INTRODUCTION

In this note, we point out a generalization of the results of [5], which will answer the following question. Let $G$ be a finite group, $p$ be a prime, $Z$ a central $p'$-subgroup of $G$, and $\lambda$ a linear character of $Z$. How many $p$-blocks of defect 0 of $G$ "lie over" $\lambda$? Although the groundwork for this result is really contained in [5], we feel that this strengthened version is worth remarking on.

As usual, we let $R$ denote a complete discrete valuation ring of characteristic 0 whose residue field, $k$ say, is algebraically closed of characteristic $p$. We let $\pi$ denote the unique maximal ideal of $R$, and let $\ast$ denote images $(\text{mod } \pi)$. We assume for technical convenience that $R$ contains a primitive $\left(\frac{|G|}{p}\right)^{\text{th}}$ root of unity ($R$ will contain all roots of unity of order prime to $p$ under the hypothesis that $k$ is algebraically closed).

One motivation for our work here is that in [4] Külshammer and the present author described a procedure for determining the number of irreducible characters of $G$ which can be afforded by $N$-projective $RG$-modules, when $N$ is a normal subgroup of $G$ (equivalently, this is the number of irreducible characters of $G$ which can be afforded by ($R$-free) $RG$-modules with vertex contained in $N$).

For an irreducible character, $\chi$, of $N$, the number of irreducible characters of $G$ which can be afforded by ($R$-free) $RG$-modules with vertex contained in $N$ and which lie over $\chi$ is shown in [4] to be the number of $p$-blocks of defect zero lying over a certain linear character of a cyclic central $p'$-subgroup of a fixed $p'$-central extension of $I(\chi)/N$ (see [4] for a more precise statement). For this purpose, it is no real loss of generality to assume that $O_{p'}(I(\chi)/N)$ is trivial, in which case the cyclic central $p'$-subgroup of the $p'$-central extension of $I(\chi)/N$ is just its centre.

The results here, together with those of [4], could be used to determine...
the number of irreducible characters of $G$ which can be afforded by $(R$-free) $RG$-modules with vertex contained in $N$. We remark that (as far as we can see) this number could not be determined from the character table of $G$ alone (except in the degenerate cases where the order or index of $N$ is prime to $p$).

If $\chi$ above is in a $p$-block of defect zero of $N$, then all irreducible characters of $G$ which can be afforded by an $N$-projective $RG$-module and which lie over $\chi$ lie themselves in $p$-blocks of defect zero of $G$. Conversely, of course, every irreducible character of $G$ in a $p$-block of defect 0 lies over a $p$-block of defect 0 of $N$, and can be afforded by an $N$-projective $RG$-module.

These remarks and the results of [4] can be used to reduce the determination of the number of blocks of defect zero of the group $G$ (assuming all relevant inertial groups can be determined) to determining the number of $p$-blocks of defect 0 (lying over the right linear characters of the centres) of $p'$-central extensions of certain simple sections of $G$. (Note that if $I(\chi)/N$ is a non-trivial $p$-group, then no $p'$-central extension of $I(\chi)/N$ can have a $p$-block of defect 0.)

The procedure is as follows: suppose that we can find a $p$-block of defect 0 of $M$, a maximal normal subgroup of $G$. Let $\mu$ be the unique irreducible character in this block. Assume for convenience (and with no real loss of generality) that $G = I(\mu)$, and that $G/M$ is non-Abelian of order divisible by $p$. Then $G$ has a $p$-block of defect zero lying over $\mu$ if and only if the $p'$-central extension of $G/M$ constructed in [4] has a $p$-block of defect zero lying over the correct linear character of its centre. The central extension of $G/M$ is of course a central product of a quasi-simple group and a cyclic $p'$-group, so the real question concerns the existence of a $p$-block of defect 0 of the quasi-simple group (over the right linear character of its centre).

For a $p$-block $B$ of Alperin type (see [3] for the definition of Alperin type (see also [1])), this reduces the problem of determining the number of simple $B$-modules (assuming all relevant inertial groups can be determined) to questions about the number of $p$-blocks of defect 0 (lying over the right linear characters of the centres) of $p'$-central extensions of certain simple sections of $G$.

We also remark that although it is the case that with very few exceptions the (known) simple groups have $p$-blocks of defect 0 for every prime divisor $p$ of their orders, the question of whether a quasi-simple group has a $p$-block of defect 0 lying over a prescribed linear character of its centre appears to be more delicate.

As in [5], we answer a more general question about blocks with normal defect group $D$ (which can be combined with Brauer’s First Main Theorem in the usual fashion).

We remark that (as is well known) those irreducible characters $\chi$ such
that \( \langle \text{Res}_G^G(\chi), \lambda \rangle \neq 0 \) form a union of \( p \)-blocks of ordinary irreducible characters of \( G \). Also, it is no loss of generality to assume that \( \lambda \) is faithful, so we do.

We recall that to say that the irreducible character, \( \chi \), of \( G \) lies over \( \lambda \) means that \( \langle \text{Res}_G^G(\chi), \lambda \rangle \neq 0 \), and to say that the \( p \)-block, \( B \), of \( G \) lies over \( \lambda \) means that one (and hence each) ordinary irreducible character of \( B \) lies over \( \lambda \).

We let \( u^G \) denote the conjugacy class of \( u \) and \( \hat{u}^G \) denote the corresponding class sum. We say that a conjugacy class \( xc \) of \( G \) is \( Z \)-good if \( x \) is not \( G \)-conjugate to \( xz \) for any \( z \) in \( Z^* \). Then it is easy to see that every irreducible character of \( G \) which lies over \( \lambda \) vanishes identically outside the \( Z \)-good conjugacy classes. We note also that \( Z \) permutes the conjugacy classes of \( G \) via \( u^G \rightarrow zu^G \) for \( u \) in \( G \), \( z \) in \( Z \), and that the \( Z \)-good conjugacy classes fall into orbits of length \( |Z| \).

We let \( D = O_p(G) \) (the possibility that \( D \) is trivial is not excluded). We let \( P \) denote a fixed Sylow \( p \)-subgroup of \( G \), \( X \) denote a full set of representatives for those \( (P, P) \)-double cosets \( PxP \) for which it is possible to choose \( x \) so that:

(i) \( x \) is \( p \)-regular, and \( x^G \) is \( Z \)-good.

(ii) \( D \) is a Sylow \( p \)-subgroup of \( C_G(x) \).

(iii) \( P \cap x^{-1}Px = D \).

Our main result is:

**Theorem 1.** The number of blocks of \( G \) which have defect group \( D \) and lie over the linear character \( \lambda \) of \( Z \) is zero if \( X \) is empty and if \( X \) is non-empty it is the rank of the matrix \( M \) whose rows and columns are indexed by representatives of the \( Z \)-orbits of \( Z \)-good \( p \)-regular conjugacy classes with defect group \( D \), and whose \( (u^G, v^G) \)-entry is

\[
\sum_{x \in X} \sum_{(z, w) \in Z \times Z} (\lambda(z^{-1}) \lambda(w) |u^G \cap wxP| |v^G \cap zxP|)^*.
\]

**Proof.** Let \( s: Z(kG) \rightarrow Z(kG) \) be the map defined as in the proof of the main theorem of [5].

Let \( e_z = |Z|^{-1} \sum_{z \in Z} \lambda(z^{-1})z \) in \( Z(RG) \) (a central idempotent of \( RG \)).

We note that (as is well known) an irreducible character \( \chi \) of \( G \) lies over \( \lambda \) if and only if \( \chi(e_z) \neq 0 \).

As in the proof of the main theorem of [5], we see that the \( k \)-subspace of \( Z(kG) \) spanned by the block idempotents of blocks with defect group \( D \) is spanned by

\[
\{(u^G) \hat{s} : u \text{ is p-regular and } D \in \text{Syl}_p(C_G(u))\}.
\]
Also from the proof of that theorem, we see that for such \( u \), we have

\[
\hat{(u^G)} = \sum_v \sum_{x \in P \setminus G/P} \left( [G : C_G(v)]_{p^-1}^{-1} |u^G \cap xP| \right) * \hat{v^G}
\]

(where \( v \) runs over a set of representatives for the \( G \)-conjugacy classes of \( p \)-regular elements \( v \) such that \( D \in \text{Syl}_p(C_G(v)) \); in fact, we may restrict attention to those double cosets containing a \( p \)-regular element \( x \) with \( D \in \text{Syl}_p(C_G(x)) \) such that \( P \cap x^{-1}Px = D \)). Then, for such \( u \), we have

\[
(e \hat{u^G} \hat{v^G}) = \sum_v \sum_{x \in P \setminus G/P} \left( [G : C_G(v)]_{p^-1}^{-1} |u^G \cap xP| \right) * \hat{v^G}.
\]

Now if the conjugacy class of \( v \) is not \( Z \)-good, then every irreducible character of \( G \) which lies over \( \lambda \) vanishes on \( v \), so we have \( e \hat{u^G} = 0 \) in \( Z(RG) \), and the inner sum may be taken over the set \( X \) defined above. Also, we need only concern ourselves with elements \( u \) such that the conjugacy class \( u^G \) is \( Z \)-good.

Now the number of blocks of \( RG \) with defect group \( D \) which lie over \( \lambda \) is the dimension of the \( k \)-span of

\[
\{ e \hat{u^G} : u \text{ is } p \text{-regular, } D \in \text{Syl}_p(C_G(u)), \hat{u^G} \text{ is } Z \text{-good} \}.
\]

However, we note that \( \lambda \) vanishes on \( v \), so that for any \( Z \)-good conjugacy class \( u^G \), \( e \hat{u^G} \hat{(wu)^G} \) is a scalar multiple of \( e \hat{u^G} \), and in the above spanning set, we only need to take one representative from each \( Z \)-orbit of \( Z \)-good conjugacy classes.

If \( v \) above lies in a \( Z \)-good conjugacy class, then so does \( zv \) for any \( z \) in \( Z \), and all the elements of \( \{ zv : z \in Z \} \) lie in different \( (Z \)-good) conjugacy classes of \( G \).

Hence for \( Z \)-good \( p \)-regular conjugacy classes with defect group \( D \), say \( u^G \) and \( v^G \), we see easily that the coefficient of \( v^G \) in \( e \hat{u^G} \) is

\[
\sum_{x \in P \setminus G/P} \sum_{z \in Z} (\lambda(z^{-1}) |Z|^{-1} [G : C_G(v)]_{p^-1}^{-1} |u^G \cap xP| \right) \cdot \hat{v^G}.
\]

On the other hand, for any \( w \) in \( Z \), this is also the coefficient of \( wv^G \) in \( e \hat{u^G} \) \( = \lambda(w) \hat{u^G} \) as remarked above), so for any such \( u \) and \( v \), the coefficient of \( wv^G \) in \( e \hat{u^G} \) is just \( \lambda(w^{-1}) \) times the coefficient of \( v^G \) in \( e \hat{u^G} \).

In fact, we could leave the entries of \( M \) in an apparently simpler form at
this point, but we prefer to illustrate that $M$ is the image \( \text{(mod } \pi) \text{) of an Hermitian matrix. For this we note that } e_x^* u^G = (|Z|^{-1})^* \sum_{w \in W} \lambda(w^{-1})^* e_x^* w^G \text{ and for any } v \text{ as above, the coefficient of } v \text{ in } e_x^*(w^G s) \text{ is given by }

\[
\sum_{x \in P \setminus G/P} \sum_{z \in Z} (\lambda(z^{-1})(|Z| [G : C_G(v)] \rho^-)^{-1} |w^G \cap xP| |v^G \cap zxP|)^*.
\]

It easily follows that the coefficient of $\hat{v}^G$ in $\hat{u}^G s$ may be written as

\[
\left( |Z|^{-2} \sum_{x \in X} \sum_{(z,w) \in Z \times Z} \lambda(z^{-1}) \lambda(w) [G : C_G(v)] \rho^- |u^G \cap wxP| |v^G \cap zxP| \right)^*.
\]

The result now follows by elementary linear algebra.

We give an application generalizing results of Brauer and Fowler [2] and Wada [6].

**Corollary 2.** Suppose that $p$ is odd. Let $u_1, \ldots, u_r$ be $p$-regular elements of $C_G(D)$ in distinct $Z$-good conjugacy classes of $G$ such that $u_iZ$ and $u_jZ$ are not conjugate in $G/Z$ for any $i \neq j$ and such that for each $i$, we have:

(i) $u_i$ inverts no non-trivial $p$-element of $C_G(D)$.

(ii) $D$ is a Sylow $p$-subgroup of $C_G(u_i)$.

(iii) $u_iZ$ is an involution of $G/Z$.

Then there are at least $r$ $p$-blocks of $G$ with defect group $D$ which lie over the linear character $\lambda$ of $Z$.

**Proof.** Let $\bar{G}$ denote $G/Z$, etc. If $t$ is an element of $G$ conjugate to one of the $u_i$'s, we claim that $P \cap t^{-1}Pt = D$. For if not, $t$ would invert some element of $P \cap t^{-1}Pt$, and this would be an element of $C_p(D)$. Now $PtP$ is a union of $[P : D]$ right cosets of $P$ in $G$, and each such coset contains exactly one conjugate of $t$ (there are $[P : D]$ conjugates of $t$ in $PtP$ as $D \in \text{Syl}_p(C_G(t))$, and if any coset contained two conjugates of $t$, then $t$ would invert an element of $P^*$). It follows that a $(P, P)$-double coset representative of a double coset which contains a conjugate of any of the $u_i$'s is in $X$. Also, the number of $(P, P)$-double coset representatives of double cosets which contain a conjugate of $t$ is $[G : C_G(t)]/[P : D]$.

Furthermore, if $t \in wxP$, $t^g \in zxP$ for some $z, w$ in $Z$, then $t^{-1}t^g = w^{-1}zs$ for some $s$ in $P$, so that $t$ and $t^g$ both invert $s$, forcing $s = 1_G$, $w = z$, as $t^G$ is a $Z$-good conjugacy class. The $(t^G, t^G)$-entry of the matrix $M$ is given by

\[
\sum_{x \in X} \sum_{(z,w) \in Z \times Z} (\lambda(z^{-1}) \lambda(w) |t^G \cap wxP| |t^G \cap zxP|)^*.
\]

By the remarks above, this is just $([G : C_G(t)]/[P : D])^*$. 

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On the other hand, if $t$ is conjugate to $u_i$ and $v$ is conjugate to $u_j$ for some $j \neq i$, and $t \in wxP$, $v \in zxP$ for some $z$, $w$ in $Z$, then $t^{-1}v = w^{-1}zs$ for some $s$ in $P$, so that $t$ and $v$ both invert $s$, forcing $s = 1_G$, contrary to the fact that the distinct $\bar{u}_i$'s are not conjugate in $\bar{G}$. Hence the $(t^G, v^G)$-entry of $M$ is 0. Thus the matrix $M$ has rank at least $r$, and the result follows.

REFERENCES