Journal of Number Theory **73**, 451–458 (1998)
Article No. NT982308

Locally Analytic Functions over Completions of $\mathbb{F}_r[U]$

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Communicated by D. Goss

Received May 14, 1998

It is well known that a continuous function $f: \mathbf{Z}_p \to \mathbf{Q}_p$ can be expanded by

compact subset of a local field. We study the function field case in this paper. The function field analogue of Mahler's basis is the Carlitz polynomials, and the corresponding result for continuous functions has already been established by Wagner (*Acta Arith.* **17** (1971), 389–406). We show that the conditions for a continuous function to be locally analytic in the function field case are completely similar to the $\mathbf{Q}_p$ case. An application to using integral calculus to analytically continue characteristic p-valued L-series is briefly mentioned at the end of the paper.

1. INTRODUCTION

Ultra-metric analysis consists of the p-adic analysis over number fields and the analysis of function fields of characteristic p. Since both the p-adic number fields and the function fields of characteristic p are equipped with similar non-archimedean absolute values, it is natural to expect that they have similar properties in analysis. For a detailed study on p-adic analysis, see [6].

In this paper, p is a fixed prime number and r is a power of p.

Let L denote a local field with a discrete valuation, $\mathcal{O}$ its valuation ring, $\mathcal{M} = \pi\mathcal{O}$ its maximal ideal, and $\mathcal{F} = \mathcal{O}/\mathcal{M}$ its residue field of q elements, where q is a power of p (later on q will also be a power of r). The valuation v of L is normalized such that $v(\pi) = 1$, and the absolute value is denoted

* I thank Professor D. Gross for suggesting this problem to me and for help in writing this paper, and Professor K. Conrad for helpful comments.

0022-314X/98 $25.00

$|?|$. In applications we will actually only consider the case where L is either $\mathbf{Q}_p$ or the completion of $\mathbb{F}_r(U)$ at some finite place $\mathbf{v}$.

An L-Banach space is a complete normed L-vector space E with the norm $\|?\|$ satisfying the ultra-metric inequality:

$$\|x+y\| \leqslant \max(\|x\|, \|y\|) \qquad \text{for} \quad x, y \in E.$$

We will only consider separable Banach spaces here, i.e., spaces with a countable subset generating a dense subspace. For example, let X be any compact subset of L, denote $C(X, L)$ the space of continuous functions from X to L and equip $C(X, L)$ with the sup norm, $\|f\| = \sup_{x \in X}\{|f(x)|\}$ for $f \in C(X, L)$. Then $C(X, L)$ is such a space according to Kaplansky's theorem (6, Theorem 43.3].

Definition 1. Let E be an L-Banach space. A sequence $\{e_n\}_{n \geqslant 0}$ in E is an orthonormal basis for E if for any element $x \in E$ we have:

(1) x can be uniquely written as $x = \sum_{n=0}^{\infty} \lambda_n e_n$ with $\lambda_n \in L$, and $\lambda_n \to 0$;
(2) $\|x\| = \sup_{n \geqslant 0}\{|\lambda_n|\}$.

For $L = \mathbf{Q}_p$, consider the Banach space $C(\mathbf{Z}_p, \mathbf{Q}_p)$. Then the binomial polynomials $\binom{x}{n}$ map $\mathbf{Z}_p$ into $\mathbf{Z}_p$ as they are obviously continuous and the p-adically dense positive integers are mapped to p-adic integers. K. Mahler [5] established the following theorem.

Theorem 1. *The sequence of binomial polynomials* $\{\binom{x}{n}\}_{n \geqslant 0}$ *is an orthonormal basis of* $C(\mathbf{Z}_p, \mathbf{Q}_p)$. *Moreover, let* $f \in C(\mathbf{Z}_p, \mathbf{Q}_p)$ *and write it as* $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$. *Then the coefficients* a_n *can be recovered as*

$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k).$$

Now we consider the corresponding Banach spaces of continuous functions over function fields. We follow the notations of Chapter 3 and Section 8.22 of [3]. Let $\mathbb{F}_r$ be the finite field of r elements, and for i a non-negative integer, set

$$\mathbf{A} = \mathbb{F}_r[U] \qquad \text{and} \qquad \mathbf{k} = \mathbb{F}(U) \text{ the quotient field of } \mathbf{A},$$

$$[i] = U^{r^i} - U,$$

$$L_i = \begin{cases} 1 & \text{if} \quad i = 0, \\ [i][i-1]\cdots[1] & \text{if} \quad i \geqslant 1, \end{cases}$$

$$D_i = \begin{cases} 1 & \text{if} \quad i = 0, \\ [i][i-1]^r \cdots [1]^{r^{i-1}} & \text{if} \quad i \geqslant 1, \end{cases}$$

$$e_d(x) = \begin{cases} x & \text{if } d=0, \\ \prod_{\deg(m)<d} (x-m) = \sum_{i=0}^{d} (-1)^{d-i} x^{r^i} \dfrac{D_d}{D_i L_{d-i}^{r^i}} & \\ \qquad \text{if } d \text{ is a positive integer.} \end{cases}$$

Let n be a non-negative integer which we write r-adically as $n = n_0 + n_1 r + \cdots + n_s r^s$, with $0 \leqslant n_i < r$. Then the Carlitz polynomials are defined as

$$G_n(x) = \prod_{i=0}^{s} \left(\frac{e_i(x)}{D_i}\right)^{n_i}, \qquad \text{and}$$

$$G'_n(x) = \prod_{i=0}^{s} G'_{n_i r^i} \qquad \text{where}$$

$$G'_{lr^i}(x) = \begin{cases} \left(\dfrac{e_i(x)}{D_i}\right)^l & \text{if } 0 \leqslant l < r-1, \\ \left(\dfrac{e_i(x)}{D_i}\right)^l - 1 & \text{if } l = r-1. \end{cases}$$

(N.B.: the polynomials $G'_n(x)$ are *not* the derivatives of $G_n(x)$.) The polynomials $G_n(x)$ and $G'_n(x)$ are of degree n, and behave like binomial polynomials, see [2]. (Notice the difference between the polynomials $G_n(x)$ and $G'_n(x)$ defined here and those of Carlitz's original paper [2], as we divide by D_i and Carlitz does not.) In [2], the next result is proved.

PROPOSITION 1. *Both $G_n(x)$ and $G'_n(x)$ map* $\mathbf{A}$ *to itself.*

Let $\mathbf{v}$ be an arbitrary monic prime polynomial of $\mathbf{A}$, hence it corresponds to a finite place of $\mathbf{k}$, which we still denote by $\mathbf{v}$; the associated valuation will be denoted by a non-bold v. The completion of $\mathbf{A}$ and $\mathbf{k}$ at $\mathbf{v}$ are denoted by $\mathbf{A_v}$ and $\mathbf{k_v}$, respectively; we will assume that $v(\pi)=1$ for any parameter π of $\mathbf{k_v}$. Then we see that the space $C(\mathbf{A_v}, \mathbf{k_v})$ of continuous functions from $\mathbf{A_v}$ to $\mathbf{k_v}$ is a $\mathbf{k_v}$-Banach space with the sup norm. An analogue of Mahler's theorem was discovered by C. Wagner [8] (see also [4]) with the binomial polynomials $\binom{x}{n}$ replaced by the Carlitz polynomials $G_n(x)$; this result is our next theorem.

THEOREM 2. *The Carlitz polynomials $G_n(x)$, $n \geqslant 0$, are an orthonormal basis of $C(\mathbf{A_v}, \mathbf{k_v})$. Moreover, for any $f \in C(\mathbf{A_v}, \mathbf{k_v})$, write it as $f(x) = \sum_{n=0}^{\infty} a_n G_n(x)$. Then the coefficients a_n can be recovered from the function f as*

$$a_n = (-1)^k \sum_{\deg(m)<k} G'_{r^k-1-n}(m)\, f(m), \qquad \textit{for any } k \textit{ such that } n < r^k.$$

Theorem 2 is proved via the techniques in [1] ([4] gives a different proof). In this paper, we discuss the conditions imposed on the coefficients in the expansion of f to make it locally analytic. For $f \in C(\mathbf{Z}_p, \mathbf{Q}_p)$, the results are already in [1]. For $f \in C(\mathbf{A}_\mathbf{v}, \mathbf{k}_\mathbf{v})$, we will show that Amice's result in [1] also applies to the coefficients of f as given in Theorem 2.

2. NEWTON TYPE INTERPOLATION POLYNOMIALS

In [1], Amice constructed Newton type interpolation polynomials to study interpolation problems over non-archimedean fields. We use a special case of Amice's construction [1] which is also used in [8]. Let $S = \{\alpha_0, \alpha_1, ..., \alpha_{q-1}\} \subset \mathcal{O}$ be a system of representatives of $\mathcal{F} = \mathcal{O}/\mathcal{M}$, and we assume that S contains 0. Then any element $x \in L$ can be written as

$$x = \sum_{k \gg -\infty}^{\infty} \beta_k \pi^k, \qquad \text{where} \quad \beta_k \in S.$$

We see that x is in $\mathcal{O}$ if and only if $\beta_k = 0$ for all $k < 0$. To any non-negative integer $n = n_0 + n_1 q + \cdots + n_s q^s$, $0 \leqslant n_i \leqslant q-1$, we assign the element u_n of $\mathcal{O}$ with $u_n = \alpha_{n_0} + \alpha_{n_1}\pi + \cdots + \alpha_{n_s}\pi^s$. We now define two sequences of polynomials $\{P_n(x)\}_{n \geqslant 0}$ and $\{Q_n(x)\}_{n \geqslant 0}$ as follows:

$$P_0(x) = 1, \qquad P_n(x) = (x-u_0)(x-u_1)\cdots(x-u_{n-1}) \qquad \text{for} \quad n \geqslant 1,$$

$$Q_0(x) = 1, \qquad Q_n(x) = P_n(x)/P_n(u_n) \qquad \text{for} \quad n \geqslant 1.$$

Notice that when $L = \mathbf{Q}_p$, $\mathcal{O} = \mathbf{Z}_p$, $\pi = p$, and $S = \{0, 1, ..., p-1\}$, then $Q_n(x)$ is just the binomial polynomial $\binom{x}{n}$. Amice [1] proved the next result.

THEOREM 3. (1) *The sup norm of $P_n(x)$ as an element of $C(\mathcal{O}, L)$ is $\|P_n\| = |\pi|^{\lambda_n} = |P_n(u_n)|$, where $\lambda_n = \sum_{i=1}^{\infty} [n/q^i]$.*

(2) *$\{Q_n(x)\}_{n \geqslant 0}$ is an orthonormal basis of $C(\mathcal{O}, L)$.*

3. LOCALLY ANALYTIC FUNCTIONS.

Let $a \in L$, $R = |\rho| > 0$ for some $\rho \in L$ and $B = B_{a, R} =$ the closed ball with center a and radius R,

$$B = \{x \in L\colon |x - a| \leqslant R\}.$$

DEFINITION 2. A function $f \in C(B, L)$ is said to be analytic on B if and only if f can be expanded as a Taylor series

$$f(x) = \sum_{n=0}^{\infty} a_n \left(\frac{x-a}{\rho}\right)^n,$$

which is convergent for all $x \in B$, i.e., $a_n \to 0$ as $n \to \infty$. The norm of f is defined as $\|f\|_B = \sup_{n \geqslant 0}\{|a_n|\}$, and the space of analytic functions on B is denoted $A(B)$.

It can be proved that the expansion property of f, and $\|f\|_B$, depend only on the closed ball B, and not on the choices of a and ρ.

DEFINITION 3. A function $f \in C(\mathcal{O}, L)$ is said to be locally analytic on $\mathcal{O}$ if and only if for each $x \in \mathcal{O}$, there exists a non-trivial closed ball B_x which contains x such that the restriction $f|_{B_x}$ is analytic on B_x. The space of locally analytic function on $\mathcal{O}$ is denoted $LA(\mathcal{O})$.

DEFINITION 4. A locally analytic function $f \in LA(\mathcal{O})$ is said to be of order h if for each $x \in \mathcal{O}$, the closed ball B_x in Definition 3 can be chosen to have radius at least $|\pi|^h$. The space of locally analytic functions of order h on $\mathcal{O}$ is denoted $LA_h(\mathcal{O})$. Write $\mathcal{O}$ as a disjoint union of balls with radius $|\pi|^h$: $\mathcal{O} = \bigcup_{1 \leqslant i \leqslant q^h} B_{a_i}$.

Then we define the norm of $f \in LA_h(\mathcal{O})$ by $\|f\|_h = \sup_i\{\|f\|_{B_{a_i}}\}$.

Remarks. (1) In Definition 2 the norm $\|f\|_B$ for $f \in A(B)$ is equal to $\sup_{z \in a+\rho\bar{\mathcal{O}}}\{|f(z)|\}$ where $\bar{\mathcal{O}}$ is the integral closure of $\mathcal{O}$ in the algebraic closure $\bar{L}$ of L, by the maximum principle for analytic functions, see [6].

(2) $LA_h(\mathcal{O})$ is a separable L-Banach space, and has an obvious orthonormal basis $\{\chi_{m,i}(x)\}_{m \geqslant 0,\, i=1,\ldots,q^h}$ with

$$\chi_{m,i}(x) = \begin{cases} \left(\dfrac{x-a_i}{\pi^h}\right)^m & \text{if } x \in B_{a_i}, \\ 0 & \text{if } x \notin B_{a_i}. \end{cases}$$

(3) We have $LA_h(\mathcal{O}) \subseteq LA_{h+1}(\mathcal{O})$ for any positive integer h, and as $\mathcal{O}$ is compact, $LA(\mathcal{O}) = \varinjlim_{h \geqslant 0} LA_h(\mathcal{O})$.

Through investigating the orthonormal basis $\{\chi_{m,i}(x)\}_{m \geqslant 0,\, i=1,\ldots,q^h}$ of $LA_h(\mathcal{O})$ and the norm $\|P_n\|_h$ in $LA_h(\mathcal{O})$, Amice [1] proved the following result.

THEOREM 4. (1) *For* $h \geqslant 1$, *the polynomials* $(1/S_{n,h})\, P_n(x)$, *with* $s_{n,h} \in L$, *form an orthonormal basis of* $LA_h(\mathcal{O})$ *if and only if* $v(s_{n,h}) = \sum_{i=1}^{h} [n/q^i]$.

(2) *Let* $f \in C(\mathcal{O}, L)$ *and expand it with respect to the orthonormal basis* $\{Q_n(x)\}_{n \geqslant 0}$ *as* $f(x) = \sum_{n=0}^{\infty} a_n Q_n(x)$. *Then* f *is locally analytic on* $\mathcal{O}$ *if and only if* $\liminf_n (v(a_n)/n) > 0$.

Corollary. *Let* $f \in C(\mathbf{Z}_p, \mathbf{Q}_p)$ *be written as* $f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$. *Then* f *is locally analytic on* $\mathbf{Z}_p$ *if and only if* $\liminf_n (v(a_n)/n) > 0$.

4. LOCALLY ANALYTIC FUNCTIONS OVER COMPLETIONS OF $\mathbb{F}_R[U]$.

Now we consider the case $L = \mathbf{k}_\mathbf{v}$ for any finite place $\mathbf{v}$ of $\mathbf{k} = \mathbb{F}_r(U)$. Thus $\mathcal{O} = \mathbf{A}_\mathbf{v}$, and we let π be an irreducible polynomial of degree d in U. Then the cardinality of the residue field is $q = r^d$, and $\mathbf{A}_\mathbf{v}$ is isomorphic to $\mathbb{F}_q[[\pi]]$. From Section 1 and Section 2, we know two sets of orthonormal bases for the space $C(\mathbf{A}_\mathbf{v}, \mathbf{k}_\mathbf{v})$: the Newton type interpolation polynomials $\{Q_n(x)\}_{n \geqslant 0}$ and the Carlitz polynomials $\{G_n(x)\}_{n \geqslant 0}$. Both $Q_j(x)$ and $G_j(x)$ are polynomials of degree j, hence for each $n \geqslant 0$, there exists $\{g_{n,j}\}_{j=0,1,\ldots,n} \subset \mathbf{k}_\mathbf{v}$, such that $G_n(x) = \sum_{j=0}^{n} g_{n,j} Q_j(x)$ with $\max_{0 \leqslant j \leqslant n} \{|g_{n,j}|\} = \|G_n\| = 1$. Write these relations using matrices. Thus for each non-negative integer n, we have

$$\begin{pmatrix} G_0(x) \\ G_1(x) \\ \vdots \\ G_n(x) \end{pmatrix} = \begin{pmatrix} g_{0,0} & 0 & 0 & \cdots & 0 \\ g_{1,0} & g_{1,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_{n,0} & g_{n,1} & g_{n,2} & \cdots & g_{n,n} \end{pmatrix} \begin{pmatrix} Q_0(x) \\ Q_1(x) \\ \vdots \\ Q_n(x) \end{pmatrix}.$$

Denote $\mathbf{G}_n = (g_{i,j})_{0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n}$ with $g_{i,j} = 0$ for $i < j$ for each non-negative integer n, then $\mathbf{G}_n \in GL(n+1, \mathbf{A}_\mathbf{v})$ and $|g_{n,n}| = \max_{0 \leqslant j \leqslant n}\{|g_{n,j}|\} = 1$, as the G_n's and Q_n's are both orthonormal bases. The matrix $\mathbf{G}_\infty = (g_{i,j})_{i \geqslant 0, j \geqslant 0}$ can be viewed as the transformation matrix from the orthonormal basis $\{Q_n(x)\}_{n \geqslant 0}$ to the orthonormal basis $\{G_n(x)\}_{n \geqslant 0}$.

Lemma 1. *Let* E *be an* L*-Banach space,* $E_0 = \{x \in E \mid \|x\| \leqslant 1\}$, *and* $\bar{E} = E_0/\pi E_0$. *Then a sequence of elements* $\{e_i\}_{i \geqslant 0}$ *in* E *is an orthonormal basis of* E *if and only if* $e_i \in E_0$ *for all* i *and the images* $\bar{e}_i$ *of* e_i *in* $\bar{E}$ *consist of a basis (in the algebraic sense) of the* $\mathcal{O}/\pi\mathcal{O}$*-vector space* $\bar{E}$.

Proof. See [7, p. 70].

Theorem 5. *Fix a non-negative integer* h. *The polynomials* $\pi^{\mu_{n,h}} G_n(x)$ *with* $n \geqslant 0$ *form an orthonormal basis of the Banach space* $LA_h(\mathbf{A}_\mathbf{v})$, *where* $\mu_{n,h} = \sum_{i=h+1}^{\infty} [n/q^i]$.

Proof. From the first part of Theorem 4, $\{\pi^{\mu_{n,h}} Q_n(x)\}_{n\geqslant 0}$ is an orthonormal basis of the Banach space $E = LA_h(\mathbf{A_v})$. For any $n \geqslant 0$, write $R_n(x) = \pi^{\mu_{n,h}} Q_n(x)$, $H_n(x) = \pi^{\mu_{n,h}} G_n(x)$, and $h_{i,j} = g_{i,j} \pi^{\mu_{i,h} - \mu_{j,h}} \in \mathbf{A_v}$, then $(h_{i,j})_{0 \leqslant i \leqslant n, 0 \leqslant j \leqslant n}$ is a lower triangular matrix with diagonal elements $h_{i,i} = g_{i,i}$ having absolute value 1, and

$$\begin{pmatrix} H_0(x) \\ H_1(x) \\ \vdots \\ H_n(x) \end{pmatrix} = \begin{pmatrix} h_{0,0} & 0 & 0 & \cdots & 0 \\ h_{1,0} & h_{1,1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{n,0} & h_{n,1} & h_{n,2} & \cdots & h_{n,n} \end{pmatrix} \begin{pmatrix} R_0(x) \\ R_1(x) \\ \vdots \\ R_n(x) \end{pmatrix}.$$

This proves $\|H_i\|_h \leqslant 1$ for all $i \geqslant 0$ and the reductions $\bar{H}_i(x)$ form a basis of the $\mathbf{A_v}/\pi\mathbf{A_v}$-vector space $\bar{E}$. Then $\pi^{\mu_{n,h}} G_n(x)$ forms an orthonormal basis by Lemma 1.

COROLLARY. *Let* $f(x) = \sum_{n=0}^{\infty} G_n(x)$ *be a continuous function on* $\mathbf{A_v}$, *and let* $\gamma = \liminf_n (v(a_n)/n)$. *Then*

(1) $f(x)$ *is locally analytic of order* h *if and only if*

$$v(a_n) - \sum_{i=h+1}^{\infty} \left[\frac{n}{q^i}\right] \to \infty \qquad as \quad n \to \infty.$$

(2) $f(x)$ *is locally analytic if and only if* $\gamma > 0$. *If* $\gamma > 0$ *and* $l = \max(0, [-(\log(q-1) + \log\gamma)/\log q] + 1)$, *then* $f(x)$ *is locally analytic of order* $h \geqslant l$.

Proof. The first part follows immediately from Theorem 5. For the second part, notice that any locally analytic function on $\mathbf{A_v}$ is locally analytic of order h for some positive integer h, since $\mathbf{A_v}$ is compact. Therefore the equivalence condition is clear because the limit of $\sum_{i=h+1}^{\infty} [n/q^i]/n$ is equal to $1/(q-1)\, q^h > 0$. If $\gamma = \liminf_n (v(a_n)/n) > 0$ and $l = \max(0, [-(\log(q-1) + \log\gamma)/\log q] + 1)$, then for any integer $h \geqslant l$, $\gamma - \lim_{n\to\infty} (\sum_{i=h+1}^{\infty} [n/q^i])/n > 0$, hence $f(x)$ is locally analytic of order h.

As an application, let $\mathbf{A} = \mathbb{F}_r[T]$ and let $f_{\mu_x}(z) = \sum_i m_x(i)\,(z^i/i!)$ be the divided power series associated to the v-adic zeta measure μ_x as in Section 8.22 of [3]. Note that μ_x is a 1-parameter family of measures when the parameter x is sufficiently large. Goss points out that for small x Thakur's calculation of f_{μ_x} (Th. 8.22.12 of [3]) implies that μ_x blows up (i.e., becomes an *unbounded* distribution) *logarithmically* as a function of i. Moreover the function on $\mathbf{A}_v$ which is

$$t \mapsto \begin{cases} t^{s_v} & \text{if} \quad |t|_v = 1, \\ 0 & \text{if} \quad |t|_v < 1, \end{cases}$$

with s_v as in Section 8.3 of [3], is obviously locally analytic. Thus Goss points out that our main result, Theorem 5, can be used to analytically continue the v-adic integral for the zeta function. In other words, our main result makes possible an *integral calculus* approach to the v-adic analytic continuation of this zeta function! This should be a very general phenomenon both v-adically and at ∞.

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