On the Cartan–Jacobson Theorem

Pablo Alberca Bjerregaard
Departamento de Matemática Aplicada, Universidad de Málaga,
29012, Málaga, Spain

Alberto Elduque
Departamento de Matemáticas, Facultad de Ciencias, Universidad de Zaragoza,
50009, Zaragoza, Spain

and

Cándido Martín González and Francisco José Navarro Márquez
Departamento de Álgebra, Geometría y Topología, Universidad de Málaga, Apdo. 59,
29080, Málaga, Spain

Communicated by Efim Zelmanov

Received March 9, 1999

The well-known Cartan–Jacobson theorem claims that the Lie algebra of derivations of a Cayley algebra is central simple if the characteristic is not 2 or 3. In this paper we have studied these two cases, with the following results: if the characteristic is 2, the theorem is also true, but, if the characteristic is 3, the derivation algebra is not simple. We have also proved that in this last case, there is a unique nonzero proper seven-dimensional ideal, which is a central simple Lie algebra of type $A_3$, and the quotient of the derivation algebra modulo this ideal turns out to be isomorphic, as a Lie algebra, to the ideal itself. The original motivation of this work was a series of computer-aided calculations which proved the simplicity of derivation algebras of Cayley algebras in the case of characteristic not 3. These computations also proved the existence of a unique nonzero proper ideal

1Supported by the Spanish AECI through the project “estructuras asociativas y no asociativas.”
2Supported by the Spanish DGES Number Pb 97-1291-C03-03.

3Supported by the Spanish DGICYT Number PB97-1497, and by the Spanish AECI through the project “estructuras asociativas y no asociativas.”
(which turns out to be seven-dimensional) in the algebra of derivations of split
Cayley algebras in characteristic 3. © 2002 Elsevier Science (USA)

1. INTRODUCTION

In any nonassociative algebra $A$, we can define the bracket $[x, y] := xy -
 yx$ for any $x, y \in A$. The left and right multiplication operators $L_x$ and $R_x$
with their usual meaning will be also used. The operator $\text{ad}_x: A \rightarrow A (x \in
A)$ defined as $\text{ad}_x := R_x - L_x$ will appear also frequently in the sequel. If
$A$ is a $K$-algebra ($K$ being a commutative unitary ring), and $f, g: A \rightarrow A$
are endomorphisms of the underlying $K$-module, then we define $[f, g] := \ f \circ g - g \circ f$.

Let $F$ be a field and consider for vectors $(u_1, u_2, u_3)$ and $(v_1, v_2, v_3)$ in
$F^3$ the scalar product

$$\langle(u_1, u_2, u_3), (v_1, v_2, v_3)\rangle := u_1 v_1 + u_2 v_2 + u_3 v_3$$

and the vector product

$$(u_1, u_2, u_3) \times (v_1, v_2, v_3) := (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

Let $O_s$ be the set of matrices $\begin{pmatrix} a & u \\ v & b \end{pmatrix}$, where $a$ and $b$ are elements in $F$
and $u, v \in F^3$. These matrices will be called, from now on, octonions. The
set $O_s$ is a vector space over $F$ under addition and scalar multiplication
componentwise. Define the product of two octonions by

$$\begin{pmatrix} a & u \\ v & b \end{pmatrix} \cdot \begin{pmatrix} a' & u' \\ v' & b' \end{pmatrix} := \begin{pmatrix} a a' + \langle u, v' \rangle & a u' + b' u - v \times v' \\ a' v + b v' + u \times u' & \langle v, u' \rangle + b b' \end{pmatrix}.$$  

It follows that $O_s$ is made into an algebra under this multiplication with
the octonion

$$1 := \begin{pmatrix} 1 & (0, 0, 0) \\ (0, 0, 0) & 1 \end{pmatrix}$$

as the unit element. In what follows, we identify $F$ with the subalgebra $F1$
of $O_s$. We make $O_s$ into a composition algebra by taking the involution

$$x = \begin{pmatrix} a & u \\ v & b \end{pmatrix} \mapsto \bar{x} = \begin{pmatrix} b & -u \\ -v & a \end{pmatrix}, \quad x \in O_s,$$

and defining the quadratic form $n(x) := x \bar{x}$.
The canonical basis $\mathcal{B}$ of $\mathfrak{O}_s$ is the basis $\{e_1, e_2, u_1, u_2, u_3, v_1, v_2, v_3\}$ given by

$$e_1 := \begin{pmatrix} 1 & (0, 0, 0) & 0 \\ (0, 0, 0) & 0 & 1 \end{pmatrix}, \quad e_2 := \begin{pmatrix} 0 & (0, 0, 0) \\ (0, 0, 0) & 1 \end{pmatrix},$$

$$u_i := \begin{pmatrix} 0 & e_i \\ (0, 0, 0) & 0 \end{pmatrix}, \quad v_i := \begin{pmatrix} 0 & (0, 0, 0) \\ e_i & 0 \end{pmatrix}, \quad i = 1, 2, 3,$$

where $\{c_1, c_2, c_3\}$ is the canonical basis of $F^3$. We compute the multiplication table of the canonical basis:

<table>
<thead>
<tr>
<th></th>
<th>$e_1$</th>
<th>$e_2$</th>
<th>$u_1$</th>
<th>$u_2$</th>
<th>$u_3$</th>
<th>$v_1$</th>
<th>$v_2$</th>
<th>$v_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_1$</td>
<td>$e_1$</td>
<td>$0$</td>
<td>$u_1$</td>
<td>$u_2$</td>
<td>$u_3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$e_2$</td>
<td>$0$</td>
<td>$e_2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$v_1$</td>
<td>$v_2$</td>
<td>$v_3$</td>
<td>$0$</td>
</tr>
<tr>
<td>$u_1$</td>
<td>$0$</td>
<td>$u_1$</td>
<td>$0$</td>
<td>$v_3$</td>
<td>$-v_2$</td>
<td>$e_1$</td>
<td>$0$</td>
<td>$0$</td>
</tr>
<tr>
<td>$u_2$</td>
<td>$0$</td>
<td>$u_2$</td>
<td>$-v_3$</td>
<td>$0$</td>
<td>$v_1$</td>
<td>$0$</td>
<td>$e_1$</td>
<td>$0$</td>
</tr>
<tr>
<td>$u_3$</td>
<td>$0$</td>
<td>$u_3$</td>
<td>$v_2$</td>
<td>$-v_1$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$e_1$</td>
</tr>
<tr>
<td>$v_1$</td>
<td>$v_1$</td>
<td>$0$</td>
<td>$e_2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$-u_3$</td>
<td>$u_2$</td>
</tr>
<tr>
<td>$v_2$</td>
<td>$v_2$</td>
<td>$0$</td>
<td>$0$</td>
<td>$e_2$</td>
<td>$0$</td>
<td>$u_3$</td>
<td>$0$</td>
<td>$-u_3$</td>
</tr>
<tr>
<td>$v_3$</td>
<td>$v_3$</td>
<td>$0$</td>
<td>$0$</td>
<td>$0$</td>
<td>$e_2$</td>
<td>$-u_2$</td>
<td>$u_1$</td>
<td>$0$</td>
</tr>
</tbody>
</table>

We can easily determine the general form of a derivation of $\mathfrak{O}_s$. Let $\mathcal{D}$ be a derivation of $\mathfrak{O}_s$ and $\mathcal{D}(e_1) := (a_1, x_1, y_1, b_1) \in \mathfrak{O}_s$. As $e_1$ is an idempotent, it follows that $\mathcal{D}(e_1) = \mathcal{D}(e_1) \cdot e_1 + e_1 \cdot \mathcal{D}(e_1)$; computing the left- and righthand sides of this identity, we have $a_1 = b_1 = 0$. By going on with the rest of the multiplication table, we see that the general form of the matrix, relative to the canonical basis, of a derivation of $\mathfrak{O}_s$ is

$$\begin{pmatrix}
0 & 0 & -t_4 & -t_5 & -t_6 & -t_1 & -t_2 & -t_3 \\
0 & 0 & t_4 & t_5 & t_6 & t_1 & t_2 & t_3 \\
t_1 & -t_1 & t_7 & t_{10} & t_{13} & 0 & -t_6 & t_5 \\
t_2 & -t_2 & t_8 & t_{11} & t_{14} & t_6 & 0 & -t_4 \\
t_3 & -t_3 & t_9 & t_{12} & -t_7 & -t_{11} & -t_5 & t_4 & 0 \\
t_4 & -t_4 & 0 & -t_3 & t_2 & -t_7 & -t_8 & -t_9 & 0 \\
t_5 & -t_5 & t_3 & 0 & -t_1 & -t_{10} & -t_{11} & -t_{12} & 0 \\
t_6 & -t_6 & -t_2 & t_1 & 0 & -t_{13} & -t_{14} & t_7 & t_{11} \\
\end{pmatrix}, \quad (1)$$

where $t_i$, $i = 1, \ldots, 14$, are arbitrary scalars from the ground field.
2. MAIN THEOREM

In order to get to our main theorem, we shall need a number of results. Thus, for instance, we have:

**LEMMA 1.** Let $C$ be a Cayley algebra over an arbitrary field $F$, and define

$$D_{x,y} := [L_x, L_y] + [L_x, R_y] + [R_x, R_y]$$

for all $x, y \in C$. Then the linear span of $\{D_{x,y}; x, y \in C\}$ is an ideal of $\text{Der}(C)$.

**Proof.** For all $a, x, y \in C$, we have $(a, x, y) = -(x, a, y)$, where $(a, x, y) = (ax)y - a(xy)$ is the associator of the elements $a, x, y$; hence $L_a(xy) = (T_a(x))y - x(L_a(y))$, where $T_a := L_a + R_a$. Also $(x, a, y) = -(x, a, y)$, implying $R_a(xy) = -(R_a(x))y + x(T_a(y))$. As a consequence we have:

(i) $[L_a, L_b](xy) = ([T_a, T_b](x))y + x([L_a, L_b](y))$,

(ii) $[L_a, R_b](xy) = -([T_a, R_b](x))y - x([L_a, T_b](y))$,

(iii) $[R_a, R_b](xy) = ([R_a, R_b](x))y + x([T_a, T_b](y))$,

and by flexibility, $[L_a, R_b] = [R_a, L_b]$. Summing (i), (ii), and (iii), we obtain

$$D_{a,b}(xy) = D_{a,b}(x)y + xD_{a,b}(y)$$

for all $x, y \in C$; that is, $D_{a,b}$ is a derivation of $C$ for any $a, b \in C$. Taking into account that for any $D \in \text{Der}(C)$, one has $[D, L_x] = L_{D(x)}$, $[D, R_x] = R_{D(x)}$, and the Jacobi identity, we obtain $[D, D_{a,b}] = D_{D(a),b} + D_{a,D(b)}$; hence the linear span of the $D_{a,b}$'s is an ideal in $\text{Der}(C)$.

Supposing now that $C$ is a split Cayley $F$-algebra, we have a Peirce decomposition $C = (Fe_1 \oplus Fe_2) \oplus U \oplus V$, where $U$ is the linear span of $\{u_1, u_2, u_3\}$ and $V$ is the linear span of $\{v_1, v_2, v_3\}$. This decomposition of $C$ is a $\mathbb{Z}_3$-grading inducing a $\mathbb{Z}_3$-grading of $\text{Der}(C)$ given by $\text{Der}(C) = \mathcal{D}_0 \oplus \mathcal{D}_1 \oplus \mathcal{D}_2$.

**PROPOSITION 1.** If $C$ is a split Cayley algebra, then:

(i) $\mathcal{D}_0 \cong \text{sl}(3, F)$,

(ii) $\mathcal{D}_1 = \text{span}(D_{v, u}; u \in U)$,

(iii) $\mathcal{D}_2 = \text{span}(D_{v, v}; v \in V)$.

Moreover, $\dim(\text{Der}(C)) = 14$, and any derivation is inner. If the characteristic is not 3, then $\text{Der}(C) = \text{span}(D_{x,y}; x, y \in C)$. 

Proof. Let \( q \) denote the norm of \( C \); since \( d(Fe_1 + Fe_2) = 0 \) for all \( d \in \mathcal{D}_0 \), we have \( d(e_1) = d(e_2) = 0 \) (see (1)). As \( q(d(x), y) + q(x, d(y)) = 0 \) for all \( x, y \in C \), we also have \( q(d(u_i), v_j) + q(u_i, d(v_j)) = 0 \); in particular, if the matrix of \( d \) relative to the basis \( \{e_1, e_2, u_1, u_2, v_1, v_2, v_3\} \) is

\[
\begin{pmatrix}
0 & A \\
A & B
\end{pmatrix},
\]

then \( A = -B' \). Moreover, the trilinear form \( (a, b, c) \mapsto q(a, bc) \) is alternate in \( U \). Consequently, \( 0 = q(d(a), bc) + q(a, d(b)c) + q(a, bd(c)) = \text{Tr}(A) \times q(a, bc) \), and therefore, \( \text{Tr}(A) = 0 \). Thus we obtain a monomorphism of Lie algebras \( \Omega: \mathcal{D}_0 \to \text{sl}(3, F) \) given by \( \Omega(d) = A \). To prove that this monomorphism is actually an isomorphism, it suffices to realize that:

- for \( i \neq j \), the linear map \( d_{ij}: C \to C \), given by \( d_{ij}(u_p) = \delta_{pj}u_j \), \( d_{ij}(v_q) = -\delta_{pq}v_j \), and \( d_{ij}(e_k) = 0 \), for all \( k \) (being \( \delta_{kl} \) the Kronecker symbols), is an element of \( \mathcal{D}_0 \).
- the set of all \( \Omega(d_{ij}) \) (with \( i \neq j \)) span \( \text{sl}(3, F) \) as a Lie algebra.

One can see that \( D_{u_i,v_j} = 3d_{ij} (i \neq j) \); hence if the characteristic of the ground field is not 3, \( \mathcal{D}_0 \) agrees with span \( \{D_{xy}: x \in U, y \in V\} \). If the characteristic is arbitrary, since \( d_{ij} = -[L_{u_i}, R_{v_j}] \), any derivation of \( \mathcal{D}_0 \) is inner. Now, the linear map \( \Omega_1: \mathcal{D}_1 \to U \) given by \( d \mapsto d(e_1) \) is a monomorphism (see (1)). Furthermore, for all \( u \in U, D_{e_1,u} \in \mathcal{D}_1 \) and \( D_{e_1,u}(e_1) = -u \); hence \( \Omega_1 \) is an isomorphism and \( \text{dim}(\mathcal{D}_1) = 3 \). A similar argument proves that \( \text{dim}(\mathcal{D}_2) = 3 \) and \( \text{dim}(\text{Der}(C)) = \text{dim}(\text{sl}(3, F)) + \text{dim}(U) + \text{dim}(V) = 8 + 3 + 3 = 14 \).

Lemma 2. Let \( F \) be a field of characteristic 3 and let \( K \) be a separable extension of \( F \) of degree 2 such that \( K = F[a] \), with \( a^2 = -\alpha \neq 0, \alpha \in F \). Let \( W \) be a three-dimensional free \( K \)-module with basis \( B = \{a_1, a_2, a_3\} \). Let \( \sigma: W \times W \to K \) be a nondegenerate hermitian form whose matrix relative to the previous basis is \( \text{diag}(\alpha_1, \alpha_2, \alpha_3) \) with \( \alpha_i \in F \). Then:

(i) For any \( \phi \) is the Lie algebra,
\[
\text{su}(W, \sigma) := \{\varphi \in \text{End}_K(W): \sigma(\varphi(w_1), w_2) + \sigma(w_1, \varphi(w_2)) = 0 = \text{Tr}(\varphi)\},
\]
we have \( (\text{ad}(\phi))^3 = -\text{Tr}(\phi^2)\text{ad}(\phi) \).

(ii) For all \( \phi \in \text{su}(W, \sigma) \), there exists \( \lambda \in F \) such that the matrix of \( \phi + \lambda aI \) relative to \( B \) is

\[
\begin{pmatrix}
ax & \alpha_2y_3 & \alpha_3y_2 \\
-\alpha_1y_3 & -ax & \alpha_3y_1 \\
-\alpha_1y_2 & -\alpha_2y_1 & 0
\end{pmatrix},
\]

(2)
with \( x \in F, \ y_1, y_2, y_3 \in K, y_i = p_i + q_i a, \) and \( y \mapsto \bar{y} \) denoting the nontrivial \( F \)-automorphism of \( K \). Moreover,
\[
\text{Tr}(\phi^2) = ax^3 + \alpha_2 \alpha_3 (y_1 \bar{y}_1) + \alpha_1 \alpha_3 (y_2 \bar{y}_2) + \alpha_1 \alpha_2 (y_3 \bar{y}_3)
\]
\[
= ax^3 + \alpha_2 \alpha_3 (p_1^2 + q_1^2 a) + \alpha_1 \alpha_3 (p_2^2 + q_2^2 a) + \alpha_1 \alpha_2 (p_3^2 + q_3^2 a).
\]

**Proof.** By the Cayley–Hamilton theorem, \( \phi^3 + \text{Tr}(\phi^2) \phi - \text{det}(\phi) I = 0 \); hence \( (\text{ad}(\phi))^3 = \text{ad}(\phi^3) = -\text{Tr}(\phi^2) \text{ad}(\phi) \). To see the second part, it suffices to take into account that \( \text{Tr}(I) = 0 = \text{Tr}(\phi) \). The form of the matrix \( \phi \) follows directly from the definition of \( \text{su}(W, \sigma) \). Then the existence of \( \lambda \) is trivial, and computing directly the trace of the square of the matrix (2), we obtain the formula for \( \text{Tr}(\phi^2) \).

**Theorem 1.** Let \( C \) be a Cayley algebra over a field \( F \). Then

(i) If the characteristic of \( F \) is not 3, then \( \text{Der}(C) \) is a central simple Lie algebra of dimension 14.

(ii) If the characteristic of \( F \) is 3, then \( \text{Der}(C) \) has a unique proper ideal
\[
I = \text{span}\{D_{xy}; \ x, y \in C\} = \text{ad}[C, C].
\]

In this case, \( I \) is central simple of dimension 7 and type \( A_2 \) (that is, it is a form of \( \text{psl}(3, F) \)) and \( I \cong \text{Der}(C)/I \) as a Lie algebra. Every central simple Lie algebra of type \( A_2 \) emerges as a proper ideal of the Lie algebra of derivations of some Cayley algebra.

**Proof.** (i) Let us suppose that the characteristic of \( F \) is not 3. We can suppose that \( C \) is a split Cayley algebra. Let \( 0 \neq I \) be an ideal of \( \text{Der}(C) = D_0 \oplus D_1 \oplus D_2 \). If \( I \cap D_0 \neq 0 \), as \( D_0 \cong \text{sl}(3, F) \) is simple, then \( D_0 \subseteq I \). Since \( [d, D_{ei, u}] = D_{ei, d(u)}, \forall u \in U, \forall d \in D_0, \) and \( D_0(U) = U \), we obtain that \( D_1 = \{D_{ei, u}; u \in U\} = [D_0, D_1] \subseteq I \). In a similar way, we can prove \( D_2 \subseteq I \). If \( D_0 \cap I = 0 \), as \( I \) is a \( D_0 \)-submodule of \( \text{Der}(C) \) and the decomposition \( \text{Der}(C) = D_0 \oplus D_1 \oplus D_2 \) is in terms of irreducible nonisomorphic \( D_0 \)-modules, we have that \( I = D_1 \) or \( I = D_2 \) or \( I = D_1 \oplus D_2 \). But none of them is an ideal of \( \text{Der}(C) \).

(ii) Let us consider now the characteristic-3 case. We denote by \( C^+ \) the symmetrization of the alternative algebra \( C \), that is, the Jordan algebra whose underlying vector space agrees with that of \( C \) but whose new product is \( x \circ y := xy + yx \). Taking into account that any alternative algebra is flexible, we have that, for all \( x \in C, \text{ad} x \in \text{Der}(C^+) \).

Moreover, \( C \) is a Lie-admissible algebra because \( \text{char}(F) = 3 \) and we have \([x, y, z] + [[y, z], x] + [[z, x], y] = (x, y, z) - (y, x, z) + (y, z, x) - (z, y, x) + (z, x, y) - (x, z, y) = 6(x, y, z) = 0 \). Consequently, the algebra \( C^- \) whose underlying vector space agrees with that of \( C \) but whose product is given by \( [x, y] := xy - yx \) is a Lie algebra. Furthermore,
ad \ x \in \text{Der}(C^{-}), \ \forall \ x \in C. \text{ As char}(F) = 3, \text{ we have } xy = \frac{1}{2}(x \circ y + [x, y]) \text{ and then } \text{ad} \ x \in \text{Der}(C) \text{ for all } x \in C.

Then \ I := \text{ad} \ C = \text{ad}[C, C] \cong [C, C] \text{ is a seven-dimensional ideal of } \text{Der}(C) \text{ (see [6, p. 173]). As char}(F) = 3, D_{a,b} = [L_a - R_a, L_b - R_b] = [\text{ad} a, \text{ad} b] = \text{ad}[a, b], \text{ for all } a, b \in C. \text{ Consequently, } \ I \text{ is the ideal of Lemma 1. Then } \ I \text{ is simple because } [C, C] \text{ is a simple Lie algebra of type } A_2, \text{ and conversely any central simple Lie algebra of type } A_2 \text{ is as the previous one (see [5, p. 352]).}

Let \ J \text{ be an ideal of } \text{Der}(C). \text{ If } I \not\subseteq J, I \cap J = 0, \text{ and then } 0 = [D, \text{ad} x] = \text{ad} D(x) \text{ for all } x \in C \text{ and for all } D \in J. \text{ Then } D(x) \in F_1 \text{ for all } x \in C, \text{ and as } D(C) = D([C, C]) \subseteq [C, C], \text{ we have that } D = 0 \text{ and } J = 0. \text{ Thus, } I \text{ is the only minimal proper ideal of } \text{Der}(C).

Let \ K \text{ be a two-dimensional composition subalgebra of } C, \text{ let } K = F[a] \text{ with } 0 \neq a^2 = -\alpha \in F, \text{ and let } Z_{0} = \{d \in \text{Der}(C); d(K) = 0\}. \text{ By extending scalars, we have that } Z_{0} \text{ is isomorphic to } \text{sl}(3). \text{ As in [7, p. 70], } Z_{0} \cong \text{su}(K, \sigma), \text{ with } \sigma(x, y) = q(x, y) - \alpha^{-1}aq(ax, y). \text{ Since } I \cap Z_{0} \text{ is a proper ideal of } Z_{0}, \text{ then } I \cap Z_{0} = Z(Z_{0}) \text{ whose dimension is } 1. \text{ Thus } \text{Der}(C)/I \cong Z_{0}/Z(Z_{0}) = \text{psu}(K, \sigma), \text{ which is a central simple Lie algebra of type } A_2, \text{ that is, a form of } \text{psl}(3, F).

If \ b \in K \text{ with } q(b) = \beta \neq 0 \text{ and } c \in (K + Kb)^\perp \text{ with } q(c) = \gamma \neq 0, \text{ then } \{b, c, bc\} \text{ is an orthogonal } K\text{-basis of } K \text{ and the coordinate matrix of } \sigma \text{ in this basis is diag}(\beta, \gamma, \beta \gamma).

On the other hand, there exists a Cayley algebra \ C, such that \ \text{psu}(K^\perp, \sigma) \cong [\ C, C]\] as a Lie algebra. \text{ As } \forall x, y \in [\ C, C]\text{ we have } [[x, y], y] = xy^2 + y^2x - 2yx^2, \text{ with } \bar{q} \text{ the norm in } \ C, \text{ then } (\text{ad} y)^3 = -\bar{q}(y)\text{ad} y. \text{ Now, as for all } \phi \in \text{su}(K^\perp, \sigma) \text{ we have } (\text{ad} \phi)^3 = -\text{Tr}(\phi^2)\text{ad} \phi, \text{ it follows that } \bar{q}|_{[\ C, C]} \text{ is equivalent to the quadratic form defined in } \text{psu}(K^\perp, \sigma) \text{ as } \bar{Q}(\phi + KI) := \text{Tr}(\phi^2).

By Lemma 2, the coordinate matrix of \ \tilde{Q} \text{ (relative to a suitable basis) is}

\text{diag}(\alpha, \beta \gamma \alpha, \beta \gamma^2 \alpha, \beta^2 \gamma \alpha, \beta \gamma, \beta \gamma \alpha)

\text{(take into account that } \{\alpha_1 = \beta, \alpha_2 = \gamma, \alpha_3 = \beta \gamma\}). \text{ Consequently, scaling the basis, we get the following coordinate matrix of } \tilde{Q}:

\text{diag}(\alpha, \beta, \alpha \beta, \gamma, \alpha \gamma, \beta \gamma, \alpha \beta \gamma).

Then \ \tilde{Q} \text{ is equivalent to } q|_{[C, C]} \text{ and } \bar{q} \text{ is equivalent to } q. \text{ This implies that } \ C \text{ is isomorphic to } C \text{ (see [7, p. 62] or [8, Theorem 3.23, p. 70]), and } \text{Der}(C)/I \cong \text{psu}(K^\perp, \sigma) \cong [C, C] \cong I.
3. AN ALTERNATIVE APPROACH

To finish this work, we include some of the computational techniques which motivated some of the results in this work. These computations prove also the simplicity of $\text{Der}(C)$ out of characteristic 3.

Once we have obtained the general form (1) of a derivation in the canonical basis, we can also compute a basis of $\text{Der}(O)$. This will be useful later to study the simplicity of $\text{Der}(O)$. Thus we can construct a canonical basis of $\text{Der}(O)$ by taking $\{d_i: i = 1, \ldots, 14\}$, where $d_i$ is obtained by setting $t_i = 1$ and $t_j = 0$ if $j \neq i$ at the previous general matrix of a derivation. The multiplication table of this basis is the following Table 1.

Next we prove that $\text{Der}(C)$ is central simple when the characteristic is not 3. We take first $C = O$. Let us denote by $[\ , \ ]$ the Lie product of $\text{Der}(O)$. Take the first canonical derivation $d_1$ and perform left multiplications by all the 14 canonical derivations; that is, consider the set

$$\{[d_1, d_1], [d_2, d_1], [d_3, d_1], \ldots, [d_{14}, d_1]\}.$$

The maximum number of linearly independent derivations in the linear span of the above set is determined by the characteristic of $F$, namely,

$$\{d_1, d_2, d_3, 2d_5, -2d_6, 2d_7 - d_{11}, 3d_{10}, 3d_{13}\}, \quad \text{if char}(F) \neq 2, 3,$$

$$\{d_1, d_2, d_3, d_{10}, d_{11}, d_{13}\}, \quad \text{if char}(F) = 2.$$

Perform now left multiplications of any of the eight (or six) derivations above by any of the 14 canonical derivations. If the characteristic of $F$ is neither 2 nor 3, we obtain 14 linearly independent derivations in the linear span of these brackets. In the case of characteristic 2, there are 13 linearly independent derivations, namely,

$$\{d_1, d_2, d_3, d_5, d_6, d_7, d_8, d_9, d_{10}, d_{11}, d_{12}, d_{13}, d_{14}\},$$

that is, all of the canonical derivations but $d_4$. As $[d_6, d_3] = d_4$, we finally see that, when the characteristic is other than 3, the principal ideal generated by $d_1$ is the whole algebra $\text{Der}(O)$. Then, by looking at the multiplication table of $\text{Der}(O)$, we see that $d_1$ is in the ideal generated by $d_i$ with $i \neq 1$; that is, the ideal generated by any of the 14 canonical derivations is the whole algebra, if the characteristic is not 3.

Now let us show that if the characteristic is not 3, any nontrivial derivation of $O$ may be transformed into a scalar multiple of a canonical derivation by performing left multiplications by, at most, three suitable canonical derivations. This will prove that any nontrivial ideal of $\text{Der}(O)$ is the whole algebra; that is, $\text{Der}(O)$ is simple. (It is clear that $[\text{Der}(O), \text{Der}(O)] \neq 0$.) Let $\tau := \sum_{i=1}^{14} \lambda_i d_i, \lambda_i \in F$, be a derivation of $O$. As $[d_{10}, [d_{12}, [d_{13}, \tau]]] = \lambda_8 d_{10}$, if $\lambda_8 \neq 0$, we are done. If $\lambda_8 = 0$, then
<table>
<thead>
<tr>
<th>Table I</th>
</tr>
</thead>
<tbody>
<tr>
<td>(d_1)</td>
</tr>
<tr>
<td>---------</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>
proceed as follows:

If \( \lambda_8 = 0 \), then \([d_{10}, [d_{13}, \tau]] = -\lambda_9 d_{10}\).
If further on \( \lambda_9 = 0 \), then \([d_{13}, [d_{14}, \tau]] = -\lambda_{12} d_{13}\).
If further on \( \lambda_{12} = 0 \), then \([d_{14}, [d_{2}, \tau]] = 3\lambda_2 d_{14}\).
If further on \( \lambda_5 = 0 \), then \([d_{13}, [d_{1}, \tau]] = 3\lambda_4 d_{13}\).
If further on \( \lambda_4 = 0 \), then \([d_8, [d_{2}, \tau]] = -\lambda_{10} d_2\).
If further on \( \lambda_{10} = 0 \), then \([d_{14}, [d_{2}, \tau]] = -\lambda_5 d_2\).
If further on \( \lambda_3 = 0 \), then \([d_8, [d_7, \tau]] = -3\lambda_6 d_{13}\).
If further on \( \lambda_6 = 0 \), then \([d_8, [d_7, \tau]] = -\lambda_7 d_2\).
If further on \( \lambda_7 = 0 \), then \([d_8, [d_7, \tau]] = -\lambda_{11} d_6\).
If further on \( \lambda_{11} = 0 \), then \([d_8, [d_7, \tau]] = -\lambda_{14} d_3\).
If further on \( \lambda_{14} = 0 \), then \([d_6, [d_7, \tau]] = 3\lambda_1 d_{13}\).
If further on \( \lambda_1 = 0 \), then \([d_7, [d_{10}, \tau]] = \lambda_2 d_1\).
If further on \( \lambda_2 = 0 \), then \( \tau = \lambda_{13} d_{13}\).

In the characteristic-3 case, we can prove with a similar straight argument that any nontrivial derivation of \( O_4 \) may be transformed into a scalar multiple of canonical derivations by performing left multiplications by canonical derivations. Now, pick the first canonical derivation \( d_1 \) and proceed to left-multiplying it by all of the canonical derivations. A basis of six derivations is obtained in the linear span of these brackets:

\[ \{d_1, d_2, d_3, 2d_5, d_6, 2d_7 + 2d_{11}\}. \]

Left-multiplying each one by the canonical derivations, we obtain a basis of seven derivations in the linear span of the previous brackets:

\[ \{d_1, d_2, d_3, 2d_4, 2d_5, d_6, 2d_7 + 2d_{11}\}. \]

Now left-multiplying each of these derivations by any canonical derivation produces no more linearly independent derivations. Thus, the ideal \( \langle d_1 \rangle \) generated by \( d_1 \) is a proper seven-dimensional ideal of \( \text{Der}(O_4) \) with the previous derivations as a basis. Let us show that the ideals generated by \( d_2, d_3, d_4, d_5, \) or \( d_6 \) are identical to that generated by \( d_1 \). As these derivations are in the basis of \( \langle d_1 \rangle \), the ideals generated by any of them are included in \( \langle d_1 \rangle \). Conversely,

\[
[d_{10}, d_2] = d_1, \quad [d_{13}, d_3] = d_1, \quad [d_{10}, [d_6, d_4]] = 2d_1,
[d_6, d_5] = -2d_1.
\]

In a similar way, it can be proved that the ideals generated by \( d_i \), with \( i = 7, 8, \ldots, 14 \), are the whole algebra. Since any derivation of \( O_4 \) may be
transformed into a scalar multiple of canonical derivations by performing left multiplications, we have proved that the ideal $\langle d_1 \rangle$ is contained in any nonzero ideal of $\text{Der}(O_s)$. On the other hand, one can find a natural isomorphism $\text{Der}(O_s)/\langle d_1 \rangle \cong \text{psl}(3, F)$ which proves that $\langle d_1 \rangle$ is maximal, hence unique.

REFERENCES