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Well-posedness of a class of perturbed optimization problems in Banach spaces ☆

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ABSTRACT

Let X be a Banach space and Z a nonempty subset of X. Let $J: Z \to \mathbb{R}$ be a lower semicontinuous function bounded from below and $p \geqslant 1$. This paper is concerned with the perturbed optimization problem of finding $z_0 \in Z$ such that $\|x - z_0\|^p + J(z_0) = \inf_{z\in Z}\{\|x - z\|^p + J(z)\}$, which is denoted by $\min_J(x, Z)$. The notions of the J-strictly convex with respect to Z and of the Kadec with respect to Z are introduced and used in the present paper. It is proved that if X is a Kadec Banach space with respect to Z and Z is a closed relatively boundedly weakly compact subset, then the set of all $x \in X$ for which every minimizing sequence of the problem $\min_J(x, Z)$ has a converging subsequence is a dense G_δ-subset of $X \setminus Z_0$, where Z_0 is the set of all points $z \in Z$ such that z is a solution of the problem $\min_J(z, Z)$. If additionally $p > 1$ and X is J-strictly convex with respect to Z, then the set of all $x \in X$ for which the problem $\min_J(x, Z)$ is well-posed is a dense G_δ-subset of $X \setminus Z_0$.

1. Introduction

Let X be a real Banach space endowed with the norm $\|\cdot\|$. Let Z be a nonempty closed subset of X, $J: Z \to \mathbb{R}$ a function defined on Z and let $p \geqslant 1$. The perturbed optimization problem considered here is of finding an element $z_0 \in Z$ such that

$$\|x - z_0\|^p + J(z_0) = \inf_{z\in Z}\left\{\|x - z\|^p + J(z)\right\} \tag{1.1}$$

which is denoted by $\min_J(x, Z)$. Any point z_0 satisfying (1.1) (if exists) is called a solution of the problem $\min_J(x, Z)$. In particular, if $J \equiv 0$, then the perturbed optimization problem $\min_J(x, Z)$ reduces to the well-known best approximation problem.

The perturbed optimization problem $\min_J(x, Z)$ was presented and investigated by Baranger in [2] for the case when $p = 1$ and by Bidaut in [6] for the case when $p \geqslant 1$. The existence results have been applied to optimal control problems governed by partial differential equations, see for example, [2–6,8,16,26].

Assume that J is lower semicontinuous and bounded from below. In the case when $p = 1$, Baranger in [2] proved that if X is a uniformly convex Banach space then the set of all $x \in X$ for which the problem $\min_J(x, Z)$ has a solution is a dense G_δ-subset of X, which clearly extends Stechkin's results in [30] on the best approximation problem. Since then, this

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problem has been studied extensively, see for example [6,8,20,28]. In particular, Cobzas extended in [9] Baranger's result to the setting of reflexive Kadec Banach space; while Ni relaxed in [27] the reflexivity assumption made in Cobzas' result.

For the general case when $p>1$, this kind of perturbed optimization problems is only founded to be studied by Bidaut in [6]. Recall from [23] that a sequence $\{z_n\}\subseteq Z$ is a minimizing sequence of the problem $\min_J(x,Z)$ if

$$\lim_{n\to\infty}\bigl(\|x-z_n\|^p+J(z_n)\bigr)=\inf_{z\in Z}\bigl(\|x-z\|^p+J(z)\bigr),$$

and that the problem $\min_J(x,Z)$ is well-posed if $\min_J(x,Z)$ has a unique solution and every minimizing sequence of the problem $\min_J(x,Z)$ converges to this solution. It was proved in [6] that if X is a uniformly convex Banach space and Z is a bounded closed subset, then the set of all $x\in X$ such that the problem $\min_J(x,Z)$ is well-posed is a dense G_δ-subset of $X\setminus Z$. Recently, for the special case when $p=2$, Fabian proved in [17] that if X is reflexive and Kadec, then the set of all $x\in X$ such that $\min_J(x,Z)$ has a solution is a residual set of X.

The purpose of the present paper is to continue to carrying out investigations in this line and to try to extend the results due to Bidaut in [6] to the general setting of nonreflexive Banach spaces. More precisely, we introduce the notions of the J-strict convexity with respect to Z and of Kadec property with respect to Z, and prove that if Z is a nonempty closed, relatively boundedly weakly compact subset of X (not necessarily bounded) and that X is a Kadec Banach space with respect to Z, then the set of all $x\in X$ for which every minimizing sequence of the problem $\min_J(x,Z)$ has a converging subsequence is a dense G_δ-subset of $X\setminus Z_0$, where Z_0 is the set of all points $z\in Z$ such that z is a solution of the problem $\min_J(z,Z)$. If X is additionally assumed to be J-strictly convex with respect to Z and $p>1$, then we further show that the set of all $x\in X$ for which the problem $\min_J(x,Z)$ is well-posed is a dense G_δ-subset of $X\setminus Z_0$. Examples are provided to illustrate that our results obtained in the present paper extend the earlier ones even in the case when $p=1$.

2. Preliminaries

We begin with some standard notations. Let X be a Banach space with the dual X^*. We use $\langle\cdot,\cdot\rangle$ to denote the inner product connecting X^* and X. The closed (respectively open) ball in X at center x with radius r is denoted by $\mathbf{B}_X(x,r)$ (respectively $\mathbf{U}(x,r)$). In particular, we write $\mathbf{B}_X=\mathbf{B}_X(0,1)$ and $\mathbf{B}^*=\mathbf{B}_{X^*}$ for short, and omit the subscript if no confusion caused. For a subset A of X, the linear hull and the closure of A are respectively denoted by $\operatorname{span}A$ and $\overline{A}$. We first recall the notation of Fréchet differentiability and a related important proposition, see for example [29].

Definition 2.1. Let A be an open subset of X and $f:A\to\mathbb{R}$ a real-valued function. Let $x\in A$. f is said to be Fréchet differentiable at x if there exists an $x^*\in X^*$ such that

$$\lim_{y\to x}\frac{f(y)-f(x)-\langle x^*,y-x\rangle}{\|y-x\|}=0.$$

x^* is called the Fréchet differential at x which is denoted by $\mathrm{D}f(x)$.

Proposition 2.1. *Let f be a locally Lipschitz continuous function on an open subset A of X. Suppose that X is a reflexive Banach space. Then f is Fréchet differentiable on a dense subset of A.*

The following notions are well-known, see for example, [7,25].

Definition 2.2. X is said to be

(i) strictly convex if, for any $x_1,x_2\in\mathbf{B}$, the condition $\|x_1+x_2\|=2$ implies that $x_1=x_2$;
(ii) uniformly convex if, for any sequences $\{x_n\},\{y_n\}\subseteq\mathbf{B}$, the condition $\lim_{n\to\infty}\|x_n+y_n\|=2$ implies that $\lim_{n\to\infty}\|x_n-y_n\|=0$;
(iii) (sequentially) Kadec if, for any sequence $\{x_n\}\subseteq\mathbf{B}$, $x_0\in\mathbf{B}$ with $\|x_n\|\to\|x_0\|$, the condition $x_n\to x_0$ weakly implies that $\lim_{n\to\infty}\|x_n-x_0\|=0$.

The notions in the following definition are the refinements and extensions of the corresponding ones in Definition 2.2, where part (i) is known in [1]. Let Z be a subset of X and J be a real-valued function on Z.

Definition 2.3. X is said to be

(i) strictly convex with respect to (w.r.t.) Z, if, for any $z_1,\ z_2\in Z$ such that $\|x-z_1\|=\|x-z_2\|$ for some $x\in X$, the condition $\|x-z_1+x-z_2\|=\|x-z_1\|+\|x-z_2\|$ implies that $z_1=z_2$;
(ii) J-strictly convex with respect to (w.r.t.) Z, if, for any $z_1,\ z_2\in Z$ such that $\|x-z_1\|=\|x-z_2\|$ for some $x\in X$, the conditions that $\|x-z_1+x-z_2\|=\|x-z_1\|+\|x-z_2\|$ and $J(z_1)=J(z_2)$ imply that $z_1=z_2$;
(iii) J-strictly convex, if X is J-strictly convex w.r.t. X;
(iv) (sequentially) Kadec with respect to (w.r.t.) Z, if, for any sequence $\{z_n\}\subseteq Z$ and $z_0\in Z$ such that there exists a point $x\in X$ satisfying $\lim_{n\to+\infty}\|x-z_n\|=\|x-z_0\|$, the condition $z_n\to z_0$ weakly implies that $\lim_{n\to\infty}\|z_n-z_0\|=0$.

In particular, in the case when $Z = X$, the strict convexity w.r.t. Z (respectively the Kadec property w.r.t. Z) reduces to the strict convexity (respectively the Kadec property), while in the case when $J \equiv 0$, the J-strict convexity w.r.t. Z reduces to the strict convexity w.r.t. Z. Moreover, the following implications are clear for any subset Z of X and real-valued function J on Z:

$$\begin{array}{ccc} \text{the strict convexity} & \Longrightarrow & \text{the strict convexity w.r.t. } Z \\ \Downarrow & & \Downarrow \\ \text{the } J\text{-strict convexity} & \Longrightarrow & \text{the } J\text{-strict convexity w.r.t. } Z \end{array} \tag{2.1}$$

and

$$\text{the Kadec property} \quad \Longrightarrow \quad \text{the Kadec property w.r.t. } Z. \tag{2.2}$$

Note that X is Kadec w.r.t. Z provided that Z is locally compact. The following example presents the cases when X is J-strictly convex w.r.t. Z and/or Kadec w.r.t. Z but not strictly convex and/or Kadec. Recall from [18,19] that X is said to be uniformly convex in every direction if, for every $z \in X \setminus \{0\}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that $|\lambda| < \epsilon$ if $\|x\| = \|y\| = 1$, $x - y = \lambda z$ and $\frac{1}{2}\|x + y\| > 1 - \delta$. From [11], it follows that X is uniformly convex in every direction if and only if, for any sequences $\{x_n\} \subseteq \mathbf{B}$ and $\{y_n\} \subseteq \mathbf{B}$, the conditions $\{x_n - y_n\} \subseteq \operatorname{span}\{z\}$ for some $z \in X$ and $\|x_n + y_n\| \to 2$ imply $\|x_n - y_n\| \to 0$.

Example 2.1. Let Y be a Banach space and let $X = l_\infty(Y)$ denote the Banach space of all sequences (x_i) of Y such that $\sup_i \|x_i\| < \infty$ with the norm $\|\cdot\|_\infty$ defined by

$$\|x\|_\infty = \sup_i \|x_i\| \quad \text{for each } x = (x_i) \in l_\infty(Y).$$

Let $X_c = l_c(Y)$ be the subspace of $l_\infty(Y)$ given by

$$l_c(Y) = \big\{x = (x_i) \in l_\infty(Y)\colon\ \{x_i\}_{i\in\mathbb{N}} \text{ is totally bounded}\big\}.$$

Clearly, Y can be isometrically embedded in X_c by the mapping $y \mapsto (x, x, \ldots,)$ for each $y \in Y$. Then the following assertions hold.

(1) If Y is Kadec, then X_c is Kadec w.r.t. Y.
(2) If Y is strictly convex, then X_c is strictly convex w.r.t. Y.
(3) If Y is uniformly convex, then X is Kadec w.r.t. Y.
(4) If Y is uniformly convex in every direction, then X is strictly convex w.r.t. Y.
(5) X_c contains an isometric copy of l_∞ and hence X and X_c are neither Kadec nor strictly convex even if Y is uniformly convex.

Proof. Recall that a subset A of a Banach space is totally bounded if and only if its closure $\overline{A}$ is compact. Thus, the assertion (5) is clear because, for some fixed $y \in Y$ with $\|y\| = 1$, the mapping $(\alpha_i) \mapsto (\alpha_i y)$ represents an isometric embedding of l_∞ in $l_c(Y)$ (noting that $\{\alpha_i y\}_{i\in\mathbb{N}}$ is totally bounded for each $(\alpha_i) \in l_\infty$).

Below we only verify the assertion (1) because the other assertions can be proved similarly. Let $\{z_n\} \subseteq Y$ and $z_0 \in Y$ be such that $\lim_{n\to\infty} \|x - z_n\|_\infty = \|x - z_0\|_\infty > 0$ for some $x = (x_i) \in l_c(Y)$ and $z_n \rightharpoonup z_0$ weakly. Let $x^* \in l_c(Y)^*$ with $\|x^*\| = 1$ be such that $\langle x^*, (x - z_0)\rangle = \|x - z_0\|_\infty$. Then

$$\|x - z_n + x - z_0\|_\infty \geqslant \big\langle x^*, (x - z_n + x - z_0)\big\rangle \to 2\|x - z_0\|_\infty.$$

Thus $\|x - z_n + x - z_0\|_\infty \to 2\|x - z_0\|_\infty$. Note that $\overline{\{x_i\}_{i\in\mathbb{N}}}$, the closure of $\{x_i\}_{i\in\mathbb{N}}$, is compact since $\{x_i\}_{i\in\mathbb{N}}$ is totally bounded. Then, by the definition of $\|\cdot\|_\infty$, there exists a sequence $\{a_n\}$ contained in $\overline{\{x_i\}_{i\in\mathbb{N}}}$ such that

$$\|2x - z_n - z_0\|_\infty = \|2a_n - z_n - z_0\| \quad \text{for each } n = 1, 2, \ldots.$$

Moreover, without loss of generality, we may assume that $a_n \to a_0$ for some $a_0 \in \overline{\{x_i\}_{i\in\mathbb{N}}}$. Since

$$\big|\|2a_n - z_n - z_0\| - \|2a_0 - z_n - z_0\|\big| \leqslant 2\|a_n - a_0\|,$$

it follows that

$$\lim_n \big\|(a_0 - z_n) + (a_0 - z_0)\big\| = \lim_n \|2a_n - z_n - z_0\| = \lim_n \|2x - z_n - z_0\|_\infty = 2\|x - z_0\|_\infty. \tag{2.3}$$

Note that $\|a_0 - z_n\| \leqslant \|x - z_0\|_\infty$ and $\|a_0 - z_0\| \leqslant \|x - z_0\|_\infty$. This together with (2.3) implies that

$$\|a_0 - z_0\| = \|x - z_0\|_\infty \quad \text{and} \quad \lim_{n\to\infty} \|a_0 - z_n\| = \|x - z_0\|_\infty.$$

Since $a_0 - z_n \to a_0 - z_0$ weakly and Y is Kadec, we have that $a_0 - z_n \to a_0 - z_0$ and hence $\|z_n - z_0\| \to 0$. This completes the proof of the first assertion. □

Note that X is J-strictly convex w.r.t. Z if J is one to one on Z. One example for which X is J-strictly convex w.r.t. Z but not strictly convex w.r.t. Z is as follows.

Example 2.2. Let X be the Banach space l_∞ with the sup-norm defined by $\|x\| = \sup_i |c_i|$ for each $x = (c_i) \in l_\infty$. Let $Z := \{z = (t, 0, \ldots) \in X\colon\ t \geqslant 0\}$ and $J : Z \to \mathbb{R}$ the function defined by $J(z) = \|z\|$ for each $z \in Z$. Then J is one to one on Z. Hence X is J-strictly convex w.r.t. Z. Let $z_1 = (1, 0, \ldots) \in Z$, $z_2 = (2, 0, \ldots) \in Z$ and $x = (1, 1, \ldots) \in l_\infty$. Then $\|x - z_1\| = 1$, $\|x - z_2\| = 1$ and $\|x - z_1 + x - z_2\| = 2$. This means that X is not strictly convex w.r.t. Z because $z_1 \neq z_2$.

We end this section with the factorization theorem due to Davis, Figiel, Johnson and Pelczynski in [10], see also [14], which will play an important role for our study in the next section.

Proposition 2.2. *Let A be a weakly compact subset of a Banach space X and let $Y = \overline{\operatorname{span} A}$. Then there exist a reflexive Banach space R and a one-to-one continuous linear mapping $T : R \to Y$ such that $T(\mathbf{B}_R) \supseteq A$, where $\mathbf{B}_R$ denotes the unit ball in R.*

3. Minimization problems

Let $p \geqslant 1$. For the remainder of the present paper, we always assume that Z is a nonempty closed subset of X, $J : Z \to \mathbb{R}$ is a lower semicontinuous function bounded from below. Without loss of generality, we may assume that

$$\inf_{z\in Z} J(z) > 0. \tag{3.1}$$

Define the function $\varphi : X \mapsto \mathbb{R}$ by

$$\varphi(x) = \inf_{z\in Z}\bigl\{\|x - z\|^p + J(z)\bigr\}^{\frac{1}{p}} \quad \text{for each } x \in X. \tag{3.2}$$

Let $x \in X$. Then $z_0 \in Z$ is a solution to the problem $\min_J(x, Z)$ if and only if z_0 satisfies that

$$\bigl(\|x - z_0\|^p + J(z_0)\bigr)^{\frac{1}{p}} = \varphi(x). \tag{3.3}$$

The set of all solutions to the problem $\min_J(x, Z)$ is denoted by $P_{Z,J}(x)$, that is,

$$P_{Z,J}(x) = \bigl\{z_0 \in Z\colon\ \bigl\{\|x - z_0\|^p + J(z_0)\bigr\}^{\frac{1}{p}} = \varphi(x)\bigr\}.$$

Lemma 3.1. *Let $\varphi : X \mapsto \mathbb{R}$ be defined by* (3.2)*. Then*

$$\bigl|\varphi(x) - \varphi(x')\bigr| \leqslant \|x - x'\| \quad \textit{for any } x, x' \in X. \tag{3.4}$$

Proof. Let $x, x' \in X$. It suffices to verify that

$$\varphi(x) - \varphi(x') \leqslant \|x - x'\|. \tag{3.5}$$

Since $J(z) > 0$ for each $x \in Z$ by (3.1), we have that, for each $z \in Z$,

$$\begin{aligned}\bigl(\|x - z\|^p + J(z)\bigr)^{\frac{1}{p}} &\leqslant \Bigl(\bigl(\|x - x'\| + \|x' - z\|\bigr)^p + \bigl(0 + J(z)^{\frac{1}{p}}\bigr)^p\Bigr)^{\frac{1}{p}}\\ &\leqslant \|x - x'\| + \bigl(\|x' - z\|^p + J(z)\bigr)^{\frac{1}{p}}.\end{aligned} \tag{3.6}$$

It follows that

$$\inf_{z\in Z}\bigl(\|x - z\|^p + J(z)\bigr)^{\frac{1}{p}} \leqslant \|x - x'\| + \inf_{z\in Z}\bigl(\|x' - z\|^p + J(z)\bigr)^{\frac{1}{p}}$$

and (3.5) is proved. □

Lemma 3.2. *Let Y be a subspace of X, $x \in Y$ and $y^* \in Y^*$. Suppose that*

$$\lim_{t\to 0^+}\left(\frac{\varphi(x + th) - \varphi(x)}{t} - \langle y^*, h\rangle\right) = 0 \quad \textit{for each } h \in Y. \tag{3.7}$$

Let $\{z_n\} \subseteq Z$ be a minimizing sequence of the problem $\min_J(x, Z)$ such that $b(x) := \lim_{n\to\infty}\|x - z_n\|$ exists. Then

$$\|y^*\| \leqslant \frac{b^{p-1}(x)}{\varphi^{p-1}(x)}. \tag{3.8}$$

Proof. Let $t > 0$ and $\epsilon > 0$. Then, there exists $N > 0$ such that

$$\bigl(\|x - z_n\|^p + J(z_n)\bigr)^{\frac{1}{p}} < \varphi(x) + t\epsilon \quad \text{for each } n \geqslant N. \tag{3.9}$$

Let $h \in Y$ and $n \geqslant N$. Then, in view of the definition of φ, one has that

$$\varphi(x+th) - \varphi(x) \leqslant \left(\|x+th-z_n\|^p + J(z_n)\right)^{\frac{1}{p}} - \left(\|x-z_n\|^p + J(z_n)\right)^{\frac{1}{p}} + t\epsilon. \tag{3.10}$$

Write $s_t = \|x+th-z_n\| - \|x-z_n\|$. Then,

$$s_t \leqslant t\|h\|. \tag{3.11}$$

Define the function $\gamma_n : [0,+\infty) \to \mathbb{R}$ by

$$\gamma_n(s) = \left[\left(\|x-z_n\|+s\right)^p + J(z_n)\right]^{\frac{1}{p}} \quad \text{for each } s \in [0,+\infty).$$

Then

$$\gamma_n'(s) = \left[\left(\|x-z_n\|+s\right)^p + J(z_n)\right]^{\frac{1-p}{p}} \left(\|x-z_n\|+s\right)^{p-1} \quad \text{for each } s \in [0,+\infty),$$

It follows from the Mean-Value Theorem that there exists $\theta \in (0,1)$ such that

$$\frac{\gamma_n(s_t) - \gamma_n(0)}{s_t} = \left[\left(\|x-z_n\|+\theta s_t\right)^p + J(z_n)\right]^{\frac{1-p}{p}} \left(\|x-z_n\|+\theta s_t\right)^{p-1}. \tag{3.12}$$

This together with (3.11) implies that

$$\frac{\gamma_n(s_t) - \gamma_n(0)}{t} \leqslant \left[\left(\|x-z_n\|+t\|h\|\right)^p + J(z_n)\right]^{\frac{1-p}{p}} \left(\|x-z_n\|+t\|h\|\right)^{p-1}\|h\|. \tag{3.13}$$

Hence

$$\lim_{n\to+\infty} \frac{\gamma_n(s_t) - \gamma_n(0)}{t} \leqslant \left[\left(b(x)+t\|h\|\right)^p + \varphi^p(x) - b^p(x)\right]^{\frac{1-p}{p}} \left(b(x)+t\|h\|\right)^{p-1}\|h\|$$

and

$$\lim_{t\to 0^+} \lim_{n\to+\infty} \frac{\gamma_n(s_t) - \gamma_n(0)}{t} \leqslant \frac{b^{p-1}(x)}{\varphi^{p-1}(x)}\|h\|. \tag{3.14}$$

By (3.10),

$$\varphi(x+th) - \varphi(x) \leqslant \gamma(s_t) - \gamma(0) + t\epsilon; \tag{3.15}$$

hence

$$\frac{\varphi(x+th) - \varphi(x)}{t} \leqslant \frac{\gamma_n(s_t) - \gamma_n(0)}{t} + \epsilon.$$

Combining this with (3.14), we get that

$$\lim_{t\to 0^+} \frac{\varphi(x+th) - \varphi(x)}{t} \leqslant \lim_{t\to 0^+} \lim_{n\to+\infty} \frac{\gamma_n(s_t) - \gamma_n(0)}{t} \leqslant \frac{b^{p-1}(x)}{\varphi^{p-1}(x)}\|h\| + \epsilon$$

and so

$$\lim_{t\to 0^+} \frac{\varphi(x+th) - \varphi(x)}{t} \leqslant \frac{b^{p-1}(x)}{\varphi^{p-1}(x)}\|h\|.$$

This together with assumption (3.7) yields that

$$\langle y^*, h\rangle \leqslant \frac{b^{p-1}(x)}{\varphi^{p-1}(x)}\|h\| \tag{3.16}$$

and (3.8) is seen to hold because $h \in Y$ is arbitrary. □

Let $q \geqslant 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$ and let $a : \mathbf{B}^* \to \mathbb{R}$ be the function defined by

$$a(x^*) = \left(1 - \|x^*\|^q\right)^{\frac{1}{q}} \quad \text{for each } x^* \in \mathbf{B}^*.$$

For $\delta > 0$, set

$$Z_J(x,\delta) = \left\{z \in Z\colon\ \left(\|x-z\|^p + J(z)\right)^{\frac{1}{p}} < \varphi(x) + \delta\right\} \tag{3.17}$$

and $Z_0 = \{z \in Z\colon\ z \in P_{Z,J}(z)\}$. Define for each $n \in \mathbb{N}$

$$H_n^{\varphi}(Z) = \left\{ x \in X \setminus Z_0\colon\ \begin{array}{c} \text{there exist } \delta > 0 \text{ and } x^* \in \mathbf{B}^* \text{ such that} \\ \inf\limits_{z \in Z_J(x,\delta)} \left\{\langle x^*, x-z\rangle + a(x^*) J^{\frac{1}{p}}(z)\right\} > \left(1 - 2^{-n}\right)\varphi(x) \end{array} \right\}. \tag{3.18}$$

Furthermore we write

$$H^{\varphi}(Z)=\bigcap_{n=1}^{\infty}H_n^{\varphi}(Z) \tag{3.19}$$

and

$$M^{\varphi}(Z)=\left\{x\in X\setminus Z_0\colon \begin{array}{l}\text{there is } x^*\in\mathbf{B}^* \text{ such that for each } \epsilon\in[0,1] \text{ there is } \delta>0\\ \text{satisfying } \inf\limits_{z\in Z_J(x,\delta)}\{\langle x^*,x-z\rangle+a(x^*)J^{\frac{1}{p}}(z)\}>(1-\epsilon)\varphi(x)\end{array}\right\}.$$

Obviously,

$$M^{\varphi}(Z)\subset H^{\varphi}(Z). \tag{3.20}$$

Lemma 3.3. *Let Z be a relatively boundedly weakly compact subset of X. Then $H^{\varphi}(Z)$ is a dense G_δ-subset of $X\setminus Z_0$.*

Proof. We first verify that $H^{\varphi}(Z)$ is a G_δ-subset of X. By (3.19), we only need to prove that $H_n^{\varphi}(Z)$ is open for each n. For this end, let $n\in\mathbb{N}$ and $x\in H_n^{\varphi}(Z)$. Then there exist $\delta>0$ and $x^*\in\mathbf{B}^*$ such that

$$\beta:=\inf_{z\in Z_J(x,\delta)}\{\langle x^*,x-z\rangle+a(x^*)J^{\frac{1}{p}}(z)\}-\left(1-2^{-n}\right)\varphi(x)>0. \tag{3.21}$$

Let $\lambda>0$ be such that $\lambda<\min\{\delta/2,\beta/2\}$. It suffices to show that $\mathbf{U}(x,\lambda)\subset H_n^{\varphi}(Z)$. To do this, let $y\in\mathbf{U}(x,\lambda)$ and $\delta^*=\delta-2\lambda$. Let $z\in Z_J(y,\delta^*)$ be arbitrary. Then

$$\left(\|y-z\|^p+J(z)\right)^{1/p}<\varphi(y)+\delta^*. \tag{3.22}$$

It follows that

$$\left(\|x-z\|^p+J(z)\right)^{1/p}\leqslant\left(\|y-z\|^p+J(z)\right)^{1/p}+\|y-x\|<\varphi(y)+\delta^*+\lambda$$

since $\|x-y\|<\lambda$. By (3.4), one has that

$$\left(\|x-z\|^p+J(z)\right)^{1/p}\leqslant\varphi(y)+\delta^*+\lambda\leqslant\varphi(x)+\delta^*+2\lambda=\varphi(x)+\delta.$$

Hence $z\in Z_J(x,\delta)$. It follows from (3.21) that

$$\langle x^*,x-z\rangle+a(x^*)J^{1/p}(z)\geqslant\beta+\left(1-2^{-n}\right)\varphi(x). \tag{3.23}$$

Therefore,

$$\begin{aligned}\langle x^*,y-z\rangle+a(x^*)J^{1/p}(z)&=\langle x^*,x-z\rangle+a(x^*)J^{1/p}(z)+\langle x^*,y-x\rangle\\&\geqslant\beta+\left(1-2^{-n}\right)\varphi(x)-\|x-y\|\\&\geqslant\beta+\left(1-2^{-n}\right)\varphi(y)-\|x-y\|-\left(1-2^{-n}\right)\|x-y\|\\&\geqslant\left(1-2^{-n}\right)\varphi(y)+\beta-2\lambda\\&\geqslant\left(1-2^{-n}\right)\varphi(y),\end{aligned}$$

where the first inequality holds because of (3.23), the second one because of (3.4) and the last two hold because $y\in\mathbf{U}(x,\lambda)$ and $\lambda<\min\{\delta/2,\beta/2\}$. Consequently,

$$\inf_{z\in Z_J(y,\delta^*)}\{\langle x^*,y-z\rangle+a(x^*)J^{1/p}(z)\}>\left(1-2^{-n}\right)\varphi(y), \tag{3.24}$$

as $z\in Z_J(y,\delta^*)$ is arbitrary. This means that $y\in H_n^{\varphi}(Z)$ and so $\mathbf{U}(x,\lambda)\subset H_n^{\varphi}(Z)$ holds.

Now we are to prove the density of $H^{\varphi}(Z)$ in $X\setminus Z_0$. By (3.20), we only need to prove that $M^{\varphi}(Z)$ is dense in X. To this end, let $x_0\in X\setminus Z_0$ and $0<\epsilon<\frac{1}{3}$. Set $N=\|x_0\|+4\varphi(x_0)+1$. Let K denote the weak closure of the set $(\mathbf{B}(0,N)\cap Z)\cup\{x_0\}$ and $Y=\operatorname{span}K$. Then K is a weakly compact subset of $\overline{Y}$. From Lemma 2.2, there exist a reflexive Banach space R and a one-to-one continuous linear mapping $T:R\to\overline{Y}$ such that $T(\mathbf{B}_R)\supseteq K$. This implies that

$$T(R)\supseteq Y. \tag{3.25}$$

Define the function $f_Z:R\to[0,+\infty)$ by

$$f_Z(u)=\varphi(x_0+Tu)\quad\text{for each } u\in R. \tag{3.26}$$

Then, by (3.4),

$$\left|f_Z(u)-f_Z(v)\right|=\left|\varphi(x_0+Tu)-\varphi(x_0+Tv)\right|\leqslant\|Tu-Tv\|\leqslant\|T\|\|u-v\| \tag{3.27}$$

for any $u, v\in R$; hence f_Z is Lipschitz continuous on R. Since R is reflexive, Lemma 2.1 is applicable to concluding that f_Z is Fréchet differentiable on a dense subset of R. Therefore, there exists a point $\bar v\in R$ such that $\|T\|\|\bar v\|<\epsilon$ and f_Z is Fréchet differentiable at $\bar v$ with the derivative $\mathrm{D}f_Z(v)=v^*$. Then

$$\lim_{u\to 0}\frac{f_Z(\bar v+u)-f_Z(\bar v)-\langle v^*,u\rangle}{\|u\|}=0. \tag{3.28}$$

Therefore, for each $r>0$,

$$\lim_{t\to 0^+}\frac{f_Z(\bar v+tv)-f_Z(\bar v)-\langle v^*,tv\rangle}{t}=0 \tag{3.29}$$

holds uniformly for all $v\in\mathbf{B}_R(0,r)$. In particular, this implies that

$$\langle v^*,u\rangle\leqslant\|Tu\|\quad\text{for each } u\in R. \tag{3.30}$$

Define a linear functional y^* on TR by

$$\langle y^*,Tu\rangle=\langle v^*,u\rangle\quad\text{for each } u\in R. \tag{3.31}$$

Then $y^*\in T(R)^*$ by (3.30) and hence $y^*\in Y^*$ by (3.25). Let $x=x_0+T\bar v$. Then $x\in\mathbf{U}(x_0,\epsilon)$ and $x\in K+Tv\subset T(R)$. Moreover,

$$\left\|T^{-1}x\right\|=\left\|T^{-1}x_0+\bar v\right\|\leqslant\left\|T^{-1}x_0\right\|+\|\bar v\|\leqslant 1+\frac{\epsilon}{\|T\|}. \tag{3.32}$$

In view of the definition of f_Z, one has by (3.29) and (3.31) that

$$\lim_{t\to 0^+}\frac{\varphi(x+tTv)-\varphi(x)-\langle y^*,tTv\rangle}{t}=0 \tag{3.33}$$

holds uniformly for all $v\in\mathbf{B}_R(0,r)$. By Hahn–Banach theorem, y^* can be extended to $x^*\in X^*$ such that

$$\|x^*\|=\|y^*\|\quad\text{and}\quad\langle x^*,Tu\rangle=\langle v^*,u\rangle\quad\text{for each } u\in R. \tag{3.34}$$

We claim that, for each $\varepsilon>0$, there is $\delta>0$ such that

$$\langle x^*,x-z\rangle+a(x^*)J^{\frac{1}{p}}(z)>(1-\varepsilon/2)\varphi(x)\quad\text{for each } z\in Z_J(x,\delta). \tag{3.35}$$

Granting this, $x\in M^{\varphi}(Z)$ and the proof is complete since $\|x-x_0\|<\epsilon$.

To verify the claim, suppose on the contrary that there exist an $\varepsilon_0>0$ and a sequence $\{z_n\}$ in Z such that

$$\lim_{n\to\infty}\left(\|x-z_n\|^p+J(z_n)\right)^{\frac{1}{p}}=\varphi(x) \tag{3.36}$$

and

$$\langle x^*,x-z_n\rangle+a(x^*)J^{\frac{1}{p}}(z_n)\leqslant(1-\varepsilon_0/2)\varphi(x)\quad\text{for each } n\in\mathbb{N}. \tag{3.37}$$

Without loss of generality, we may assume that $b(x):=\lim_n\|x-z_n\|$ exists and

$$\varphi(x)\leqslant\left(\|x-z_n\|^p+J(z_n)\right)^{\frac{1}{p}}\leqslant\varphi(x)+\epsilon\quad\text{for each } n\in\mathbb{N}. \tag{3.38}$$

Hence, by (3.4), we get that, for each $n\in\mathbb{N}$,

$$\|x_0-z_n\|\leqslant\left(\|x-z_n\|^p+J(z_n)\right)^{\frac{1}{p}}+\|x-x_0\|\leqslant\varphi(x_0)+2\|x-x_0\|+\epsilon\leqslant\varphi(x_0)+1$$

(noting that $\|x-x_0\|<\epsilon$ and $\epsilon\leqslant\frac{1}{3}$). Hence, $\|z_n\|\leqslant\varphi(x_0)+\|x_0\|+1<N$ and $\{z_n\}\subseteq K$. Since $K\subseteq T(\mathbf{B}_R)$, it follows that $\|T^{-1}z_n\|\leqslant 1$ for each $n\in\mathbb{N}$. This together with (3.32) implies that $\{T^{-1}(x-z_n)\}\subseteq\mathbf{B}_R(0,r)$, where $r=\frac{\epsilon}{\|T\|}+2$. Take $\{t_n\}\in(0,1)$ such that $t_n^2\geqslant(\|x-z_n\|^p+J(z_n))^{\frac{1}{p}}-\varphi(x)$ and $t_n\to 0$. Then, by (3.33), one gets that

$$\lim_{n\to\infty}\left(\frac{\varphi(x+t_n(z_n-x))-\varphi(x)}{t_n}-\langle x^*,z_n-x\rangle\right)=0. \tag{3.39}$$

For notational convenience, we write

$$M(z,t)=\left\|(1-t)(x-z)\right\|^p+J(z)\quad\text{for each } z\in Z \text{ and } t\in(0,1). \tag{3.40}$$

Let $n \in \mathbb{N}$. Then,

$$\left(\left\|x+t_n(z_n-x)-z_n\right\|^p+J(z_n)\right)^{\frac{1}{p}}=\frac{\|(1-t_n)(x-z_n)\|^p+J(z_n)}{(M(z_n,t_n))^{\frac{p-1}{p}}}=\frac{(1-t_n)\|(1-t_n)(x-z_n)\|^{p-1}\|x-z_n\|+J(z_n)}{(M(z_n,t_n))^{\frac{p-1}{p}}}.$$

Consequently,

$$\begin{aligned}\varphi\big(x+t_n(z_n-x)\big)-\varphi(x)&\leqslant\left(\left\|x+t_n(z_n-x)-z_n\right\|^p+J(z_n)\right)^{\frac{1}{p}}-\varphi(x)\\&=\frac{\|(1-t_n)(x-z_n)\|^{p-1}\|x-z_n\|+J(z_n)}{(M(z_n,t_n))^{\frac{p-1}{p}}}-\varphi(x)-t_n\frac{\|(1-t_n)(x-z_n)\|^{p-1}\|x-z_n\|}{(M(z_n,t_n))^{\frac{p-1}{p}}}.\end{aligned}\tag{3.41}$$

By Hölder inequality, we have

$$\begin{aligned}\left\|(1-t_n)(x-z_n)\right\|^{p-1}\|x-z_n\|+J(z_n)&=\|x-z_n\|\left\|(1-t_n)(x-z_n)\right\|^{\frac{p}{q}}+J^{\frac{1}{p}}(z_n)J^{\frac{1}{q}}(z_n)\\&\leqslant\left(\|x-z_n\|^p+J(z_n)\right)^{\frac{1}{p}}\left(\left\|(1-t_n)(x-z_n)\right\|^p+J(z_n)\right)^{\frac{1}{q}}\\&=\left(\|x-z_n\|^p+J(z_n)\right)^{\frac{1}{p}}\left(M(z_n,t_n)\right)^{\frac{p-1}{p}}.\end{aligned}\tag{3.42}$$

Hence,

$$\frac{\|(1-t_n)(x-z_n)\|^{p-1}\|x-z_n\|+J(z_n)}{(M(z_n,t_n))^{\frac{p-1}{p}}}-\varphi(x)\leqslant\left(\|x-z_n\|^p+J(z_n)\right)^{\frac{1}{p}}-\varphi(x)\leqslant t_n^2.\tag{3.43}$$

Combing this and (3.41), we obtain that

$$\limsup_{n\to\infty}\left(\frac{\varphi\big(x+t_n(z_n-x)\big)-\varphi(x)}{t_n}+\frac{\|(1-t_n)(x-z_n)\|^{p-1}\|x-z_n\|}{(M(z_n,t_n))^{\frac{p-1}{p}}}\right)\leqslant 0.$$

By (3.39), one has that

$$\liminf_{n\to\infty}\left(\langle x^*,x-z_n\rangle-\frac{(1-t_n)^{p-1}\|x-z_n\|^p}{(M(z_n,t_n))^{\frac{p-1}{p}}}\right)\geqslant 0.\tag{3.44}$$

Note that

$$\lim_{n\to\infty}M(z_n,t_n)=\varphi^p(x)\quad\text{and}\quad\lim_{n\to\infty}\|x-z_n\|=b(x).\tag{3.45}$$

It follows from (3.44) that

$$\|x^*\|\geqslant\frac{b^{p-1}(x)}{\varphi^{p-1}(x)}\tag{3.46}$$

and

$$\liminf_{n\to\infty}\big(\langle x^*,x-z_n\rangle+a(x^*)J^{\frac{1}{p}}(z_n)\big)\geqslant\frac{b^p(x)}{\varphi^{p-1}(x)}+a(x^*)\big(\varphi^p(x)-b^p(x)\big)^{\frac{1}{p}}\tag{3.47}$$

because

$$\lim_{n\to\infty}J(z_n)=\lim_{n\to\infty}\big(\|x-z_n\|^p+J(z_n)\big)-\lim_{n\to\infty}\|x-z_n\|^p=\varphi^p(x)-b^p(x).\tag{3.48}$$

On the other hand, by (3.25) and (3.33), one sees that (3.7) holds. Note that $\{z_n\}\subseteq Z$ is a minimizing sequence of the problem $\min_J(x,Z)$. Hence we can apply Lemma 3.2 to get that $\|y^*\|\leqslant\frac{b^{p-1}(x)}{\varphi^{p-1}(x)}$. Hence $\|x^*\|\leqslant\frac{b^{p-1}(x)}{\varphi^{p-1}(x)}$ thanks to (3.34). Combing this with (3.46), we have that

$$\|x^*\|=\frac{b^{p-1}(x)}{\varphi^{p-1}(x)}.\tag{3.49}$$

Thus, by definition,

$$a(x^*)=\big(1-\|x^*\|^q\big)^{\frac{1}{q}}=\frac{(\varphi^p(x)-b^p(x))^{\frac{1}{q}}}{\varphi^{p-1}(x)}.$$

It follows from (3.47) that

$$\liminf_{n\to\infty}\left(\langle x^*,x-z_n\rangle+a(x^*)J^{\frac{1}{p}}(z_n)\right)\geqslant\varphi(x),$$

which contradicts (3.37) and completes the proof. □

Lemma 3.4. *Let Z be a relatively boundedly weakly compact subset of X. Suppose that X is a Kadec Banach space w.r.t. Z. Let $x \in H^{\varphi}(Z)$. Then, any minimizing sequence of the problem* $\min_J(x, Z)$ *has a converging subsequence.*

Proof. In view of the definition of $H^{\varphi}(Z)$ in (3.19), there exist a positive sequence $\{\delta_n\}$ and a sequence $\{x_m^*\} \subseteq \mathbf{B}^*$ such that

$$\inf_{z\in Z_J(x,\delta_m)} \left\{\left\langle x_m^*, x - z\right\rangle + a\left(x_m^*\right) J^{\frac{1}{p}}(z)\right\} > \left(1 - 2^{-m}\right)\varphi(x) \quad \text{for each } m \in \mathbb{N}. \tag{3.50}$$

Let $\{z_n\}$ be any minimizing sequence of the problem $\min_J(x, Z)$, i.e.,

$$\lim_{n\to\infty} \left(\|x - z_n\|^p + J(z_n)\right)^{\frac{1}{p}} = \varphi(x). \tag{3.51}$$

Without loss of generality, assume that

$$\delta_n \leqslant \delta_m \quad \text{and} \quad z_n \in Z_p(x, \delta_m) \quad \text{if } n > m, \tag{3.52}$$

and that $b(x) = \lim_{n\to\infty} \|x - z_n\|$ exists. Then $\lim_{n\to\infty} J(z_n)$ exists by (3.51). Note that $\{z_n\}$ is bounded and Z is relatively boundedly weakly compact. We also assume that, without loss of generality, $z_n \to z_0$ weakly as $n \to \infty$ for some $z_0 \in X$. Then we have that

$$\left(\|x - z_0\|^p + \lim_{n\to\infty} J(z_n)\right)^{\frac{1}{p}} \leqslant \lim_{n\to\infty} \left(\|x - z_n\|^p + J(z_n)\right)^{\frac{1}{p}} = \varphi(x). \tag{3.53}$$

Let $m, n \in \mathbb{N}$ satisfy $n > m$. Then, by (3.50) and (3.52),

$$\left\langle x_m^*, x - z_n\right\rangle + a\left(x_m^*\right) J^{\frac{1}{p}}(z_n) > \left(1 - 2^{-m}\right)\varphi(x) \tag{3.54}$$

and so

$$\left\langle x_m^*, x - z_0\right\rangle + a\left(x_m^*\right) \lim_{n\to\infty} J^{\frac{1}{p}}(z_n) \geqslant \left(1 - 2^{-m}\right)\varphi(x). \tag{3.55}$$

Using Hölder inequality, we have

$$\left\|x_m^*\right\| \|x - z_0\| + a\left(x_m^*\right) \lim_{n\to\infty} J^{\frac{1}{p}}(z_n) \leqslant \left(\left\|x_m^*\right\|^q + \left(a\left(x_m^*\right)\right)^q\right)^{\frac{1}{q}} \cdot \left(\|x - z_0\|^p + \lim_{n\to\infty} J(z_n)\right)^{\frac{1}{p}}. \tag{3.56}$$

Since

$$\left\langle x_m^*, x - z_0\right\rangle + a\left(x_m^*\right) \lim_{n\to\infty} J^{\frac{1}{p}}(z_n) \leqslant \left\|x_m^*\right\| \|x - z_0\| + a\left(x_m^*\right) \lim_{n\to\infty} J^{\frac{1}{p}}(z_n), \tag{3.57}$$

it follows from (3.56) that

$$\left\langle x_m^*, x - z_0\right\rangle + a\left(x_m^*\right) \lim_{n\to\infty} J^{\frac{1}{p}}(z_n) \leqslant \left(\left\|x_m^*\right\|^q + \left(a\left(x_m^*\right)\right)^q\right)^{\frac{1}{q}} \cdot \left(\|x - z_0\|^p + \lim_{n\to\infty} J(z_n)\right)^{\frac{1}{p}}. \tag{3.58}$$

Noting that $(\|x_m^*\|^q + (a(x_m^*))^q = 1$ and (3.53), we get that

$$\left\langle x_m^*, x - z_0\right\rangle + a\left(x_m^*\right) \lim_{n\to\infty} J^{\frac{1}{p}}(z_n) \leqslant \left(\|x - z_0\|^p + \lim_{n\to\infty} J(z_n)\right)^{\frac{1}{p}} \leqslant \varphi(x).$$

This together with (3.55) implies that

$$\left(\|x - z_0\|^p + \lim_{n\to\infty} J(z_n)\right)^{\frac{1}{p}} = \varphi(x). \tag{3.59}$$

Combining this with (3.51), one sees that

$$\lim_{n\to\infty} \|x - z_n\| = \|x - z_0\|. \tag{3.60}$$

Noting that X is Kadec w.r.t. Z and $z_n \to z_0$ weakly, it follows that $\lim_{n\to\infty} \|z_0 - z_n\| = 0$ and so $z_0 \in Z$, which completes the proof. □

Note that, for any $x \in X$, if every minimizing sequence of the problem $\min_J(x, Z)$ has a converging subsequence, then $P_{Z,J}(x) \neq \emptyset$. Thus, the following theorem is a direct consequence of Lemmas 3.3 and 3.4.

Theorem 3.1. *Let Z be a relatively boundedly weakly compact subset of X. Suppose that X is Kadec w.r.t. Z. Then the set of all $x \in X$ such that $P_{Z,J}(x) \neq \emptyset$ and every minimizing sequence of the problem* $\min_J(x, Z)$ *has a converging subsequence is a dense G_δ-subset of $X \setminus Z_0$.*

The following corollary is direct from (2.2) and Theorem 3.1.

Corollary 3.1. *Let Z be a relatively boundedly weakly compact subset of X. Suppose that X is Kadec. Then the set of all $x \in X$ such that $P_{Z,J}(x) \neq \emptyset$ and every minimizing sequence of the problem $\min_J(x, Z)$ has a converging subsequence is a dense G_δ-subset of $X \setminus Z_0$.*

Theorem 3.2. *Let Z be a relatively boundedly weakly compact subset of X. Suppose that X is both Kadec w.r.t. Z and J-strictly convex w.r.t. Z. Suppose further that $p > 1$. Then the set of all $x \in X$ such that the problem $\min_J(x, Z)$ is well-posed is a dense G_δ-subset of $X \setminus Z_0$.*

Proof. By Lemma 3.3, $H^\varphi(Z)$ is a G_δ-subset of $X \setminus Z_0$; while, by Lemma 3.4, for each $x \in H^\varphi(Z)$ and any minimizing sequence for the problem $\min_J(x, Z)$ has a converging subsequence and so $P_{Z,J}(x) \neq \emptyset$. Thus, we only need to prove that $P_{Z,J}(x)$ is a singleton for each $x \in H^\varphi(Z)$. To this purpose, let $x \in H^\varphi(Z)$ and $z_1, z_2 \in P_{Z,J}(x)$. Then, by the definition of $H^\varphi(Z)$, for each $n \in \mathbb{N}$, there exists $x_n^* \in \mathbf{B}^*$ such that

$$\langle x_n^*, x - z_i \rangle + a(x_n^*) J^{\frac{1}{p}}(z_i) > (1 - 2^{-n})\varphi(x) \quad \text{for each } i = 1, 2. \tag{3.61}$$

Without loss of generality, we may assume that $\{x_n^*\}$ converges weakly* to some $x^* \in \mathbf{B}^*$. Then $a(x^*) \geqslant \lim_{n\to\infty} a(x_n^*)$. Hence

$$\langle x^*, x - z_i \rangle + a(x^*) J^{\frac{1}{p}}(z_i) \geqslant \varphi(x) \quad \text{for each } i = 1, 2. \tag{3.62}$$

It follows that

$$\langle x^*, x - z_1 + x - z_2 \rangle + a(x^*)\left(J^{\frac{1}{p}}(z_1) + J^{\frac{1}{p}}(z_2)\right) \geqslant 2\varphi(x).$$

Using Hölder inequality and the fact that $\|x^*\|^q + a(x^*)^q = 1$, one has that

$$\begin{aligned} 2\varphi(x) &\leqslant \left(\|x - z_1 + x - z_2\|^p + \left(J^{\frac{1}{p}}(z_1) + J^{\frac{1}{p}}(z_2)\right)^p\right)^{\frac{1}{p}} \\ &\leqslant \left(\left(\|x - z_1\| + \|x - z_2\|\right)^p + \left(J^{\frac{1}{p}}(z_1) + J^{\frac{1}{p}}(z_2)\right)^p\right)^{\frac{1}{p}} \\ &\leqslant \left(\|x - z_1\|^p + J(z_1)\right)^{\frac{1}{p}} + \left(\|x - z_2\|^p + J(z_2)\right)^{\frac{1}{p}} \\ &= 2\varphi(x). \end{aligned} \tag{3.63}$$

Consequently,

$$\|x - z_1 + x - z_2\| = \|x - z_1\| + \|x - z_2\|. \tag{3.64}$$

Furthermore, since $p > 1$, (3.63) implies that

$$\|x - z_1\| = \|x - z_2\| \quad \text{and} \quad J(z_1) = J(z_2). \tag{3.65}$$

Thus the assumed J-strict convexity of X together with (3.64) and (3.65) implies that $x - z_1 = x - z_2$; hence $z_1 = z_2$. This completes the proof. □

The following corollary is a direct consequence of (2.1), (2.2) and Theorem 3.2.

Corollary 3.2. *Let Z be a relatively boundedly weakly compact subset of X. Suppose that X is Kadec and strictly convex. Suppose further that $p > 1$. Then the set of all $x \in X$ such that the problem $\min_J(x, Z)$ is well-posed is a dense G_δ-subset of $X \setminus Z_0$.*

The following example illustrates that our results obtained in the present paper are proper extensions of earlier results in [9,27] even in the case when $p = 1$.

Example 3.1. Let Y be a uniformly convex Banach space and let $X = l_\infty(Y)$ be the Banach space defined as in Example 2.1. Let Z be a nonempty closed subset of Y and $J : Z \to \mathbb{R}$ a lower semicontinuous function bounded from below. Then Z is a relatively boundedly weakly compact subset of X. Furthermore, X is both strictly convex and Kadec w.r.t. Z by Example 2.1. Thus Theorems 3.1 and 3.2 are applicable. Therefore, the set of all $x \in l_\infty(Y)$ such that $P_{Z,J}(x) \neq \emptyset$ and every minimizing sequence of the problem $\inf_{z\in Z}\{\|x - z\|^p + J(z)\}$ has a converging subsequence is a dense G_δ-subset of $l_\infty(Y) \setminus Z_0$. Moreover, if $p > 1$, then the set of all $x \in l_\infty(Y)$ such that $\min_J(x, Z)$ is well-posed is a dense G_δ-subset of $l_\infty(Y) \setminus Z_0$. Note that in the case when $p = 1$, the corresponding results in [9,27] are not applicable because X is not Kadec.

The following example provides the case when Theorem 3.2 is applicable but not Corollary 3.2.

Example 3.2. Let $X = l_\infty$ be the Banach space as in Example 2.2. Let Z be a nonempty closed subset of the subspace $\{z = (z, 0, \ldots) \in l_\infty\colon\ z > 0\}$. Then Z is locally compact and so X is Kadec w.r.t. Z. Let $J : Z \to \mathbb{R}$ be the function defined as in Example 2.2. Then X is J-strictly convex w.r.t. Z by Example 2.2. Suppose that $p > 1$. Then, Theorem 3.2 is applicable and so the set of all $x \in X$ such that the problem $\inf_{z\in Z}\{\|x - z\|^p + J(z)\}$ is well-posed is a dense G_δ-subset of $X_c \setminus Z_0$. Note that Corollary 3.2 is not applicable.

4. Concluding remarks

Let G and E be subsets of X. Recall that G is said to be porous in E if there exist $t \in (0,1]$ and $r_0 > 0$ such that for every $x \in E$ and $r \in (0, r_0]$ there is a point $y \in E$ such that $\mathbf{B}(y, tr) \subseteq \mathbf{B}(x, r) \cap (E \setminus G)$. A subset G is said to be σ-porous in E if it is a countable union of sets which are porous in E. The notion of σ-porousity was introduced by E.P. Dolzhenko in [15] to describe a certain class of exceptional sets which appear in the study of boundary behavior of complex function. This notion was applied in [13] by Blasi, Myjak and Papini to the study of the existence and uniqueness problem of the best approximation. For the further applications in approximation theory, the reader is refereed to [12,21,22,24]. In the case when $p = 1$, we proved in [23] that if X is uniformly convex then the set of all points $x \in X \setminus Z_0$ for which the problem $\min_J(x, Z)$ fails to be approximatively compact (recalling that the problem $\min_J(x, Z)$ is approximatively compact if every minimizing sequence of the problem $\min_J(x, Z)$ has a converging subsequence) is a σ-porous set in $X \setminus Z_0$. One key fact used in the proof of this result is that

$$z_0 \in P_{Z,J}(x) \quad \Rightarrow \quad z_0 \in P_{Z,J}\big(z_0 + \alpha(x - z_0)\big) \quad \text{for each } \alpha \in [0,1]. \tag{4.1}$$

However, in the case when $p > 1$, (4.1) is no longer valid in general. For example, let $X = \mathbb{R}$, $Z = [0,1]$ and $J : Z \to \mathbb{R}$ be defined by $J(z) = z$ for each $z \in Z$. Take $x = 2$, $z_0 = 1$ and $p = 2$. Then $z_0 \in P_{Z,J}(x)$. However, for $\alpha = \frac{3}{4}$, one has that $P_{Z,J}(x_\alpha) = \{\frac{3}{4}\}$ and so $z_0 \notin P_Z(x_\alpha)$. We do not know whether the set of all points $x \in X \setminus Z_0$ for which the problem $\min_J(x, Z)$ fails to be well-posed is a σ-porous subset of $X \setminus Z_0$ in the case when $p > 1$ and X is uniformly convex.

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