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A Result on q-Series and Its Application to Quadratic Forms

GEORGE E. ANDREWS*

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

AND

YOSHIYUKI KITAOKA

Department of Mathematics, Nagoya University, Chikusa-ku, Nagoya 464, Japan

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The purpose of this paper is to prove a conjectured q-identity. The result is then applied to estimating the local density of solutions of certain systems of quadratic form identities.

1. INTRODUCTION

Our object here is to consider the solvability of quadratic form identities. In particular, we consider the identity

$$S[X] \equiv {}^tXSX = T, \tag{1.1}$$

where S and T are integral positive definite matrices of degrees m and n, respectively. The goal is to find sufficient conditions for the existence of integral solutions. When $m \geqslant 2n+3$ we know that the solvability of (1.1) over all rings of p-adic integers $\mathbb{Z}_p$ together with the minimum of T being sufficiently large implies the solvability over $\mathbb{Z}$, the ring of rational integers [5]. However, no such result is known for $m \leqslant 2n+2$ $(n \geqslant 2)$.

On the other hand, a famous result of Siegel asserts that the weighted average of the numbers of integral solutions of $S_i[X] = T$, where S_i runs

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over a complete set of representatives of the classes in the genus of S, is, roughly speaking, the infinite product of local densities $\alpha_p(T, S)$, where

$$\alpha_p(T, S) = \lim_{t \to \infty} (p^{-t})^{mn - n(n+1)/2} \# \{X \bmod p^t \mid S[X] \equiv T \bmod p^t\}.$$

Making this weighted average large is important for global solvability.

Consequently we wish to bound the local densities away from zero. If $m \geqslant 2n+3$, then $\alpha_p(T, S) > c(S)$ holds for some positive constant $c(S)$ provided $\alpha_p(T, S) \neq 0$ [6]. Unfortunately things become much more difficult if $m \leqslant 2n+2$. By [6, Th. 1, p. 442], the behavior of local densities is ruled by the primitive solutions for $m = 2n+1$, $2n+2$. However, this is not the case for $m \leqslant 2n$. Indeed part (a) of [6, Th. 1, p. 442] gives a bound from below for the local densities in terms of primitive solutions. But if, for example, $n+1 \leqslant m < 2n$, $m-n-1 \neq$ Witt index over $\mathbb{Q}_p$ of S, then $\alpha_p(p^t T, S) \to \infty$ as $t \to \infty$, and for large t there are no primitive solutions at all. Thus the primitive solutions are too strong and special sufficient conditions; so the discussion of primitive solutions is not really relevant in this instance. To begin with we consider $\alpha_p(p^t T, S)$ as $t \to \infty$ with T and S fixed. Except for the case $m = 2n$, $n = r+2$ with r the Witt index over $\mathbb{Q}_p$ of S, we have for some positive constant $c(T, S)$ (assuming $m \leqslant 2n$)

$$\alpha_p(p^t T, S) > c(T, S)$$

provided $\alpha_p(p^t T, S) \neq 0$ from [6]. In the remaining case, it is also shown in [6] that

$$\alpha_p(p^t T, S) > c(T, S)\, p^{-t}. \tag{1.2}$$

In [6], the important question of reversing the inequality in (1.2) is also treated for $p \neq 2$ which is assumed hereafter. Namely, in the remaining case,

$$\alpha_p(p^t T, S) < c'(T, S)\, p^{(\varepsilon - 1)t} \tag{1.3}$$

holds for any $\varepsilon > 0$ providing the following is true:

CONJECTURE [6]. *Let*

$$\begin{bmatrix} k \\ g \end{bmatrix} = \begin{cases} \dfrac{(1-q^k)(1-q^{k-1}) \cdots (1-q^{k-g+1})}{(1-q^g)(1-q^{g-1}) \cdots (1-q)}, & 0 \leqslant g \leqslant k \\ 0 & \text{otherwise} \end{cases} \tag{1.4}$$

$$H_n(x) = \sum_{r=0}^{r} \begin{bmatrix} n \\ r \end{bmatrix} x^r, \tag{1.5}$$

and define $F(a, k, z)$ inductively:

$$F(0, k, z) = -1 + \sum_{g=0}^{k} (-1)^{k-g} H_{k-g}(-q) \begin{bmatrix} k \\ g \end{bmatrix} q^{g(g+3)/2-k} z^g, \tag{1.6}$$

$$F(a+1, k, z) = \sum_{g=a+1}^{k} F(a, g, z)(-1)^{k-g} H_{k-g}(-q) \begin{bmatrix} k \\ g \end{bmatrix} q^{g(g+3)/2-k} z^g \tag{1.7}$$

$$- F(a, k, z)\, q^{(a+1)(a+2)/2} z^{a+1}.$$

Then (1.3) *holds provided $F(n-2, n-1, q^{-n}) = F(n-2, n, q^{-n}) = 0$ where $n = rkT$.*

The details of how these identities imply (1.3) are given in [6]. This conjecture is an immediate consequence of Corollary 1 given in Section 3 and is, in effect, restated as Corollary 2.

From the proof of the conjecture we may summarize the behavior of $\alpha_p(p^t T, S)$. If $m = 2n$ and $n =$ (Witt index of S over Q_p) + 2, then

$$\alpha_p(p^t T, S) \to 0 \qquad \text{as} \quad t \to \infty. \tag{1.8}$$

If $n + 1 \leqslant m \leqslant 2n$ and the above case is excluded, then there exists a positive constant $c(T, S)$ such that

$$\alpha_p(p^t T, S) > c(T, S) \tag{1.9}$$

provided $\alpha_p(p^t T, S) \neq 0$.

Our local results should be contrasted with the global observation that the smaller $m - n$ is the harder it is to solve $S[X] = T$. What are the implications of our local results for the global situation?

2. Background Lemmas

We require four lemmas. We begin with an evaluation of $H_n(-q)$.

Lemma 1. *$H_n(-q) = (1-q)(1-q^3) \cdots (1-q^v)$ where v is the largest odd number $\leqslant n$.*

Proof. This is proved in quite disguised form in [2, p. 20]. Here is a simple generating function proof. We shall use the standard notation:

$$(A)_n = (A; q)_n = (1-A)(1-Aq)\cdots(1-Aq^{n-1}), \tag{2.1}$$

$$\begin{aligned}\sum_{n\geqslant 0} \frac{H_n(-q)t^n}{(q)_n} &= \sum_{n\geqslant 0}\sum_{r=0}^{n} \frac{(-q)^r t^n}{(q)_r (q)_{n-r}} \\ &= \sum_{n,r\geqslant 0} \frac{(-q)^r t^{n+r}}{(q)_r (q)_n} = \frac{1}{(t)_\infty (-tq)_\infty} \\ &\qquad \text{(by Euler's sum, [1, p. 19, Eq. (2.2.5)])} \\ &= \frac{1+t}{(t^2; q^2)_\infty} = (1+t) \sum_{m=0}^{\infty} \frac{t^{2m}}{(q^2; q^2)_\infty}. \end{aligned} \tag{2.2}$$

Hence comparing coefficients of t^n on both sides, we get

$$H_n(-q) = \frac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q^2)(1-q^4)\cdots(1-q^{2m})}, \tag{2.3}$$

where $2m$ is the largest even number $\leqslant n$. This is equivalent to the assertion of Lemma 1. ∎

We must now define polynomials in two variables which specialize to the $F(a, k; z)$ given in the Introduction:

$$F(0, k; y, z) = -1 + \sum_{g=0}^{k} (-1)^{k-g} H_{k-g}(-q) \begin{bmatrix} k \\ g \end{bmatrix} q^{\binom{g+1}{2} - (k-g)} y^g, \tag{2.4}$$

and for $a \geqslant 0$

$$\begin{aligned} F(a+1, k; y, z) = &-z^{a+1} q^{\binom{g+1}{2}} F(a, k; y, z) \\ &+ \sum_{g=a+1}^{k} F(a, g; y, z)(-1)^{k-g} \\ &\times H_{k-g}(-q) \begin{bmatrix} k \\ g \end{bmatrix} q^{\binom{g+1}{2} - (k-g)} z^g. \end{aligned} \tag{2.5}$$

We note in passing that $F(0, k; y, z)$ is actually independent of z and that

$$F(a, k, z) = F(a, k; z, z). \tag{2.6}$$

LEMMA 2. *For* $a \geqslant 0$,

$$F(a, a; y, z) = 0. \tag{2.7}$$

Proof. By (2.4), $F(0, 0; y, z) = 0$, and by (2.5) for $a \geqslant 0$

$$F(a+1, a+1; y, z) = -z^{a+1} q^{\binom{a+1}{2}} F(a, a+1; y, z) + F(a, a+1; y, z)\, q^{\binom{a+1}{2}} z^{a+1} = 0. \quad ▮$$

LEMMA 3. *For $a \geqslant 0$, $k \geqslant 0$*

$$F(a, k; q^{-2}, z) = 0. \tag{2.8}$$

Proof. Clearly by (2.5) we need only prove that for $k \geqslant 0$

$$F(0, k; q^{-2}, z) = 0. \tag{2.9}$$

Now by reversing the sum

$$\begin{aligned}
&F(0, k; q^{-2}, z)\\
&= -1 + \sum_{g=0}^{k} (-1)^g H_g(-q) \begin{bmatrix} k \\ g \end{bmatrix} q^{\binom{k-g+1}{2} - g} q^{-2(k-g)}\\
&= -1 + \sum_{m \geqslant 0} (q; q^2)_m \begin{bmatrix} k \\ 2m \end{bmatrix} q^{\binom{k-2m+1}{2} - 2m} q^{-2(k-2m)}\\
&\quad - \sum_{m \geqslant 0} (q; q^2)_{m+1} \begin{bmatrix} k \\ 2m+1 \end{bmatrix} q^{\binom{k-2m}{2} - 2m - 1} q^{-2(k-2m-1)}\\
&\qquad \text{(by Lemma 1)}\\
&= -1 + \sum_{m \geqslant 0} (q; q^2)_m \frac{(q^{-k})_{2m}}{(q)_{2m}} q^{\binom{k+1}{2} - 2m} q^{-2(k-2m)}\\
&\quad + \sum_{m \geqslant 0} (q; q^2)_{m+1} \frac{(q^{-k})_{2m+1}}{(q)_{2m+1}} q^{\binom{k+1}{2} - 2m - 1} q^{-2(k-2m-1)}\\
&= -1 + q^{-2k + \binom{k+1}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-k}, q^{-k+1}; q^2, q^2 \\ 0 \end{matrix} \right)\\
&\quad + q^{-2(k-1) + \binom{k+1}{2} - 1} (1 - q^{-k}) {}_2\phi_1 \left(\begin{matrix} q^{-k+1}, q^{-k+2}; q^2, q^2 \\ 0 \end{matrix} \right)\\
&\qquad \text{(in standard } q\text{-hypergeometric series notation [7, p. 90])}\\
&= -1 + q^{\binom{k+1}{2} - 2k - \binom{k}{2}} + q^{-2k+2+\binom{k+1}{2} - 1} (1 - q^{-k}) q^{-\binom{k-1}{2}}\\
&\qquad \text{(by } q\text{-Vandermonde summation, [7, p. 97, (3.3.2.7)])}\\
&= -1 + q^{-k} + 1 - q^{-k}\\
&= 0,
\end{aligned} \tag{2.10}$$

as desired. ▮

LEMMA 4. *For* $a \geq 0$

$$F(a, k; y, z) = F(a, k; yq, z) + yz^a q^{a+1}(1-q^k)\{F(a, k-1; yq, zq) + F(a-1, k-1; zq, zq) - (1 - z^a q^{\binom{a+2}{2}-1})\, F(a-1, k-1; yq, zq)\}, \tag{2.11}$$

where we take $F(-1, k; y, z) = 1$ *identically.*

Proof. We proceed by induction on a. For $a = 0$,

$$F(0, k; y, z) - F(0, k; yq, z)$$
$$= \sum_{g=0}^{k-1} (-1)^{k-1-g} H_{k-g-1}(-q)(1-q^k) \begin{bmatrix} k-1 \\ g \end{bmatrix} q^{(g+1)(g+4)/2-k} y^{g+1}$$
$$= yq(1-q^k) \sum_{g=0}^{k-1} (-1)^{k-1-g} H_{k-1-g}(-q) \begin{bmatrix} k-1 \\ g \end{bmatrix} q^{g(g+3)/2-(k-1)}(yq)^g$$
$$= yq(1-q^k)(1 + F(0, k-1; yq, z)),$$

which is (2.11) with $a = 0$.

Assume the result is true now up to and including a given a. Then we see that by (2.5) and the induction hypothesis

$$F(a+1, k; y, z)$$
$$= -z^{a+1} q^{\binom{a+2}{2}} \{F(a, k; yq, z) + yz^a q^{a+1}(1-q^k)[F(a, k-1; yq, zq) + F(a-1, k-1; zq, zq) - (1 - z^a q^{\binom{a+2}{2}-1})\, F(a-1, k-1; yq, zq)]\}$$
$$+ \sum_{g=a+1}^{k} (-1)^{k-g} H_{k-g}(-q) \begin{bmatrix} k \\ g \end{bmatrix} q^{\binom{g+1}{2}-(k-g)} z^g$$

$$\{F(a, g; yq, z) + yz^a q^{a+1}(1-q^g) \times [F(a, g-1; yq, zq) + F(a-1, g-1; zq, zq) - (1 - z^a q^{\binom{a+2}{2}-1})\, F(a-1, g-1; yq, zq)]\}$$

$$
\begin{aligned}
&= F(a+1, k, yq, z) \\
&\quad + yz^a q^{a+1} \Big\{ -z^{a+1} q^{\binom{a+2}{2}} (1-q^k) \\
&\quad \times [F(a, k-1; yq, zq) + F(a-1, k-1; zq, zq) \\
&\quad - (1 - z^a q^{\binom{a+2}{2}-1}) F(a-1, k-1; yq, zq)] \\
&\quad + \sum_{g=a}^{k-1} (-1)^{k-1-g} H_{k-1-g}(-q) \\
&\quad \times \begin{bmatrix} k-1 \\ g \end{bmatrix} (1-q^k)\, q^{\binom{g+2}{2} - (k-g-1)} z^{g+1} \\
&\quad \times [F(a, g; yq, zq) + F(a-1, g; zq, zq) \\
&\quad - (1 - z^a q^{\binom{a+2}{2}-1}) F(a-1, g; yq, zq)] \Big\} \\
&= F(a+1, k; yq, z) \\
&\quad + yz^{a+1} q^{a+2} (1-q^k) \{ F(a+1, k-1; yq, zq) \\
&\quad + F(a, k-1; zq, zq) - (1 - z^a q^{\binom{a+2}{2}-1}) F(a, k-1; yq, zq) \\
&\quad - z^a q^{a(a+3)/2} F(a, k-1; yq, zq) \\
&\quad + (zq)^{a+1} q^{\binom{a+2}{2}} F(a, k-1; yq, zq) \} \\
&\qquad \text{(where we have applied Lemma 2)} \\
&= F(a+1, k; yq, z) \\
&\quad + yz^{a+1} q^{a+2} (1-q^k) \{ F(a+1, k-1; yq, zq) \\
&\quad + F(a, k-1; zq, zq) - (1 - z^{a+1} q^{\binom{a+3}{2}-1}) F(a, k-1; yq, zq) \}. \quad (2.12)
\end{aligned}
$$

Hence the truth at a implies the truth at $a+1$; therefore, Lemma 4 is true for all $a \geqslant 0$. ∎

3. The Main Result

We are now prepared to prove the conjecture described in the Introduction.

Theorem 1. *For* $2 \leqslant j \leqslant s \leqslant a+2$,

$$F(a, k; q^{-j}, q^{-s}) = 0. \quad (3.1)$$

Proof. We proceed by a double induction first on a and then on j. Initially $a=0$. Therefore $j=s=2$, and Lemma 3 tells us that (3.1) is true in this case.

Now let us assume (3.1) is true up to but not including a particular a. Clearly by Lemma 3

$$F(a, k; q^{-2}, q^{-s})=0; \tag{3.2}$$

so we now assume that (3.1) also is true at a up to but not including a particular $j(>2)$. Hence by (2.11)

$$\begin{aligned} F(a, k; q^{-j}, q^{-s}) = {} & F(a, k; q^{-(j-1)}, q^{-s}) \\ & + q^{-j-sa+a+1}(1-q^k)\{F(a, k-1; q^{-(j-1)}, q^{-(s-1)}) \\ & + F(a-1, k-1; q^{-(s-1)}, q^{-(s-1)}) \\ & - (1-q^{-sa+\binom{a+2}{2}-1}) F(a-1, k-1; q^{-(j-1)}, q^{-(s-1)})\} \\ = {} & 0, \end{aligned} \tag{3.3}$$

since the first two F's on the right of (3.3) are zero by the hypothesis on j and the remainder are zero by the hypothesis on a. (Note that since $2<j\leqslant s\leqslant a+2$, therefore $2\leqslant j-1\leqslant s-1\leqslant (a-1)+2$.) Thus (3.1) follows by our double induction. ∎

COROLLARY 1. *For all* $a\geqslant 0$, $k\geqslant 0$,

$$F(a, k, q^{-a-2})=0. \tag{3.4}$$

Proof. This is just Theorem 1 with $j=s=a+2$. ∎

The conjecture stated in the Introduction is just Corollary 1 with $k=a+1$ and $k=a+2$.

Now let $M\supset N$ be regular quadratic lattices over $\mathbb{Z}_p$ with $rkM=m$, $rkN=n$, $r=$ Witt index of M, and suppose $n+1\leqslant m\leqslant 2n$. From [6] we know $0\leqslant r\leqslant n$, $0\leqslant m-2r\leqslant 4$, and for $t\geqslant 0$

$$\alpha_p(p^tN, M)>c(M, N)\, p^{t(n-r)(n+r+1-m)}. \tag{3.5}$$

The only instance in which the exponent on p is negative occurs for $m=2n$, $n=r+2$.

COROLLARY 2. *If* $m=2n$, $n=r+2$, $p\neq 2$, *then for* $\varepsilon>0$

$$\alpha_p(p^tN, M)<c(\varepsilon)\, p^{(\varepsilon-2)t}. \tag{3.6}$$

Proof. This assertion was proved in [6] subject to the conjecture stated in Section 1 which is a special case of Corollary 1. ∎

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