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# On Constructing the Solution to the Exterior Dirichlet Problem for the Helmholtz Equation*

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In this paper a method is given for constructing the solution to the exterior Dirichlet problem for the Helmholtz equation in three dimensions. This method is modeled after the procedure of Colton and Kleinman (Proc. Roy. Soc. Edin. **86A**(1980), 29–42) for solving the corresponding two-dimensional problem. The scattering problem is reformulated as an integral equation and it is shown that its solution can be represented as a convergent Neumann series for small values of the wave number. Comparisons are made between the present method and known results. Examples are given which illustrate the method.

## 1. INTRODUCTION

Recently, several methods have been given (see [1, 4, 5, 8]) for constructing the solution of the exterior Dirichlet problem for the Helmholtz equation in both two and three dimensions. In [8], Kleinman obtains the solution to the scattering problem explicitly in terms of the Green's function for the corresponding potential problem. In [1], by using a subsidiary condition obtained from the Helmholtz integral representation, Ahner reformulates the problem as a boundary integral equation and shows that it can be solved by iteration. In this approach it is necessary to compute the first eigenfunction of the integral equation associated with the problem. In [4], Colton and Kleinman consider both the direct and inverse scattering problem in two dimensions. Their results are based on the use of conformal mapping techniques (for the case of the inverse scattering problem) and the fact that the integral, over the boundary of the obstacle, of the normal derivative of the solution to Laplace's equation satisfying a boundedness condition at infinity, vanishes. Instead of using the free space Green's function as the kernel function, Colton and Kress [5] choose a kernel which in the limiting case $k = 0$ becomes the Green's function for Laplace's equation in three

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dimensions in the exterior of some ball which is contained in the interior of the scatterer. Their work also has applications to both the direct and inverse scattering problem.

In this paper another method is given for solving the exterior Dirichlet scattering problem in three dimensions. The method here is based on the work of Colton and Kleinman [4]. Their method, as indicated earlier, makes excellent use of a certain property of the solution to the potential problem in the plane. Unfortunately, there is no result analogous to this in three dimensions. Nonetheless, motivated by their work, a direct extension is made here of their results to the three-dimensional case. The resulting integral equation, however, involves the same eigenfunction which appeared in [1]; consequently, the present method is domain dependent.

Besides introducing to the literature another constructive method for solving the Dirichlet scattering problem in three dimensions, it is hoped that the present note will serve to demonstrate a connection between the work in [1] and that in [4] which otherwise might go unnoticed.

In the next section the solution to the exterior Dirichlet problem for Laplace's equation in three dimensions is considered. There the problem is reformulated as an integral equation, which parallels a similar integral equation in [4] for the corresponding two-dimensional case. It is shown that this integral equation can be solved by iteration. In Section 3, the exterior Dirichlet scattering problem is considered and it is shown that the solution can also be found by iteration, provided the wave number $k$ is sufficiently small. In both Sections 2 and 3 comparisons are made between the present method and the results in [1, 4]. In the last section one potential and one scattering problem are solved by using the results in Sections 2 and 3, respectively.

## 2. Constructing the Solution to the Exterior Dirichlet Problem for Laplace's Equation

Let $V_i$ be a bounded domain in $R^3$ containing the origin, with a closed, simply connected $C^2$ boundary $S$ and let $V_e$ denote the region exterior to $\bar{V}_i$. Let $\hat{\mathbf{n}}$ denote the outward unit normal to $S$; let $\mathbf{x}$ denote a typical point in $R^3$ and let $r = |\mathbf{x}|$. Consider the exterior Dirichlet problem for Laplace's equation

$$u_0(\mathbf{x}) = u_0^i(\mathbf{x}) + u_0^s(\mathbf{x}) \qquad \text{in} \quad R^3 \backslash V_i, \tag{2.1a}$$

$$\Delta u_0^s = 0 \qquad \text{in} \quad V_e, \tag{2.1b}$$

$$u_0(\mathbf{x}) = 0 \qquad \text{for} \quad \mathbf{x} \in S, \tag{2.1c}$$

$$\lim_{r \to \infty} u_0^s(\mathbf{x}) = 0. \tag{2.1d}$$

It is to be noted that the more familiar condition at infinity, namely that $u_0^s$ is regular there (see [7, p. 217]), is automatically guaranteed by (2.1b) and (2.1d) (see [6, p. 179]). It is assumed that $u_0^i$ is a given solution to Laplace's equation in all of $R^3$, except possibly in some set of measure zero contained in $V_e$. In any event, $u_0^i$ is assumed to be twice continuously differentiable on $S$. Under these conditions, a unique solution $u_0^s(x)$ to (2.1) exists and $u_0^s \in C^2(R^3\backslash \bar{V}_i) \cap C^1(R^3\backslash V_i)$ (cf. [6]).

From Green's identities we have

$$\begin{aligned} \frac{1}{4\pi}\int_S \left\{ u_0^s(\mathbf{y}) \frac{\partial}{\partial n}\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}|}\frac{\partial u_0^s(\mathbf{y})}{\partial n} \right\} dS_{\mathbf{y}} \\ &= 0, \qquad \mathbf{x} \in V_i, \\ &= \tfrac{1}{2}u_0^s(\mathbf{x}), \qquad \mathbf{x} \in S, \\ &= u_0^s(\mathbf{x}), \qquad \mathbf{x} \in V_e. \end{aligned} \tag{2.2}$$

Similarly, since $u^i$ satisfies Laplace's equation in $V_i$ we have

$$\begin{aligned} \frac{1}{4\pi}\int_S \left\{ u_0^i(\mathbf{y}) \frac{\partial}{\partial n}\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{|\mathbf{x}-\mathbf{y}|}\frac{\partial u_0^i(\mathbf{y})}{\partial n} \right\} dS_{\mathbf{y}} \\ &= -u_0^i(\mathbf{x}), \qquad \mathbf{x} \in V_i, \\ &= -\tfrac{1}{2}u_0^i(\mathbf{x}), \qquad \mathbf{x} \in S, \\ &= 0, \qquad \mathbf{x} \in V_e. \end{aligned} \tag{2.3}$$

Using boundary condition (2.1b), it follows from (2.2) and (2.3) that

$$\begin{aligned} u_0^i(\mathbf{x}) - \frac{1}{4\pi}\int_S \frac{1}{|\mathbf{x}-\mathbf{y}|}\frac{\partial u_0(\mathbf{y})}{\partial n} dS_{\mathbf{y}} &= 0, \qquad \mathbf{x} \in \bar{V}_i, \\ &= u_0(\mathbf{x}), \qquad \mathbf{x} \in V_e. \end{aligned} \tag{2.4}$$

Taking the normal derivative as the field point $\mathbf{x}$ approaches $S$ from the exterior region we obtain

$$v_0(\mathbf{x}) = 2v_0^i(\mathbf{x}) - K_0^* v_0(\mathbf{x}), \tag{2.5}$$

where $v_0(\mathbf{x}) = \partial u_0(\mathbf{x})/\partial n$, $v_0^p(\mathbf{x}) = \partial u_0^p(\mathbf{x})/\partial n$, $p = i, s$; and where

$$K_0^* v_0(\mathbf{x}) \equiv \frac{1}{2\pi}\int_S \frac{\partial}{\partial n_{\mathbf{x}}}\frac{1}{|\mathbf{x}-\mathbf{y}|} v_0(\mathbf{y})\, dS_{\mathbf{y}}. \tag{2.6}$$

Let

$$\langle \phi, \psi \rangle = \int_S \phi(\mathbf{y})\, \psi(\mathbf{y})\, dS_{\mathbf{y}}. \tag{2.7}$$

Let the constant $\beta$ be defined by

$$\beta = \langle v_0, 1 \rangle = \langle v_0^s, 1 \rangle, \tag{2.8}$$

where the last equality can be established by using Green's second identity to verify that $\langle v_0^i, 1 \rangle = 0$. (In fact this last result is essential in order that a solution to (2.5) exist.) Motivated by the work of Colton and Kleinman [4] we obtain from (2.5) and (2.8) the following integral equation for $v_0(\mathbf{x})$:

$$v_0(\mathbf{x}) = 2v_0^i(\mathbf{x}) - \frac{\beta}{4\pi} \frac{\partial}{\partial n_{\mathbf{x}}} \frac{1}{|\mathbf{x}|} - \frac{1}{2\pi} \int_S \left\{ \frac{\partial}{\partial n_{\mathbf{x}}} \frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{2} \frac{\partial}{\partial n_{\mathbf{x}}} \frac{1}{|\mathbf{x}|} \right\} v_0(\mathbf{y})\, dS_{\mathbf{y}}. \tag{2.9}$$

It should be noted that integral equation (2.9) is analogous to integral equations (2.6) and (2.9) in [4]. Letting $C(S)$ denote the Banach space of continuous complex valued functions defined on $S$ and equipped with the sup norm, the following result may be proven:

THEOREM 2.1. *Let $L_0\psi = K_0^*\psi - (1/4\pi)(\partial/\partial n_{\mathbf{x}})(1/|\mathbf{x}|)\, \langle \psi, 1 \rangle$. Then $L_0$ is a compact operator from $C(S)$ into $C(S)$ and $\sigma(L_0) \subset (-1, 1)$, where $\sigma(L_0)$ denotes the spectrum of $L_0$.*

That $L_0$ is a compact operator from $C(S)$ into $C(S)$ follows from the analogous property for the operator $K_0^*$ (see [12]), and since the argument used to show $\sigma(L_0) \subset (-1, 1)$ is the same here as used in [4, Theorem 1], we shall omit the proof. It follows that integral equation (2.9) may be solved iteratively. Thus we need only determine the constant $\beta$.

In two dimensions it is known (e.g., see [14, p. 609]) that the integral over the boundary of the normal derivative of the solution to Laplace's equation in the exterior region must vanish. This observation was crucial in the work of Colton and Kleinman [4]. There is no analogous condition for the solvability of the exterior Neumann problem in three-dimensional space. For this reason, it is perhaps not surprising that, unlike the two-dimensional case, $\int_S \partial u_0^s(\mathbf{y})/\partial n\, dS_{\mathbf{y}}$ is domain dependent and as we shall show it also depends upon the nature of $u_0^s$ on the surface as well.

Let $\xi_0(\mathbf{x})$ be the solution to

$$\xi_0(\mathbf{x}) = -K^*\xi_0(\mathbf{x}). \tag{2.10}$$

It can be shown (e.g., see [3; 10, p. 229]) that the eigenfunction $\xi_0$ can be obtained, up to a multiplicative constant, by iteration. Furthermore, it can be shown (e.g., see [12, pp. 376–377]) that

$$\frac{1}{2\pi}\int_S \frac{1}{|\mathbf{x}-\mathbf{y}|}\,\xi_0(\mathbf{y})\,dS_{\mathbf{y}} = \text{const.} \tag{2.11}$$

Assume that $\xi_0$ has been normalized so that the constant appearing on the right-hand side of (2.11) is one. From (2.4) for $\mathbf{x} \in S$ we have

$$\frac{1}{2\pi}\int_S \frac{1}{|\mathbf{x}-\mathbf{y}|}\,v_0(\mathbf{y})\,dS_{\mathbf{y}} = 2u_0^i(\mathbf{x}). \tag{2.12}$$

Multiplying (2.12) by $\xi_0(\mathbf{x})$ and integrating, we have from (2.8)

$$\beta = \langle v_0, 1\rangle = 2\langle u_0^i, \xi_0\rangle. \tag{2.13}$$

A result similar to (2.13) was established by the author in [1, Eq. (3.14)]. (In [1, Eq. (3.14)], the term $\gamma_0$ is later shown to be a constant and there the normalization for $\xi(\mathbf{x})$ is slightly different from the one for $\xi_0(\mathbf{x})$ here.)

It should be pointed out that there is a similarily between the derivation of integral equation (2.9) with $\beta$ given in (2.13) here and integral equation (3.16) in [1] for the exterior Dirichlet potential problem. Both are based on integral equation (2.5) and in both use is made of the subsidiary condition (2.12) and the eigenfunction in (2.10).

## 3. Constructing the Solution to the Exterior Dirichlet Problem for the Helmholtz Equation

We now consider the exterior Dirichlet problem for the Helmholtz equation. This problem is to determine the scattered field $u^s \in C^2(R^3\backslash\bar{V}_i) \cap C^1(R^3\backslash V_i)$ such that

$$u(\mathbf{x}) = u^i(\mathbf{x}) + u^s(\mathbf{x}), \qquad \mathbf{x} \in R^3\backslash V_i, \tag{3.1a}$$

$$(\Delta + k^2)\,u^s(\mathbf{x}) = 0 \qquad \text{in} \quad V_e, \tag{3.1b}$$

$$u(\mathbf{x}) = 0 \qquad \text{on} \quad S, \tag{3.1c}$$

$$\lim_{r\to\infty} r\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \tag{3.1d}$$

where the incident field $u^i(\mathbf{x})$ is known and satisfies the Helmholtz equation in all of $R^3$; where $k$ is real; where $u(\mathbf{x})$ denotes the total field; and where the *radiation condition* is assumed to hold uniformly in all directions. Using the same argument as used in Section 2 for obtaining (2.4) and using the boundary condition (3.1c), the following representation may be obtained

$$u^i(\mathbf{x}) - \frac{1}{4\pi}\int_S \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} v(\mathbf{y})\, dS_{\mathbf{y}} = 0, \qquad \mathbf{x} \in \bar{V}_i, \\ = u(\mathbf{x}), \qquad \mathbf{x} \in V_e, \tag{3.2}$$

where $v(\mathbf{y}) = \partial u(\mathbf{y})/\partial n$. Taking the normal derivative of (3.2) as the field point approaches the surface from $V_e$, we obtain

$$2v^i(\mathbf{x}) - \frac{1}{2\pi}\int_S \frac{\partial}{\partial n_{\mathbf{x}}} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} v(\mathbf{y})\, dS_{\mathbf{y}} = v(\mathbf{x}), \qquad \mathbf{x} \in S. \tag{3.3}$$

From (3.2) for $\mathbf{x} \in S$ we get

$$2\langle u^i, \xi_0\rangle = \int_S \gamma_1(\mathbf{y}; k)\, v(\mathbf{y})\, dS_{\mathbf{y}} = \langle v, \gamma_1\rangle, \tag{3.4}$$

where $\xi_0(\mathbf{x})$ is the same function as in Section 2 and, where

$$\gamma_1(\mathbf{y}; k) = \frac{1}{2\pi}\int_S \xi_0(\mathbf{z}) \frac{e^{ik|\mathbf{z}-\mathbf{y}|}}{|\mathbf{z}-\mathbf{y}|} dS_{\mathbf{z}}. \tag{3.5}$$

Furthermore from (2.11) and the normalization for $\xi_0$, it can be shown

$$2\langle u^i, \xi_0\rangle = \langle v, \gamma_1\rangle = \langle v, 1\rangle + \langle v, \gamma\rangle, \tag{3.6}$$

where

$$\gamma(\mathbf{y}; k) = \frac{1}{2\pi}\int_S \xi_0(\mathbf{z}) \frac{e^{ik|\mathbf{z}-\mathbf{y}|} - 1}{|\mathbf{z}-\mathbf{y}|} dS_{\mathbf{z}}. \tag{3.7}$$

Multiplying (3.4) by $(1/4\pi)(\partial/\partial n_{\mathbf{x}})(1/|\mathbf{x}|)$ (we could also have used $(1/4\pi)$ $(\partial/\partial n_{\mathbf{x}})(e^{ik|\mathbf{x}|}/|\mathbf{x}|)$ but we have opted for using the simpler kernel) we get

$$-\frac{\langle u^i, \xi_0\rangle}{2\pi} \frac{\partial}{\partial n_{\mathbf{x}}} \frac{1}{|\mathbf{x}|} + \frac{1}{4\pi} \frac{\partial}{\partial n_{\mathbf{x}}} \frac{1}{|\mathbf{x}|} \langle \gamma_1, v\rangle = 0. \tag{3.8}$$

Substituting (3.8) into (3.6) we obtain

$$v(\mathbf{x}) = g(\mathbf{x}) - L_k v(\mathbf{x}), \tag{3.9}$$

where

$$L_k v(x) = \frac{1}{2\pi}\int_S \left\{ \frac{\partial}{\partial n_{\mathbf{x}}} \frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{2} \frac{\partial}{\partial n_{\mathbf{x}}} \frac{1}{|\mathbf{x}|} \gamma_1(\mathbf{y}, k) \right\} v(\mathbf{y})\, dS_{\mathbf{y}}, \tag{3.10}$$

and, where

$$g(\mathbf{x}) = 2v^i(\mathbf{x}) - \frac{\langle u^i, \xi_0 \rangle}{2\pi} \frac{\partial}{\partial n_{\mathbf{x}}} \frac{1}{|\mathbf{x}|}. \tag{3.11}$$

We next show that for sufficiently small $k$, integral equation (3.9) can be solved by iteration. From the definitions of $L_0$ and $L_k$ and using (3.6) and (3.7) it can be shown that

$$\|L_k - L_0\| = O(k), \tag{3.12}$$

where $\|\cdot\|$ denotes the operator norm taken with respect to our Banach space $C(S)$. Using the result in (3.12) we now show that the spectral radius of $L_k$ is less than one. The proof we have chosen to use here follows the argument used in [9, Theorem 2.2] and in [4, Theorem 2].

THEOREM 3.1. *There exists a positive constant $k_0$ such that for $|k| \leqslant k_0$, the spectral radius of $L_k$ is less than unity.*

*Proof.* From (3.12) we have

$$\|(\lambda I + L_k) - (\lambda I + L_0)\| = \|L_k - L_0\| = O(k). \tag{3.13}$$

From Theorem 2.1 we have that $(\lambda I + L_0)^{-1}$ exists for $|\lambda| \geqslant \lambda_0$, where $\lambda_0 < 1$ and as a consequence of the bounded inverse theorem (see [13, p. 63]) it is a bounded linear operator. Let

$$M = \max_{|\lambda| \geqslant \lambda_0} \|(\lambda I + L_0)^{-1}\|. \tag{3.14}$$

From (3.13) and (3.14) it follows that there is a positive number $k_0$ such that for $|k| \leqslant k_0$

$$\|(\lambda I + L_k) - (\lambda I + L_0)\| < M^{-1} \leqslant \|(\lambda I + L_0)^{-1}\|^{-1} \tag{3.15}$$

for all $|\lambda| \geqslant \lambda_0$. Thus from [16, p. 164] it follows that $(\lambda I + L_k)^{-1}$ exists for all $|\lambda| \geqslant \lambda_0$ and $|k| \leqslant k_0$. Hence the spectral radius of $L_k$ is less than one.

From this result we have

THEOREM 3.2. *There exists a $k_0 > 0$ such that for $|k| \leqslant k_0$, the solution to integral equation* (3.9) *is given by*

$$v(\mathbf{x}) = \sum_{n=0}^{\infty} (-L_k)^n g(\mathbf{x}). \tag{3.16}$$

There is a difference between the derivation of integral equation (3.9) and analogous integral equation (2.23) in [4] for the corresponding scattering problem in two-dimensional space. To illustrate this point, let us proceed as in [4]. In (3.2) set $\mathbf{x}=\mathbf{0}\in V_i$ and then multiply the resulting equation by $\alpha(\partial/\partial n_{\mathbf{x}})(e^{ik|\mathbf{x}|}/|\mathbf{x}|)$, where $\alpha$ is at the moment an arbitrary constant; we obtain

$$2\alpha u^i(\mathbf{0})\frac{\partial}{\partial n_{\mathbf{x}}}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}=\frac{\alpha}{2\pi}\int_S v(\mathbf{y})\frac{e^{ik|\mathbf{y}|}}{|\mathbf{y}|}\frac{\partial}{\partial n_{\mathbf{x}}}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\,dS_{\mathbf{y}}. \tag{3.17}$$

From (3.3) and (3.17) we obtain for $\mathbf{x}\in S$

$$v(\mathbf{x})=g(\mathbf{x};\alpha)-M_k^\alpha v(\mathbf{x}), \tag{3.18}$$

where

$$g(\mathbf{x};\alpha)=2v^i(\mathbf{x})-2\alpha u^i(\mathbf{0})\frac{\partial}{\partial n_{\mathbf{x}}}\frac{1}{|\mathbf{x}|} \tag{3.19}$$

and where

$$M_k^\alpha v(\mathbf{x})=\frac{1}{2\pi}\int_S\left\{\frac{\partial}{\partial n_{\mathbf{x}}}\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|}-\alpha\frac{e^{ik|\mathbf{y}|}}{|\mathbf{y}|}\frac{\partial}{\partial n_{\mathbf{x}}}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\right\}v(\mathbf{y})\,dS_{\mathbf{y}}. \tag{3.20}$$

It is crucial for our purposes here that the difference $\|M_k^\alpha-L_0\|$ be small for $|k|$ sufficiently small. Taking $k=0$, we obtain

$$(M_0^\alpha-L_0)\,v(\mathbf{x})=\frac{1}{2\pi}\frac{\partial}{\partial n_{\mathbf{x}}}\frac{1}{|\mathbf{x}|}\int_S\left(\frac{1}{2}-\frac{\alpha}{|\mathbf{y}|}\right)v(\mathbf{y})\,dS_{\mathbf{y}}. \tag{3.21}$$

Unfortunately, there is no *proper* choice for the parameter $\alpha$, for which the operator $M_0^\alpha-L_0$ will vanish for all $v\in C(S)$.

## 4. Examples

We now provide some explicit examples of the iteration procedures developed in the previous sections. Consider the problem in (2.1), where $S$ is a sphere of radius $a$ and

$$u_0^i(\mathbf{x})=1/|\mathbf{x}-\mathbf{x}_e|,\qquad \mathbf{x}_e\in V_e. \tag{4.1}$$

From [15, p. 85] we have

$$\frac{1}{|\mathbf{x}-\mathbf{x}_e|}=\sum_{n=0}^{\infty}\frac{r_<^n}{r_>^{n+1}}P_n(\cos\alpha), \tag{4.2}$$

where $\cos\alpha = \hat{\mathbf{r}}(\mathbf{0},\mathbf{x})\cdot(\hat{\mathbf{r}}(\mathbf{0},\mathbf{x}_e)$, where $\hat{\mathbf{r}}(\mathbf{0},\mathbf{x})$ denotes a unit vector from $\mathbf{0}$ to $\mathbf{x}$. For a sphere, $\xi=$ const. satisfies (2.10) and normalizing $\xi$ we find $\xi_0(\mathbf{x}) = (2a)^{-1}$. From (2.13) and using the orthogonality of the Legendre polynomials we obtain

$$\beta = 2\int_S \frac{1}{|\mathbf{y}-\mathbf{x}_e|}\frac{1}{2a}\,dS_{\mathbf{y}} = \frac{4\pi a}{r_e}. \tag{4.3}$$

For $\mathbf{x}\in S$,

$$\begin{aligned} g_0(\mathbf{x}) &\equiv 2v_0^i(\mathbf{x}) - \frac{\beta}{4\pi}\frac{\partial}{\partial n_{\mathbf{x}}}\frac{1}{|\mathbf{x}|} \\ &= \frac{1}{ar_e} + 2\sum_{n=1}^{\infty} n\frac{a^{n-1}}{r_e^{n+1}}P_n(\cos\alpha). \end{aligned} \tag{4.4}$$

For $\mathbf{x},\mathbf{y}\in S$, it can be shown

$$\frac{\partial}{\partial n_{\mathbf{x}}}\left\{\frac{1}{|\mathbf{x}-\mathbf{y}|} - \frac{1}{2}\frac{1}{|\mathbf{x}|}\right\} = -\frac{1}{2a}\frac{1}{|\mathbf{x}-\mathbf{y}|} + \frac{1}{2a^2}. \tag{4.5}$$

Since $|\mathbf{x}|=|\mathbf{y}|=a$, we employ an average of two expansions of $1/|\mathbf{x}-\mathbf{y}|$, similar to (4.2) with $|\mathbf{x}|=a_> > |\mathbf{y}|=a$ in one and $|\mathbf{x}|=a_< < |\mathbf{y}|=a$ in the other. We find

$$\begin{aligned} L_0\,g(\mathbf{x}) = -\frac{1}{4\pi a}\int_S &\left\{\frac{1}{2}\left[\sum_{m=0}^{\infty}\frac{a^m}{a_>^{m+1}}P_m(\cos\gamma) - \frac{1}{a}\right]\right. \\ &\left. + \frac{1}{2}\sum_{m=1}^{\infty}\frac{a_<^m}{a^{m+1}}P_m(\cos\gamma)\right\} \\ &\times\left\{\frac{1}{ar_e} + 2\sum_{n=1}^{\infty} n\frac{a^{n-1}}{r_e^{n+1}}P_n(\cos\delta)\right\}dS, \end{aligned} \tag{4.6}$$

where $\cos\gamma = \hat{\mathbf{r}}(\mathbf{0},\mathbf{x})\cdot\hat{\mathbf{r}}(\mathbf{0},\mathbf{y})$ and $\cos\delta = \hat{\mathbf{r}}(\mathbf{0},\mathbf{x}_e)\cdot\hat{\mathbf{r}}(\mathbf{0},\mathbf{y})$. Using the orthogonality of the Legendre polynomials and subsequently letting $a_> = a_< = a$, it follows that

$$L_0\,g_0(\mathbf{x}) = -2\sum_{n=1}^{\infty}\frac{n}{(2n+1)}\frac{a^{n-1}}{r_e^{n+1}}P_n(\cos\alpha). \tag{4.7}$$

From an induction argument it can be shown

$$L_0^m\,g_0(x) = (-1)^m\,2\sum_{n=1}^{\infty}\frac{n}{(2n+1)^m}\frac{a^{n-1}}{r_e^{n+1}}P_n(\cos\alpha) \tag{4.8}$$

for $m = 1, 2,\ldots$. It follows that

$$v_0(\mathbf{x}) = \sum_{m=0}^{\infty} (-L_0)^m g_0(\mathbf{x}) = \sum_{n=0}^{\infty} (2n+1)\frac{a^{n-1}}{r_e^{n+1}} P_n(\cos\alpha), \tag{4.9}$$

where the geometric series in powers of $1/(2n+1)$ has been summed. Substituting (4.9) into (2.4) we get

$$u_0(\mathbf{x}) = \sum_{n=0}^{\infty} \left\{\frac{r_<^n}{r_>^{n+1}} - \frac{a^{2n+1}}{r_e^{n+1} r^{n+1}}\right\} P_n(\cos\alpha), \tag{4.10}$$

where $r_> = \max\{r, r_e\}$ and $r_< = \min\{r, r_e\}$. This agrees with the standard result using separation of variables.

Next consider problem (3.1) for a plane wave incident on a sphere of radius $a$. Let the coordinate system be oriented so that the origin $\mathbf{0}$ coincides with the center and the $z$ axis is aligned with the direction $\hat{\mathbf{k}}$ of the plane wave (so that $\hat{\mathbf{z}} = \hat{\mathbf{k}}$). Let the point $\mathbf{x}$ have spherical coordinates $(r, \theta, \phi)$. From [11, pp. 107–108] for $r > a$ we have

$$u^i(\mathbf{x}) = e^{ikr\cos\theta} = \sum_{m=0}^{\infty} i^m(2m+1) j_m(kr) P_m(\cos\theta), \tag{4.11}$$

$$\frac{e^{ik|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} = ik \sum_{m=0}^{\infty} (2m+1) j_m(ka) h_m^{(1)}(kr) P_m(\cos\gamma). \tag{4.12}$$

From Eqs. (3.11) and (4.11), keeping $\xi_0(\mathbf{x}) = (2a)^{-1}$, and using the orthogonality of the Legendre polynomials we obtain

$$g(\mathbf{x}) = \frac{j_0(ka)}{a} + 2k \sum_{m=0}^{\infty} i^m(2m+1) j_m(ka)' P_m(\cos\theta), \qquad \mathbf{x} \in S, \tag{4.13}$$

where the prime denotes differentiation with respect to the argument. From (3.5) and (4.12) we have

$$\gamma_1(\mathbf{y}; k) = ikaj_0(ka) h_0^{(1)}(ka). \tag{4.14}$$

From (4.13), (4.14), and (3.10) we have

$$L_k g(\mathbf{x}) = \frac{1}{2\pi} \int_S \left\{\frac{j_0(ka)}{a} + 2k \sum_{m=0}^{\infty} i^m(2m+1) j_m(ka)' \right.$$
$$\left. \times P_m(\cos\theta_{\mathbf{y}})\right\} \left\{\frac{ik}{2a} j_0(ka) h_0^{(1)}(ka)\right.$$

$$+\frac{ik^2}{2}\sum_{n=0}^{\infty}(2n+1)[j_n(ka)'\,h_n^{(1)}(ka_>)$$

$$+j_n(ka_<)\,h_n^{(1)}(ka)']\,P_n(\cos\gamma)\Big\}\,dS_y. \tag{4.15}$$

From the orthogonality of the Legendre polynomials and subsequently letting $a_> = a_< = a$, it follows that

$$L_k\,g(\mathbf{x}) = \left[\frac{j_0(ka)}{a} + 2kj_0(ka)'\right] b_0(ka)$$

$$+\,2k\sum_{m=1}^{\infty} i^m(2m+1)\,j_m(ka)'\,b_m(ka)\,P_m(\cos\theta), \tag{4.16}$$

where

$$b_0(ka) = ikaj_0(ka)\,h_0^{(1)}(ka) + ik^2a^2[\,j_0(ka)'\,h_0^{(1)}(ka)$$

$$+\,j_0(ka)\,h_0^{(1)}(ka)'\,], \tag{4.17a}$$

$$b_m(ka) = ik^2a^2[\,j_m(ka)'\,h_m^{(1)}(ka) + j_m(ka)\,h_m^{(1)}(ka)'\,],$$

$$m \geqslant 1. \tag{4.17b}$$

From an induction argument it can be shown that

$$L_k^n\,g(\mathbf{x}) = \left[\frac{j_0(ka)}{a} + 2kj_0(ka)'\right][b_0(ka)]^n$$

$$+\,2k\sum_{m=1}^{\infty} i^m(2m+1)\,j_m(ka)'\,[b_m(ka)]^n\,P_m(\cos\theta). \tag{4.18}$$

From (3.16) it follows that

$$v(\mathbf{x}) = \left[\frac{j_0(ka)}{a} + 2kj_0(ka)'\right]\sum_{n=0}^{\infty}[-b_0(ka)]^n$$

$$+\,2k\sum_{m=1}^{\infty}\sum_{n=0}^{\infty} i^m(2m+1)\,j_m(ka)'\,[-b_m(ka)]^n\,P_m(\cos\theta). \tag{4.19}$$

It will presently be shown that $|b_m(ka)| < 1$ for $m \geqslant 0$ for sufficiently small values of the wave number $k$; consequently, the geometric series in (4.19) may be summed

$$v(\mathbf{x}) = \left[\frac{j_0(ka)}{a} + 2kj_0(ka)'\right]\frac{1}{1+b_0(ka)}$$
$$+ 2k\sum_{m=1}^{\infty} i^m(2m+1)\frac{j_m(ka)'}{1+b_m(ka)}P_m(\cos\theta). \qquad (4.20)$$

Making use of the Wronskian identity (see [11, p. 68])

$$W\{j_m, h_m^{(1)}\} = i/k^2a^2$$

it can be shown

$$v(\mathbf{x}) = \frac{-ik}{(ka)^2}\sum_{m=0}^{\infty} i^m(2m+1)\frac{1}{h_m^{(1)}(ka)}P_m(\cos\theta), \qquad \mathbf{x}\in S. \quad (4.21)$$

Substituting (4.21) into (3.2) the following result is obtained

$$u(\mathbf{x}) = \sum_{m=0}^{\infty} i^m(2m+1)\left[j_m(kr) - \frac{j_m(ka)}{h_m^{(1)}(ka)}h_m^{(1)}(kr)\right]P_m(\cos\theta), \qquad \mathbf{x}\in V_e \quad (4.22)$$

which agrees with the classical result (e.g., see [2, p. 358]).

Now let us demonstrate that $|b_m(ka)| < 1$ for $m \geqslant 0$ for sufficiently small values of $k$. From the series representations for the Bessel and Hankel functions [11] it can be shown

$$b_m(ka) = -e^{ika}/(2m+1) + O(ka), \qquad m \geqslant 1 \qquad (4.23)$$

and from [11, p. 73] we get

$$b_0(ka) = e^{ika}\{\cos ka - (\sin ka/ka) + i\sin ka\}. \qquad (4.24)$$

Hence, for $k$ sufficiently small, it is seen that the desired inequality is valid.

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