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Bounds for Quasiconformal Distortion Functions

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Several new inequalities are proved for the distortion function $\varphi_K(r)$ appearing in the quasiconformal Schwarz lemma. Other related special functions are studied and applications are given to quasiconformal maps in the plane. Some open problems are solved, too. © 1997 Academic Press

1. INTRODUCTION

The so-called quasiconformal φ-distortion function is defined as

$$\varphi_K(r) = \mu^{-1}\left(\frac{1}{K}\mu(r)\right), \qquad 0 < r < 1,\ 0 < K < \infty, \tag{1.1}$$

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where

$$\mu(r) \equiv \frac{\pi}{2}\frac{\mathscr{K}'(r)}{\mathscr{K}(r)} \tag{1.2}$$

and [BF], [WW]

$$\mathscr{K}(r) \equiv \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - r^2 \sin^2\theta}}, \qquad \mathscr{K}'(r) \equiv \mathscr{K}(r'),$$

$r' = \sqrt{1 - r^2}$. As suggested by its name, this function has found important applications in the theory of quasiconformal mappings [LV], [P], [AVV4, Q], and it has been the subject matter of the recent papers [AVV1, AVV2, VV]. In [VV, V], it is also pointed out that the function $\varphi_K(r)$ occurs in number theory, in particular, in Ramanujan's work on modular equations and singular values of elliptic integrals [B, BB, V].

Some of the function-theoretic applications rely on explicit estimates for the rather involved function $\varphi_K(r)$. An example of such an explicit inequality is [LV, He]

$$r^{1/K} < \varphi_K(r) < 4^{1-1/K} r^{1/K}, \tag{1.3}$$

for all $K > 1$ and $r \in (0, 1)$. Some other special functions of the geometric theory of quasiconformal maps can be expressed in terms of $\varphi_K(r)$. For example,

$$\eta_K(r) \equiv \left[\varphi_K\left(\sqrt{\frac{r}{1+r}}\right)\Big/\varphi_{1/K}\left(\frac{1}{\sqrt{1+r}}\right)\right]^2, \qquad r, K \in (0, \infty), \tag{1.4}$$

is a function occurring in the study of both quasisymmetric functions and quasiconformal mappings [A, LV].

The goal of this paper is to sharpen some existing inequalities and to derive concavity and convexity results for the functions $\varphi_K(r)$ and $\eta_K(r)$ from which new functional inequalities follow. We now state some of our main results.

1.5. THEOREM. *For each* $r \in (0, 1)$, *define* $f(K) = \varphi_K(r)(1 + r')^{2/K}/r^{1/K}$ *and* $g(K) = \varphi_{1/K}(r)[2(1 + r')]^K/r^K$. *Then* $f(K)$ *and* $g(K)$ *are strictly decreasing and increasing on* $[1, \infty)$, *respectively, with* $f(\infty) = 1$ *and* $g(\infty) = \infty$. *In particular,*

$$\varphi_K(r) < (1 + r')^{2(1-1/K)} r^{1/K} \tag{1}$$

and

$$\varphi_{1/K}(r) > [2(1+r')]^{1-K} r^K \tag{2}$$

for all $K > 1$ *and* $r \in (0,1)$.

1.6. THEOREM. *For each* $K > 1$, *the function* $f(x) = \operatorname{arth} \varphi_K(\operatorname{th} x)$ *is strictly increasing and concave from* $(0,\infty)$ *onto* $(0,\infty)$. *In particular, for* $a, b \in (0,1)$,

$$\varphi_K\left(\frac{a+b}{1+ab+a'b'}\right) \geq \frac{\varphi_K(a)+\varphi_K(b)}{1+\varphi_K(a)\varphi_K(b)+\varphi_{1/K}(a')\varphi_{1/K}(b')}, \tag{1}$$

with equality if and only if $K = 1$ *or* $a = b$,

$$\varphi_K\left(\frac{a+b}{1+ab}\right) \leq \frac{\varphi_K(a)+\varphi_K(b)}{1+\varphi_K(a)\varphi_K(b)} \leq \frac{2\varphi_K((a+b)/(1+ab+a'b'))}{1+\left(\varphi_K((a+b)/(1+ab+a'b'))\right)^2}, \tag{2}$$

with equality in the first if and only if $K = 1$ *and in the second if and only if* $K = 1$ *or* $a = b$.

1.7. THEOREM. *For* $K > 1$, *let* f *and* g *be defined on* R *by* $f(x) = \log \eta_K(e^x)$ *and* $g(x) = (\eta_K(e^x))^{-1/2}$. *Then* f *is strictly increasing and convex with* $f(R) = R$ *and* $1/K < f'(x) < K$ *for all* $x \in (-\infty,\infty)$, *while* g *is strictly decreasing and convex with* $g(R) = (0,\infty)$. *In particular,*

$$\frac{4\eta_K(a^2)\eta_K(b^2)}{\left(\sqrt{\eta_K(a^2)}+\sqrt{\eta_K(b^2)}\right)^2} \leq \eta_K(ab) \leq \sqrt{\eta_K(a^2)\eta_K(b^2)}, \tag{1}$$

with equality in the first if and only if $a = b$ *and in the second if and only if* $K = 1$ *or* $a = b$, *and*

$$\left(\frac{b}{a}\right)^{1/K} \leq \frac{\eta_K(b)}{\eta_K(a)} \leq \left(\frac{b}{a}\right)^K \tag{2}$$

for $0 < a \leq b$, *with equality if and only if* $K = 1$ *or* $a = b$.

In the sequel we shall prove several further inequalities of this kind, solve some open problems on $\varphi_K(r)$, and use these inequalities and solutions to sharpen known distortion results for plane quasiconformal mappings. Our notation is fairly standard, as in [LV], except for our abbreviation for hyperbolic functions. We let th denote the hyperbolic tangent function and let arth denote its inverse. Whenever x is in $[0, 1]$, we let $x' = \sqrt{1 - x^2}$.

2. PRELIMINARY RESULTS

In this section, we establish some lemmas which are needed in the proofs of our main results.

2.1. LEMMA. *There exists a unique* $r_0 \in (0, 1)$ *such that the function* $F(r) \equiv (2/\pi)(1 - r^2)\mathscr{K}(r)\mathscr{K}'(r) + \log r - 2\log(1 + r')$ *is strictly decreasing on* $(0, r_0\,]$ *and strictly increasing on* $[\,r_0, 1)$. *In particular,*

$$0 \le \frac{2}{\pi}(1 - r^2)\mathscr{K}(r)\mathscr{K}'(r) + \log r \le 2\log(1 + r') \tag{2.2}$$

for each $r \in [0, 1]$. *The first equality holds if and only if* $r = 1$, *the second equality holds if and only if* $r = 0$ *or* $r = 1$.

Proof. By differentiating and using Legendre's relation [BF, 110.10], we get

$$\frac{\pi}{4} rr' \frac{dF}{dr} = \frac{\pi}{2} - r'\mathscr{K}(r)\mathscr{E}'(r) \equiv F_1(r), \tag{2.3}$$

where

$$\mathscr{E}(r) \equiv \int_0^{\pi/2} \sqrt{1 - r^2 \sin^2\theta}\, d\theta \qquad \text{and} \qquad \mathscr{E}'(r) \equiv \mathscr{E}(r') \tag{2.4}$$

are the complete elliptic integrals of the second kind. Then

$$-\frac{r'}{r}\frac{dF_1}{dr} = F_2(r), \tag{2.5}$$

where

$$F_2(r) = r'\mathscr{K}(r)\{[\mathscr{K}'(r) - \mathscr{E}'(r)]/r'\} + \{\mathscr{E}'(r)[\mathscr{E}(r) - \mathscr{K}(r)]/r^2\}.$$

By [AVV3, Theorem 2.2(3), (7)], it follows that F_2 is strictly decreasing from $(0, 1)$ onto $(-\infty, \infty)$. Since $F_1(0) = F_1(1^-) - \pi/2 = 0$, the equation $F_1(r) = 0$, for $r \in (0, 1)$, has a unique solution $r_0 \in (0, 1)$. Moreover, $F_1(r) < 0$ if $r \in (0, r_0)$, and $F_1(r) > 0$ if $r \in (r_0, 1)$. Hence F is strictly decreasing on $(0, r_0\,]$ and strictly increasing on $[\,r_0, 1)$.

Since $F(0^+) = F(1^-) = 0$, the second inequality in (2.2) follows. The first inequality in (2.2) appears in [W]. It is also implied by [AVV3, Lemma 4.2(1)]. ∎

2.6. *Remarks*. (1) The second inequality in (2.2) is a significant improvement of the previously known inequality [W; AVV3, Lemma 4.2(1)]

$$\frac{2}{\pi} r'^2 \mathscr{K}(r)\mathscr{K}'(r) + \log r \le \log 4.$$

(2) Unfortunately, the function $F(r)$ in Lemma 2.1 is not globally convex on $(0, 1)$. In fact, we have

$$\frac{\pi}{4} r^2 r'^3 \frac{d^2F}{dr^2} \equiv F_3(r),$$

where $F_3(r) = \pi r^2 + r'\{\mathscr{K}(r)[\mathscr{E}'(r) - r^2\mathscr{K}'(r)] - \mathscr{E}'(r)[\mathscr{E}(r) - r'^2\mathscr{K}(r)]\} - (\pi/2)$.

Clearly, $F_3(r)$ tends to $\pi/2$ as $r \to 1$, so that $F''(r)$ tends to ∞ as r tends to 1. On the other hand, we can show that $F''(r)$ tends to $-\infty$ as r tends to 0. In fact,

$$\frac{\pi}{4}F''(r) - \frac{\pi}{2(1+r')r'^3} + \frac{\mathscr{E}(r')}{r'^2}\,\frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{r^2}$$
$$= \frac{1}{r'^2}\,\frac{\mathscr{K}(r)\left[\mathscr{E}(r') - r^2\mathscr{K}(r')\right] - (\pi/2)}{r^2}.$$

Letting $r \to 0$, we see that on the left side, by [AVV3, Theorem 2.2(7)], the second and third expressions have finite limits $(-\pi/4) + (\pi/4) = 0$. By l'Hôpital's Rule, the right side tends to $-\infty$ as $r \to 0$. Hence $F''(r) \to -\infty$ as $r \to 0$. Thus F is concave near 0 and convex near 1.

Now, we recall that [He; AVV3, Lemma 2.1]

$$\frac{d\mu(r)}{dr} = -\frac{\pi^2}{4}\,\frac{1}{rr'^2\mathscr{K}(r)^2}, \tag{2.7}$$

$$\frac{\partial s}{\partial r} = \frac{ss'^2\mathscr{K}(s)^2}{Krr'^2\mathscr{K}(r)^2}, \tag{2.8}$$

and

$$\frac{\partial s}{\partial K} = \frac{2}{\pi K} s s'^2 \mathscr{K}(s) \mathscr{K}'(s), \tag{2.9}$$

where $s = \varphi_K(r)$, $0 < r < 1$, $0 < K < \infty$.

2.10. LEMMA. *The functions*

$$f(r) \equiv \frac{(2 - r^2)\mathscr{K}(r) - 2\mathscr{E}(r)}{(2\log(1/r')) - r^2} \qquad \text{and} \qquad g(r) \equiv \frac{\mathscr{E}(r)^2 - (r'\mathscr{K}(r))^2}{r^4}$$

are strictly increasing from $(0,1)$ *onto* $(\pi/8, 1/2)$ *and* $(\pi^2/32, 1)$, *respectively.*

Proof. Let $F(r)$ be the numerator in $f(r)$. Then

$$\frac{r'^2 F'(r)}{2r^3} = \frac{\mathscr{E}(r) - r'^2\mathscr{K}(r)}{2r^2}$$

so that the result for f follows from [AVV3, Theorem 2.2(7); AVV5, Lemma 2.2].

Next, let $G(r)$ be the numerator in $g(r)$. Then

$$\frac{G'(r)}{4r^3} = \frac{[\mathscr{K}(r) - \mathscr{E}(r)]^2}{2r^4},$$

so that the result for g follows from [AVV6, Theorem 2.1(6); AVV5, Lemma 2.2]. ∎

2.11. LEMMA. *For* $K > 1$, $r \in (0,1)$, *let* $s = \varphi_K(r)$. *Then the functions* $f(r) \equiv (s'/r')[\mathscr{K}(s)/\mathscr{K}(r)]^2$, $g(r) \equiv (s/r)[\mathscr{K}'(s)/\mathscr{K}'(r)]^2$, $h(r) \equiv (s/r)[\mathscr{K}(s)/\mathscr{K}(r)]^2$, *and* $k(r) \equiv (s'/r')[\mathscr{K}'(s)/\mathscr{K}'(r)]^2$ *are all strictly decreasing on* $(0,1)$, *with ranges* $(0,1)$, $(1,\infty)$, (K^2,∞), *and* $(0, 1/K^2)$, *respectively.*

Proof. By differentiation and simplification, we see that $f'(r) < 0$, if and only if $F(s) > F(r)$, where

$$F(x) = \mathscr{K}'(x)\left[(2 - x^2)\mathscr{K}(x) - 2\mathscr{E}(x)\right].$$

But this inequality is true by Lemma 2.10 and [AVV3, Theorem 2.2(3)].

The proof for $g(r)$ is similar.

Next, since $h(r) = K^2 g(r)$ and $k(r) = f(r)/K^2$, the remaining results follow immediately. Finally, the limiting values are clear. ∎

2.12. LEMMA. *For $K > 1$, $r \in (0,1)$, let $s = \varphi_K(r)$. Then the functions $f(r) \equiv \mathscr{K}(s)/\mathscr{K}(r)$ and $g(r) \equiv \mathscr{K}'(s)/\mathscr{K}'(r)$ are both strictly increasing from $(0,1)$ onto $(1, K)$, and $(1/K, 1)$, respectively.*

Proof. By logarithmic differentiation,

$$rr'^2\mathscr{K}(r)\mathscr{K}'(r)\frac{f'(r)}{f(r)} = \mathscr{K}'(s)\left[\mathscr{E}(s) - s'^2\mathscr{K}(s)\right] - \mathscr{K}'(r)\left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right],$$

which is positive by [AVV3, Theorem 2.2(3), (7)], so that the result for f follows. Next, since $g(r) = f(r)/K$, the result for g follows immediately. ▮

2.13. LEMMA. *Suppose that $f(x, y)$ is positive, increasing, and log-concave in x and y, separately, for (x, y) in the plane domain D, that $(1/f(x,y))(\partial f/\partial x)$ is decreasing in y, and that $(1/f(x,y))(\partial f/\partial y)$ is decreasing in x. Then, for each segment $I \subset D$ parallel to the vector $(\cos\varphi, \sin\varphi)$, $\varphi \in (0, \pi/2)$, $f|I$ is log-concave.*

Proof. Let $F(t) = \log f(x(t), y(t))$, $f_1(x, y) = (1/f(x,y))(\partial f/\partial x)$, $f_2(x, y) = (1/f(x,y))(\partial f/\partial y)$. Here, $x(t) = tx_1 + (1-t)x_2$ and $y(t) = ty_1 + (1-t)y_2$ are simultaneously increasing or decreasing. Then

$$\frac{d^2F}{dt^2} = (x_1 - x_2)^2\frac{\partial f_1}{\partial x} + (y_1 - y_2)^2\frac{\partial f_2}{\partial y} + (x_1 - x_2)(y_1 - y_2)\left(\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial x}\right),$$

which is negative by the hypotheses. ▮

For convenience, we also recall the following result (cf. [AVV2, Lemma 2.12]).

2.14. LEMMA. *For an interval $I \subset [0, \infty)$, suppose that $f, g : I \to [0, \infty)$ are functions such that $f(x)/g(x)$ is decreasing on $I \setminus \{0\}$ and $g(0) = 0$, $g(x) > 0$ for $x > 0$. Then*

$$f(x+y)(g(x) + g(y)) \le g(x+y)(f(x) + f(y))$$

for $x, y, x+y \in I$. Moreover, if the monotonicity of $f(x)/g(x)$ is strict, then the above inequality is also strict on $I \setminus \{0\}$.

3. PROOFS

3.1. *Proof of Theorem* 1.5. By logarithmic differentiation

$$K^2\frac{f'(K)}{f(K)} = \frac{2}{\pi}s'^2\mathscr{K}(s)^2\frac{\mathscr{K}'(r)}{\mathscr{K}(r)} + \log\frac{r}{(1+r')^2}, \tag{3.2}$$

where $s = \varphi_K(r)$. Since $s'\mathscr{K}(s)$ is strictly decreasing in K by [AVV3, Theorem 2.2(3)], it follows from (3.2) that

$$K^2\frac{f'(K)}{f(K)} \le \frac{2}{\pi}r'^2\mathscr{K}(r)\mathscr{K}'(r) + \log\frac{r}{(1+r')^2},$$

which is negative by Lemma 2.1. Hence f is strictly decreasing on $[1,\infty)$.

Next, by logarithmic differentiation, we have

$$\frac{g'(K)}{g(K)} = -\frac{2}{\pi}[u'\mathscr{K}(u)]^2\frac{\mathscr{K}'(r)}{\mathscr{K}(r)} - \log\frac{r}{2(1+r')}, \tag{3.3}$$

where $u = \varphi_{1/K}(r)$. Since $u'\mathscr{K}(u)$ is strictly increasing in K by [AVV3, Theorem 2.2(3)], it follows from (3.3) that

$$\frac{g'(K)}{g(K)} > \log 2 - \left[\mu(r) + \log\frac{r}{1+r'}\right],$$

which is positive by the inequality [LV, p. 62]

$$\mu(r) < \log\frac{2(1+r')}{r}.$$

Hence g is strictly increasing on $[1,\infty)$.

The limiting values $\lim_{K\to\infty}f(K) = 1$ and $\lim_{K\to\infty}g(K) = \infty$ and inequalities (1) and (2) are clear. ∎

3.4. *Proof of Theorem* 1.6. Let $r = \text{th}\, x$, $s = \varphi_K(r)$. Then

$$f'(x) = \frac{1}{K}\frac{s\mathscr{K}(s)^2}{r\mathscr{K}(r)^2},$$

which is strictly decreasing and positive, by Lemma 2.11, so that f is strictly increasing and concave. The limiting values are clear.

Equalities in (1) and (2) are all clear. For the strict inequality in (1) let $K > 1$ and $a \neq b$, $\text{th}\, x = a$, $\text{th}\, y = b$, $A = \varphi_K(a)$, $B = \varphi_K(b)$, $u = \text{arth}\, \varphi_K(\text{th}\, x)$, $v = \text{arth}\, \varphi_K(\text{th}\, y)$. Then

$$\text{th}\frac{x+y}{2} = \frac{\text{th}(x+y)}{1+\sqrt{1-\text{th}^2(x+y)}} = \frac{a+b}{1+ab+a'b'}.$$

Similarly,

$$\operatorname{th}\frac{u+v}{2} = \frac{A+B}{1+AB+A'B'}.$$

By concavity of f, we have arth $\varphi_K(\operatorname{th}((x+y)/2)) > (u+v)/2$, so that $\varphi_K(\operatorname{th}((x+y)/2)) > \operatorname{th}((u+v)/2)$, hence (1) follows.

For (2), [AVV1, Lemma 3.15] implies that $f(x)/x$ is strictly decreasing, hence $f(x+y) < f(x) + f(y)$ by Lemma 2.14. Hence

$$\begin{aligned}\operatorname{arth}\varphi_K(\operatorname{th}(x+y)) &< \operatorname{arth}\varphi_K(\operatorname{th}x) + \operatorname{arth}\varphi_K(\operatorname{th}y) \\ &= \operatorname{arth}\left(\frac{\varphi_K(\operatorname{th}x)+\varphi_K(\operatorname{th}y)}{1+\varphi_K(\operatorname{th}x)\varphi_K(\operatorname{th}y)}\right),\end{aligned}$$

so that

$$\varphi_K\left(\frac{\operatorname{th}x+\operatorname{th}y}{1+\operatorname{th}x\cdot\operatorname{th}y}\right) < \frac{\varphi_K(\operatorname{th}x)+\varphi_K(\operatorname{th}y)}{1+\varphi_K(\operatorname{th}x)\varphi_K(\operatorname{th}y)}.$$

Now set $a = \operatorname{th}x$, $b = \operatorname{th}y$, and the first inequality in (2) follows. The second inequality in (2) holds by (1). ∎

3.5. *Remark.* By inverting the function in Theorem 1.6, one sees that $g(x) \equiv \operatorname{arth}\varphi_{1/K}(\operatorname{th}x)$, $K > 1$, is strictly increasing and convex from $(0,\infty)$ onto $(0,\infty)$, so that the inequalities in the theorem are reversed for $0 < K \le 1$.

3.6. *Proof of Theorem* 1.7. First, writing $f(x) = 2\log(s/s')$, where $s = \varphi_K(r)$, $r = \sqrt{e^x/(e^x+1)}$, we have $dr/dx = rr'^2/2$ and

$$\begin{aligned}f'(x) &= 2\left(\frac{1}{s}+\frac{s}{s'^2}\right)\frac{ds}{dx} \\ &= \frac{2\mathscr{K}(s)^2}{Krr'^2\mathscr{K}(r)^2}\frac{dr}{dx} = \frac{1}{K}\frac{\mathscr{K}(s)^2}{\mathscr{K}(r)^2},\end{aligned}$$

which is positive and strictly increasing in r from $(0,1)$ onto $(1/K, K)$ by Lemma 2.12. Hence f is strictly increasing and convex, with

$$1/K < f'(x) < K. \tag{3.7}$$

Next, if $p, q \in (0,1)$ with $p+q = 1$, then

$$\log\eta_K(e^{px+qy}) = f(px+qy) \le pf(x) + qf(y)$$

by the convexity of f. Thus

$$\eta_K(e^{px}e^{qy}) \le \eta_K(e^x)^p \eta_K(e^y)^q.$$

If we set $e^{x/2} = a$, $e^{y/2} = b$, and $p = q = 1/2$, then the upper bound in (1) follows.

If $x > y$ and $K \ge 1$, it follows from (3.7) that

$$(x-y)/K \le f(x) - f(y) \le (x-y)K,$$

by integration from y to x. Now (2) follows if we set $e^x = b$ and $e^y = a$.

Next, by differentiation and simplification, we get

$$-g'(x) = \frac{s'}{Ks}\left[\frac{\mathscr{K}(s)}{\mathscr{K}(r)}\right]^2,$$

which is positive and decreasing by [AVV3, Theorem 2.2(3)], so that the assertion for g follows. The lower bound in (1) now follows from the convexity of g if one sets $e^x = a^2$ and $e^y = b^2$.

Finally, the equality cases are all clear. ∎

4. ON SOME CONJECTURES

The following conjectures appear in [AVV4, p. 25]:

Conjecture 1. $\operatorname{th} \varphi_K(r) \le \varphi_K(\operatorname{th} r)$ for $K \in [1, \infty)$, $r \in [0, 1]$.

Conjecture 2. For each $K > 1$ define f_K on $(0, 1)$ by

$$f_K(r) = r^{-1/K}\varphi_K(r)\frac{\mathscr{K}(\varphi_K(r))}{\mathscr{K}(r)}.$$

For $1 < K < 2$, $f_K(r)$ is strictly decreasing, $f_2(r)$ is the constant function 2, and, for $K > 2$, $f_K(r)$ is strictly increasing.

Conjecture 3. For $K > 2$, $g_K(r) \equiv \varphi_{1/K}(r)^{1/K}\mathscr{K}(\varphi_{1/K}(r))/[r\mathscr{K}(r)]$ is strictly decreasing from $(0, 1)$ onto $(1/K, 4^{(1/K)-1})$.

Conjecture 4. For $K \ge 1$, $r \in (0, 1)$,

$$\frac{\varphi_K(\sqrt{r})^2}{\varphi_K(r)} \le (1 + r')^{2(1-1/K)}.$$

Conjecture 5. $f(K,r) \equiv \log \varphi_K(r)$ is jointly concave as a function of two variables (K,r) on $(1,\infty)\times(0,1)$.

In this section we obtain results which imply the truth of Conjectures 1–4 and give a partial solution to Conjecture 5.

The following theorem generalizes the result on Conjecture 1.

4.1. THEOREM. *Let $a \in [1,\infty]$ be a constant, and let $f:[0,a) \to [0,1)$ be an increasing differentiable function satisfying the conditions* (i) $f(0)=0$, (ii) $\lim_{x\to a} f(x)=1$, *and* (iii) $f(x) < x \le f(x)/f'(x)$ *for all* $x\in(0,a)$. *Then, for each fixed $r\in(0,1)$, the function*

$$g(K) \equiv f(\varphi_K(r))/\varphi_K(f(r))$$

is strictly decreasing from $[1,\infty)$ *onto* $(f(1),1]$. *In particular,*

$$f(1)\,\varphi_K(f(r)) \le f(\varphi_K(r)) \le \varphi_K(f(r))$$

for all $K\in[1,\infty)$ and $r\in[0,1]$.

Proof. Let $t=f(r)$, $s=\varphi_K(r)$, $u=\varphi_K(t)$, and $h(x)=x'^2\mathscr{K}(x)\mathscr{K}'(x)$. Then $g(K)=f(s)/u$ and

$$\frac{\pi K u}{2f(s)}g'(K) = \frac{sf'(s)}{f(s)}h(s) - h(u) \le h(s) - h(u) < 0,$$

since (iii) implies that $sf'(s)/f(s) \le 1$, $h(x)$ is a strictly decreasing function [AVV3, Theorem 2.2(3)], and $u<s$ by (iii). ∎

4.2. COROLLARY. *For $r\in(0,1)$, we have*

(1) $g_1(K) \equiv \operatorname{th}\varphi_K(r)/\varphi_K(\operatorname{th} r)$ *is strictly decreasing from* $[1,\infty)$ *onto* $((e^2-1)/(e^2+1), 1]$. *In particular,*

$$\left[(e^2-1)/(e^2+1)\right]\varphi_K(\operatorname{th} r) \le \operatorname{th}\varphi_K(r) \le \varphi_K(\operatorname{th} r),$$

with equality in the first if and only if $r=0$ or K tends to ∞, and in the second if and only if $r=0$ or $K=1$.

(2) $g_2(K) \equiv \sin\varphi_K(r)/\varphi_K(\sin r)$ *is strictly decreasing from* $[1,\infty)$ *onto* $(\sin 1, 1]$. *In particular*

$$\varphi_K(\sin r)\sin 1 \le \sin\varphi_K(r) \le \varphi_K(\sin r),$$

with equality in the first if and only if $r=0$ or K tends to ∞ and in the second if and only if $r=0$ or $K=1$.

(3) $g_3(K) \equiv [\log(1+\varphi_K(r))]/\varphi_K(\log(1+r))$ *is strictly decreasing from* $[1,\infty)$ *onto* $(\log 2, 1]$. *In particular,*

$$(\log 2)\,\varphi_K(\log(1+r)) \le \log(1+\varphi_K(r)) \le \varphi_K(\log(1+r)),$$

for all $K \in [1, \infty)$ *and* $r \in [0,1]$. *The first inequality reduces to equality if and only if* $r = 0$ *or* K *tends to* ∞. *The second inequality reduces to equality if and only if* $r = 0$ *or* $K = 1$.

Proof. (1) Take $f(x) = \operatorname{th} x$ and $a = \infty$. Then, the conditions (i), (ii), and (iii) hold and hence the result follows.

(2) Take $f(x) = \sin x$ and $a = \pi/2$. Then the conditions (i), (ii), and (iii) also hold, and hence the result follows.

(3) Take $f(x) = \log(1+x)$ and $a = e - 1$. Then again $f(x)$ satisfies all the conditions of Theorem 4.1, so that the result follows. ∎

The next theorem indicates that Conjecture 2 is true.

4.3. THEOREM. *For each fixed* $K > 1$, *define function* f_K *by*

$$f_K(r) = r^{-1/K}\varphi_K(r)\frac{\mathscr{K}(\varphi_K(r))}{\mathscr{K}(r)}.$$

Then f_K *is strictly decreasing from* $(0,1)$ *onto* $(K, 4^{1-1/K})$ *for* $K \in (1,2)$, *strictly increasing from* $(0,1)$ *onto* $(4^{1-1/K}, K)$ *if* $K > 2$, *and* $f_2(r) \equiv 2$.

Proof. For $(r, K) \in (0,1) \times (1, \infty)$, define

$$F(r,K) = \log f_K(r) = \log s - \frac{1}{K}\log r + \log \mathscr{K}(s) - \log \mathscr{K}(r),$$

where $s = \varphi_K(r)$. Employing (2.8), we have

$$Krr'^2\mathscr{K}(r)^2\frac{\partial F}{\partial r} = g(r,K), \tag{4.4}$$

where $g(r,K) = \mathscr{K}(s)\mathscr{E}(s) - K\mathscr{K}(r)[\mathscr{E}(r) - r'^2\mathscr{K}(r)] - [r'\mathscr{K}(r)]^2$, and

$$\mu(r)\frac{\partial g}{\partial K} = G(r,K), \tag{4.5}$$

where $G(r,K) = \mathscr{K}'(s)^2[\mathscr{E}(s)^2 - s'^2\mathscr{K}(s)^2] - (\pi/2)\mathscr{K}'(r)[\mathscr{E}(r) - r'^2\mathscr{K}(r)]$.

Next, by Lemma 2.10 and [AVV3, Theorem 2.2(3)], it follows that for each $r, G(r,K)$ is an increasing function of K on $[1, \infty)$.

Because the function $\mathscr{K}'(r)[\mathscr{E}(r) - (r')^2\mathscr{K}(r)]$ is strictly increasing from $(0,1)$ onto $(0, \pi/2)$ (see [AVV3, p. 539]),

$$\lim_{K\to\infty} G(r,K) = \frac{\pi}{2}\left\{\frac{\pi}{2} - \mathscr{K}'(r)\left[\mathscr{E}(r) - r'^2\mathscr{K}(r)\right]\right\} > 0. \tag{4.6}$$

On the other hand,

$$\mathscr{E}(r)^2 - [r'\mathscr{K}(r)]^2 < [\mathscr{E}(r) - r'^2\mathscr{K}(r)]^2 \tag{4.7}$$

for $r \in (0,1)$ since

$$\begin{aligned}\mathscr{E}(r)^2 - [r'\mathscr{K}(r)]^2 - [\mathscr{E}(r) - r'^2\mathscr{K}(r)]^2 \\ = r'^2\mathscr{K}(r)[2\mathscr{E}(r) - (2-r^2)\mathscr{K}(r)] < 0,\end{aligned} \tag{4.8}$$

by Lemma 2.10. Hence

$$\begin{aligned}G(r,1) &= \mathscr{K}'(r)\left\{\mathscr{K}'(r)[\mathscr{E}(r)^2 - r'^2\mathscr{K}(r)^2] - \frac{\pi}{2}[\mathscr{E}(r) - r'^2\mathscr{K}(r)]\right\} \\ &< \mathscr{K}'(r)[\mathscr{E}(r) - r'^2\mathscr{K}(r)]\left\{\mathscr{K}'(r)[\mathscr{E}(r) - r'^2\mathscr{K}(r)] - \frac{\pi}{2}\right\} < 0.\end{aligned} \tag{4.9}$$

From (4.6), (4.9), the monotonicity of $G(r,K)$ in K, and (4.5), it follows that, for any fixed $r \in (0,1)$, there exists a unique K_0 such that

$$\frac{\partial g}{\partial K}\begin{cases} < 0, & \text{if } 1 < K < K_0, \\ = 0, & \text{if } K = K_0, \\ > 0, & \text{if } K_0 < K < \infty. \end{cases}$$

Thus, for any fixed $r \in (0,1)$, $g(r,K)$, as a function of K, is strictly decreasing on $[1, K_0]$ and strictly increasing on (K_0, ∞). Note that $\varphi_2(r) = 2\sqrt{r}/(1+r)$, and $g(r,1) = g(r,2) = 0$ by the Landen transformation [BB, p. 12]. Hence, $1 < K_0 < 2$ for any fixed $r \in (0,1)$, and $g(r,K) < 0$ if $K \in (1,2)$, $g(r,K) > 0$ for $K > 2$, so that the result follows from (4.4).

Finally, the limiting values are clear. ∎

The following result indicates that Conjecture 3 is actually contained in Conjecture 2.

4.10. COROLLARY. *For* $K > 2$, $g_K(r) \equiv \varphi_{1/K}(r)^{1/K}\mathscr{K}(\varphi_{1/K}(r))/(r\mathscr{K}(r))$ *is strictly decreasing from* $(0,1)$ *onto* $(1/K, 4^{(1/K)-1})$. *For* $K \in (1,2)$, g_K *is strictly increasing from* $(0,1)$ *onto* $(4^{(1/K)-1}, 1/K)$.

Proof. Set $t = \varphi_{1/K}(r)$. Then $g_K(r) = 1/f_K(t)$, where f_K is as in Theorem 4.3. Hence the result follows from Theorem 4.3. ∎

We now turn to the study of Conjecture 4.

4.11. THEOREM. *For* $r \in (0,1)$, *the function* $H(K) \equiv \varphi_K(\sqrt{r})^2(1+r')^{2/K}/\varphi_K(r)$ *is strictly decreasing from* $[1,\infty)$ *onto* $(1,(1+r')^2]$. *In particular, for* $r \in (0,1)$ *and* $K \in [1,\infty)$,

$$1 \le \frac{\varphi_K(\sqrt{r})^2}{\varphi_K(r)} \le (1+r')^{2(1-1/K)}.$$

The equality holds if and only if $K = 1$.

Proof. Let $t = \sqrt{r}$, $s = \varphi_K(r)$, $u = \varphi_K(t)$. Then, by (2.9),

$$\frac{\pi^2 K^2}{8[s'\mathscr{K}(s)]^2} \frac{H'(K)}{H(K)} \equiv H_1(r,K), \tag{4.12}$$

where

$$H_1(r,K) = \mu(t)\left[\frac{u'\mathscr{K}(u)}{s'\mathscr{K}(s)}\right]^2 - \frac{1}{2}\mu(r) - \frac{\pi^2}{4}\frac{\log(1+r')}{[s'\mathscr{K}(s)]^2}.$$

Let $H_2(r,K) = \{u'\mathscr{K}(u)/[s'\mathscr{K}(s)]\}^2$, $h(x) = \mathscr{K}'(x)[\mathscr{K}(x) - \mathscr{E}(x)]$. Then

$$\frac{\pi K}{4}\frac{1}{H_2(r,K)}\frac{\partial H_2}{\partial K} = h(s) - h(u). \tag{4.13}$$

The function $h(x)$ is strictly increasing from $(0,1)$ onto $(0,\infty)$, by [AVV3, Theorem 2.2(3), (7)]. Hence it follows from (4.13) that

$$\frac{\partial H_2}{\partial K} < 0 \qquad \text{for } r \in (0,1) \text{ and } K \in (1,\infty),$$

since $s < u$ for $r \in (0,1)$ and $K \in (1,\infty)$. So $H_2(r,K)$, as a function of K, is strictly decreasing on $[1,\infty)$, and so is $H_1(r,K)$.

Now it follows from (4.12) that

$$\begin{aligned}
&\frac{\pi^2K^2}{8[s'\mathscr{K}(s)]^2}\,\frac{H'(K)}{H(K)}\\
&\quad\le H_1(r,1)\\
&\quad=\frac{\pi^2}{4}\,\frac{1}{[r'\mathscr{K}(r)]^2}\left\{\frac{2}{\pi}t'^2\mathscr{K}(t)\mathscr{K}'(t)-\frac{1}{\pi}r'^2\mathscr{K}(r)\mathscr{K}'(r)\right.\\
&\qquad\qquad\left.-\log(1+r')\right\}\\
&\quad=\frac{\pi^2}{4[r'\mathscr{K}(r)]^2}\left\{\left[\frac{2}{\pi}t'^2\mathscr{K}(t)\mathscr{K}'(t)+\log t\right]\right.\\
&\qquad\qquad\left.-\frac{1}{2}\left[\frac{2}{\pi}r'^2\mathscr{K}(r)\mathscr{K}'(r)+\log r\right]-\log(1+r')\right\}\\
&\quad<\frac{\pi^2}{8[r'\mathscr{K}(r)]^2}\left\{\frac{2}{\pi}r'^2\mathscr{K}(r)\mathscr{K}'(r)+\log r-2\log(1+r')\right\},
\end{aligned}$$

which is negative for $r\in(0,1)$ by [AVV3, Lemma 4.2(1)] and Lemma 2.1, since $t>r$. Hence H is strictly decreasing in K on $[1,\infty)$.

The limiting values and the remaining conclusions in the theorem are clear. ∎

By the same method used in the proof of Theorem 4.11, we can obtain the following improvement of the inequality stated in Conjecture 4.

4.14. THEOREM. *For each $r\in(0,1)$, $g_r(K)\equiv\varphi_K(\sqrt{r})^2(1+\sqrt{1-r})^{2/K}/\varphi_K(r)$ is strictly decreasing from $[1,\infty)$ onto $(1,(1+\sqrt{1-r})^2]$. In particular, for any $r\in(0,1)$ and $K\in[1,\infty)$,*

$$\frac{\varphi_K(\sqrt{r})^2}{\varphi_K(r)}\le\left(1+\sqrt{1-r}\right)^{2(1-1/K)},$$

with equality if and only if $K=1$.

Proof. Let $t=\sqrt{r}$, $s=\varphi_K(r)$, $u=\varphi_K(t)$. Then, by differentiation,

$$\frac{\pi^2K^2}{8[s'\mathscr{K}(s)]^2g_r(K)}\,\frac{\partial g_r}{\partial K}=G_r(K),$$

where

$$G_r(K) = \mu(t)\frac{u'^2\mathscr{K}(u)^2}{s'^2\mathscr{K}(s)^2} - \frac{1}{2}\mu(r) - \frac{\pi^2}{4}\frac{\log(1+t')}{[s'\mathscr{K}(s)]^2}.$$

By a method similar to the proof of Theorem 4.11, we can get

$$\begin{aligned} G_r(K) &\le \frac{\pi^2}{4[r'\mathscr{K}(r)]^2}\left\{\frac{2}{\pi}t'^2\mathscr{K}(t)\mathscr{K}'(t) - \frac{1}{\pi}r'^2\mathscr{K}(r)\mathscr{K}'(r)\right. \\ &\qquad\left. -\log(1+t')\right\} \\ &< \frac{\pi^2}{8[r'\mathscr{K}(r)]^2}\left\{\frac{2}{\pi}t'^2\mathscr{K}(t)\mathscr{K}'(t) + \log t - 2\log(1+t')\right\} < 0, \end{aligned}$$

yielding the monotonicity of $g_r(K)$.

The remaining conclusions are clear. ∎

Finally, we prove a result about the Conjecture 5.

4.15. THEOREM. *Define the function f on $(1,\infty)\times(0,1)$ by $f(r,K)\equiv \log\varphi_K(r)$. Then, for each segment $I\subset D$ parallel to the vector $(\cos\varphi, \sin\varphi)$ with $\varphi\in(0,\pi/2)$, $f|I$ is concave. In particular, if $t, r_1, r_2\in(0,1)$, $K_1, K_2\in[1,\infty)$ with $r_1<r_2$ and $K_1<K_2$ or with $r_1>r_2$ and $K_1>K_2$, then*

$$\varphi_{K_1}(r_1)^t\varphi_{K_2}(r_2)^{1-t} \le \varphi_{tK_1+(1-t)K_2}(tr_1+(1-t)r_2).$$

Proof. Clearly, $\varphi_K(r)$ is increasing in r as well as in K. By [Q, Lemma 1; AVV4, Corollary 2.2(1) and (2)], $\varphi_K(r)$ is log-concave in r and in K, separately. We can see from (2.8) and (2.9) that $(1/s)(\partial s/\partial r)$ is decreasing in K, and $(1/s)(\partial s/\partial K)$ is decreasing in r, where $s=\varphi_K(r)$. Hence the result follows from Lemma 2.13. ∎

4.16. THEOREM. *For $r\in(0,1)$, $c\in(1,\infty)$, let f be defined on $[1,\infty)$ by $f(K)=u'\mathscr{K}(u)/(s'\mathscr{K}(s))$, where $u=\varphi_K(t)$, $s=\varphi_K(r)$, $t=r^{1/c}$. Then f is decreasing, with $\lim_{K\to\infty}f(K)=0$.*

Proof. By differentiation and simplification, we see that $f'(K)<0$ on $(1,\infty)$ if and only if $h(s)<h(u)$, where $h(x)=\mathscr{K}'(x)[\mathscr{E}(x)-x'^2\mathscr{K}(x)]$. But this is true since h is strictly increasing [AVV3, Theorem 2.2(3)] and since $s<u$. ∎

5. PROPERTIES OF THE λ-DISTORTION FUNCTION

As is well known, the special function

$$\lambda(K) \equiv \eta_K(1) = \left(\frac{\varphi_K(1/\sqrt{2})}{\varphi_{1/K}(1/\sqrt{2})} \right)^2 \tag{5.1}$$

introduced by Lehto, Virtanen, and Väisälä [LVV] also plays an important role in the distortion theory of plane quasiconformal mappings. It yields, for example, a sharp upper bound for the linear dilatation of a plane quasiconformal mapping, and measures the distortion of the boundary values of a K-quasiconformal self-mapping of the upper half plane preserving the point ∞. In this section, we study some properties of $\lambda(K)$.

5.2. THEOREM. *The function* $f(K) \equiv [\log \lambda(K)]/\log K$ *is decreasing on* $(0,1)$ *and increasing on* $(1,\infty)$, *onto* (a,∞), *where* $a = (4/\pi)\mathscr{K}(1/\sqrt{2})^2 = 4.3768\cdots$. *In particular, for* $K \in (0,\infty)$,

$$\lambda(K) \geq K^a.$$

Proof. Since $f(1/K) = f(K)$, it is enough to prove the theorem for $K \in (1,\infty)$. For this, set $r = \varphi_K(1/\sqrt{2}) = \mu^{-1}(\pi/(2K))$. Then $1/\sqrt{2} < r < 1$ and

$$f(K) = \frac{2(\log r - \log r')}{\log \mathscr{K}(r) - \log \mathscr{K}'(r)}.$$

Let $h(r) = 2\log(r/r')$, $p(r) = \log(\mathscr{K}(r)/\mathscr{K}'(r))$, $g(r) = h(r)/p(r)$. Then $h(1/\sqrt{2}) = p(1/\sqrt{2}) = 0$, and

$$\frac{h'(r)}{p'(r)} = \frac{4}{\pi}\mathscr{K}(r)\mathscr{K}'(r),$$

which is strictly increasing in r on $[1/\sqrt{2}, 1)$ by [AVV3, Theorem 2.2(8)]. Hence g is also increasing there by the Monotone l'Hôpital's Rule [AVV5, Lemma 2.2], and so is f.

The limiting values are clear. ∎

5.3. COROLLARY. *For* $K \in (0,\infty)$, *define* F *on* $(0,\infty)$ *by* $F(p) = \lambda(K^p)^{1/p}$. *Then*

(1) *For* $K \in (1,\infty)$, $F(p)$ *is increasing from* $(0,\infty)$ *onto* (K^a,∞), *where* $a = (4/\pi)\mathscr{K}(1/\sqrt{2})^2$. *In particular, for* $K \in (1,\infty)$,

$$K^a < \lambda(K^p)^{1/p} < \lambda(K) \qquad \text{if } p \in (0,1), \tag{5.4}$$

and

$$K^a < \lambda(K) < \lambda(K^p)^{1/p} \qquad \text{if } p \in (1,\infty). \tag{5.5}$$

(2) *For $K \in (0,1)$, $F(p)$ is decreasing from $(0,\infty)$ onto $(0, K^a)$, where a is as in* (1). *In particular, for $K \in (0,1)$, the inequalities* (5.4) *and* (5.5) *are both reversed.*

Proof. For (1), since

$$\log F(p) = \frac{\log \lambda(K^p)}{\log K^p} \cdot \log K,$$

the results follow from Theorem 5.2.

For (2), set $L = 1/K$. Then $F(p) = 1/\lambda(L^p)^{1/p}$, and the results follow by (1). ∎

5.6. THEOREM. *For $p \in (0,\infty)$, the function $G(K) \equiv \lambda(K)^p/\lambda(K^p)$ is strictly decreasing on $(0,\infty)$ if $p \in (1,\infty)$, and strictly increasing on $(0,\infty)$ if $p \in (0,1)$, and it has range $(0,\infty)$. In particular,*

$$\lambda(K)^p \begin{cases} > \lambda(K^p), & \text{if } 1/K, p \in (1,\infty) \text{ or } 1/K, p \in (0,1), \\ = \lambda(K^p), & \text{if } K = 1 \text{ or } p = 1, \\ < \lambda(K^p), & \text{if } K, p \in (1,\infty) \text{ or } K, p \in (0,1). \end{cases}$$

Proof. That the range is $(0,\infty)$, follows from [AVV1, Theorem 2.13]. Next, set $r_p = \mu^{-1}(\pi/(2K^p))$, $r = r_1$. Then, by (5.1), (1.2), and (2.7),

$$\lambda(K^p) = r_p^2/\left(1 - r_p^2\right), \qquad \mu(r_p) = \pi/(2K^p), \tag{5.7}$$

$$\frac{dr_p}{dK} = \frac{2p}{\pi K} r_p\left(1 - r_p^2\right)\mathscr{K}(r_p)\mathscr{K}'(r_p). \tag{5.8}$$

Clearly,

$$r_p \begin{cases} > r > \dfrac{1}{\sqrt{2}}, & \text{if } K, p \in (1,\infty), \\ = r, & \text{if } p = 1, \\ = r = \dfrac{1}{\sqrt{2}}, & \text{if } K = 1, \\ < r < \dfrac{1}{\sqrt{2}}, & \text{if } 1/K, p \in (1,\infty), \end{cases} \tag{5.9}$$

and

$$r \begin{cases} > r_p > 1/\sqrt{2}, & \text{if } 1/K, p \in (0,1), \\ < r_p < 1/\sqrt{2}, & \text{if } K, p \in (0,1). \end{cases} \tag{5.10}$$

By (5.7), (5.8), and differentiation, we get

$$\frac{\pi K}{4p}\frac{G'(K)}{G(K)} = \mathscr{K}(r)\mathscr{K}'(r) - \mathscr{K}(r_p)\mathscr{K}'(r_p).$$

The results now follow from this equality, (5.9), (5.10), and [AVV3, Theorem 2.2(8)]. ∎

5.11. THEOREM. *For each $K \in (1,\infty)$ let f be defined on $(0,\infty)$ by $f(t) = \eta_K(t)\eta_K(1/t)$. Then f is strictly decreasing on $(0,1]$ and strictly increasing on $[1,\infty)$. In particular, for $t \in (0,\infty)$ and $K \in [1,\infty)$.*

$$\lambda(K)^2 \le \eta_K(t)\eta_K(1/t),$$

with equality if and only if $t = 1$ or $K = 1$.

Proof. Denote $r = \sqrt{t/(t+1)}$, $s = \varphi_K(r)$, $u = \varphi_K(r')$, and let $g(r) = (1/2)\log f(t) = \log s' - \log s' + \log u - \log u'$. Note that $t = 1$ if and only if $r = 1/\sqrt{2}$. Then, by differentiation,

$$\begin{aligned} g'(r) &= \left(\frac{1}{s} + \frac{s}{s'^2}\right)\frac{ds}{dr} + \left(\frac{1}{u} + \frac{u}{u'^2}\right)\frac{du}{dr} \\ &= \frac{1}{Krr'^2}\left\{\left[\frac{\mathscr{K}(s)}{\mathscr{K}(r)}\right]^2 - \left[\frac{\mathscr{K}(u)}{\mathscr{K}(r')}\right]^2\right\}, \end{aligned}$$

which is negative for $r < r'$ and positive for $r > r'$, by Lemma 2.12. ∎

5.12. *Remark.* It is easy to show that, for each $K \in (1,\infty)$, the function $f(p) \equiv (K^p)\lambda'(K^p)/\lambda(K^p)$, is increasing from $(0,\infty)$ onto (a,∞), where $a = (4/\pi)\mathscr{K}(1/\sqrt{2})^2$. In particular, $a < f(1) < f(p)$, for all $K, p \in (1,\infty)$.

6. APPLICATIONS

We can make use of the above theorems to improve some existing results in the distortion theory of quasiconformal mappings. In this section, we only show some examples.

6.1. THEOREM. *Suppose that f is a K-quasiconformal mapping of the unit disk B onto itself with $f(0) = 0$. Then*

(1) $[2(1+\sqrt{1-|z|^2})]^{1-K}\ |z|^K \le |f(z)| \le (1+\sqrt{1-|z|^2})^{2(1-1/K)}$ $|z|^{1/K}$, *for all $z \in B$;*

(2) $\varphi_{1/K}(\sin|z|) \le \sin|f(z)| \le \varphi_K(\sin|z|)$, *for all $z \in B$;*

(3) $\varphi_{1/K}(\text{th}|z|) \le \text{th}|f(z)| \le \varphi_K(\text{th}|z|)$, *for all* $z \in B$;

(4) $\varphi_{1/K}(\log(1+|z|)) \le \log(1+|f(z)|) \le \varphi_K(\log(1+|z|))$, *for all* $z \in B$;

(5) $|w_1 - w_2| \le \{[2(1+x_1')(1+x_2')]^2\, [(1+\sqrt{1-(2x_2)^2})/2]^{1/2}\}^{1-1/K} |z_1 - z_2|^{1/K}$ *for* $z_1, z_2 \in B$, *where* $w_j = f(z_j)$, $x_j' = (1-x_j^2)^{1/2}$, $j = 1, 2$, $x_1 = (|z_1| + |z_2|)/2$, *and* $x_2 = |z_1 - z_2|/(2(|z_1| + |z_2|))$.

Proof. Part (1) follows from Theorem 1.5 and the well-known quasiconformal Schwarz lemma

$$\varphi_{1/K}(|z|) \le |f(z)| \le \varphi_K(|z|).$$

Parts (2), (3), and (4) follow from Corollary 4.2, so long as we note that all the right inequalities in Corollary 4.2 are reversed if K is replaced by $1/K$.

For (5), we employ the following inequality [AVV4, Lemma 4.6]

$$|w_1 - w_2| \le 2(|w_1| + |w_2|)\varphi_K(x_2)\left\{\frac{1+\sqrt{1-(2x_2)^2}}{2}\right\}^{(1/2)(1-1/K)},$$

from which we obtain, by Theorem 1.5,

$$\begin{aligned}
|w_1 - w_2| &\le 2(\varphi_K(|z_1|) + \varphi_K(|z_2|))(1+x_2')^{2(1-1/K)} \\
&\qquad \times \left\{\frac{1+\sqrt{1-(2x_2)^2}}{2}\right\}^{(1/2)(1-1/K)} x_2^{1/K} \\
&\le \left\{[2(1+x_2')]^2\left[\frac{1+(1-(2x_2)^2)^{1/2}}{2}\right]^{1/2}\right\}^{1-1/K} \\
&\qquad \times \frac{\varphi_K(x_1)}{x_1^{1/K}}|z_1 - z_2|^{1/K} \\
&\le \left\{[2(1+x_1')(1+x_2')]^2\left[\frac{1+(1-(2x_2)^2)^{1/2}}{2}\right]^{1/2}\right\}^{1-1/K} \\
&\qquad \times |z_1 - z_2|^{1/K}.
\end{aligned}$$

■

6.2. *Remark*. Theorem 1.5 can also be used to obtain explicit bounds for $\sin|f(z)|$, $\operatorname{th}|f(z)|$, and $\log(1 + |f(z)|)$ in Corollary 6.1(2), (3), and (4).

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