2}})$.
2. $d\mu_{(W_1+F_{12})^2}(z) = \frac{1}{\pi}\frac{1}{(1-r^2)^2}\,dr\,d\theta$ *for* $z - 1 = re^{i\theta}$ *and* $0 \leqslant r \leqslant \frac{1}{\sqrt{2}}$.

Since the joint *-distribution of $W_1 + F_{12}$, $(W_1 + F_{12})^*$ and the joint *-distribution of $-(W_1 + F_{12})$, $-(W_1 + F_{12})^*$ are same, we have the following corollary.

Corollary 5.13. *Let* $\mu_{(W_1+F_{12})}$ *be the Brown measure of* $W_1 + F_{12}$. *Then the support of* $\mu_{(W_1+F_{12})}$ *is* $\{z \in \mathbb{C}\colon |z^2 - 1| \leqslant \frac{1}{\sqrt{2}}\}$.

5.3. The case of $\mathfrak{A}_3$

Theorem 5.14. $\mathfrak{A}_3$ *is not a transitive algebra. Indeed, for any* $0 \leqslant r \leqslant 1$, *there is a projection* $E_r \in \mathcal{M}$ *such that* $\tau(E_r) = r$ *and* $E_r \in \operatorname{Lat}\mathfrak{A}_3$.

Proof. $\mathfrak{A}_3$ is the algebra generated by W_1, F_{12} and $\mathcal{M}'$. Simple computation shows that

$$W_1(W_1+F_{12})^2=(W_1+F_{12})^2V_1 \quad \text{and} \quad F_{12}(W_1+F_{12})^2=(W_1+F_{12})^2F_{12}.$$

So

$$\mathfrak{A}_3 \subseteq \{(W_1+F_{12})^2\}'.$$

Let $0 \leqslant r \leqslant 1$. By Proposition 5.12, there is s, $0 \leqslant s \leqslant \frac{1}{\sqrt{2}}$, such that $\mu_{(W_1+F_{12})^2}(\mathbf{B}(1,s))=r$. By [14, Theorem 7.1], there is a hyperinvariant subspace E_r in $\mathcal{M}$ of $(W_1+F_{12})^2$ such that $\tau(E_r)=r$. $\square$

5.4. The case of $\mathfrak{A}_4$

Theorem 5.15. $\mathfrak{A}_4$ *is not a transitive algebra. Indeed, for any* $0 \leqslant r \leqslant 1$*, there is a projection* $F_r \in \mathcal{M}$ *such that* $\tau(F_r)=r$ *and* $F_r \in \operatorname{Lat}\mathfrak{A}_4$.

Proof. $\mathfrak{A}_4$ is the algebra generated by E_{12}, F_{12} and $\mathcal{M}'$. Simple computation shows that $E_{12}(E_{12}+F_{12})^2=(E_{12}+F_{12})^2E_{12}$ and $F_{12}(E_{12}+F_{12})^2=(E_{12}+F_{12})^2F_{12}$. So $\mathfrak{A}_4 \subseteq \{(E_{12}+F_{12})^2\}'$. By Corollary 5.11, there is s, $0 \leqslant s \leqslant \frac{1}{2}$, such that $\mu_{(E_{12}+F_{12})^2}(\mathbf{B}(1,s))=r$. By [14, Theorem 7.1], there is a hyperinvariant subspace F_r in $\mathcal{M}$ of $(E_{12}+F_{12})^2$ such that $\tau(F_r)=r$. $\square$

5.5. On the case of $\mathfrak{A}_2$

Let $\widetilde{\mathfrak{A}}_2 \subseteq \mathcal{B}(L^2(\mathcal{M},\tau))$ be the algebra generated by $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1$, F_{12} and $\mathcal{M}'$. Then $\mathfrak{A}_2 \subseteq \widetilde{\mathfrak{A}}_2$. By the following proposition, we only need to consider if $\widetilde{\mathfrak{A}}_2$ is transitive and the strong closure of $\widetilde{\mathfrak{A}}_2$ is $\mathcal{B}(L^2(\mathcal{M},\tau))$ or not.

Proposition 5.16. $\mathfrak{A}_2$ *is transitive if and only if* $\widetilde{\mathfrak{A}}_2$ *is transitive*; *the strong closure of* $\mathfrak{A}_2$ *is* $\mathcal{B}(L^2(\mathcal{M},\tau))$ *if and only if the strong closure of* $\widetilde{\mathfrak{A}}_2$ *is* $\mathcal{B}(L^2(\mathcal{M},\tau))$.

Proof. Suppose $\widetilde{\mathfrak{A}}_2$ is transitive. Let $P \in \operatorname{Lat}\mathfrak{A}_2$. Then $P \in \mathcal{M}$. With respect to matrix units of $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1$, $\mathcal{M} \cong M_2(\mathbb{C}) \bar{\otimes} \mathcal{M}_1$. In the following, we identify $\mathcal{M}$ with $M_2(\mathbb{C}) \bar{\otimes} \mathcal{M}_1$. Since $PE_{11}=E_{11}P$, $P=\left(\begin{smallmatrix} P_1 & 0 \\ 0 & P_2 \end{smallmatrix}\right)$, where $P_1, P_2 \in \mathcal{M}_1$. By $PE_{12}P=E_{12}P$ and simple computation, we have $P_2 \leqslant P_1$. Write $F_{12}=\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where $A, B, C, D \in \mathcal{M}_1$. By $PF_{12}P=F_{12}P$ and simple computation, we have $(I-P_2)CP_1=0$. It can be proved that C is a one-to-one operator with dense range (see Appendix A). So $\tau_{\mathcal{M}_1}(P_1)=\tau_{\mathcal{M}_1}(R(CP_1)) \leqslant \tau_{\mathcal{M}_1}(P_2)$, where $\tau_{\mathcal{M}_1}$ is the trace induced by τ on $\mathcal{M}_1$. Since $P_2 \leqslant P_1$, this implies that $P_1=P_2$. Therefore, $P \in \operatorname{Lat}\widetilde{\mathfrak{A}}_2$. Since $\widetilde{\mathfrak{A}}_2$ is transitive, $P=0$ or I. So $\mathfrak{A}_2$ is transitive.

Now suppose the strong closure of $\widetilde{\mathfrak{A}}_2$ is $\mathcal{B}(L^2(\mathcal{M},\tau))$. Then $\widetilde{\mathfrak{A}}_2$ is transitive and therefore $\mathfrak{A}_2$ is transitive. To prove the strong closure of $\mathfrak{A}_2$ is $\mathcal{B}(L^2(\mathcal{M},\tau))$, by Theorem 3.1, we need to prove that $\mathfrak{A}_2$ is 2-fold transitive. Let $Q \in \operatorname{Lat}\mathfrak{A}_2^{(2)}$. Then similar arguments as above show that $Q \in \operatorname{Lat}\widetilde{\mathfrak{A}}_2^{(2)}=\operatorname{Lat}\mathcal{B}(L^2(\mathcal{M},\tau))^{(2)}$. By Proposition 1.2, $\mathfrak{A}_2$ is 2-fold transitive. $\square$

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Appendix A. C is a one-to-one operator with dense range

Let $\mathcal{M} = (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$ be the reduced free product of $M_2(\mathbb{C})$ with $M_2(\mathbb{C})$. Then $\mathcal{M}$ is a type II_1 factor with the faithful normal tracial state τ. Let $\{E_{ij}\}_{i,j=1,2}$ and $\{F_{ij}\}_{i,j=1,2}$ be the matrix units of $(M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr}) * 1$ and $1 * (M_2(\mathbb{C}), \frac{1}{2}\operatorname{Tr})$, respectively. Then there is a unitary operator U in $\mathcal{M}$ such that $U^*E_{ij}U = F_{ij}$ for $i, j = 1, 2$. Indeed, let V in $\mathcal{M}$ be a partial isometry from E_{11} onto F_{11} and $W = F_{21}VE_{12} : E_{22} \to F_{22}$, then $U = V + W$ is a unitary operator in $\mathcal{M}$ such that $U^*E_{ij}U = F_{ij}$ for all $i, j = 1, 2$. Write

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with respect to matrix units $\{E_{ij}\}_{i,j=1,2}$, then

$$F_{11} = U^*E_{11}U = \begin{pmatrix} a^*a & a^*b \\ b^*a & b^*b \end{pmatrix},$$

$$F_{22} = U^*E_{22}U = \begin{pmatrix} c^*c & c^*d \\ d^*c & d^*d \end{pmatrix},$$

$$F_{12} = U^*E_{12}U = \begin{pmatrix} a^*c & a^*d \\ b^*c & b^*d \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

$$F_{21} = U^*E_{21}U = \begin{pmatrix} c^*a & c^*b \\ d^*a & d^*b \end{pmatrix}.$$

Since $\{E_{ij}\}_{i,j=1,2}$ and $\{F_{ij}\}_{i,j=1,2}$ are free, the distributions of a^*a, b^*b, c^*c, d^*d are same as the distribution of $E_{11}F_{11}E_{11}$ with respect to $\mathcal{M}_{E_{11}} \triangleq E_{11}\mathcal{M}E_{11}$ and the induced trace $\tau_{E_{11}}(E_{11}TE_{11}) = \frac{1}{\tau(E_{11})}\tau(E_{11}TE_{11})$. By [23], the distribution of $E_{11}F_{11}E_{11}$ is non-atomic and the density function is $\rho(t) = \frac{1}{\pi}\frac{1}{\sqrt{\frac{1}{4}-(\frac{1}{2}-t)^2}}$, $0 \leqslant t \leqslant 1$. In particular, $E_{11}F_{11}E_{11}$ is a one-to-one operator in $\mathcal{M}_{E_{11}}$ with dense range. So a^*a, b^*b, c^*c, d^*d are one-to-one operators with dense ranges. Therefore, a, b, c, d are one-to-one operators. Since a, b, c, d are in $\mathcal{M}_{E_{11}}$, which is a finite von Neumann algebra, a, b, c, d are operators with dense ranges. So a^*, b^*, c^*, d^* are also one-to-one operators with dense ranges. Since $C = b^*c$, C is a one-to-one operator with dense range.

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