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Generalized double affine Hecke algebras of rank 1 and quantized del Pezzo surfaces

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Abstract

Let D be a simply laced Dynkin diagram of rank r whose affinization has the shape of a star (i.e., D_4, E_6, E_7, E_8). To such a diagram one can attach a group G whose generators correspond to the legs of the affinization, have orders equal to the leg lengths plus 1, and the product of the generators is 1. The group G is then a 2-dimensional crystallographic group: $G = \mathbb{Z}_\ell \ltimes \mathbb{Z}^2$, where ℓ is 2, 3, 4, and 6, respectively. In this paper, we define a flat deformation $H(t, q)$ of the group algebra $\mathbb{C}[G]$ of this group, by replacing the relations saying that the generators have prescribed orders by their deformations, saying that the generators satisfy monic polynomial equations of these orders with arbitrary roots (which are deformation parameters). The algebra $H(t, q)$ for D_4 is the Cherednik algebra of type $C^\vee C_1$, which was studied by Noumi, Sahi, and Stokman, and controls Askey–Wilson polynomials. We prove that $H(t, q)$ is the universal deformation of the twisted group algebra of G, and that this deformation is compatible with certain filtrations on $\mathbb{C}[G]$. We also show that if q is a root of unity, then for generic t the algebra $H(t, q)$ is an Azumaya algebra, and its center is the function algebra on an affine del Pezzo surface. For generic q, the spherical subalgebra $eH(t, q)e$ provides a quantization of such surfaces. We also discuss connections of $H(t, q)$ with preprojective algebras and Painlevé VI.

Keywords: Generalized double affine Hecke algebras of rank 1; Quantized del Pezzo surfaces

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1. Introduction

Double affine Hecke algebras for reduced root systems were introduced by Cherednik [4] in order to prove Macdonald conjectures. Double affine Hecke algebras of type $C^\vee C_n$ were introduced in the works of Noumi, Sahi, and Stokman [24,29,31] as a generalization of Cherednik algebras of types B_n and C_n, in order to prove Macdonald conjectures for Koornwinder polynomials.

The goal of this paper is to define and study new algebras $H(t,q)$, which are generalizations of double affine Hecke algebras of type $C^\vee C_n$ in the case $n=1$. To be more specific, fix a star-shaped simply laced affine Dynkin diagram $\widehat{D}$ (i.e., $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$). Let m be the number of legs of $\widehat{D}$ and d_j-1, $j=1,\dots,m$, be the length of the jth leg. Then we define a family of algebras $H(t,q)$ depending on parameters $q\in\mathbb{C}^*$ and $t=(t_{kj})$, $t_{kj}\in\mathbb{C}^*$, $k=1,\dots,m$, $j=1,\dots,d_k$, by generators T_k, $k=1,\dots,m$, with defining relations

$$\prod_{j=1}^{d_k}\bigl(T_k-e^{2\pi ij/d_k}t_{kj}\bigr)=0,\quad k=1,\dots,m;\qquad \prod_{k=1}^{m}T_k=q. \tag{1}$$

It follows from this definition that for $\widehat{D}=\tilde{D}_4$ we get exactly the double affine Hecke algebra of type $C^\vee C_1$; on the other hand, in the case $\widehat{D}=\tilde{E}_{6,7,8}$ we get new algebras, which are the main subject of this paper.

It is obvious that if $t_{kj}=1$ and $q=1$, the algebra $H(t,q)=H(1,1)$ is a group algebra of some group G. The group G is defined by generators and relations, and is well known to be isomorphic to a 2-dimensional crystallographic group $\mathbb{Z}_\ell\ltimes\mathbb{Z}^2$, where $\ell=2,3,4,6$ in the cases $\widehat{D}=\tilde{D}_4,\tilde{E}_6,\tilde{E}_7,\tilde{E}_8$, respectively. Moreover, the algebra $H(1,q)$ is a twisted group algebra of G. Thus, $H(t,q)$ is a deformation of the twisted group algebra of G. We prove that if we regard $\log(t_{kj})$ as formal parameters then this deformation is flat (*the formal PBW theorem*), and $H(t,qe^\varepsilon)$ is the universal deformation of $H(1,q)$ if q is not a root of unity. We also prove a more delicate *algebraic PBW theorem*, which claims that (for numerical t,q) some filtrations on $H(t,q)$ have certain explicit Poincaré series, independent of t and q.

It was shown by the second author [25] that for $\widehat{D}=\tilde{D}_4$ and $q=1$ the algebra $H(t,q)$ is finite over its center $Z(t,q)$, and the spectrum of $Z(t,q)$ is an affine cubic surface, obtained from a projective one by removing three lines forming a triangle. Here we show that this result is valid also for q being a root of unity, and generalize it to the cases $\widehat{D}=\tilde{E}_6,\tilde{E}_7,\tilde{E}_8$. In these cases, the spectrum of $Z(t,q)$ (q being a root of unity) turns out to be an affine surface $S(t,q)$ obtained from a projective del Pezzo surface $\overline{S(t,q)}$ of degrees 3, 2, 1 respectively by removing a nodal $\mathbb{P}^1$. This means that for $q\neq 1$, the spherical subalgebra $eH(t,q)e$ in the algebra $H(t,q)$ (where e is the idempotent in $H(t,q)$ projecting to an eigenspace of the element T_3 corresponding to the longest leg of $\widehat{D}$) should be viewed as an algebraic quantization of the surface $S(t,1)$ (with its unique up to scaling symplectic structure). Moreover, the algebraic PBW theorem for $H(t,q)$ implies that the Rees algebra of $eH(t,q)e$ with respect to an appropriate filtration provides a quantization (in the sense of noncommutative algebraic geometry) of the projective Poisson surface $\overline{S(t,q)}$.

The structure of the paper is as follows.

In Section 2, we recall the basics about crystallographic groups in the plane.

In Section 3, we consider the twisted group algebra $B(q)=H(1,q)$ of a planar crystallographic group G, and deform it into an algebra $\widehat{\mathbf{H}}(q)$, which is a version of $H(t,q)$ in which

$\log t_{kj}$ are formal parameters. We prove the formal PBW theorem for $\widehat{\mathbf{H}}(q)$, and formulate the results on the cohomology of $B(q)$ and on its universal deformation.

In Section 4, we prove the results stated in Section 3. Namely, we compute the Hochschild cohomology of the twisted group algebra $B(q)$ and use the result to prove that $H(t, qe^{\varepsilon})$ is a universal deformation of $B(q)$ if q is not a root of unity.

In Section 5, we define an increasing filtration on $H(t, q)$, called the length filtration; its definition is based on a connection of $H(t, q)$ with a certain deformation of group algebras of affine Weyl groups (with lattice of rank 2). We show that the Poincaré series for this filtration is independent of t, q (the algebraic PBW theorem). We then use this result to establish some general properties of $H(t, q)$, e.g. that the Gelfand–Kirillov dimension of $H(t, q)$ is 2. We also show that if q is a root of unity then the algebras $H(t, q)$ are PI. Finally, we show that if in addition t is generic then $H(t, q)$ is an Azumaya algebra, and the spectrum of the center $Z(t, q)$ of $H(t, q)$ is a smooth surface.

To study finer structure of the algebras $H(t, q)$ at roots of unity, one needs to define other filtrations on $H(t, q)$, and prove the PBW theorem for them. This is done in Section 6. The proof of the PBW theorem is technical and relies on computer calculations; it is given in Section 8. We also compute the associated graded algebras attached to some of the filtrations; the proof is again postponed till Section 8. In the second half of Section 6, we proceed to show that the spectrum of the center of $H(t, q)$ is an affine del Pezzo surface. This shows that the spherical subalgebra $eH(t, q)e$ for $q \neq 1$ is a quantization of an affine del Pezzo surface, and also yields a linear algebra application given at the end of Section 6.

In Section 7, using the Riemann–Hilbert correspondence, we define a homomorphism from a formal version of the generalized double affine Hecke algebra to the completion of the deformed preprojective algebra of the quiver associated to the graph $\widehat{D}$. This construction is similar to those used in [7]. It allows us to define a holomorphic (but not algebraic) map from the universal deformation of the Kleinian singularity $\mathbb{C}^2/\Gamma$ (where $\Gamma \subset SL_2(\mathbb{C})$ is the finite subgroup corresponding to the diagram $\widehat{D}$ via the McKay correspondence) to the family of surfaces $S(t, 1)$. This is a local isomorphism of analytic varieties near $0 \in \mathbb{C}^2/\Gamma$, which in the case $\widehat{D} = \tilde{D}_4$ encodes generic solutions of the Painlevé VI equation.

In Section 8, we prove the results of Section 5, by writing presentations of $H(t, q)$ which are compatible with the filtrations.

Finally, in Section 9, we study more closely the surfaces $S(t, 1)$. Namely, let $\mathbf{G}$ be the simple Lie group corresponding to the diagram D. We show that the algebra $H(t, q)$ depends only on the projection of t to the maximal torus of $\mathbf{T} \subset \mathbf{G}$, and that the map $t \mapsto S(t, 1)$ from $\mathbf{T}$ to the moduli space of affine del Pezzo surfaces is Galois and has Galois group isomorphic to the Weyl group W of $\mathbf{G}$. Given the results of Section 7, this fact is in good agreement with the Arnold–Brieskorn theorem, saying that the monodromy group of a simple singularity is the Weyl group of the corresponding Lie algebra. This also implies that the coefficients of the equation of $S(t, 1)$, as functions of t, are polynomials of characters of irreducible representations of $\mathbf{G}$, and we compute these polynomials explicitly.

We note that the computations of Sections 8 and 9 are too complicated to be done by hand and were performed using a computer. More specifically, the third author wrote a Magma code [21] for computations in $H(t, q)$, which can be found at [28]. We also remark that the results of Sections 1–5 and 7 are independent of computer calculations.

Finally, the paper contains two appendices. In Appendix A.1, written by W. Crawley-Boevey and P. Shaw, it is shown that the algebra $H(t, q)$ is the "spherical subalgebra" (corresponding to the nodal vertex idempotent) of the multiplicative preprojective algebra introduced in the paper [9]. In Appendix A.2, we use this result to describe the structure of the multiplicative preprojective algebras for affine starlike quivers. In particular, we show that if q^ℓ is a root of unity of degree N and t is generic then the corresponding multiplicative preprojective algebra is Azumaya of rank hN, where h is the Coxeter number of the corresponding Dynkin diagram.

Remark 1. Some results of this paper can be extended to the case $n > 1$, giving a generalization of double affine Hecke algebras of type $C^\vee C_n$. This involves considering a flat 1-parameter deformation $H_n(t,q,k)$ of the algebra $\mathbb{C}[S_n] \ltimes H(t,q)^{\otimes n}$. This is done in the subsequent paper [15]; the deformations considered there appear to provide quantizations of Hilbert schemes of del Pezzo surfaces.

Remark 2. In [16], Gan and Ginzburg define and study rank n analogs of preprojective algebras of quivers. It would be interesting to define such analogs in the multiplicative situation. In the case of affine quivers, they should have Gelfand–Kirillov dimension $2n$ and be finite over center for special parameters. Also, for affine starlike quivers they should be Morita equivalent to the algebras studied in [15] (see the previous remark).

Remark 3. Finite dimensional representations of the algebra $H(t,q)$ is essentially the same thing as solutions of the multiplicative Deligne–Simpson problem, considered in [7,9]. Thus the methods of [7,9] can be used to obtain a classification of finite dimensional representations of $H(t,q)$. For double affine Hecke algebras of type A_1, this problem is solved in [5], and for the (more general) type $C^\vee C_1$ in [26].

Remark 4. In [33], M. Van den Bergh constructed quantizations of del Pezzo surfaces with a (possibly singular) genus 1 curve removed, using the method of noncommutative blowup; in the E_6 (= degree 3) case it was already done in [19]. We expect that when the removed curve is a nodal $\mathbb{P}^1$, the algebras constructed in [33] are isomorphic to the spherical subalgebras $eH(t,q)e$ for $D = E_{9-d}$, where d is the degree of the del Pezzo surface ($d = 3, 2, 1$). Checking this should involve presenting both algebras by generators and relations.

Remark 5. Let S be a del Pezzo surface of degree $d \leqslant 3$ with a genus 1 curve E removed. Let q be an automorphism of E. In a forthcoming paper we plan to show that one can define an algebra $H_{S,E}(q)$ (depending continuously on S, E, q) with an idempotent e such that $eH_{S,E}(q)e$ is a quantization of S (with its unique up to scaling symplectic structure) and $H_{S,E}(q) = H(t,q)$ for a suitable t when the curve E is a nodal rational curve. Such algebras would provide elliptic deformations of generalized double affine Hecke algebras.

Remark 6. The algebras $H(t,q)$ are a special case of a much more general class of algebras, which are flat deformations of group algebras of discrete groups, introduced in [12] (for any group acting discretely on a complex manifold with vanishing first and second Betti numbers), and in [14] (for subgroups of even elements in Coxeter groups). These more general deformations appear to be quite interesting (see e.g. [15]), but are rather poorly understood at the moment. The authors plan to study them in subsequent papers.

2. Crystallographic groups in the plane and their twisted group algebras

Let D be a simply laced Dynkin diagram, whose affinization $\widehat{D}$ has the structure of a star. That is, $\widehat{D}$ has a node with m legs growing out of it. Such diagrams D are D_4, E_6, E_7, and E_8 (the number m is 4, 3, 3, 3, respectively).

Let d_i be the length of the ith leg of $\widehat{D}$ plus 1. Consider the group G generated by T_i, $i = 1, \ldots, m$, with defining relations

$$T_i^{d_i} = 1, \quad i = 1, \ldots, m, \qquad \prod_{i=1}^{m} T_i = 1$$

(fixing a cyclic ordering of legs). Thus, for D_4 the group G is generated by a, b, c, d with

$$a^2 = b^2 = c^2 = d^2 = 1, \qquad abcd = 1,$$

for E_6 by a, b, c with

$$a^3 = b^3 = c^3 = 1, \qquad abc = 1,$$

for E_7 by a, b, c with

$$a^2 = b^4 = c^4 = 1, \qquad abc = 1,$$

for E_8 by a, b, c with

$$a^2 = b^3 = c^6 = 1, \qquad abc = 1.$$

It is well known (and easy to check) that G is isomorphic to the crystallographic group $\mathbb{Z}_\ell \ltimes \mathbb{Z}^2$, where $\ell = 2, 3, 4, 6$ for D_4, E_6, E_7, E_8, respectively, and the cyclic group $\mathbb{Z}_\ell$ acts on the lattice by rotations.

More specifically, we can view the group G as a group of affine transformations of $\mathbb{C}$ using the following formulas for the action of the generators.

In the D_4 case,

$$a(z) = -z + 1 + \mathrm{i}, \qquad b(z) = -z + 1, \qquad c(z) = -z, \qquad d(z) = -z + \mathrm{i}.$$

In the E_6 case,

$$a(z) = \zeta(z + \zeta) - \zeta, \qquad b(z) = \zeta(z - 1) + 1, \qquad c(z) = \zeta z,$$

where $\zeta = e^{2\pi \mathrm{i}/3}$.

In the E_7 case,

$$a(z) = -z + 1 - \mathrm{i}, \qquad b(z) = \mathrm{i}(z - 1) + 1, \qquad c(z) = \mathrm{i}z.$$

In the E_8 case,

$$a(z) = -z + 1 - \xi^2, \qquad b(z) = \xi^2(z - 1) + 1, \qquad c(z) = \xi z,$$

where $\xi = e^{\pi \mathrm{i}/3}$.

Now let q be an invertible variable, and denote by B the algebra generated over $\mathbb{C}[q, q^{-1}]$ by T_i with defining relations

$$T_i^{d_i} = 1, \quad i = 1, \ldots, m, \qquad \prod_{i=1}^{m} T_i = q.$$

If $q_0 \in \mathbb{C}^*$, we can define $B(q_0) = B/(q - q_0)$.

Let $q \in \mathbb{C}^*$ and $\widehat{G}$ be the central extension of G by $\mathbb{Z}$: $\widehat{G} := \mathbb{Z}_\ell \ltimes H$, where H is the Heisenberg group consisting of 3-by-3 upper triangular matrices with integer entries and ones on the diagonal. Let C be a generator of the center of $\widehat{G}$. It is shown using the formulas above (see Section 4.1) that $B(q)$ is a twisted group algebra of G – the quotient of $\mathbb{C}[\widehat{G}]$ by the relation $C = q^\ell$. In other words, $B(q) = \mathbb{C}[\mathbb{Z}_\ell] \ltimes A_{q^\ell}$, where A_Q is the Q-Weyl algebra generated by $X^{\pm 1}$, $P^{\pm 1}$ with the relation $PX = QXP$. Similar statements are true if q is a variable (i.e. if we work over $\mathbb{C}[q, q^{-1}]$).

3. Generalized double affine Hecke algebras over formal series

3.1. Cohomology

We denote by $H^i(A, A)$ or simply $H^i(A)$ the Hochschild cohomology of an algebra A. The Hochschild cohomology of an algebra A with coefficients in a bimodule M is denoted by $H^i(A, M)$.

Theorem 3.1. *Assume that $q \in \mathbb{C}^*$ is not a root of unity. Then $H^0(B(q)) = \mathbb{C}$, $H^1(B(q)) = 0$, $H^2(B(q)) = \mathbb{C}^{r+1}$, $H^i(B(q)) = 0$ for $i > 2$, where r is the rank of the Dynkin diagram D.*

Let us now recall the definition of a *universal deformation.* A flat R-algebra A_R (with R being a local commutative Artinian algebra and $\mathfrak{m} \subset R$ the maximal ideal) together with an isomorphism $A_R/\mathfrak{m} \simeq A$ is called a flat deformation of A over $S = Spec(R)$. A similar definition is made if R is pro-Artinian. A flat deformation A_R is a universal deformation of A if for every flat deformation $A_{\mathcal{O}(S)}$ of A over an Artinian base S there exists a unique map $\tau : S \to Spec(R)$ such that the isomorphism $A \simeq A_R/\mathfrak{m}$ lifts to an isomorphism $A_{\mathcal{O}(S)} \simeq \tau^* A_R$.

It is well known that if $H^2(A, A) = E$ is a finite dimensional vector space, and $H^3(A, A) = 0$ then there exists a universal deformation of A parametrized by E (i.e. with $R = \mathbb{C}[[E]]$). Therefore, we have the following corollary.

Corollary 3.2. *The universal deformation of $B(q)$ has $r + 1$ parameters.*

Now we will describe the universal deformation of $B(q)$ explicitly.

3.2. Deformations of $B(q)$

Let t_{ij}, $j = 1, \ldots, d_i$, $i = 1, \ldots, m$, be variables such that $\prod_j t_{ij} = 1$. Define u_{kj} by the formula $u_{kj} = e^{2\pi j \mathrm{i}/d_k} t_{kj}$ for $k = 1, \ldots, m$. We assume that t_{ij} are formal, in the sense that $t_{ij} = e^{\tau_{ij}}$, where τ_{ij} are formal parameters. Let t denote the collection of the variables t_{ij}. Clearly, the number of independent variables among them is r.

Define the algebra $\widehat{\mathbf{H}}$ to be (topologically) generated over $\mathbb{C}[q, q^{-1}][[\tau]]$ (where τ stands for the collection of variables τ_{ij}) by T_k, $k = 1, \ldots, m$, with defining relations

$$\prod_{j=1}^{d_k} (T_k - u_{kj}) = 0, \quad k = 1, \ldots, m; \qquad \prod_{k=1}^{m} T_k = q. \tag{2}$$

Sometimes we will use the notation a, b, c, d for T_1, T_2, T_3, T_4.

This algebra of course depends on the Dynkin diagram D, but in order to simplify notation we will not write this dependence explicitly. In the D_4 case, it is the double affine Hecke algebra of type $C^\vee C_1$ of Sahi, Noumi and Stokman [24,29,31]. So in the cases E_6, E_7, E_8, we get a generalization of the double affine Hecke algebra. If $q_0 \in \mathbb{C}^*$, we can also define the algebra $\widehat{\mathbf{H}}(q_0) := \widehat{\mathbf{H}}/(q - q_0)$ over $\mathbb{C}[[\tau]]$.

Theorem 3.3 *(The formal PBW theorem). The algebra $\widehat{\mathbf{H}}$ is a flat formal deformation of B.*

This immediately implies

Corollary 3.4. *For any $q \in \mathbb{C}^*$, the algebra $\widehat{\mathbf{H}}(q)$ is a flat formal deformation of $B(q)$.*

We will show that if q is not a root of unity then this is the most general deformation. Namely, we have

Theorem 3.5. *If q is not a root of unity then $\widehat{\mathbf{H}}(qe^{\varepsilon})$ (where ε is a new formal parameter) is a universal deformation of $B(q)$.*

Remark. In the D_4 case, Theorem 3.3 follows from the papers [24,29,31].

3.3. Proof of Theorem 3.3

The proof is based on the following simple fact, which is often used for proving flatness of formal deformations.

Lemma 3.6. *If A_0 is an algebra over $\mathbb{C}$, A a formal deformation of A_0 (over $\mathbb{C}[[t_1, \dots, t_n]]$), and M_0 is a faithful A_0-module which can be flatly deformed to an A-module, then A is a* flat *formal deformation of A_0.*

For every element $g \in G$, fix its presentation as a product of T_i, and denote by b_g^0 the same product in the algebra B. Then $\{b_g^0\}$ is a basis of B labeled by group elements $g \in G$ (it is independent of the choice of the presentations up to scaling by powers of q). Let b_g be some lifts of b_g^0 to $\widehat{\mathbf{H}}$. Let J be the maximal ideal in $\mathbb{C}[[\tau]]$.

Assume that $\widehat{\mathbf{H}}$ is not flat. Then there exists $n > 0$ and $g_1, \dots, g_k \in G$, $\alpha_1, \dots, \alpha_k \in \mathbb{C}[q, q^{-1}][[\tau]]/J^n$ (not all zero) such that $\sum \alpha_j b_{g_j} = 0$ in $\widehat{\mathbf{H}}/J^n$. This relation is nontrivial if we reduce it modulo $q - q_0$ for all but finitely many q_0. Hence $\widehat{\mathbf{H}}(q)$ is not a flat deformation of $B(q)$ for all but finitely many q.

Thus it is sufficient to establish that $\widehat{\mathbf{H}}(q)$ is flat in the case when q is a root of unity. We will assume that q^ℓ is a root of unity of degree N, where N is a positive integer.

In this case, the algebra $B(q)$ is a semidirect product of $\mathbb{Z}_\ell$ with the q-Weyl algebra (or quantum torus) A_{q^ℓ}. The algebra A_{q^ℓ} is well known to be an Azumaya algebra of rank N. In particular, the center of $B(q)$ is $Z = \mathbb{C}[T/\mathbb{Z}_\ell]$, and $B(q) \otimes_Z \overline{Q}_Z = \mathrm{Mat}_{\ell N}(\overline{Q}_Z)$ (where Q_Z is the field of fractions of Z, and $\overline{Q}_Z$ is the algebraic closure of Q_Z). So $B(q)$ admits a 2-parameter family of irreducible ℓN-dimensional representations, whose direct sum is faithful. By Lemma 3.6, to prove our theorem, it is sufficient to show that these representations can be deformed to representations of $\widehat{\mathbf{H}}(q)$.

Let V be an irreducible ℓN-dimensional matrix representation of $B(q)$. Let $\mathbf{C}_k$, $k=1,\dots,m$, be the conjugacy class of $\mathrm{diag}(u_{kj})\otimes \mathrm{Id}_{N\ell/d_k}$ in $GL_{\ell N}$; it is a smooth algebraic variety defined over $\mathbb{C}[[\tau]]$. Consider the scheme $\mathbf{Y}$ of m-tuples $(T_1,\dots,T_m)$ lying in the formal neighborhood of the orbit of V (under changes of basis), such that $T_k\in\mathbf{C}_k$ and $\prod_k T_k = q\,\mathrm{Id}_{\ell N}$. Our job is to show that the structure ring $O_{\mathbf{Y}}$ is flat over $\mathbb{C}[[\tau]]$.

Let Y be the reduction of $\mathbf{Y}$ modulo the maximal ideal in $\mathbb{C}[[\tau]]$. By Schur's lemma, Y admits a free action of $PGL_{\ell N}(\mathbb{C})$, and the quotient is a 2-dimensional formal polydisk. Thus Y is smooth and has dimension ℓ^2N^2+1.

On the other hand, let us compute the "expected dimension" of $\mathbf{Y}$, i.e. the dimension of the ambient space minus the number of equations. Fixing a matrix $T_k\in\mathbf{C}_k$ amounts to fixing d_k subspaces in an ℓN-dimensional linear space of dimension $\ell N/d_k$ which add up to the whole space. Thus, $\mathbf{C}_k$ has dimension

$$D_k=\ell^2N^2\left(1-\frac{1}{d_k}\right).$$

On the other hand, the number of equations in the condition $\prod_k T_k = q\,\mathrm{Id}_{\ell N}$ is ℓ^2N^2-1 (since the determinant of the product is fixed). Thus the expected dimension is

$$\mathbb{D}=\sum_k D_k-\ell^2N^2+1=\ell^2N^2\sum_{k=1}^{m}\left(1-\frac{1}{d_k}\right)-\ell^2N^2+1.$$

But $\sum_{k=1}^m(1-\frac{1}{d_k})=2$ (as $\widehat{D}$ is an affine diagram). Thus, $\mathbb{D}=\ell^2N^2+1$. The expected dimension of Y is obviously the same.

Thus, the expected dimension of Y coincides with its actual dimension. This implies that Y is a complete intersection, and therefore so is $\mathbf{Y}$. Since $\mathbf{Y}$ is obtained from Y by deforming its equations, and Y is a complete intersection, we conclude that $\mathbf{Y}$ is a flat deformation of Y (in fact, it is, moreover, a trivial deformation, since Y is smooth). The theorem is proved.

4. Proofs of Theorems 3.1, 3.5

4.1. Homology and cohomology of $B(q)$

In this section we prove Theorems 3.1, 3.5. We will use arguments similar to the arguments from [25].

Let us describe the isomorphism between the semidirect product $B=\mathbb{C}[\mathbb{Z}_\ell]\ltimes A_{q^\ell}$ and the algebra $B(q)$. For brevity we use the symbol D_q for the algebra A_{q^ℓ}.

In the case $\ell=2$ in the formulas defining $B(q)$ we have $d_1=d_2=d_3=d_4=2$ and we can choose the alternative set of generators P, X, s:

$$X=T_1T_2,\qquad P=T_2T_3,\qquad s=T_4.$$

These elements generate $B(q)$ modulo the relations

$$PX=q^2XP,\qquad s^{-1}Ps=P^{-1},\qquad s^{-1}Xs=X^{-1},\qquad s^2=1.$$

In the case $\ell = 3$ in the definition of $B(q)$ we have $d_1 = d_2 = d_3 = 3$, and the alternative system of generators is

$$X = T_3 T_1^{-1}, \qquad P = T_1 T_2^{-1}, \qquad s = T_3. \tag{3}$$

These elements generate $B(q)$ modulo the relations

$$PX = q^3 XP, \qquad s^{-1}Xs = q^{-1}P^{-1}X^{-1}, \qquad s^{-1}Ps = q^2 X, \qquad s^3 = 1. \tag{4}$$

In the case $\ell = 4$ in the definition of $B(q)$ we have $d_1 = 2$, $d_2 = d_3 = 4$, and the alternative system of generators is

$$X = T_2^2 T_1, \qquad P = T_3^2 T_1, \qquad s = T_3. \tag{5}$$

These elements generate $B(q)$ modulo the relations

$$PX = q^4 XP, \qquad s^{-1}Xs = q^{-2}P^{-1}, \qquad s^{-1}Ps = q^2 X, \qquad s^4 = 1. \tag{6}$$

In the case $\ell = 6$ in the definition of $\mathbb{C}_q[G]$ we have $d_1 = 2$, $d_2 = 3$, $d_3 = 6$, and the alternative system of generators is

$$X = T_3^3 T_1, \qquad P = T_3^{-2} T_2, \qquad s = T_3. \tag{7}$$

These elements generate $\mathbb{C}_q[G]$ modulo the relations

$$PX = q^6 XP, \qquad s^{-1}Xs = q^{-2}XP^{-1}, \qquad s^{-1}Ps = qX, \qquad s^6 = 1. \tag{8}$$

Theorem 4.1. *If q is not a root of unity then for $\ell = 2, 3, 4, 6$ we have*

$$\begin{aligned}
H^2(\mathbb{Z}_\ell \ltimes D_q) &= H_0(\mathbb{Z}_\ell \ltimes D_q) = \mathbb{C}^{r+1}, \\
H^1(\mathbb{Z}_\ell \ltimes D_q) &= H_1(\mathbb{Z}_\ell \ltimes D_q) = 0, \\
H^0(\mathbb{Z}_\ell \ltimes D_q) &= H_2(\mathbb{Z}_\ell \ltimes D_q) = \mathbb{C}, \\
H_{>2}(\mathbb{Z}_\ell \ltimes D_q) &= H^{>2}(\mathbb{Z}_\ell \ltimes D_q) = 0,
\end{aligned}$$

with $r = 4, 6, 7, 8$, respectively.

We prove this theorem using the technique from [1].

Recall [13] that $D_q \in VB(2)$, i.e., there exists an isomorphism of bimodules $\zeta : H^2(D_q, D_q \otimes D_q^{\mathrm{opp}}) \to D_q$, where D_q^{opp} is the algebra D_q with the opposite multiplication.

Lemma 4.2. *The isomorphism ζ is $\mathbb{Z}_\ell$-equivariant.*

Proof. Since the center of D_q is trivial, an isomorphism ζ is unique up to scaling, hence $\mathbb{Z}_\ell$ must act on ζ by a character $\chi : G \to \mathbb{C}^*$. Then by Van den Bergh's theorem [32,34], $H_0(D_q, D_q) = H^2(D_q, D_q) \otimes \chi$ as $\mathbb{Z}_\ell$-modules. But $H_0(D_q, D_q)$ is the trivial $\mathbb{Z}_\ell$-module, since D_q has a unique trace (up to scaling), sending $X^i P^j$ to $\delta_{i0}\delta_{j0}$, and this trace is clearly fixed under $\mathbb{Z}_\ell$. On the other hand, $H^2(D_q, D_q)$ is 1-dimensional (as is easily seen from the Koszul resolution, see below), and spanned by the class defined by the deformation of D_q into D_{qe^ε}. Since $\mathbb{Z}_\ell$ acts on D_{qe^ε}, we see that this class is also fixed under $\mathbb{Z}_\ell$. Thus, $\chi = 1$ and the lemma is proved. $\square$

Lemma 4.2 and Proposition 3.5 from [13] imply that there is an isomorphism between the Hochschild homology $H_i(\mathbb{Z}_\ell \ltimes D_q)$ and Hochschild cohomology $H^{2-i}(\mathbb{Z}_\ell \ltimes D_q)$. Thus it suffices to calculate the Hochschild homology $H_*(\mathbb{Z}_\ell \ltimes D_q)$.

4.2. *The decomposition of the Hochschild homology*

There is a natural structure of a $\mathbb{Z}_\ell$-module on the homology $H_i(D_q, gD_q)$, where $g \in \mathbb{Z}_\ell$. More precisely, there is an action of $\mathbb{Z}_\ell$ on the standard Hochschild complex for $H_i(D_q, gD_q)$ by the formulas:

$$g \cdot (m \otimes a_1 \otimes \cdots \otimes a_r) = m^g \otimes a_1^g \otimes \cdots \otimes a_r^g.$$

Proposition 3.1 from the paper [1] implies:

Proposition 4.3. *There is a decomposition*:

$$H_*(\mathbb{Z}_\ell \ltimes D_q) = \bigoplus_{g \in \mathbb{Z}_\ell} H_*(D_q, gD_q)^{\mathbb{Z}_\ell}.$$

For calculation of $H_*(D_q, gD_q)$ we will use the Koszul resolution.

4.3. *Calculation of $H_*(D_q, gD_q)$*

Let us denote by D_q^e the algebra $D_q \otimes D_q^{\mathrm{opp}}$. The elements $p = P \otimes P^{-1} - 1$, $x = X \otimes X^{-1} - 1$ commute and $D_q^e/\mathbf{I} = D_q$, where $\mathbf{I} = (x, p)$ is the D_q^e-submodule generated by these elements. Hence the corresponding Koszul complex yields a free resolution W_* of the D_q^e-module D_q:

$$D_q^e \xrightarrow{\mathbf{d}_1} D_q^e \oplus D_q^e \xrightarrow{\mathbf{d}_0} D_q^e \xrightarrow{\mu} D_q,$$

where $\mu(X^i P^j \otimes P^{j'} X^{i'}) = X^i P^{j+j'} X^{i'}$ and for $z = z_1 \otimes z_2$ we have $\mathbf{d}_0(z, 0) = zp = z_1 P \otimes P^{-1} z_2 - z_1 \otimes z_2$, $\mathbf{d}_0(0, z) = zx = z_1 X \otimes X^{-1} z_2 - z_1 \otimes z_2$, $\mathbf{d}_1(z) = (zx, -zp) = (z_1 X \otimes X^{-1} z_2 - z_1 \otimes z_2, -z_1 P \otimes P^{-1} z_2 + z_1 \otimes z_2)$.

Using the Koszul complex for D_q we prove the following

Lemma 4.4. *Suppose that we have an automorphism g of the algebra D_q given by the formulas*:

$$X^g = q^{b_1} X^{g_{11}} P^{g_{21}},$$
$$P^g = q^{b_2} X^{g_{12}} P^{g_{22}}$$

(*where* $\det(g_{ij}) = 1$), *and suppose that the map $g - 1 : \mathbb{Z}^2 \to \mathbb{Z}^2$ is injective. Let $v^1, \dots, v^k \in \mathbb{Z}^2$ be vectors such that $v^i - v^j \notin (g-1)\mathbb{Z}^2$ for $i \neq j$ and $k = |\mathbb{Z}^2/(g-1)\mathbb{Z}^2|$. Then we have*

$$H_{>0}(D_q, gD_q) = 0$$

and

$$H_0(D_q, gD_q) = gD_q/[D_q, gD_q] = \bigoplus_{i=1}^{k} \mathbb{C}\langle X^{v_1^i} P^{v_2^i} + [D_q, gD_q]\rangle.$$

Proof. Identifying the vector spaces gD_q and D_q by sending $gx \in gD_q$ to $x \in D_q$ and taking the tensor product of the resolution W_* and gD_q over D_q^e, we get the complex

$$D_q \xrightarrow{\hat{\mathbf{d}}_1} D_q \oplus D_q \xrightarrow{\hat{\mathbf{d}}_0} D_q,$$

where $\hat{\mathbf{d}}_1(z) = (X^g z X^{-1} - z, -P^g z P^{-1} + z)$, $\hat{\mathbf{d}}_0(w, z) = P^g w P^{-1} - w + X^g z X^{-1} - z$. The homology of this complex is exactly the homology $H_*(D_q, gD_q)$.

It is easy to check that $\hat{\mathbf{d}}_1$ is injective, so we have $H_2(D_q, gD_q) = 0$. So it remains to compute H_1 and H_0.

Let us write the maps $\hat{\mathbf{d}}_0$, $\hat{\mathbf{d}}_1$ in terms of the PBW basis in D_q. A direct calculation shows that

$$\hat{\mathbf{d}}_0(w, y) = \sum_{i,j} \bigl(\delta^{(1)} c_2 + \delta^{(2)} c_1\bigr)(i, j) X^i P^j,$$

$$\hat{\mathbf{d}}_1(z) = \left(\sum_{i,j} \delta^{(1)} c(i, j) X^i P^j, -\sum_{i,j} \delta^{(2)} c(i, j) X^i P^j \right),$$

where $w = \sum_{i,j} c_1(i, j) X^i P^j$, $y = \sum_{i,j} c_2(i, j) X^i P^j$, $z = \sum_{i,j} c(i, j) X^i P^j$ and

$$\bigl(\delta^{(1)} c\bigr)(i, j) = \exp\bigl(h\bigl(g_{21} i - j + (1 - g_{11}) g_{21} + b_1\bigr)\bigr) c\bigl((i, j) - w_1\bigr) - c(i, j),$$
$$\bigl(\delta^{(2)} c\bigr)(i, j) = \exp\bigl(h(g_{22} i - g_{22} g_{12} + b_2)\bigr) c\bigl((i, j) - w_2\bigr) - c(i, j),$$

with $w_1 = (g_{11} - 1, g_{21})$, $w_2 = (g_{12}, g_{22} - 1)$ being a basis of the lattice $(g - 1)\mathbb{Z}^2$ and $q = \exp(h)$.

The operations $\delta^{(i)}$, $i = 1, 2$, preserve the space F_{fin} of the functions on $\mathbb{Z}^2$ with finite support, and they obviously commute. These operations are discrete analogues of partial differentiations, and the image of $\delta^{(i)}$ could be described in terms of the discrete analog of integration:

$$\bigl(I^{(1)} c\bigr)(i, j) = \sum_{(m,n) \in (i,j) + k w_1,\, k \in \mathbb{Z}} \exp\bigl(hs(m, n)\bigr) c(m, n),$$
$$\bigl(I^{(2)} c\bigr)(i, j) = \sum_{(m,n) \in (i,j) + k w_2,\, k \in \mathbb{Z}} \exp\bigl(hs(m, n)\bigr) c(m, n),$$

where $s : \mathbb{Z}^2 \to \mathbb{Z}$ is any function such that $I^{(i)}$ satisfy the equations

$$I^{(1)} \delta^{(1)} c = 0, \qquad I^{(2)} \delta^{(2)} c = 0,$$

for any function $c \in F_{\mathrm{fin}}$. The last equations are equivalent to the system:

$$s\big((i,j)+w_1\big) = -ig_{21}+g_{21}+j+s(i,j)-b_1,$$
$$s\big((i,j)+w_2\big) = s(i,j)-ig_{22}-b_2.$$

This system has a k-dimensional (affine) space of solutions. Indeed, if we fix the values of $s(v_i)$, $i=1,\dots,k$, then the value of s at the point $v_i+mw_1+nw_2$ could be found from the system. In particular the solution of the system normalized by the condition $s(v_i)=0$ has the form

$$s(v)=s_i(m,n), \quad \text{for } v=v_i+mw_1+nw_2,$$

where

$$s_i(m,n) = \frac{(2-g_{11})g_{21}m^2}{2} + g_{22}(1-g_{11})mn - \frac{g_{11}g_{22}n^2}{2} + \left(\frac{g_{12}g_{22}}{2} - b_2 - g_{22}v_i^1\right)n + \big(g_{21}\big(g_{11}-v_i^1\big)+v_i^2-b_1\big)m.$$

Thus we have the following description of the image of $\delta^{(i)}$:

$$\operatorname{Im}\delta^{(i)} = \operatorname{Ker} I^{(i)}.$$

Having this description we can show that $\operatorname{Ker}\hat{\mathbf{d}}_0 \subset \operatorname{Im}\hat{\mathbf{d}}_1$. Indeed if $(w,y)\in\operatorname{Ker}\hat{\mathbf{d}}_0$, $w=\sum_{i,j}c_1(i,j)X^iP^j$, $y=\sum_{i,j}c_2(i,j)X^iP^j$, then $\delta^{(1)}c_2=-\delta^{(2)}c_1$. As $I^{(1)}$ commutes with $\delta^{(2)}$, we have

$$0=I^{(1)}\delta^{(1)}c_2=-I^{(1)}\delta^{(2)}c_1=-\delta^{(2)}I^{(1)}c_1.$$

As $c_1\in F_{\text{fin}}$, the equation implies $I^{(1)}c_1=0$, hence $c_1=\delta^{(1)}c$ for some $c\in F_{\text{fin}}$. A similar calculation shows that $c_2=\delta^{(2)}c'$ for some $c'\in F_{\text{fin}}$. Moreover, the equation

$$\delta^{(1)}\delta^{(2)}(c+c')=\delta^{(2)}c_1+\delta^{(1)}c_2=0$$

implies $c+c'=0$. Thus $H_1(D_q,gD_q)=0$.

Let us now prove that $H_0(D_q,gD_q)=\mathbb{C}^k$. Indeed, it is easy to see that $\operatorname{Im}\hat{\mathbf{d}}_0=\bigcap_{s=1}^k\operatorname{Ker}I_s$, where $I_i: F_{\text{fin}}\to\mathbb{C}$:

$$I_i=\sum_{(m,n)\in\mathbb{Z}^2} q^{s(m,n)}f(v_i+mw_1+nw_2).$$

Thus $\dim H_0(D_q,gD_q)\leqslant k$.

On the other hand, the vectors $gX^{v_1^s}P^{v_2^s}$, $s=1,\dots,k$, are linearly independent modulo the subspace $[D_q,gD_q]\subset gD_q$ because the subspace $[D_q,gD_q]$ is spanned by the vectors of the form $g(X^iP^j-q^{f(i,j,u)}X^{i+u_1}P^{j+u_2})$ where $u\in(g-1)\mathbb{Z}^2$ and $f(i,j,u)$ is some function. Thus $H_0(D_q,gD_q)=gD_q/[D_q,gD_q]=\mathbb{C}^k$, and the lemma is proved. $\square$

Corollary 4.5. *If q is not a root unity and $g \in \mathbb{Z}_\ell$ is not the unit element then $H_{>0}(D_q, gD_q) = 0$ and*

$$H_0(D_q, sD_q) = \mathbb{C}^4, \quad \text{for } \ell = 2,$$
$$H_0(D_q, sD_q) = H_0(D_q, s^2 D_q) = \mathbb{C}^3, \quad \text{for } \ell = 3,$$
$$H_0(D_q, sD_q) = H_0(D_q, s^3 D_q) = \mathbb{C}^2, \qquad H_0(D_q, s^2 D_q) = \mathbb{C}^4, \quad \text{for } \ell = 4,$$
$$H_0(D_q, sD_q) = H_0(D_q, s^5 D_q) = \mathbb{C}, \qquad H_0(D_q, s^2 D_q) = H_0(D_q, s^4 D_q) = \mathbb{C}^3,$$
$$H_0(D_q, s^3 D_q) = \mathbb{C}^4, \quad \text{for } \ell = 6,$$

where s is a generator of $\mathbb{Z}_\ell$.

4.4. Calculation of $H_*(D_q, D_q)^{\mathbb{Z}_\ell}$

Using the Koszul resolution we can easily calculate the homology of D_q (see for example Section 4 of [25]).

Lemma 4.6. *If q is not a root of unity then*

$$H_0(D_q, D_q) = H_2(D_q, D_q) = \mathbb{C}, \qquad H_1(D_q, D_q) = \mathbb{C}^2.$$

Moreover, for all ℓ we have

$$H_0(D_q, D_q)^{\mathbb{Z}_\ell} = H_2(D_q, D_q)^{\mathbb{Z}_\ell} = \mathbb{C}, \qquad H_1(D_q, D_q)^{\mathbb{Z}_\ell} = 0.$$

Proof. Indeed the first statement follows from a simple calculation with Koszul resolution; this calculation is done for example in Section 4 of [25].

Let us now prove the second statement. Recall that by Lemma 4.2 and the results of [32,34] the space $H_1(D_q, D_q)$ is $\mathbb{Z}_\ell$ equivariantly isomorphic to the space

$$H^1(D_q, D_q) = \mathrm{Der}(D_q)/\langle [x, \cdot],\ x \in D_q \rangle,$$

where $\mathrm{Der}(D_q)$ is the space of the derivations of D_q, i.e. the space of the $\mathbb{C}$-linear maps $d : D_q \to D_q$ with the property $d(xy) = d(x)y + xd(y)$. From the last description we see that $H^1(D_q, D_q)$ is spanned[1] by two derivations $\mathbf{d}_1, \mathbf{d}_2$: $\mathbf{d}_1(X^i P^j) = iX^i P^j$, $\mathbf{d}_2(P^j X^i) = jP^j X^i$. The action of the group $\mathbb{Z}_\ell$ on the space $\mathrm{Der}(D_q)$ is given by the formula $(g \cdot d)(x) = g^{-1}(d(g(x)))$. Thus the action of element $g \in \mathbb{Z}_\ell$, $\ell = 2, 4$, of order 2 is given by the formula:

$$g(\mathbf{d}_1) = -\mathbf{d}_1, \qquad g(\mathbf{d}_2) = -\mathbf{d}_2,$$

hence $H^1(D_q, D_q)^{\mathbb{Z}_4} \subset H^1(D_q, D_q)^{\mathbb{Z}_2} = 0$. The action of the element $g \in \mathbb{Z}_\ell$, $\ell = 3, 6$, of order 3 is given by the formula:

$$g(\mathbf{d}_1) = -\mathbf{d}_1 + \mathbf{d}_2, \qquad g(\mathbf{d}_2) = -\mathbf{d}_1,$$

hence $H^1(D_q, D_q)^{\mathbb{Z}_6} \subset H^1(D_q, D_q)^{\mathbb{Z}_3} = 0$.

[1] Note that this is only true if q is not a root of unity; if q is a root of unity then the spaces $H^i(D_q, D_q)$ are infinite dimensional for $i = 0, 1, 2$.

As $H^0(D_q, D_q) = Z(D_q) = \mathbb{C}$ is isomorphic to $H_2(D_q, D_q) = \mathbb{C}$ and the action of $\mathbb{Z}_\ell$ on the center is trivial, we get $H_2(D_q, D_q)^{\mathbb{Z}_\ell} = \mathbb{C}$. Analogously, we have seen in the proof of Lemma 4.2 that $H^2(D_q, D_q)$ and $H_0(D_q, D_q) = D_q/[D_q, D_q]$ are trivial $\mathbb{Z}_\ell$-modules. □

4.5. Proof of Theorem 4.1

To complete the proof we only need to calculate the action of $\mathbb{Z}_\ell$ on the cohomology $H_0(D_q, s^i D_q)$, $i = 1, \ldots, \ell - 1$. Let us recall that the action of the group $\mathbb{Z}_\ell$ on $H_0(D_q, s^i D_q) = s^i D_q/[s^i D_q, D_q]$ is induced by the conjugation action of $\mathbb{Z}_\ell$ on $s^i D_q$. As $s^i(x) - x \in [s^i D_q, D_q]$ we get that the action of s^i on $H_0(D_q, s^i D_q)$ is trivial and we only need to calculate $H_0(D_q, s^2 D_q)^{\mathbb{Z}_4}$ in the case $\ell = 4$ and $H_0(D_q, s^2 D_q)^{\mathbb{Z}_6}$, $H_0(D_q, s^3 D_q)^{\mathbb{Z}_6}$, $H_0(D_q, s^4 D_q)^{\mathbb{Z}_6}$. Let us prove $H_0(D_q, s^2 D_q)^{\mathbb{Z}_4} = \mathbb{C}^3$. Indeed, this follows by calculation from the fact that we have the equalities:

$$s \cdot s = s, \qquad s \cdot \left(s^2 X\right) = q^{-2} s^2 P, \qquad s \cdot \left(s^2 P\right) = q^2 s^2 X, \qquad s \cdot \left(s^2 X P\right) = s^2 X P$$

modulo $[D_q, s^2 D_q]$. An analogous calculation shows that

$$H_0\left(D_q, s^2 D_q\right)^{\mathbb{Z}_6} = H_0\left(D_q, s^4 D_q\right)^{\mathbb{Z}_6} = \mathbb{C}^2, \qquad H_0\left(D_q, s^3 D_q\right)^{\mathbb{Z}_6} = \mathbb{C}^2.$$

Thus Proposition 4.3 implies the theorem.

4.6. Infinitesimal deformations

In this subsection we prove Theorem 3.5.

Let $q \in \mathbb{C}^*$, and let (τ, h), where $\tau = (\tau_{kj}) \in \mathbb{C}^r$ and $h \in \mathbb{C}$, be a nonzero vector. Let H' be the algebra over the ring $\mathbb{C}[\varepsilon]/\varepsilon^2$ of dual numbers, generated by T_i with defining relations (2), with $t_{ij} = e^{\varepsilon \tau_{ij}}$, and q replaced by $q e^{h\varepsilon}$. Theorem 3.5 follows from Theorem 4.1 and the following lemma (in which q is allowed to be a root of unity).

Lemma 4.7. *There is no isomorphism of $\mathbb{C}[\varepsilon]/(\varepsilon^2)$-algebras between H' and $B(q) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]/(\varepsilon^2)$ which is equal to the identity map modulo the ideal (ε).*

In the case $\ell = 2$ this lemma was proved in [25]. We show how we can modify this proof for the cases $\ell = 3, 4, 6$.

In the proof of this lemma we use the following description of the algebra H': it is generated by T_i with defining relations

$$T_i^{d_i} = 1 + \varepsilon \sum_{k=1}^{d_i - 1} L_k^i(\tau) T_i^{d_i - k}, \quad i = 1, 2, 3,$$

$$T_1 T_2 T_3 = q e^{h\varepsilon},$$

where $L_k^i(\tau)$ are appropriate formal series in τ.

Proof of Lemma 4.7. First we explain the proof in the case $\ell = 3$. Let us denote by ϕ the natural isomorphism of vector spaces

$$B(q) \to \varepsilon H'.$$

Assume that there is an isomorphism between H' and $B(q) \otimes_{\mathbb{C}} \mathbb{C}[\varepsilon]/(\varepsilon^2)$ lifting the identity. A direct calculation shows that the following equation holds in H':

$$\begin{aligned} PX - q^3XP = \varepsilon\big(&3hq^3XP - s\big(q^{-2}L_2^1X + q^2L_2^2P^{-1}X + L_3^3PX\big) \\ &+ s^2\big(q^2L_1^1 + qL_1^2X + L_1^3PX\big)\big), \end{aligned} \tag{9}$$

where P, X are given by formulas (3). Hence there exist elements $X', P' \in B(q)$ such that the elements $P + \varepsilon P'$ and $X + \varepsilon X'$ of $B(q)[\varepsilon]/\varepsilon^2$ satisfy Eq. (9) modulo ε^2. If we write the elements X', P' in the PBW basis:

$$X' = \sum_{i=0}^{2} s^i \Phi_i^X, \qquad P' = \sum_{i=0}^{2} s^i \Phi_i^P,$$

with $\Phi_i^X, \Phi_i^P \in \mathbb{C}_q[X^{\pm 1}, P^{\pm 1}]$, then the equality of the coefficients before ε in Eq. (9) yields

$$\Phi_0^P X - q^3 X \Phi_0^P = 3hq^3XP, \tag{10}$$

$$\Phi_1^P X - q^3 X^s \Phi_1^P + P^s \Phi_1^X - q^3 \Phi_1^X P = -\big(q^{-2}L_2^1X + q^2L_2^2P^{-1}X + L_2^3PX\big), \tag{11}$$

$$\Phi_2^P - q^3 X^{s^2} \Phi_2^P + P^{s^2} \Phi_2^X - q^3 \Phi_2^X P = q^2L_1^1 + q^2L_1^2X + L_1^3PX. \tag{12}$$

Here $X^{s^j} := s^{-j}Xs^j$ and $P^{s^j} := s^{-j}Ps^j$.

If we expand the LHS of (10) in terms of the PBW basis X^iP^j, we see that the coefficient before XP is zero, hence $h = 0$. Similarly to the description of $\operatorname{Im}\hat{\mathbf{d}}_0$ from the proof of Theorem 4.1 we see that the coefficients c_{ij} of expansion $\sum c_{ij}X^iP^j$ of LHS of (11) satisfy the equations

$$I_n(c) = \sum_{(i,j) \in v_n + (s-1)\mathbb{Z}^2} q^{f_n(i,j)} c_{i,j} = 0,$$

$n = 1, 2, 3$, where $v_i \in \mathbb{Z}^2$ are distinct modulo $(s-1)\mathbb{Z}^2$ and f_n is a quadratic expression of i, j. At the same time, we see that the RHS of (11) has the form $\sum_{n=1}^{3} c_n(q, \tau) X^{w_n^1} P^{w_n^2}$, and vectors w_i, $i = 1, 2, 3$, are distinct modulo $(s-1)\mathbb{Z}^2$. Thus we proved that $L_2^1 = L_2^2 = L_2^3 = 0$ for all i. Analogously, considering Eq. (12) we prove that $L_1^1 = L_1^2 = L_1^3 = 0$.

Thus we get that the vectors $\tau_i = (\tau_{i1}, \tau_{i2}, \tau_{i3})$ satisfy the equation:

$$L_j^i(\tau_i) = 0, \quad j = 1, 2, 3.$$

It is easy to see that Jacobi matrix of this system is nondegenerate at zero, so we have $\tau_i = 0$, $i = 1, 2, 3$.

The proof in the case $\ell = 4, 6$ is the same, except that we need to use instead of formula (9) the formulas:

$$\begin{aligned} PX - q^4XP = \varepsilon\bigl(4hq^4XP &+ s\bigl(q^{m_{11}}L_3^3 - q^{m_{12}}L_3^2P\bigr) \\ &+ s^2\bigl(q^{m_{21}}L_2^2P^2 - q^{m_{22}}L_2^3P + L_1^1\bigl(q^{m_{23}}P - q^{m_{24}}X^{-1}P^2\bigr)\bigr) \\ &+ s^3\bigl(q^{m_{31}}L_1^3P^2 - q^{m_{32}}L_1^2X^{-1}P^2\bigr)\bigr), \end{aligned}$$

for the case $\ell = 4$, and

$$\begin{aligned} PX - q^6XP = \varepsilon\bigl(&6hq^6XP + sq^{n_{11}}L_5^3 + s^2\bigl(-q^{n_{21}}L_4^3PX - q^{n_{22}}L_2^2X^3 - q^{n_{23}}L_2^2X^2P^{-1}\bigr) \\ &+ s^3\bigl(3q^{n_{31}}L_1^1PX - q^{n_{32}}L_1^1P + q^{n_{33}}L_3^3PX\bigr) \\ &+ s^4\bigl(q^{n_{41}}L_1^2P + q^{n_{42}}X^{-2}L_1^2PX^{-1} - q^{n_{43}}L_2^3XP\bigr) + s^5q^{n_{51}}L_1^3XP\bigr), \end{aligned}$$

for the case $\ell = 6$. In the last two formulas m_{ij}, n_{ij} are integers, whose exact values play no role in the proof. □

Proof of Theorem 3.5. The second Hochschild cohomology $H^2(\mathbb{Z}_\ell \ltimes D_q) = \mathbb{C}^{r+1}$ is the tangent space to the moduli space of all deformations. The deformations coming from the family $\{H(t,q)\}_{t\in(\mathbb{C}^*)^r,\, q\in\mathbb{C}^*}$ yield a subspace in this tangent space. Lemma 4.7 shows that this subspace is of dimension $r+1$, i.e., coincides with the entire tangent space. This implies that the above family furnishes a universal deformation. □

5. Generalized double affine Hecke algebras over $\mathbb{C}$

5.1. *Definition*

Now we let t_{kj} be complex numbers such that $\prod_j t_{kj} = 1$ for all k. Such collections of numbers form an algebraic torus $\mathbb{T}$. For $t \in \mathbb{T}$, define an algebra $H(t,q)$ in the same way as $\widehat{\mathbf{H}}(q)$, except that this is now an algebra over $\mathbb{C}$. That is, the algebra $H(t,q)$ is generated over $\mathbb{C}$ by T_i, $i = 1,\dots,m$, with defining relations (2). This family of algebras can be obtained by specializations of a single (obviously defined) algebra $\mathbf{H}(q)$ over $\mathbb{C}[\mathbb{T}]$, in which t_{kj} are central elements, and, yet more universally, of the algebra $\mathbf{H}$ over $\mathcal{R} := \mathbb{C}[\mathbb{T}] \otimes \mathbb{C}[q,q^{-1}]$, in which both q and t_{kj} are central elements.

5.2. *The length filtration and PBW theorem*

We will now introduce an important length filtration on the algebras $\mathbf{H}$, $\mathbf{H}(q)$, and $H(t,q)$. To do so, let us note that the group G is the group W_+ of even elements of an affine Weyl group W of rank m, of types $\hat{A}_1 \times \hat{A}_1$, $\hat{A}_2$, $\hat{B}_2$, and $\hat{G}_2$, if G is of types D_4, E_6, E_7, E_8, respectively.[2]

The group W is generated by $s_1,\dots,s_m$ with the defining relations $s_i^2 = 1$ and $(s_ks_{k+1})^{d_k} = 1$, $k \in \mathbb{Z}_m$. The isomorphism of $\eta_0 : G \to W_+$ is given by the formula $T_k \to s_ks_{k+1}$.

[2] Here, by rank we mean the number of nodes of the affine Dynkin diagram.

Let $\sigma : G \to G$ be the automorphism defined by the formulas $\sigma(T_1) = T_1^{-1}$, $\sigma(T_2) = T_2^{-1}$, and in the D_4 case $\sigma(T_3) = T_2 T_3^{-1} T_2^{-1}$. It is easy to see that the homomorphism η_0 extends to an isomorphism $\eta : \{1, \sigma\} \ltimes G \to W$, given by the formula $\eta_0(\sigma) = s_2$.

Let us now construct a deformation of η_0. Let $f \mapsto \overline{f}$ be the automorphism of $\mathcal{R}$ defined by the formula $\bar{q} = q^{-1}$, $\overline{t_{kj}} = t_{k,-j}^{-1}$ (where $-j$ is taken modulo d_k). Let the algebra $\mathbf{A}$ be generated by $\mathcal{R}$ and additional generators s_k with defining relations

$$s_k^2 = 1, \qquad s_k f = \overline{f} s_k, \quad f \in \mathcal{R},$$

and

$$\prod_{j=1}^{d_k} \left(s_k s_{k+1} - q^{-\delta_{km}} t_{kj}\right) = 0.$$

It is clear that we have a homomorphism of algebras $\eta : \mathbf{H} \to \mathbf{A}$ defined by the formula $\eta(T_k) = q^{\delta_{km}} s_k s_{k+1}$.

Let $\sigma : \mathbf{H} \to \mathbf{H}$ be the automorphism of algebras defined by the formulas $\sigma(f) = \overline{f}$ for $f \in \mathcal{R}$, $\sigma(T_1) = T_1^{-1}$, $\sigma(T_2) = T_2^{-1}$, and in the D_4 case $\sigma(T_3) = T_2 T_3^{-1} T_2^{-1}$ (clearly, $\sigma^2 = 1$). It is easy to see that the homomorphism η extends to an isomorphism $\eta : (\mathbb{C}[\sigma]/(\sigma^2 = 1)) \ltimes \mathbf{H} \to \mathbf{A}$, given by the formula $\eta(\sigma) = s_2$. To see that this is an isomorphism, it suffices to construct the inverse, which is given by the formulas $\eta^{-1}(s_2) = \sigma$, $\eta^{-1}(s_1) = T_1 \sigma$, $\eta^{-1}(s_3) = \sigma T_2$, and for type D_4, $\eta^{-1}(s_4) = \sigma T_2 T_3$.

Define the filtration $F_L^\bullet$ on $\mathbf{A}$ and $\mathbf{H}$ by the formulas $\deg(\mathcal{R}) = 0$, $\deg(s_i) = 1$. We call this filtration *the length filtration*.

For any element $x \in W$, fix a reduced decomposition $w(x)$ of x. Also, for any word w in the letters s_i, define the element $T_w \in \mathbf{A}$ to be the product of the generators s_i according to the word w.

The next theorem (which can also be found in [14]) is a PBW theorem for $\mathbf{H}$. It is formulated in terms of the length filtration.

Theorem 5.1.

(i) *The elements $T_{w(x)}$ form a basis of $\mathbf{A}$ as a left $\mathcal{R}$-module. The elements $T_{w(x)}$ for even x form a basis of $\mathbf{H}$ over $\mathcal{R}$.*
(ii) *The elements $T_{w(x)}$ with* $\operatorname{length}(x) \leqslant N$ *form a basis of $F_L^N \mathbf{A}$ as a left $\mathcal{R}$-module. The elements $T_{w(x)}$ for even x of length $\leqslant N$ form a basis of $F_L^N \mathbf{H}$ over $\mathcal{R}$.*
(iii) *The $\mathcal{R}$-modules $F_L^N \mathbf{H} / F_L^{N-1} \mathbf{H}$, $F_L^N \mathbf{A} / F_L^{N-1} \mathbf{A}$ are free. The Hilbert series of $\mathbf{H}, \mathbf{A}$ under the length filtration is the same as those of W, W_+.*

Proof. It is clear that (i) and (iii) follow from (ii), so it suffices to prove (ii).

First of all, the elements $T_{w(x)}$ are linearly independent after reduction modulo the ideal $t_{kj} = e^{2\pi \mathrm{i} j/d_k}$, so by Theorem 3.3 they are linearly independent in $\mathbf{A}$. It remains to show that $T_{w(x)}$ with length of x being $\leqslant N$ is a spanning set of $F_L^N \mathbf{A}$.

Let us write the relation

$$\prod_{j=1}^{d_k} \left(s_k s_{k+1} - q^{-\delta_{km}} t_{kj}\right) = 0$$

as a deformed braid relation:

$$s_k s_{k+1} \cdots + \text{S.L.T.} = t_k s_{k+1} s_k \cdots + \text{S.L.T.},$$

where $t_k := (-1)^{d_k+1} q^{-\delta_{km} d_k} t_{k1} \cdots t_{kd_k}$, S.L.T. mean "smaller length terms," and the products on both sides have length d_k. This can be done by multiplying the relation by $s_k s_{k+1} \cdots$ (d_k factors).

Clearly, T_w for all words w of length $\leqslant N$ span $F_L^N \mathbf{A}$. So we just need to take any word w of length $\leqslant N$ and express T_w via $T_{w(x)}$ for $x \in W$, length$(x) \leqslant N$.

It is well known from the theory of Coxeter groups (see e.g. [2]) that using the braid relations, one can turn any non-reduced word into a word that is not square free, and any reduced expression of a given element of W into any other reduced expression of the same element. Thus, if w is non-reduced, then by using the deformed braid relations and the relations $s_i^2 = 1$, we can reduce T_w to a linear combination of T_u with words u of smaller length than w. On the other hand, if w is a reduced expression for some element $x \in W$, then using the deformed braid relations we can reduce T_w to a linear combination of T_u with u shorter than w, and $T_{w(x)}$. Thus $T_{w(x)}$ is a spanning set. The theorem is proved. □

Corollary 5.2. *The Gelfand–Kirillov dimension of* $\mathbf{H}(q)$ *as an algebra over* $\mathbb{C}[\mathbb{T}]$*, and of* $H(t,q)$ *is* 2.

5.3. The general properties of $H(t,q)$ when q is a root of unity

Recall that an algebra A is PI of degree K if all polynomial identities of the matrix algebra of size K are satisfied in A.

Theorem 5.3. *Let q be a root of unity such that q^ℓ has degree N. Then $H(t,q)$ and $\mathbf{H}(q)$ are PI algebras of degree $\leqslant \ell N$.*

Proof. According to the proof of Theorem 3.3, the algebra $B(q)$ has an embedding into $\mathrm{Mat}_{\ell N}(R)$, where R is the ring of regular functions on a formal 2-dimensional polydisk (obtained by considering the formal neighborhood of an irreducible ℓN-dimensional representation of $B(q)$). It is shown in the proof of Theorem 3.3 that this embedding can be deformed into an embedding of $\widehat{\mathbf{H}}(q)$ into $\mathrm{Mat}_{\ell N}(R)[[\tau]]$. This implies the result for the formal algebra $\widehat{\mathbf{H}}(q)$.

Now let us establish the result for the algebra $\mathbf{H}(q)$. Theorems 3.3, 5.1 imply that the algebra $\widehat{\mathbf{H}}(q)$ is the formal completion of $\mathbf{H}(q)$ with respect to the ideal defined by the equation $t = 1$, and the natural map $\mathbf{H}(q) \to \widehat{\mathbf{H}}(q)$ is injective. This implies the result for $\mathbf{H}(q)$ and hence for $H(t,q)$ for special t. □

Let $U = U_N \subset \mathbb{T}$ be the Zariski open set, defined by the following condition: one cannot choose nonnegative integers p_{kj} with $\sum_j p_{kj} = p < \ell N$ for all $k = 1, \ldots, m$ such that $\prod_{k,j} (u_{kj})^{p_{kj}} = q^p$. It is clear that U is nonempty.

Theorem 5.4. *Let q be a root of unity such that q^ℓ has degree N. Then every irreducible representation of $H(t,q)$ has dimension $\leqslant \ell N$. For $t \in U$, the dimension is exactly ℓN.*

Proof. Let V be a nonzero finite dimensional representation of $H(t,q)$ of dimension $p < \ell N$. Then the product of determinants of T_k is equal to q^p. Thus q^p should equal the product of u_{kj} with multiplicities. For $t \in U$, this leads to a contradiction, so for such t we have $p \geqslant \ell N$.

It remains to prove the opposite inequality for irreducible representations V. This follows from Theorem 5.3 and the following well-known theorem is due to Kaplansky. □

Theorem 5.5. *If A is a PI algebra of degree $\leqslant K$ then any irreducible A-module is finite dimensional and has dimension $\leqslant K$.*

Remark. Note that if q is not a root of unity, then $H(t,q)$ does not have finite dimensional representations for generic t. This can be seen by taking the determinant of the relation $\prod_k T_k = q$.

5.4. The algebras $H(t,q)$ when q is a root of unity, for generic t

Theorem 5.6. *Let q be a root of unity such that q^ℓ has degree N. Then for $t \in U$, the algebra $H(t,q)$ is an Azumaya algebra of rank ℓN over an affine 2-dimensional $\mathbb{C}$-scheme $S(t,q)$ of finite type.*

Proof. We first recall the following theorem about PI algebras.

Theorem 5.7. (M. Artin, see [18].) *Let A be a finitely generated algebra over $\mathbb{C}$, which is PI of degree $\leqslant K$. If the dimension of every irreducible finite dimensional A-module is K then A is an Azumaya algebra, and its center is finitely generated.*

This result and Theorem 5.4 implies that for $t \in U$, the algebra $H(t,q)$ is Azumaya, with finitely generated center $Z(t,q)$.

Let $S(t,q)$ be the spectrum of $Z(t,q)$. Since the Gelfand–Kirillov dimension of $H(t,q)$ is 2, $\dim S(t,q) = 2$. □

Corollary 5.8. *Let $q = 1$, $t \in U = U_1$, and the minimal polynomial of the generator $c = T_3$ has a simple root. Then $H(t,q)$ is the endomorphism algebra of a vector bundle over $S(t,q)$.*

Proof. Let e be the projection to the eigenspace of c corresponding to the simple root of the minimal polynomial (it is a polynomial of degree ℓ of the generator $c = T_3$). If $q = 1$ and $t \in U$ then $H(t,q)$ is an Azumaya algebra, so the map $Z(t,q) \to eH(t,q)e$ given by $z \to ze$ is an isomorphism, and $H(t,q)e$ is a projective module over $eH(t,q)e$. This gives rise to the required vector bundle. □

5.5. Smoothness of $S(t,q)$

Theorem 5.9. *If q is a root of unity and $t \in U$ with $u_{kj} \neq u_{kj'}$ for $j \neq j'$ then $S(t,q)$ is a smooth surface.*

Proof. Let $K = \ell N$. Since $H(t,q)$ is an Azumaya algebra, $S(t,q)$ is isomorphic to the moduli space $M(t,q)$ of irreducible (K-dimensional) representations of $H(t,q)$. The space $M(t,q)$ is a quotient of the affine variety of K-dimensional matrix representations of $H(t,q)$ by the free PGL_K-action, so it is an affine variety.

More specifically, each connected component[3] of the space $M(t,q)$ is a quotient by the free PGL_K-action of the subvariety Y in the product of m special semisimple conjugacy classes

[3] A priori, the connected components of $M(t,q)$ correspond to various multiplicities of eigenvalues of T_j. However, we will see later in Proposition 6.6 that in fact the varieties $M(t,q)$ and $S(t,q)$ are connected, i.e. eigenvalues of T_j always occur with equal multiplicities.

$C_1, \dots, C_m$ in GL_K (those of T_j) defined by the equation $T_1 \cdots T_m = q$. Thus we just need to show that this subvariety is smooth.

Let us first deal with the cases E_6, E_7, E_8. In these cases, let us fix the matrix c to be diagonal and consider the map $\mu: C_1 \times C_2 \to GL_K$, $\mu(a,b) = ab$. It lands in a fixed coset of SL_K in GL_K. Our job is to show that qc^{-1} is a regular value of μ. To do so, it suffices to show that the differential of μ: $d\mu(a,b) = (da)b + a(db)$, is surjective onto sl_K. Let $v = [x,a]$, $w = [y,b]$ be two tangent vectors to C_1, C_2 at a, b, respectively. Then we get

$$d\mu(a,b)(v,w) = [x,a]b + a[y,b].$$

To show that the map $x, y \to [x,a]b + a[y,b]$ is surjective, let us assume that $z \in gl_K$ is orthogonal to its image. Then we have $\mathrm{Tr}([x,a]bz) = 0$, $\mathrm{Tr}(a[y,b]z) = 0$ for all matrices x, y. So $[a,bz] = [b,za] = 0$. Set $u = bza$. Then $[a,u] = [b,u] = 0$. Hence u is a scalar (since a, b is an irreducible collection of matrices), and $z = \lambda b^{-1}a^{-1}$, where λ is a scalar. So the map in question is surjective (onto the tangent space of the SL_K-coset).

The case of D_4 is similar. In this case we have a map $\nu: C_1 \times C_2 \times C_4 \to GL_K$ given by $\nu(a,b,d) = dab$. We need to show that the value qc^{-1} is regular. The computation of the orthogonal complement of the image of the differential of ν will give equations $[d,abz] = 0$, $[a,bzd] = 0$, $[b,zda] = 0$. Set $u = bzda$. Then $[a,u] = [b,u] = [d,u] = 0$. Since a, b, d are irreducible, u is a scalar, and $z = \lambda b^{-1}a^{-1}d^{-1}$, as desired. □

6. The geometry of the algebras $H(t,q)$

6.1. Other filtrations

In this subsection we will study filtrations on $\mathbf{H}(q)$ which are, in reality, simpler and more useful that F_L, but whose study requires computer calculations. They will be used to study the geometry of the algebras $H(t,q)$, in particular their intimate connection with del Pezzo surfaces. To be more specific, these filtrations correspond to the simplest compactifications of affine del Pezzo surfaces, which are described below.

Namely, we introduce increasing filtrations $F^\bullet_{jkl}\mathbf{H}(q)$ on $\mathbf{H}(q)$ as follows: $\deg \mathbb{C}[\mathbb{T}] = 0$, $\deg(c) = 0$, and

$F^\bullet_{111}$ for D_4: $\deg(b) = 1, \deg(d) = 1, \deg(a) = 1$;
$F^\bullet_{112}$ for D_4: $\deg(b) = 1, \deg(d) = 1$;
$F^\bullet_{111}$ for E_6: $\deg(b) = 1, \deg(b^2) = 1, \deg(b^2cb) = 1$;
$F^\bullet_{112}$ for E_6: $\deg(b) = 1, \deg(b^2) = 1$;
$F^\bullet_{123}$ for E_6: $\deg(b) = 1$;
$F^\bullet_{112}$ for E_7: $\deg(b) = 1, \deg(b^2) = 1$;
$F^\bullet_{123}$ for E_7: $\deg(b) = 1$;
$F^\bullet_{123}$ for E_8: $\deg(b) = 1$.

Since these elements are generators, these filtrations are well defined. Similarly one defines a filtration $F^\bullet_{jkl}H(t,q)$ for specific points t.

Remark. The subscripts are the degrees of the generators of the center of $H(t,1)$ with respect to the corresponding filtration; this will be discussed below.

Theorem 6.1 *(The PBW theorem). The spaces $F^N_{jkl}\mathbf{H}(q)/F^{N-1}_{jkl}\mathbf{H}(q)$ are free $\mathbb{C}[\mathbb{T}]$-modules.*

Proof. Theorem 6.1 is proved in Sections 8.1, 8.2. The structure of the proof is as follows.

Let $F = F_{jkl}$. Let m_r be the dimension of $F^r H(1,q)$, and $\{g_s,\ s \geqslant 1\}$ be a labeling of elements of G by positive integers, such that $g_1, \ldots, g_{m_r}$ is a basis of $F^r H(1,q)$.

We will find a collection of "legal" monomials $\{h_s \in \mathbf{H}(q),\ s \geqslant 1\}$ in generators, which satisfy the following conditions:

(i) If $m_{r-1} < s \leqslant m_r$ then $h_s \in F^r\mathbf{H}(q)$, and h_s with $s \leqslant m_r$ span $F^r\mathbf{H}(q)$ over $\mathbb{C}[\mathbb{T}]$;
(ii) h_s specialize to g_s at $t = 1$.

Property (i) implies that for $m_{r-1} < s \leqslant m_r$ the images h'_s of h_s in $F^r\mathbf{H}(q)/F^{r-1}\mathbf{H}(q)$ span this module over $\mathbb{C}[\mathbb{T}]$. Thus, it remains to show that they are linearly independent.

To do so, we assume that we have a nontrivial linear relation

$$\sum_{s=m_{r-1}+1}^{m_r} f_s(t)h'_s = 0, \qquad f_s \in \mathbb{C}[\mathbb{T}].$$

Then we have a linear relation

$$\sum_{s=1}^{m_r} f_s(t)h_s = 0.$$

Expanding this relation in a power series near $t = 1$, we get a linear relation in $\widehat{\mathbf{H}}(q)$. This relation is nontrivial because of condition (ii). Thus we obtain a contradiction with Theorem 3.3. This completes the proof. □

In the case $t = 1$, the filtrations $F^\bullet_{jkl}$ are very easy to understand. In this case, we have a natural (up to scaling by powers of q) basis of $H(t,q) = B(q)$ corresponding to group elements, and it is easy to see that our filtrations are compatible with this basis. So let us say which basis elements have degree $\leqslant N$.

Realizing G as $\mathbb{Z}_\ell \ltimes \mathbb{Z}^2$, we can write any group element as a product $c^j Y$, where $0 \leqslant j \leqslant \ell - 1$ and Y belongs to the lattice. The basis elements of degree $\leqslant N$ are then those products $c^j Y$ for which $\|Y\| \leqslant N$, where $\|Y\|$ is a certain norm on $\mathbb{R}^2$, depending on a particular filtration. We will describe these norms in all cases.

D_4, F_{111}. The lattice is hexagonal (generated by two vectors v, w of equal length making angle 60° with each other) and the norm is such that the unit ball is the hexagon whose vertices are v and its images under $\mathbb{Z}_6$.

D_4, F_{112}. The lattice is rectangular, and the norm is $|x| + |y|$.

E_6. The lattice is hexagonal, and

(i) for F_{111}, the norm is such that the unit ball is the triangle whose vertices are $v + w$ and its two images under $\mathbb{Z}_3$;

(ii) for F_{112}, the norm is such that the unit ball is the hexagon whose vertices are v and its images under $\mathbb{Z}_6$;

(iii) for F_{123}, the norm is such that the unit ball is the triangle whose vertices are v and its images under $\mathbb{Z}_3$.

E_7. The lattice is rectangular, and

(i) for F_{112}, the norm is $\max(|x|, |y|)$;

(ii) for F_{123}, the norm is $|x| + |y|$.

E_8, F_{123}. The lattice is hexagonal, and the norm is such that the unit ball is the hexagon whose vertices are v and its images under $\mathbb{Z}_6$.

This implies the following result.

Proposition 6.2. *The Poincaré series of $H(t,q)$ and $\mathbf{H}(q)$ with respect to the filtration $F^{\bullet}_{jkl}$ is*

$$P_{D_4,111}(z) = 2\left(1 + \frac{6z}{(1-z)^2}\right),$$

$$P_{D_4,112}(z) = 2\left(1 + \frac{4z}{(1-z)^2}\right),$$

$$P_{E_6,111}(z) = 3\left(1 + \frac{9z}{(1-z)^2}\right),$$

$$P_{E_6,112}(z) = 3\left(1 + \frac{6z}{(1-z)^2}\right),$$

$$P_{E_6,123}(z) = 3\left(1 + \frac{3z}{(1-z)^2}\right),$$

$$P_{E_7,112}(z) = 4\left(1 + \frac{8z}{(1-z)^2}\right),$$

$$P_{E_7,123}(z) = 4\left(1 + \frac{4z}{(1-z)^2}\right),$$

$$P_{E_8,123}(z) = 6\left(1 + \frac{6z}{(1-z)^2}\right).$$

6.2. The associated graded algebras of $\mathbf{H}(q)$ and $H(t,q)$

Let $\mathbf{H}_0^{jkl}(q)$ and $H_0^{jkl}(t,q)$ be the associated graded algebras of $\mathbf{H}(q)$ and $H(t,q)$ with respect to the filtration F_{jkl}.

Let us give a description of some of the algebras $H_0^{jkl}(t,q)$ by generators and relations. A similar description (with t being variables) is valid for $\mathbf{H}_0^{jkl}(q)$.

For $\ell \geqslant 3$ and a monic polynomial p of degree ℓ, define an algebra $K_\ell(p)$ to be the free algebra generated by elements c, d modulo the ideal generated by elements

$$p(c),\ dc^2d,\ dc^3d, \ldots,\ dc^{\ell-2}d,\ dcd - q^{-\ell}cdc^{\ell-1}dc,\ dcdcd. \tag{13}$$

Theorem 6.3. *In the cases E_6, E_7, E_8, the algebra $\mathbf{H}_0^{123}(t,q)$ is isomorphic to the algebra $K_\ell(p)$, where $p(x) = \prod_{j=1}^{\ell}(x - u_{3j})$, and $\ell = 3, 4, 6$ for E_6, E_7, E_8, respectively.*

Proof. The proof is given in Section 8.3. Note that in Section 8.3 we obtain the same relations with $-q^{-\ell}$ replaced with $(-1)^{\ell}q^{-\ell}$, due to the fact that generators have been rescaled. $\square$

A similar description of the graded algebra exists in the D_4 case. Namely, we have

Proposition 6.4. *In the case D_4, the algebra $H_0^{112}(t,q)$ is generated by c, z_1, z_2 with defining relations*

$$p(c)=0, \qquad z_1cz_1=0, \qquad z_2cz_2=0, \qquad cz_1cz_2c=q^2z_2cz_1, \qquad z_1z_2=q^{-2}z_2c^2z_1.$$

On the other hand, the algebra $H_0^{111}(t,q)$ for D_4 is generated by c, w_1, w_2, w_3 with defining relations

$$p(c)=0, \qquad w_1w_3=0, \qquad w_1w_2=0, \qquad w_2w_3=0,$$
$$w_3c^2w_1=0, \qquad w_2c^2w_1=0, \qquad w_3c^2w_2=0,$$
$$w_1cw_1=0, \qquad w_2cw_2=0, \qquad w_3cw_3=0,$$
$$w_3cw_1=q^2cw_1cw_3c, \qquad w_2cw_1=q^{-2}cw_1cw_2c, \qquad w_3cw_2=q^{-2}cw_2cw_3c.$$

Proof. The proof is analogous to the proof of Theorem 6.3, using the presentations of $H(t,q)$ in Section 8; the elements z_1 and z_2 are the images in the graded algebra of the elements $c^{-1}b$ and dc^{-1}, while the elements w_1, w_2, w_3 are the images of $c^{-1}d^{-1}$, db, $b^{-1}c^{-1}$. $\square$

These results show that the algebra $H_0^{jkl}(t,q)$ for the considered filtrations does not depend on t_{ij} with $i \neq 3$.

Remark. These results imply that the PI degree in Theorem 5.3 is exactly ℓN (since it is so for the corresponding associated graded algebras).

6.3. The geometric characterization of the associated graded algebras

To characterize the associated graded algebras geometrically, let us recall the theory of noncommutative curves [30].

Let X be a projective algebraic curve, σ an automorphism of X, and $\mathcal{L}$ an ample line bundle on X. Then one can define the twisted homogeneous coordinate ring $B(X,\sigma,\mathcal{L})$ as follows. This is a $\mathbb{Z}_+$-graded ring, and $B(X,\sigma,\mathcal{L})[n]=\mathrm{Hom}(\mathcal{O}, \bigotimes_{j=0}^{n-1}\mathcal{L}^{\sigma^j})$. The multiplication is defined by the formula $a*b=\sigma_*^{\deg b}(a)\otimes b$ for homogeneous a,b. In noncommutative algebraic geometry, this twisted homogeneous coordinate ring is viewed as the homogeneous coordinate ring of a noncommutative projective curve.

Similarly, if E is a vector bundle on X equivariant under σ then one can define the graded algebra $B(X,\sigma,\mathcal{L},E)$ by

$$B(X,\sigma,\mathcal{L},E)[n]=\mathrm{Hom}\left(E, E\otimes\left(\bigotimes_{j=0}^{n-1}\mathcal{L}^{\sigma^j}\right)\right),$$

with multiplication as above.

Let $X_n' = \mathbb{P}_1^1 \cup \mathbb{P}_2^1 \cup \cdots \cup \mathbb{P}_n^1$ be a chain of projective lines, i.e. the point 0 of $\mathbb{P}_i^1$ is identified with the point ∞ of $\mathbb{P}_{i+1}^1$ for $1 \leqslant i \leqslant n-1$.

Let X_n be the union of n projective lines forming an n-gon (i.e. each two consecutive ones in a cyclic order intersect at a point). Clearly, X_n is obtained from X_n' by gluing the point ∞ of $\mathbb{P}_1^1$ with the point 0 of $\mathbb{P}_n^1$. For $n=1$, X_n is a single $\mathbb{P}^1$ with a node, and we will denote it simply by X.

Let P_i be a smooth point lying on the ith component of X_n. The group $\mathbb{C}^*$ acts naturally on X_n; let σ be the action of $q^\ell \in \mathbb{C}^*$. Set $\mathcal{L} = \mathcal{O}(\sum_i P_i)$.

Let $p_t(x) = x^\ell + \alpha_1 x^{\ell-1} + \cdots + \alpha_\ell$ be the monic polynomial of degree ℓ annihilating c ($p_t(x) = \prod_j (x - u_{3j})$). Let A_t be the companion matrix corresponding to the polynomial p_t. This is the ℓ-by-ℓ matrix defined by the formula

$$A_t v_i = v_{i+1}, \quad 1 \leqslant i < \ell, \qquad A_t v_\ell = -\alpha_1 v_\ell - \alpha_2 v_{\ell-1} - \cdots - \alpha_\ell v_1,$$

where v_i is the standard basis of $\mathbb{C}^\ell$; thus $p_t(A_t) = 0$.

Consider the trivial vector bundle of rank ℓ on X_n'. Let V_0 be its fiber at $0 \in \mathbb{P}_n^1$, and V_∞ its fiber at $\infty \in \mathbb{P}_1^1$.

Let A be an invertible ℓ-by-ℓ matrix. Let $E(A)$ be the vector bundle on X_n obtained from the trivial bundle of rank ℓ on X_n' by gluing the fibers V_0, V_∞ using the map $A: V_\infty \to V_0$. Thus, if Δ is an effective divisor on X_n not containing the gluing points then sections of $E(A)$ with poles at Δ are collections of $\mathbb{C}^\ell$-valued rational functions $\phi_1, \ldots, \phi_n$ of one variable z with poles at Δ which satisfy the conditions

$$\phi_1(0) = \phi_2(\infty), \quad \ldots, \quad \phi_{n-1}(0) = \phi_n(\infty), \quad \phi_n(0) = A\phi_1(\infty).$$

Obviously, the bundle $E(A)$ is $\mathbb{C}^*$-equivariant.

Theorem 6.5.

(i) *The algebra $H_0^{jkl}(t,q)$ is isomorphic to $B(X_n, \sigma, \mathcal{L}, E(A_t^\varepsilon))$, where*:
for D_4 and $jkl = 111$, $n = 3$, $\varepsilon = -1$;
for D_4 and $jkl = 112$, $n = 2$, $\varepsilon = -1$;
for E_6, E_7, E_8, $jkl = 123$, $n = 1$, $\varepsilon = 1$.

(ii) *Let u_{31} be a simple root of the minimal polynomial of c. Let e be the projector to the u_{31}-eigenspace of c in $\mathbb{C}[c] \subset H_0^{jkl}(t,q)$. Then the "spherical subalgebra" $eH_0^{jkl}(t,q)e$ is isomorphic to $B(X_n, \sigma, \mathcal{L})$ (for n as above).*

Proof. The second statement follows from the first one, so it suffices to prove (i). Let us do it first in the cases E_6, E_7, E_8.

We start with an explicit description of the algebra $B(X, \sigma, \mathcal{L}, E(A_t))$.

By the definition, $B(X, \sigma, \mathcal{L}, E(A_t))[n]$ is the space of rational functions f of one variable z with values in $\mathrm{Mat}_\ell(\mathbb{C})$ with divisor of poles dominated by the divisor $(1) + (q^\ell) + \cdots + (q^{(n-1)\ell})$, and such that $f(0) = A_t f(\infty) A_t^{-1}$. Furthermore, the multiplication law in $B(X, \sigma, \mathcal{L}, E(A_t))$ is given by the formula $(f * g)(z) = f(q^{-\ell m} z) g(z)$ for homogeneous elements f, g such that g has degree m. Thus, for example, $B(X,\sigma,\mathcal{L},E(A_t))[0]$ has basis $1, A_t, A_t^2, \dots, A_t^{\ell-1}$, while $B(X,\sigma,\mathcal{L},E(A_t))[1]$ is the space of functions of the form

$$f(z) = \frac{Mz - A_t M A_t^{-1}}{z-1},$$

where M is any matrix of size ℓ by ℓ.

Now we will define a homomorphism of graded algebras $\xi : H_0^{123}(t,q) \to B(X,\sigma,\mathcal{L},E(A_t))$. It is defined by the formulas

$$\xi(c) = A_t, \qquad \xi(d) = \frac{E_{11}z - A_t E_{11} A_t^{-1}}{z-1},$$

where E_{ij} is an elementary matrix. It is straightforward to check that the relations of $H_0^{123}(t,q)$ given in Theorem 6.3 are satisfied, hence ξ is well defined. Moreover, it is easy to check that $B(X,\sigma,\mathcal{L},E(A_t))$ is generated by degrees 0 and 1, which implies that ξ is surjective. By comparing the Poincaré series we find that ξ is bijective. This proves (i) for E_6, E_7, E_8.

Now let us handle the case D_4, the 112 filtration. By the definition, $B(X_2,\sigma,\mathcal{L},E(A_t^{-1}))[n]$ is the space of pairs rational functions (f_1, f_2) of one variable z with values in $\mathrm{Mat}_2(\mathbb{C})$ with divisor of poles dominated by the divisor $(1) + (q^2) + \dots + (q^{2(n-1)})$, and such that $f_1(0) = f_2(\infty)$, $A_t f_2(0) A_t^{-1} = f_1(\infty)$. Thus $B(X_2,\sigma,\mathcal{L},E(A_t^{-1}))[0]$ has basis $(1,1), (A_t, A_t)$, while $B(X_2,\sigma,\mathcal{L},E(A_t^{-1}))[1]$ is the space of pairs of functions (f_1, f_2) of the form

$$f_1(z) = \frac{A_t M_1 A_t^{-1} z - M_2}{z-1}, \qquad f_2(z) = \frac{M_2 z - M_1}{z-1},$$

where M_i are any 2-by-2 matrices.

Now we will define a homomorphism of graded algebras $\xi : H_0^{112}(t,q) \to B(X_2,\sigma,\mathcal{L}, E(A_t^{-1}))$. It is defined by the formulas

$$\xi(c) = A_t, \qquad \xi(z_1) = \left(-\frac{E_{11}}{z-1}, \frac{E_{11}z}{z-1}\right), \qquad \xi(z_2) = \left(\frac{A_t E_{11} A_t^{-1} z}{z-1}, \frac{-E_{11}}{z-1}\right).$$

Similarly to the $E_{6,7,8}$ cases, ξ is well defined and is an isomorphism.

Finally, we consider the 111 filtration for D_4. By the definition, $B(X_3,\sigma,\mathcal{L},E(A_t^{-1}))[n]$ is the space of triples of rational functions (f_1, f_2, f_3) of one variable z with values in $\mathrm{Mat}_2(\mathbb{C})$ with divisor of poles dominated by the divisor $(1) + (q^2) + \dots + (q^{2(n-1)})$, and such that $f_1(0) = f_2(\infty)$, $f_2(0) = f_3(\infty)$, $A_t f_3(0) A_t^{-1} = f_1(\infty)$. Thus $B(X_3,\sigma,\mathcal{L},E(A_t^{-1}))[0]$ has basis $(1,1,1), (A_t, A_t, A_t)$, while $B(X_3,\sigma,\mathcal{L},E(A_t^{-1}))[1]$ is the space of triples of functions (f_1, f_2, f_3) of the form

$$f_1(z) = \frac{A_t M_1 A_t^{-1} z - M_3}{z-1}, \qquad f_2(z) = \frac{M_2 z - M_1}{z-1}, \qquad f_3(z) = \frac{M_3 z - M_2}{z-1},$$

where M_i are any 2-by-2 matrices.

Now we will define a homomorphism of graded algebras $\xi : H_0^{111}(t,q) \to B(X_3,\sigma,\mathcal{L}, E(A_t^{-1}))$. It is defined by the formulas

$$\xi(c)=A_t,\qquad \xi(w_1)=\left(-\frac{E_{11}}{z-1},\frac{E_{11}z}{z-1},0\right),$$
$$\xi(w_2)=\left(\frac{A_tE_{11}A_t^{-1}z}{z-1},0,\frac{-E_{11}}{z-1}\right),\qquad \xi(w_3)=\left(0,-\frac{E_{11}}{z-1},\frac{E_{11}z}{z-1}\right).$$

Similarly to the $E_{6,7,8}$ cases, ξ is well defined and is an isomorphism. □

6.4. The center of $H(t,q)$

We return to the study of the center $Z(t,q)$ and the scheme $S(t,q)$ when q is a root of unity, such that q^ℓ has order N.

Proposition 6.6. *For any t, the scheme $S(t,q)$ is an irreducible affine algebraic surface.*

Proof. In the D_4 case, the theorem follows from [25]. Thus consider the cases E_6, E_7, E_8. The associated graded algebra $\operatorname{gr} Z(t,q)$ is a subalgebra in the center $Z(\operatorname{gr} H(t,q))$ of $\operatorname{gr} H(t,q)$. As follows from our description of $\operatorname{gr} H(t,q)=H_0^{123}(t,q)$ in Theorem 6.5, the center $Z(\operatorname{gr} H(t,q))$ is the function algebra of the cone of a nodal $\mathbb{P}^1$. So this algebra has no zero divisors, which implies the result. □

Theorem 6.7. *Let q be a root of unity such that q^ℓ has order N. Consider the filtration F_{jkl} on $H(t,q)$, where $jkl=111$ for D_4 and $jkl=123$ for $E_{6,7,8}$. Then for any t:*

(i) $\operatorname{gr} Z(t,q)=Z(\operatorname{gr} H(t,q))$ (*where $Z(A)$ denotes the center of an algebra A, and* gr *is taken with respect to F_{jkl}*);
(ii) *The Poincaré series of $Z(\operatorname{gr} H(t,q))$ is*

$$Q_{D_4}(z)=1+\frac{3z^N}{(1-z^N)^2};$$
$$Q_{E_{6,7,8}}(z)=1+\frac{z^N}{(1-z^N)^2}.$$

Proof. Statement (ii) follows from the description of $\operatorname{gr} H(t,q)$given in Theorem 6.5. So let us prove (i).

A priori, $\operatorname{gr} Z(t,q)$ is a subalgebra in $Z(\operatorname{gr} H(t,q))$. We must show that in fact, these two algebras coincide.

The coincidence of the two algebras is easy in the case $t=1$, by a direct computation. Therefore, it suffices to establish the coincidence of the two algebras for generic (or even Weil generic) t.

In the D_4 case, it is easy to produce three generators in degree N of $\operatorname{gr} Z(t,q)$: one should consider the Demazure–Lusztig realization of $H(t,q)$ as reflection–difference operators on functions of x (see [25,29,31] for a definition) and take the element $Z_1=x^N+x^{-N}$ and its images under the $SL_2(\mathbb{Z})$-action. This implies the desired result for D_4, so we can now focus on E_6, E_7, E_8.

Recall that for a $\mathbb{Z}_+$-filtered algebra A, the Rees algebra $R(A)$ is the graded algebra $\bigoplus_n F^nA$ with the obvious degree preserving multiplication.

The algebra $R(\widehat{\mathbf{H}}(q))$ is a formal graded deformation of $R(H(1,q))$. Let $Z'_R(1,q)$ be the quotient of the center of this deformation by the ideal generated by τ_{kj}. Obviously, $Z'_R(1,q)$ is a graded subalgebra of $Z_R(1,q) := R(Z(1,q))$, and our job is to show that they coincide.

We will use a simple lemma from commutative algebra.

Lemma. *Let A be a finitely generated $\mathbb{Z}_+$-graded algebra over $\mathbb{C}$ without zero divisors, and $B \subset A$ be a graded subalgebra. Suppose that r is a positive integer, and there exists a constant $C > 0$ such that $\dim B[n] > Cn^{r-1}$ for large enough n. Then the transcendence degree of B is at least r.*

Proof. Let d be the transcendence degree of B, and $x_1, \dots, x_d \in B$ be algebraically independent homogeneous elements. Let $E = \mathbb{C}[x_1, \dots, x_d]$. Let $X = \mathit{Proj}(A)$, and $Y = \mathit{Proj}(E)$ (Y is a weighted projective space). We have a natural dominant morphism $f : \mathrm{Cone}(X) \to \mathrm{Cone}(Y)$, defined by the embedding $E \subset A$. Any element $x \in B$ is algebraic over E, which means that when regarded as a function on $\mathrm{Cone}(X)$, it is locally constant on a generic fiber of f. Let Z be a closed subvariety of X of dimension equal to the dimension of Y such that $\mathrm{Cone}(Z)$ is transversal to a generic fiber of f. In this case, the function $x \in B$ on $\mathrm{Cone}(X)$ is completely determined by its restriction to $\mathrm{Cone}(Z)$. Thus if R is the homogeneous coordinate ring of Z then the natural map $B \to R$ is an embedding of graded algebras. This implies that $\dim R[n] > Cn^{r-1}$ for large n, and hence $\dim Z \geqslant r - 1$. But $\dim Z = \dim Y = d - 1$, so $d \geqslant r$, as desired. □

Now, the algebras $Z_R(1,q)$ and $Z'_R(1,q)$ are domains. Also, recall that for generic t, $H(t,q)$ is an Azumaya algebra. Thus $Z'_R(1,q)$ has quadratic growth and hence by the lemma has transcendence degree 3 over $\mathbb{C}$. This means that $Z_R(1,q)$ is algebraic over $Z'_R(1,q)$.

Now we will need the following lemma.

Lemma 6.8. *Let A_0 be an algebra over $\mathbb{C}$ with center Z_0. Assume that Z_0 is a domain, and any derivation of A_0 which vanishes on Z_0 is inner. Let A be a flat deformation of A_0 over $\mathbb{C}[[\hbar]]$. Let Z be the center of A and $Z'_0 = Z/\hbar Z$. Assume that Z_0 is algebraic over Z'_0. Then $Z_0 = Z'_0$.*

Remark. Note that this lemma is false over a filed k of positive characteristic. The classical counterexample: A is the Weyl algebra with generators x, y and defining relation $xy - yx = 1$, $A_0 = k[x, y]$.

Proof of Lemma 6.8. The Hochschild complex of $A[\hbar^{-1}]$ is filtered by degrees in $\hbar$. There is a Brylinski spectral sequence [3] attached to this filtration. The E_1 term of this sequence is the Hochschild cohomology of A_0: $E_1^{p,q} = H^{p+q}(A_0)$. In particular, $E_1^{p,-p} = Z_0$. Thus our job is to show that all the differentials $d_i^{p,-p}$ are zero.

Assume this is not the case, and let $d_n = d_n^{p,-p}$ be the first nonzero differential. For any $z \in Z_0$, $d_n(z)$ is a coset of derivations of A_0 modulo inner derivations. In particular, $d_n(z)$ gives rise to a well defined derivation $\tilde{d}_n(z)$ of Z_0.

We claim that $\tilde{d}_n(z) = 0$. Indeed, $\tilde{d}_n(z)$ is a derivation with respect to z. If $z \in Z'_0$, then $\tilde{d}_n(z) = 0$. If $z \in Z_0$ and P is a minimal polynomial of z over Z'_0 then we find $0 = \tilde{d}_n(P(z))(w) = P'(z)\tilde{d}_n(z)(w)$. Since Z_0 is a domain, we find $\tilde{d}_n(z) = 0$, as desired.

Thus $d_n(z)$ acts trivially on the center, and hence by our assumption $d_n(z) = 0$. The lemma is proved. □

Let $\tau = \tau(\hbar)$ be a formal path. We apply Lemma 6.8 to $A_0 = R(H(1,q))$, and $A = R(\widehat{\mathbf{H}}(q))|_{\tau=\tau(\hbar)}$. The conditions of the lemma hold because A_0 is an Azumaya algebra everywhere over $Spec\, Z_R(1,q)$ except a subset of codimension 2, and any derivation of an Azumaya algebra which is zero on the center is inner. The conclusion of the lemma implies claim (i). $\square$

Let u_{31} be a simple root of the minimal polynomial of c, and e be the idempotent defined in Theorem 6.5(ii).

Corollary 6.9 *("Satake isomorphism"). The natural map $\phi : Z(t,q) \to eH(t,q)e$ from the center to the spherical subalgebra given by $z \to ze$ is injective. It defines an isomorphism of $Z(t,q)$ onto the center of the spherical subalgebra. If $q = 1$, ϕ is an isomorphism.*

Proof. The first statement holds because it holds at the graded level by Theorem 6.5. The second and third statements follow from the first one and the Poincaré series consideration. $\square$

Now fix $q \in \mathbb{C}^*$ (not necessarily a root of unity), and let t be generic in the Zariski sense.

Proposition 6.10.

(i) *For E_6, $\mathrm{gr}_{F_{111}}(eH(t,q)e)$ is isomorphic to $B(X,\sigma,\mathcal{L}^{\otimes 3})$.*
(ii) *For E_7, $\mathrm{gr}_{F_{112}}(eH(t,q)e)$ is isomorphic to $B(X,\sigma,\mathcal{L}^{\otimes 2})$.*

Proof. The statements are easy in the group case $t = 1$. On the other hand, in both cases the family of algebras $\mathrm{gr}(eH(t,q)e)$ is flat, since by Theorem 6.1, their Poincaré series is independent of t, q. Therefore, it follows from the theory of noncommutative curves (see [30]) that for generic t, the algebra $\mathrm{gr}(eH(t,q)e)$ is isomorphic to $B(X_{t,q},\sigma_{t,q},\mathcal{L}^{\otimes p})$, where $p = 3$ in (i) and $p = 2$ in (ii), $X_{t,q}$ is a genus 1 curve, $\sigma_{t,q}$ its automorphism, and $\mathcal{L} = \mathcal{O}(P)$, where P is a smooth point of $X_{t,q}$. Here $(X_{t,q},\sigma_{t,q})$ depends algebraically on t, q, and $(X_{1,q},\sigma_{1,q}) = (X, q^{\ell})$, as explained above.

If q is a root of unity, such that q^{ℓ} has degree N, then $H(t,q)$ is an Azumaya algebra of rank ℓN, and hence $\sigma_{t,q}$ has order N.

Let Σ_N be the modular curve parametrizing elliptic curves with points of order N. We see that if q is a root of unity as above, we get a regular map $t \mapsto \psi_q(t)$ from an open subset of $\mathbb{T}$ to Σ_N, given by $\psi_q(t) = (X_{t,q},\sigma_{t,q})$. For large enough N, the curve Σ_N is not rational, and hence the map ψ_q must be constant. Thus, $X_{t,q} = X$ for all t, q. So we can think of $\sigma_{t,q}$ as an element of $\mathbb{C}^*$.

If q is a root of unity, so must be $\sigma_{t,q}$. Therefore, for q being a root of unity, $\sigma_{t,q}$ is independent of t. Thus, for such q, $\sigma_{t,q} = \sigma_{1,q} = q^{\ell}$. Since roots of unity are Zariski dense, this equality holds for all q. We are done. $\square$

This allows one to give the following descriptions of the associated graded algebras of $Z(t,q)$ for Zariski generic t.

Corollary 6.11.

(i) *For D_4, $\mathrm{gr}_{F_{111}}(Z(t,q))$ is isomorphic to $B(X_3,1,\mathcal{L})$, with grading multiplied by N. So the Poincaré series of $Z(t,q)$ under this filtration is*

$$1 + \frac{3z^N}{(1-z^N)^2}.$$

(ii) *For E_6, $\mathrm{gr}_{F_{111}}(Z(t,q))$ is isomorphic to $B(X,1,\mathcal{L}^{\otimes 3})$, with grading multiplied by N. So the Poincaré series of $Z(t,q)$ under this filtration is*

$$1+\frac{3z^N}{(1-z^N)^2}.$$

(iii) *For E_7, $\mathrm{gr}_{F_{112}}(Z(t,q))$ is isomorphic to $B(X,1,\mathcal{L}^{\otimes 2})$, with grading multiplied by N. So the Poincaré series of $Z(t,q)$ under this filtration is*

$$1+\frac{2z^N}{(1-z^N)^2}.$$

(iv) *For E_8, $\mathrm{gr}_{F_{123}}(Z(t,q))$ is isomorphic to $B(X,1,\mathcal{L})$, with grading multiplied by N. So the Poincaré series of $Z(t,q)$ under this filtration is*

$$1+\frac{z^N}{(1-z^N)^2}.$$

Proof. Statements (i) and (iv) follow from Theorems 6.5 and 6.7. Statements (ii) and (iii) follow from Proposition 6.10 and Corollary 6.9. □

6.5. *Quantization of del Pezzo surfaces*

Recall that a del Pezzo surface is a smooth projective surface with ample anticanonical bundle K^{-1}. Apart from $\mathbb{P}^1\times\mathbb{P}^1$ such surfaces are obtained from $\mathbb{P}^2$ by blowing up n sufficiently generic points ($n\leqslant 8$). The degree of a del Pezzo surface is the dimension of the projective space of $\Gamma(K^{-1})$. For example, a del Pezzo surface of degree 3 is a cubic surface in $\mathbb{P}^3$. The degree of a projective plane with n generic points blown up is $9-n$.

Let q be a root of unity, such that the order of q^ℓ is N. Let t be generic in the Zariski sense.

Theorem 6.12.

(i) *In the D_4 case, $Z(t,q)$ is generated by degree N elements x,y,z with defining relation*

$$xyz+x^2+y^2+z^2+a_2x+a_3y+a_4z+a_1=0,$$

where a_m are functions of t_{kj}.

(ii) *In the E_6 case, $Z(t,q)$ is generated by degree N elements x,y,z with defining relation*

$$xyz+x^3+y^3+z^2+a_1x^2+a_2y^2+a_3x+a_4y+a_5z+a_6=0,$$

where a_m are functions of t_{kj}.

(iii) *In the E_7 case, $Z(t,q)$ is generated by degree N elements x,y and degree $2N$ element z with defining relation*

$$xyz+x^4+y^3+z^2+a_1x^3+a_2x^2+a_3y^2+a_4x+a_5y+a_6z+a_7=0,$$

where a_m are functions of t_{kj}.

(iv) *In the E_8 case, $Z(t,q)$ is generated by degree N element x, degree $2N$ element y, and degree $3N$ element z with defining relation*

$$xyz + x^5 + y^3 + z^2 + a_1x^4 + a_2y^2 + a_3x^3 + a_4z + a_5x^2 + a_6y + a_7x + a_8 = 0,$$

where a_m are functions of t_{kj}.

Here the degree in cases (i), (ii), (iii), (iv) *is computed with respect to the filtrations F_{111}, F_{111}, F_{112}, F_{123}, respectively.*

Proof. The proof is based on Corollary 6.11.

Namely, in the D_4 case, $Z(t,q)$ is generated by three elements x, y, z of degree N (since this is true already for the graded algebra, by Theorem 6.5). From looking at the Poincaré series it is clear that these elements satisfy a cubic defining relation, whose homogeneous part (as we know from studying the graded algebras) is $xyz = 0$ (triangle at infinity). Thus, this relation has the form

$$xyz + Q(x,y,z) = 0,$$

where Q is an inhomogeneous quadratic form. Using affine transformations (i.e., shifts of x, y, z by constants), this equation can be brought to the form

$$xyz + A(x) + B(y) + C(z) = 0$$

where A, B, C are (at most) quadratic polynomials. For generic t these polynomials have nonvanishing quadratic terms, since this is so for $t = 1$. Thus the coefficients in these terms can be normalized to be 1, which gives the result.

In the E_6 case, $Z(t,q)$ is also generated by three elements x, y, z of degree N (since this is true already for the graded algebra). From looking at the Poincaré series it is clear that these elements satisfy a cubic defining relation. However, now the curve at infinity is a nodal $\mathbb{P}^1$ rather than triangle, so the homogeneous part of the cubic relation can be brought by linear transformations to the form $xyz + x^3 + y^3 = 0$. Thus, the cubic relation has the form

$$xyz + x^3 + y^3 + Q(x,y,z) = 0,$$

where Q is an inhomogeneous quadratic form. Using affine transformations, this equation can be brought to the form

$$xyz + A(x) + B(y) + C(z) = 0$$

where A, B are cubic polynomials and C is at most quadratic. For generic t, the polynomial C has nonvanishing quadratic coefficient, since it is so for $t = 1$. So the leading coefficients of A, B, C can be normalized to be 1.

In the E_7 case, $Z(t,q)$ is generated by three elements x, y, z of degree $N, N, 2N$ (since this is true already for the graded algebra). From looking at the Poincaré series it is clear that these elements satisfy a defining relation in degree 4. Since the curve at infinity is a nodal $\mathbb{P}^1$, the homogeneous part of this relation can be brought by linear transformations to the form $xyz +$

$x^4 + z^2 = 0$. Thus, after linear transformation the inhomogeneous relation can be brought to the form

$$xyz + A(x) + B(y) + C(z) = 0$$

where A is a quartic polynomial, C a quadratic polynomial, and B is at most cubic. For generic t the polynomial B has nonvanishing cubic term, since it is so for $t = 1$. So the leading coefficients of A, B, C can be normalized to be 1.

In the E_8 case, $Z(t,q)$ is generated by three elements x, y, z of degree $N, 2N, 3N$ (since this is true already for the graded algebra). From looking at the Poincaré series it is clear that these elements satisfy a defining relation in degree 6. Since the curve at infinity is a nodal $\mathbb{P}^1$, the homogeneous part of this relation can be brought by linear transformations to the form $xyz + y^3 + z^2 = 0$. Thus, after linear transformation the inhomogeneous relation can be brought to the form

$$xyz + A(x) + B(y) + C(z) = 0$$

where A is at most a quintic polynomial, C a quadratic polynomial, and B is a cubic polynomial. For generic t the polynomial A has nonvanishing quintic term, since it is so for $t = 1$. So the leading coefficients of A, B, C can be normalized to be 1. □

The theorem shows that we can view $S(t,q)$ of types D_4, E_6, E_7, E_8 as sitting inside the weighted projective space with weights $(1,1,1)$, $(1,1,1)$, $(1,1,2)$, $(1,2,3)$, respectively. Denote by $\overline{S(t,q)}$ its closure (= *Proj* of the Rees algebra of $Z(t,q)$ for the filtrations $F_{111}, F_{111}, F_{112}, F_{123}$, respectively), and by $C(t,q)$ the curve at infinity (= *Proj* of the graded algebra of $Z(t,q)$ with respect to these filtrations).

Corollary 6.13. *In the cases* (i)–(iv)*, the curve at infinity $C(t,q)$ consists of smooth points of the compact surface $\overline{S(t,q)}$. Therefore, the surface $\overline{S(t,q)}$ is smooth for generic t.*

Proof. This follows from the nonvanishing of the highest coefficients of A, B, C. □

Thus we get the following result on the structure of $S(t,q)$ for generic t.

Theorem 6.14. *$S(t,q)$ is isomorphic to*:

(i) [25] *in the D_4 case, a del Pezzo surface of degree* 3 *with a triangle removed*;
(ii) *in the E_6 case, a del Pezzo surface of degree* 3 *with a nodal $\mathbb{P}^1$ removed*;
(iii) *in the E_7 case, a del Pezzo surface of degree* 2 *with a nodal $\mathbb{P}^1$ removed*;
(iv) *in the E_8 case, a del Pezzo surface of degree* 1 *with a nodal $\mathbb{P}^1$ removed.*

Proof. The theorem follows from the well-known fact in algebraic geometry (see e.g. [10, pp. 60–71]) that equations of the form $xyz + A(x) + B(y) + C(z)$ in weighted projective space give realizations of del Pezzo surfaces. Namely, such realizations of del Pezzo surfaces S are obtained by considering the *Proj* of the ring $\bigoplus_{k\geqslant 0} H^0(S, (K^*)^{\otimes k})$, where K^* is the anticanonical bundle on S. □

Abusing terminology, we will use the name "del Pezzo surface" for all, and not only smooth, members of the family $S(t,q)$.

Note that any smooth surface S in $\mathbb{C}^3$ has a natural symplectic structure up to scaling. If the equation of the surface is $F(x,y,z)=0$ then the symplectic form is $\omega=\frac{dx\wedge dy\wedge dz}{dF}$. If S is singular, this symplectic form becomes singular at singular points of S, but still defines a regular Poisson structure on S. It is defined by the formulas

$$\{x,y\}=\frac{\partial F}{\partial z},\qquad \{y,z\}=\frac{\partial F}{\partial x},\qquad \{z,x\}=\frac{\partial F}{\partial y}.$$

Theorem 6.14 and Corollary 6.9 implies

Theorem 6.15. *Let $q=e^{\hbar}$, where $\hbar$ is a formal parameter. Then the algebra $eH(t,q)e$ is a deformation quantization of the del Pezzo surface $S(t,1)$, with its natural Poisson structure* (*with an appropriate normalization*).

Proof. It is sufficient to prove the result for generic t. In this case the surface $S(t,1)$ is smooth and the natural Poisson structure is symplectic, given by the formulas above.

Now, the algebra $eH(t,q)e$ is a quantization of some (maybe different) Poisson structure $\{\,,\}'$ on $S(t,1)$. This Poisson structure must have the form $\{\,,\}'=f\{\,,\}$, where f is a polynomial function on $S(t,1)$. But $\{\,,\}'$ must preserve the filtration on $eH(t,1)e$, which implies that f has to be constant. The theorem is proved. □

Proposition 6.16. *For generic t, $H^1(S(t,1),\mathbb{C})=0$, and $H^2(S(t,1),\mathbb{C})=\mathbb{C}^{r+1}$, where r is the rank of the Dynkin diagram D* (*i.e.,* 4, 6, 7, 8).

Proof. The identity $H^1(S(t,1),\mathbb{C})=0$ obviously holds for $t=1$, since in this case $S(t,1)$ is $T/\mathbb{Z}_\ell$. Thus the only way $H^1(S(t,1),\mathbb{C})$ could be nonzero for generic $t\neq 1$ is if there were vanishing 1-cycles as $t\to 1$ near the singular points of $S(1,1)$ (this follows from the fact that the deformation $S(t,1)$ near $t=1$ is topologically trivial everywhere including infinity, except the singular points of $T/\mathbb{Z}_\ell$). But it is clear that at the singular points, there could only be vanishing 2-cycles and not 1-cycles. Thus, $H^1(S(t,1),\mathbb{C})=0$ for generic t.

Now, it is easy to see that the Euler characteristic of $S(t,1)$ is $r+2$ for generic t. This implies the result, since $S(t,1)$ is affine and cannot have cohomology above degree 2. □

Remark 1. This proposition shows that the symplectic structure on $S(t,1)$ for generic t is unique up to scaling, since by Proposition 6.16, $H^1(S(t,1),\mathbb{C})=0$ and hence $S(t,1)$ does not have nonconstant nowhere vanishing functions (the logarithmic differential of such a function would represent a nontrivial class in H^1).

Remark 2. Since our algebras are equipped with filtrations, we also get quantum surfaces in the sense of noncommutative algebraic geometry, which are quantum deformations of commutative compact surfaces. Namely, the homogeneous coordinate rings of these quantum surfaces are the Rees algebras of $eH(t,q)e$ equipped with filtrations F_{ijk}.

Let us specify which commutative surfaces are quantized in this way. Obviously, these commutative surfaces are the *Proj*'s of the Rees algebras of $eH(t,1)e=Z(t,1)$. So let us describe (omitting the proofs) what surfaces we get (for generic t). The cases of F_{111} for D_4, F_{111} for E_6,

F_{112} for E_7, and F_{123} for E_8 are covered by Theorem 6.14. The remaining cases are F_{112} for D_4, F_{112} and F_{123} for E_6 and F_{123} for E_7.

- F_{112} for D_4: $\overline{S(t,1)}$ is a singular del Pezzo surface of degree 2, the divisor at infinity consists of two rational curves intersecting at two points, and both intersection points carry A_1 singularities of the surface.
- F_{112} for E_6: $\overline{S(t,1)}$ is a smooth del Pezzo surface of degree 2 (i.e., of type E_7), the divisor at infinity consists of two rational curves intersecting at two points.
- F_{123} for E_6: $\overline{S(t,1)}$ is a singular del Pezzo surface of degree 1 (i.e., of type E_8), the divisor at infinity is a rational curve intersecting itself at a point. The surface has a singularity of type A_2 at this point.
- F_{123} for E_7: $\overline{S(t,1)}$ is a singular del Pezzo surface of degree 1 (i.e., of type E_8), the divisor at infinity is a rational curve intersecting itself at a point. The surface has a singularity of type A_1 at this point.

Finally, we derive a corollary about the Hochschild cohomology of the generalized Cherednik algebras.

Proposition 6.17. *Let $q = e^{\hbar}$, where $\hbar$ is a formal parameter. Then the Betti numbers of the Hochschild cohomology of the algebra $H(t,q)[\hbar^{-1}]$ for generic t are $b_0 = 1$, $b_1 = 0$, $b_2 = r+1$, $b_i = 0$ for $i > 2$.*

Proof. The proof is analogous to the proof in the case of D_4, given in [25]. Namely, the result follows from Proposition 6.16, Theorem 6.15, and the theorem on cohomology of quantizations of symplectic manifolds (see e.g. [11] and references therein). □

6.6. Nondegeneracy of the map $t \to S(t,q)$

Del Pezzo surfaces given by the equations $xyz + A(x) + B(y) + C(z)$ form a moduli space $\mathcal{M}$, coordinates on which are the coefficients of A, B, C (stipulating that the highest coefficients are 1 and there is only one independent coefficient among the constant terms of A, B, C). Thus the dimension of $\mathcal{M}$ is equal to the rank r of the corresponding Dynkin diagram D (i.e., 4, 6, 7, 8).

Let q be a root of unity, and $\zeta_q : \mathcal{U} \to \mathcal{M}$ be the map attaching $S(t,q)$ to t (here $\mathcal{U}$ is some open set in $\mathbb{T}$).

Theorem 6.18. *The map ζ_q is dominant.*

Proof. Consider a formal path $t = t(s)$ such that $t(0) = 1$ and $S(t(s),q) = S(1,q)$. Thus $H(t(s),q)$ is a formal deformation of $H(1,q)$ over a fixed center $Z(1,q)$. We will show that this deformation is trivial.

Consider the first nontrivial order of the s-expansion. In this order our deformation defines a Hochschild 2-cocycle γ of $H(1,q)$ as an algebra over $Z(1,q)$. It suffices to show that this cocycle is trivial; then we can make a gauge transformation to insure that the lowest nontrivial order in s is becomes one step higher, and obtain the result by induction.

The group $H^2_{Z(1,q)}(H(1,q))$ is a finitely generated module over $Z(1,q)$, i.e. a coherent sheaf on $S(1,q)$. The surface $S(1,q)$ is isomorphic to $T/\mathbb{Z}_\ell$, and hence has m isolated singular points ($m = 4, 3, 3, 3$). Near any other point, $H(1,q)$ (and $H(t,q)$ for t close to 1) is an Azumaya algebra. This shows that γ vanishes outside of the singular points. Hence in showing that $\gamma = 0$, we may replace $H(1,q)$ and $Z(1,q)$ by their completions near the singular points.

Now, let p be a singular point of $S(1,q)$, and $\mathbb{Z}_k$ be the stabilizer of this point in $\mathbb{Z}_\ell$. It is clear that the completion of $H(1,q)$ near p is Morita equivalent to $A = \mathbb{C}[\mathbb{Z}_k] \ltimes \mathbb{C}[[x,y]]$, where the generator of $\mathbb{Z}_k$ multiplies x by the primitive kth root of unity, and y by the inverse of this root. The cocycle $\gamma|_A$ comes from a formal deformation $A(s)$ of A that keeps the center fixed.

Now, it is easy to compute using Koszul resolutions that

$$H^2\big(\mathbb{C}[\mathbb{Z}_k] \ltimes \mathbb{C}[[x,y]]\big) = H^2\big(\mathbb{C}[[x,y]]\big)^{\mathbb{Z}_k} \oplus \mathbb{C}^{k-1}.$$

If γ had a nontrivial projection to the first summand, then the center would collapse under deformation. Thus, γ belongs to the second summand. This means that our deformation falls into the family of algebras from [8]. For this family, it is known that the deformation of the center is a versal deformation of the singularity A_{k-1}. This means that if the center is fixed then the deformation is trivial. Thus we see that $A(s)$ is a trivial deformation, and hence $\gamma|_A = 0$. Thus $\gamma = 0$.

Now Lemma 4.7 implies that the path $t = t(s)$ must be trivial: $t(s) = 1$. This implies the statement. □

6.7. A linear algebra application

The results of Section 6 imply the following result from linear algebra, which appears to be new. Fix a diagram D of type D_4, E_6, E_7, E_8.

Theorem 6.19. *Let q be a root of unity such that q^ℓ has order N, and t be generic* (*more specifically, $t \in U$ with $u_{kj} \neq u_{kj'}$ for $j \neq j'$*). *Let $S(t,q)$ be the space of conjugacy classes of collections of diagonalizable matrices $T_1, \ldots, T_m$ of size ℓN, such that eigenvalues of T_k are u_{kj}, $j = 1, \ldots, d_k$* (*with some multiplicities*), *and $T_1 \cdots T_m = q$. Then the multiplicities of the eigenvalues are all the same* (*i.e., equal $\ell N/d_k$*), *and $S(t,q)$ is an affine del Pezzo surface described in Theorem* 6.14. *Moreover, a generic surface of this kind is obtained in this way.*

7. The Riemann–Hilbert map

7.1. Preprojective algebras

Let Q be a quiver, and let $E(Q)$ be the set of edges of Q. The path algebra of Q is spanned by paths in Q with multiplication given by concatenation of paths. In particular, it contains the idempotents p_i corresponding to the paths of length 0 at the vertices i of Q.

Let $\mathbb{D}(Q)$ be the double of Q, obtained by adding, for any edge $h \in E(Q)$, a new edge h^* in the opposite direction. Setting $h^{**} = h$, we get an involution of the set of edges $E(\mathbb{D}(Q))$.

The Gelfand–Ponomarev deformed preprojective algebra Π_μ is the quotient of the path algebra of $\mathbb{D}(Q)$ by the relation

$$\sum_{h \in E(Q)} \big[h, h^*\big] = \sum_i \mu_i p_i,$$

where μ_i are complex numbers.

Let Γ be a finite subgroup of $SL(2,\mathbb{C})$. To such a group, Crawley-Boevey and Holland [8] assigned an algebra $Q_\Gamma(c)$ generated by Γ and its tautological 2-dimensional representation V with defining relations

$$gvg^{-1} = v^g, \quad g \in \Gamma,\ v \in V,$$

and

$$[v,w] = (v,w)\sum_g c_g g,$$

where c is a class function on Γ and (v,w) is the symplectic inner product. Let e_i be primitive idempotents in $\mathbb{C}\Gamma$ attached to irreducible representations V_i of Γ (they are unique up to conjugation). We denote by $Q'_\Gamma(c)$ the algebra $\bigoplus_{i,j} e_i Q_\Gamma(c) e_j$.

Let Q be an affine quiver. McKay's correspondence assigns to Q a finite subgroup Γ of $SL(2)$, whose irreducible representations are labeled by vertices of Q. Let χ_i be the character of the ith irreducible representation.

Proposition 7.1. *(See [8].) The algebra Π_μ is isomorphic to $Q'_\Gamma(c)$ if $\sum c_g \chi_i(g) = \mu_i$.*

Now assume that Q is starlike (i.e., D_4, E_6, E_7, E_8). In this case, let i_0 be the nodal vertex, and $p = p_{i_0}$ be the correspondent idempotent of Π_μ. Let $K(\mu) = p\Pi_\mu p$.

Proposition 7.2. *(See [6,22,23].) The algebra $K(\mu)$ is generated by elements U_k, $k = 1,\dots,m$, corresponding to the legs of Q, subject to defining relations*

$$U_k(U_k - \mu_{i_1(k)})(U_k - \mu_{i_1(k)} - \cdots - \mu_{i_{d_k-1}(k)}) = 0,$$

where $i_1(k),\dots,i_{d_k-1}(k)$ are the vertices of the kth leg of Q enumerated from the nodal vertex, and

$$\sum_{k=1}^m U_k = -\mu_{i_0}.$$

Proof. The elements U_k are just the elements $h_k^* h_k$, where h_k are the edges of $\mathbb{D}(Q)$ starting at i_0 and going along the kth leg. They obviously generate $K(\mu)$. It is easy to compute that U_k satisfy the relations above, and it is not hard to check that these relations are defining. $\square$

Thus, the algebra $K(\mu)$ is an additive analog of the algebra $H(t,q)$.

Remark. A recent paper [9] introduces a multiplicative analog of the preprojective algebra—the multiplicative preprojective algebra Π_μ^{mult} of a quiver Q. Like the usual preprojective algebra, this algebra has idempotents p_i attached to the vertices i of the quiver. It can be shown (see Appendix A.1 below) that the algebra $H(t,q)$ is isomorphic to $p\Pi_\lambda^{\mathrm{mult}} p$ (for appropriate λ).

7.2. The Riemann–Hilbert map

Let $\mathbf{K}$ be the algebra $K(\mu)$ where μ are formal parameters (i.e. it is an algebra over $\mathbb{C}[[\mu]]$). Let $\mathbf{H}$ be the algebra $H(t,q)$ with $q = e^{\hbar}$ and $u_{kj} = e^{\beta_{kj}}$ where β_{kj} are formal parameters. Note that $\mathbf{H}$ is different from the algebras considered in Sections 2–6, since now we take completion near the point $u = 1$ (unipotent case) rather than $t = 1$ (infinite group case).

Representations of the algebra $\mathbf{K}$ are solutions of the additive Deligne–Simpson problem, while representations of $\mathbf{H}$ are solutions of the multiplicative one. Thus we have a Riemann–Hilbert map between completions of these algebras, defined as follows.

Let $z_1, \ldots, z_m$ be distinct points on $\mathbb{CP}_1$. Consider the flat connection ∇ on the trivial bundle over $X = \mathbb{P}_1 \setminus \{z_1, \ldots, z_m\}$ with fiber $\mathbf{K}$ which has first order poles with residues $U_k + \mu_{i_0}/m$ at z_k. Let $z_0 \neq z_k$ for any k and γ_k be the standard generators of $\pi_1(X, z_0)$ going around z_k such that $\prod_k \gamma_k = 1$. Let $\widetilde{\mathbf{K}}$ be the completion of $\mathbf{K}$ with respect to the ideal defined by $U_1, \ldots, U_m$, and $T_k' \in \widetilde{\mathbf{K}}$ be the monodromies of the connection ∇ around γ_k. Let $\bar{T}_k = e^{2\pi i \lambda_k} T_k'$, where

$$\lambda_k = -\mu_{i_0}/m - \sum_{j=1}^{d_k-1} \frac{j}{d_k} \mu_{i_{d_k-j}(k)}.$$

Denote by $e^{\bar{\beta}_{kj}}$ the roots of the polynomial equation satisfied by $\bar{T}_k$, and by $e^{\bar{\hbar}}$ the product of T_k. These are exponentials of linear functions of μ.

Let $\widetilde{\mathbf{H}}$ be the completion of $\mathbf{H}$ with respect to the ideal generated by the elements $T_k - 1$.

Proposition 7.3. *The map $T_k \to \bar{T}_k$, $\beta_{kj} \to \bar{\beta}_{kj}$, $\hbar \to \bar{\hbar}$ is an embedding $\phi_0 : \mathbf{H} \to \widetilde{\mathbf{K}}$ which extends by continuity to an isomorphism $\phi : \widetilde{\mathbf{H}} \to \widetilde{\mathbf{K}}$.*

Proof. The fact that the given formulas define a homomorphism of algebras is obtained by an easy direct computation. The fact that ϕ_0 is injective may be checked for $\hbar = 0$ (i.e., modulo $\hbar$), since the target algebra $\widetilde{\mathbf{K}}$ is flat over $\mathbb{C}[[\hbar]]$. If $\hbar = 0$, then the injectivity of ϕ_0 follows from the fact that the spectrum $S(t, 1)$ of the center of $H(t, 1)$ is irreducible (Proposition 6.6), and therefore the center embeds into its completion at every point. The fact that ϕ_0 extends to an isomorphism of completions is now straightforward. □

Remark. The isomorphism ϕ is independent on the choice of the point z_0 up to inner automorphisms. Thus the corresponding isomorphism between the centers $\phi_c : Z(\widetilde{\mathbf{H}}) \to Z(\widetilde{\mathbf{K}})$ is independent on the choice of z_0. As a function of $z_1, \ldots, z_m$, ϕ_c is projectively invariant. Therefore, it is completely canonical in the cases E_6, E_7, E_8 and depends on one parameter (cross ratio) for D_4.

For numerical values of μ and t, q, the map ϕ is not well defined, since we cannot compute monodromies of connections with infinite dimensional fiber. However, we have the following proposition.

Proposition 7.4. *For parameters related as above, we have a "pullback" functor between the categories of finite dimensional representations $\phi^* : \mathrm{Rep}_f(K(\mu)) \to \mathrm{Rep}_f H(t,q)$.*

This functor is obviously far from being an equivalence, since the 2-parameter family of finite dimensional representations at q being a root of unity (with $q^\ell \neq 1$) does not belong to its image. However, the restriction of ϕ^* to the situation when $c_1 = 0$ and $q = 1$ produces a canonical holomorphic (but not algebraic) mapping $\phi^* : S_0(\mu) \to S(t, 1)$, where $S_0(\mu)$ is the versal deformation of $\mathbb{C}^2/\Gamma$ (= the spectrum of the center of $K(\mu)$); this map is an isomorphism near the origin. More precisely, as we mentioned above, in the case of D_4 the map ϕ^* depends on the cross ratio s of the points $z_1, \dots, z_4$, while in types E_6, E_7, E_8 it is independent of any choices and completely canonical.

Remark 1. In the case of D_4, consider the inverse map $(\phi^*)^{-1}$, and the point $g(s) = (\phi^*)^{-1}(y, t, s) \in S_0(\mu)$, where $y \in S(t, 1)$ (here μ is a linear transformation of $\log t$ as explained above). Clearly, $g(s)$, regarded as a function of the cross-ratio s, represents an isomonodromic deformation of 2-dimensional local systems on $\mathbb{C}P^1$ wit 4 regular singular points. Thus, in appropriate coordinates it satisfies the differential equation Painlevé VI (with general parameters), and generic solutions of Painlevé VI are obtained in this way (see [27]). The known symmetry of Painlevé VI under the affine Weyl group $\tilde{W}(D_4)$ of type D_4 [27] is combined from the usual $W(D_4)$ symmetry on the deformation of $\mathbb{C}^2/\Gamma$ and the lattice of rank 4 coming from the map $t \to \log t$.

Remark 2. As above, let Γ be the group attached to the diagram $\widehat{D}$ via McKay's correspondence. Let Γ_+ be the quotient of Γ by ± 1. Then Γ_+ has generators a, b, c and the following defining relations:

D_4: $a^2 = b^2 = c^2 = 1$, $abc = 1$.
E_6: $a^3 = b^3 = c^2 = 1$, $abc = 1$.
E_7: $a^2 = b^4 = c^3 = 1$, $abc = 1$.
E_8: $a^2 = b^3 = c^5 = 1$, $abc = 1$.

For D_4 it is convenient to add a new generator d and write the relations as $a^2 = b^2 = c^2 = d^1 = 1$, $abcd = 1$.

Then in all cases the relations are exactly the same as for the group G, except that the order of the last generator equals its order in G minus 1.

This shows that any irreducible representation V of Γ can be viewed as a representation of the algebra $H(t, q)$ for appropriate t and q. Indeed, the above implies that the action of Γ in V is generated by operators a, b, c satisfying the above relations up to sign. But then (rescaled versions of) a, b, c define an action in V of a 1-parameter family of algebras $H(t, q)$. Indeed, the relations for $H(t, q)$ are obtained if for $E_{6,7,8}$ one replaces the relation $c^p = 1$ $(p = 2, 3, 5)$ by the relation $(c^p - 1)(c - \lambda) = 0$, and for D_4 one replaces the relation $d = 1$ with the relation $(d - 1)(d - \lambda) = 0$.

8. Proofs of Theorems 6.1 and 6.5

8.1. *Efficient presentations*

The main difficulty in doing computations on noncommutative algebras given a presentation is that there are no general algorithms for such computations; indeed, most natural problems about finitely presented algebras are known to be undecidable. In the commutative case, the primary tool is the notion of Gröbner basis; while this notion has been extended to the noncommutative case [17], noncommutative Gröbner bases need not be finite in general, and their finiteness depends highly on the choice of presentation. Thus the first task in computing in our algebras is to find "efficient" presentations, i.e., presentations admitting finite Gröbner bases.

For computational purposes, it turns out to be convenient to first rescale the generators slightly. We thus obtain the following presentations of our algebras. In each case, we observe that a can be expressed easily as a polynomial in $a^{-1} = bc/q$ (bcd/q for D_4), so it suffices to take b and c (and d) as generators. This observation, plus some mild rescaling of the generators and parameters, gives rise to the following (not yet efficient) versions of the algebras.

For D_4:

$$\langle b, c, d \mid c^2 - g_1 c + 1, b^2 - f_1 b + 1, d^2 - h_1 d + 1, (bcd)^2 - e_1(bcd) + Q\rangle,$$

where $Q = q^2$. (Note that b and c have been rescaled to change the last coefficients of their minimal polynomials.)

For E_6:

$$\langle b, c \mid c^3 - g_1 c^2 + g_2 c - 1, b^3 - f_1 b^2 + f_2 b - 1, (bc)^3 - e_2(bc)^2 + e_1(bc) - Q\rangle,$$

where $Q = q^3$.

For E_7:

$$\langle b, c \mid c^4 - g_1 c^3 + g_2 c^2 - g_3 c + 1, b^4 - f_1 b^3 + f_2 b^2 - f_3 b + Q, (bc)^2 - e_1(bc) + Q\rangle,$$

where $Q = q^4$.

For E_8:

$$\langle b, c \mid c^6 - g_1 c^5 + g_2 c^4 - g_3 c^3 + g_4 c^2 - g_5 c + 1, b^3 - f_1 b^2 + f_2 b + Q, (bc)^2 - e_1(bc) + Q\rangle,$$

where $Q = q^6$. (Again, b and c have been rescaled for E_6 and E_8.)

For E_6, the above filtration already admits a finite Gröbner basis, with respect to the "shortlex" term order. We first note that

$$b^3 = f_1 b^2 - f_2 b + 1,$$
$$c^3 = g_1 c^2 - g_2 c + 1.$$

This allows us to compute inverses in the algebra; multiplying the third relation by $(bc)^{-1}$ gives

$$bcbc = e_2 bc - e_1 + Q\big(c^2 - g_1 c + g_2\big)\big(b^2 - f_1 b + f_2\big);$$

conjugating by b gives

$$cbcb = e_2 cb - e_1 + Q\big(b^2 - f_1 b + f_2\big)\big(c^2 - g_1 c + g_2\big),$$

allowing us to expand b^2c^2 in smaller monomials. The noncommutative analogue of the Buchberger algorithm shows that these four relations generate a Gröbner basis, with leading terms b^3, c^3, $bcbc$, b^2c^2. In particular, any element in the algebra can be expressed as a linear combination of words for which none of these four words is a subword. Since the elements Gröbner basis are compatible with the filtration (the degree of the leading term is at least as high as the degree of the remaining terms), and the coefficients are in $\mathbb{C}[\mathbb{T}]$, we conclude that Theorem 6.1 holds for the 123 filtration of the E_6 algebra. A similar short computation gives a Gröbner basis for the above presentation of E_7 compatible with the 123 filtration, proving Theorem 6.1 for that case as well. (In that case, the leading terms of the Gröbner basis are c^4, b^3, bcb, $bccbc$, $bbccc$.)

For E_8, it appears that the above presentation does not admit a finite Gröbner basis for any choice of term order compatible with the filtration. We must therefore use a different generating set. For $0 \leqslant i \leqslant 5$, we let $d_i = bc^{-i}$, and take as generators c together with d_0 through d_5. We then obtain (via a much longer computation, although still quite short on a computer) a shortlex Gröbner basis, with respect to the variable ordering $c, d_1, d_3, d_5, d_0, d_2, d_4$, having leading terms as follows:

(1) c^6.
(2) $d_i c$, $0 \leqslant i \leqslant 5$.
(3) $d_i d_j$ with $2 \leqslant i \leqslant 4$, $0 \leqslant j \leqslant 5$.
(4) $d_i d_j$ with $i \in \{1, 5\}$, $j \in \{0, 2, 4\}$.
(5) $d_i d_i d_j$ with $i \in \{1, 5\}$, $j \in \{1, 3, 5\}$.

Similarly for D_4, we take as generators $z_1 = dc^{-1}$, $z_2 = c^{-1}b$, which satisfy relations

$$\begin{aligned} c^2 - g_1 c + 1 &= 0, \\ z_1 c z_1' &= c^{-1}, \\ z_2' c z_2 &= c^{-1}, \\ Q z_2' z_1' + z_1 c^2 z_2 &= e_1 c^{-1}, \\ c z_2 c z_1 c + Q z_1' c z_2' &= e_1, \end{aligned}$$

where

$$\begin{aligned} z_1' &:= h_1 c^{-1} - z_1, \\ z_2' &:= f_1 c^{-1} - z_2; \end{aligned}$$

again, this is a Gröbner basis for a suitable term order, compatible with the 112 filtration.

8.2. *Proof of Theorem 6.1*

The efficient presentations given above are each compatible with the corresponding $F^\bullet_{123}$ filtration (except for D_4, for which the filtration is $F^\bullet_{112}$); this has the immediate consequence that Theorem 6.1 holds for those filtrations, since we have given a presentation with relations integral over $\mathbb{C}[\mathbb{T}]$ such that the corresponding Poincaré series uniformly agrees with the infinite group case.

It remains to consider the filtrations $F^\bullet_{111}$ for D_4, $F^\bullet_{111}$ and $F^\bullet_{112}$ for E_6 and $F^\bullet_{112}$ for E_7. In these cases, we have been unable to find presentations with compatible Gröbner bases, and must therefore use more ad hoc arguments.

For $F^{\bullet}_{112}(E_6)$, we may take generators c, $d = bc^{-1}$, $e = c^{-1}b^{-1}c^{-1}$, with $\deg(c) = 0$, $\deg(d) = \deg(e) = 1$. Using the efficient presentation for E_6, we can easily solve for the relations these satisfy in degree 2; using these relations (which again have integral coefficients), we find that the monomials

$$c^i,\ c^i e^j c^m,\ c^i d^l c^m,\ c^i e^j c^k d^l c^m$$

with $i, m \in \{0, 1, 2\}$, $k \in \{0, 2\}$, $j, l \geqslant 1$, span the algebra, and thus must form a basis (since they do so in the infinite group case, and thus generically). The corresponding case of Theorem 6.1 follows.

Similarly, for $F^{\bullet}_{111}(D_4)$, we take generators c, $w_1 = c^{-1}d^{-1}$, $w_2 = db$, $w_3 = b^{-1}c^{-1}$, and find that the monomials

$$c^i,\ c^i w_1^j c^l,\ c^i w_2^j c^l,\ c^i w_3^j c^l,\ c^i w_1^j c w_2^k c^l,\ c^i w_1^j c w_3^k c^l,\ c^i w_2^j c w_3^k c^l$$

with $i, l \in \{0, 1\}$, $j, k \geqslant 1$, span the algebra, so again must form a basis, giving that case of Theorem 6.1.

For $F^{\bullet}_{111}(E_6)$ and $F^{\bullet}_{112}(E_7)$, the proof is similar; in each case we can order the monomials at degree 2 in such a way as to obtain monic relations with coefficients in $\mathbb{C}[\mathbb{T}]$. Since degree 2 relations suffice for the infinite group cases, they suffice generically, and thus suffice in general.

8.3. Proof of Theorem 6.3

It is straightforward to use our efficient presentations to obtain presentations of the associated graded algebra $H_0^{123}(t, q)$: simply remove low-order terms from the relations. These can be simplified considerably by removing redundant relations; we thus obtain the following presentations. Note that for E_6 and E_7 we follow the lead of E_8 in replacing the generator b by a generator $d = bc^{-1}$ (i.e., by an appropriate translation in the infinite group).

For E_6:

$$\langle c, d \mid c^3 - g_1c^2 + g_2c - 1, dcd - q^{-\ell}cdc^2dc, dcdcd\rangle.$$

For E_7:

$$\langle c, d \mid c^4 - g_1c^3 + g_2c^2 - g_3c + 1, dc^2d, dcd + q^{-\ell}cdc^3dc\rangle.$$

For E_8:

$$\langle c, d \mid c^6 - g_1c^5 + g_2c^4 - g_3c^3 + g_4c^2 - g_5c + 1, dc^2d, dc^3d, dc^4d, dcd + q^{-\ell}cdc^5dc\rangle.$$

Since $dcdcd = 0$ in the E_7 and E_8 algebras, we thus obtain the claimed uniform presentation for the algebras $H_0^{123}(t, q)$. The theorem is proved.

Remark. Similar considerations apply for the case $F^{\bullet}_{112}$ for E_6. The associated graded algebra of $H_0^{112}(t, q)$ in this case has presentation

$$\begin{aligned}&\langle c, d \mid c^3 - g_1c^2 + g_2c - 1 = 0, dccd = dcd = dcce = ecd = ece = ecce = 0,\\ &\quad de = Qed + g_2dce, dce = Q^{-1}ceccdc\rangle.\end{aligned}$$

9. The surface for $q = 1$

We can use our efficient presentations for the 123 filtrations of E_6, E_7, E_8 to give explicit equations for the surfaces $S(t, 1)$ with coordinate ring $Z(t, 1)$. Since the associated graded algebra of $Z(t, 1)$ is just the center of the associated graded algebra of $H(t, 1)$, we can write down the leading terms of the generators of the center, and then simply solve for the coefficients of the lower degree terms. We can then solve for the resulting equation, and put it into canonical form as in Theorem 6.12. The generators x, y, z that we obtain from this process are unfortunately rather complicated, even in the simplest case E_6; it turns out, however, that the equation itself can be described much more simply. We consider each case in turn.

For E_6, we obtain the equation

$$xyz = x^3 + a_1x^2 + a_3x + y^3 + a_2y^2 + a_4y + z^2 + a_5z + a_6,$$

where a_i are polynomials in the parameters u_{kj}. The simplest way to specify these polynomials is as follows. Each of the triples u_{k1}, u_{k2}, u_{k3} multiplies to 1, and thus specifies a point on the maximal torus $\mathbf{T}(SL_3)$; as the equation is invariant under permutations of each triple, the coefficients a_i are actually (virtual) characters of SL_3^3. In fact, it turns out that these virtual characters factor through the natural map from SL_3^3 to the simply-connected group E_6 (mapping SL_3^3 to a locally isomorphic semisimple subgroup of E_6); equivalently, they are functions on the maximal torus $\mathbf{T}(E_6)$ invariant under the action of the Weyl group. In particular, we can express them in terms of the fundamental characters of E_6, and thus obtain the following equation:

$$\begin{aligned} xyz = {} & x^3 + (\chi_1)x^2 + (\chi_3 - \chi_2)x + y^3 + (\chi_2)y^2 + (\chi_4 - \chi_1)y \\ & + z^2 + (\chi_5 - 6)z + (\chi_6 - 3\chi_5 + 9), \end{aligned}$$

where χ_1 through χ_6 are the fundamental characters. We note that each coefficient has a different fundamental character as its leading term, which is associated to the coefficient as follows. We assign a simple root of E_6 to each monomial with nonconstant coefficient in such a way that each of the three polynomials corresponds to a different leg of the Coxeter diagram, in order of increasing degree. In particular, the constant term corresponds to the central root, the coefficients of degree 1 in x, y, z correspond to the roots adjacent to the center, and so forth. (The labeling we have chosen for the roots can thus be read off from the equation.) Note that although the coefficients above are characters on the simply connected group, the isomorphism class of the surface depends only on the image in the adjoint group; the center of E_6 acts by $(x, y, z) \mapsto (\zeta_3 x, \zeta_3^{-1} y, z)$. Thus the surfaces $S(t, 1)$ are parametrized by the quotient by the Weyl group of the adjoint torus; this agrees with Theorem 8.4 of [20].

As a corollary, we find that the map from $\mathbf{T}(E_6)$ to the canonical equation is Galois with Galois group $W(E_6)$; similarly the Galois group of the normal closure of the map from $\mathbf{T}(SL_3)^3$ is the semidirect product of $W(E_6)$ by an elementary abelian group $(1/3)Q/P$ of order 3^5 (the quotient by the weight lattice P of $1/3$ times the root lattice Q, the latter being the closure of the weight lattice of A_2^3 under $W(E_6)$).

Similarly, for E_7, the parameters u_{kj} naturally specify elements of $\mathbf{T}(SL_2 \times SL_4^2)$, and the coefficients of the equation, virtual characters of $SL_2 \times SL_4^2$, factor through the natural map to the simply-connected group E_7. We obtain the equation

$$\begin{aligned}
xyz = {} & x^4 + \chi_1 x^3 + (\chi_2 - 2\chi_3 + 23)x^2 + (\chi_4 - \chi_1\chi_3 - 5\chi_6 + 29\chi_1)x \\
& + y^3 + (\chi_3 - 25)y^2 + (\chi_5 - \chi_2 - 16\chi_3 + 206)y \\
& + z^2 + (\chi_6 - 6\chi_1)z \\
& + \chi_7 + \chi_3^2 - 3\chi_6\chi_1 + 9\chi_1^2 - 10\chi_5 + 9\chi_2 + 62\chi_3 - 558,
\end{aligned}$$

where the χ_i are fundamental characters of the simply connected group of type E_7, associated to the corresponding terms in the same way as for E_6. The map from $\mathbf{T}(E_7)$ to the canonical equation is Galois with Galois group $W(E_7)$; the map from $\mathbf{T}(SL_2 \times SL_4^2)$ has normal closure with Galois group $W(E_7) \ltimes [(1/4)Q/P]$, where P is the weight lattice of E_7 and Q is the root lattice.

Finally, for E_8, the algebra is parametrized by $\mathbf{T}(SL_2 \times SL_3 \times SL_6)$, but the center is parametrized by $\mathbf{T}(E_8)$; we obtain the equation

$$xyz = x^5 + a_1x^4 + a_3x^3 + a_5x^2 + a_7x + y^3 + a_2y^2 + a_6y + z^2 + a_4z + a_8,$$

where

$$\begin{aligned}
a_1 &= \chi_1 - 248; \\
a_2 &= \chi_2 - 25\chi_1 + 2325; \\
a_3 &= \chi_3 - 3\chi_2 - 170\chi_1 + 23405; \\
a_4 &= \chi_4 - 6\chi_3 - 35\chi_2 + 920\chi_1 - 57505; \\
a_5 &= \chi_5 - 7\chi_4 - 135\chi_3 - 2\chi_2\chi_1 + 580\chi_2 + 23\chi_1^2 + 7652\chi_1 - 955978; \\
a_6 &= \chi_6 - \chi_5 - 28\chi_4 + 170\chi_3 - 16\chi_2\chi_1 + 2006\chi_2 \\
&\quad + 206\chi_1^2 - 51436\chi_1 + 2401694; \\
a_7 &= \chi_7 - 13\chi_6 - 104\chi_5 - 5\chi_4\chi_1 + 1045\chi_4 - \chi_3\chi_2 + 29\chi_3\chi_1 + 4145\chi_3 \\
&\quad + 2\chi_2^2 + 359\chi_2\chi_1 - 45708\chi_2 - 4444\chi_1^2 + 275989\chi_1 + 4532634; \\
a_8 &= \chi_8 - 58\chi_7 - 10\chi_6\chi_1 + 1245\chi_6 + 9\chi_5\chi_1 + 2177\chi_5 - 3\chi_4\chi_3 \\
&\quad - 17\chi_4\chi_2 + 741\chi_4\chi_1 - 65323\chi_4 + 9\chi_3^2 + 161\chi_3\chi_2 - 4405\chi_3\chi_1 \\
&\quad + 189168\chi_3 + \chi_2^2\chi_1 + 192\chi_2^2 + 62\chi_2\chi_1^2 - 38134\chi_2\chi_1 + 2537119\chi_2 \\
&\quad - 558\chi_1^3 + 494091\chi_1^2 - 52476655\chi_1 + 1484285983.
\end{aligned}$$

The Galois groups are $W(E_8)$ from $\mathbf{T}(E_8)$ and $W(E_8) \ltimes [(1/6)Q/P]$ (where $P = Q$ is the weight and root lattice of E_8) from $\mathbf{T}(SL_2 \times SL_3 \times SL_6)$.

Note that a similar expression holds for the D_4 case.

Remark. The fact that the Galois groups of coverings defined above are Weyl groups of E_6, E_7, E_8 also follows (without computation) from the results of Section 7 and the Arnold–Brieskorn theorem, saying that the monodromy group of a simple singularity is the corresponding Weyl group.

In addition to suggesting that the canonical equations should have a group-theoretical interpretation, the above form for the equations has a particularly interesting consequence. One of the most important structures on a del Pezzo surface is the collections of lines on the surface (e.g., the 27 lines on a cubic surface). Normally, these lines are only defined over an extension field; in our case, however, it turns out that every line on the surface is actually rational over $\mathbf{T}(E_k)$, so in particular is rational in the roots of the three minimal polynomials. Each line intersects the curve at infinity; since the smooth part of the curve at infinity has a natural multiplicative group structure (more precisely, a natural divisor class of degree 3, 2, 1 for E_6, E_7, E_8), we obtain an element of this group for each line (up to global inversion and, for E_6, multiplication by a 3rd root of unity (respectively multiplication by -1 for E_7)), each of which is a Laurent monomial in the roots of the minimal polynomials.

For E_6, we obtain 27 lines, corresponding to the ratios (shortest weights of E_6)

$$b_i/a_j,\ c_i/b_j,\ a_i/c_j, \quad 1 \leqslant i, j \leqslant 3,$$

where a_i, b_i, c_i are the roots of the minimal polynomials of a, b, c, respectively.

For E_7, we obtain 56 lines, corresponding to the ratios (shortest weights of E_7)

$$a_i/b_jb_k,\ a_i/c_jc_k, \quad 1 \leqslant i \leqslant 2,\ 1 \leqslant j < k \leqslant 4;$$
$$b_i/c_j,\ c_i/b_j, \quad 1 \leqslant i, j \leqslant 4.$$

Finally, for E_8, we obtain 240 lines, corresponding to the ratios (roots of E_8)

$$a_i/a_j, \quad 1 \leqslant i, j \leqslant 2,\ i \neq j;$$
$$b_i/b_j, \quad 1 \leqslant i, j \leqslant 3,\ i \neq j;$$
$$c_i/c_j, \quad 1 \leqslant i, j \leqslant 6,\ i \neq j;$$
$$a_i/c_jc_kc_l, \quad 1 \leqslant i \leqslant 2,\ 1 \leqslant j < k < l \leqslant 6;$$
$$b_i/c_jc_k,\ c_jc_k/b_i, \quad 1 \leqslant i \leqslant 3,\ 1 \leqslant j < k \leqslant 6;$$
$$a_ib_jc_k,\ 1/a_ib_jc_k, \quad 1 \leqslant i \leqslant 2,\ 1 \leqslant j \leqslant 3,\ 1 \leqslant k \leqslant 6.$$

Furthermore, we find that the surface is singular if and only if one or more of the roots of E_k vanishes. The most singular case corresponds to the identity element of the torus, in which case a, b, and c are all required to be unipotent; we readily verify in each case that the surface has a singularity of type E_k in this case. Note in particular that for E_8 the surface for the unipotent case has equation

$$xyz = x^5 + y^3 + z^2;$$

all of the lower degree terms vanish.

We conjecture that when Q is a root of unity of order k prime to l, the coefficients of the corresponding canonical equation are simply given by composing the above functions with the kth power map on the original torus; we have checked this for E_6, $k=2$, as well as the corresponding statement for D_4, $k=3$.

Any ℓ-dimensional representation of the algebra corresponds to a point on the corresponding surface; this thus gives rise to two natural (open) questions. First, which representations correspond to singular points? (Reducibility appears to be sufficient, but is not necessary.) Second, which representations correspond to points on a line of the surface? For the latter question, it appears from experiments that the family of representations associated to a line can be parametrized in such a way that a, b, c, and all their powers are linear functions in the parameter, but it is unclear why this should be so.

Appendix A.1. Generalized double affine Hecke algebras and multiplicative preprojective algebras, by W. Crawley-Boevey and P. Shaw

Let K be a field. Let $w=(w_1,\dots,w_k)$ be a collection of positive integers, let $\mu\in K^*$, and let $\xi_{ij}\in K^*$ ($1\leqslant i\leqslant k$, $1\leqslant j\leqslant w_i$). Following a question of Etingof, we show that the associative K-algebra $A_{w,\mu,\xi}$ with generators $x_1,\dots,x_k$ and relations

$$x_1x_2\cdots x_k=\mu 1,$$
$$(x_i-\xi_{i1}1)(x_i-\xi_{i2}1)\cdots(x_i-\xi_{i,w_i}1)=0\quad (i=1,\dots,k),$$

is isomorphic to $e\Lambda^q e$ for a suitable multiplicative preprojective algebra Λ^q and a suitable idempotent e. Note that the generalized double affine Hecke algebra $H(t,q)$ is isomorphic to $A_{w,\mu,\xi}$ for suitable (w,μ,ξ). There is a corresponding result for the algebra with relation $x_1+\dots+x_k=\mu 1$ in terms of the deformed preprojective algebra, see [23]. Note that by rescaling one of the x_i, and the corresponding ξ_{ij}, one may assume that $\mu=1$. We make this assumption from now on.

We use the notation of [9, §8], introducing a quiver Q_w with vertex set $I=\{0\}\cup\{[i,j]\mid 1\leqslant i\leqslant k,\ 1\leqslant j\leqslant w_i-1\}$, an element $q\in(K^*)^I$, and an ordering $<$ on the arrows of $\overline{Q}_w$. Let Λ^q be the corresponding multiplicative preprojective algebra. We denote by e_v the idempotent corresponding to the trivial path at a vertex $v\in I$.

Lemma 10.1. $A_{w,1,\xi}\cong e_0\Lambda^q e_0$.

Proof. By [9, Lemma 8.1], $e_0\Lambda^q e_0$ is spanned by the paths in $\overline{Q}$ with head and tail at 0. In fact it is generated by the paths $a_{i1}a_{i1}^*$ ($i=1,\dots,k$), because any arrows a_{ij},a_{ij}^* with j maximal which occur in a path, must occur as part of a product $a_{ij}a_{ij}^*$, and if $j>1$ this can be rewritten as $q_{i,j-1}a_{i,j-1}^*a_{i,j-1}+(q_{i,j-1}-1)e_{[i,j-1]}$. The elements $y_i=\xi_{i1}(a_{i1}a_{i1}^*+e_0)$ clearly satisfy $y_1y_2\cdots y_k=e_0$, and a calculation similar to [6, §3] shows that $(y_i-\xi_{i1}e_0)(y_i-\xi_{i2}e_0)\cdots(y_i-\xi_{i,w_i}e_0)=0$. Thus there is a surjective algebra homomorphism $\theta: A_{w,1,\xi}\to e_0\Lambda^q e_0$ sending x_i to y_i. To show that θ is an isomorphism, it suffices to show that any $A_{w,1,\xi}$-module M can be obtained by restriction from an $e_0\Lambda^q e_0$-module. Let X be the representation of $\overline{Q}$ with $X_0=M$,

$$X_{[i,j]}=(x_i-\xi_{i1}1)(x_i-\xi_{i2}1)\cdots(x_i-\xi_{ij}1)M,$$

and with a_{ij} the inclusion and a_{ij}^* multiplication by $(x_i - \xi_{ij}1)/\xi_{ij}$. Clearly X defines a Λ^q-module, and $e_0X = M$, as desired. $\square$

Appendix A.2. Generalized double affine Hecke algebras and multiplicative preprojective algebras, continued

In this appendix we will use the results of the main part of the paper and of Appendix A.1 to prove some results about the structure of multiplicative preprojective algebras of [9], in particular in the case of an affine quiver Q of type $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$.

11.1. Starlike quivers

In this subsection we give some results about multiplicative preprojective algebras for starlike quivers; they can be rather easily deduced from the paper [9].

We retain the notation of [9]. In particular, given a function q on the set of vertices of Q with values in $\mathbb{C}^*$, Λ^q denotes the corresponding multiplicative preprojective algebra.

Proposition 11.1. *Let Q be the quiver of type A_m, with vertices labeled $1, \ldots, m$. In this case the algebra Λ^q is finite dimensional, and it is zero unless $\prod_{p=i}^{j} q_p = 1$ for some $i \leqslant j$.*

Proof. Let a_i be the edges from i to $i+1$ and a_i^* from $i+1$ to i. The algebra Λ^q is the quotient of the path algebra of the double of Q by the relations

$$1 + a_1^* a_1 = q_1, \quad q_2\bigl(1 + a_2^* a_2\bigr) = 1 + a_1 a_1^*, \quad \ldots, \quad q_m = 1 + a_{m-1} a_{m-1}^*.$$

This algebra is isomorphic to a usual (deformed) preprojective algebra for the quiver Q. In particular, it is finite dimensional (as Q is of finite Dynkin type; see [8]).

Now, if $\Lambda^q \neq 0$, then it must have a finite dimensional irreducible representation. Therefore, by Theorem 1.9 in [9], $q^\alpha = 1$ for some positive root α, as desired. $\square$

Let Q be any starlike quiver, with vertices labeled by pairs (j, k), where $k = 1, \ldots, m$ is the number of the leg, and $j = 1, \ldots, d_k - 1$ the number of the vertex on the kth leg, enumerated from the nodal vertex.

Proposition 11.2. *The algebra $B := \Lambda^q / \Lambda^q e_0 \Lambda^q$ is finite dimensional. Furthermore, if $\prod_{p=i}^{j} q_{(p,k)} \neq 1$ for any $k = 1, \ldots, m$ and $1 \leqslant i \leqslant j < d_k$, then this algebra is zero and hence we have a natural Morita equivalence between Λ^q and $e_0 \Lambda^q e_0$.*

Proof. The algebra B is a cyclic Λ^q-module in which e_0 acts by zero. Thus B is supported at the non-nodal vertices of Q. Hence B can be regarded as a cyclic module over the direct sum of the multiplicative preprojective algebras for quivers of types A_{d_k-1}. By Proposition 11.1, this implies that B is finite dimensional. Also, if the condition on parameters holds, these algebras are zero, and hence $B = 0$, as desired. $\square$

Let L^q be the quotient of the path algebra of the double of Q by the relations of the multiplicative preprojective algebra except the nodal relation.

Lemma 11.3.

(i) *Let i be a vertex of Q. Then $e_i L^q e_0$ is a cyclic right module over $e_0 L^q e_0$ generated by the straight* (= *shortest*) *path connecting the nodal vertex* 0 *with i.*
(ii) *The same is true for $e_i \Lambda^q e_0$ as a right module over $e_0 \Lambda^q e_0$.*

Proof. It is clear that (ii) follows from (i), so it suffices to prove (i). Let γ be any path leading from 0 to the vertex i. We need to show that γ is a linear combination of straight paths over $e_0 L^q e_0$. If γ visits more than one leg, then it is a product of a shorter path with an element of $e_0 L^q e_0$, so by induction (in the length of the path) we may assume that γ is entirely contained in one leg, and hence, without loss of generality, that Q has only one leg to begin with. We will assume that the orientation of Q is toward the nodal vertex. If γ is not straight, it contains a subpath aa^* for some edge a. If a ends at the vertex 0, then γ is a product of a shorter path and a path starting and ending at e_0, and by induction we are done. Otherwise, we may use the relation at the head of a to replace aa^* with $c_1 b^* b + c_2$ where b is the edge that begins at the head of a and is directed toward the nodal vertex, and c_1 and c_2 are constants. This allows us to represent γ as a linear combination of a shorter path and a path of the same length but smaller sum of distances of the vertices passed to the nodal vertex. This proves that the straight path is a generator, as desired. □

Corollary 11.4. *If the condition on q in Proposition* 11.2 *holds, then for each i, $e_i \Lambda^q e_0 = p_i e_0 \Lambda^q e_0$, where $p_i \in e_0 \Lambda^q e_0$ are certain idempotents. Hence $\Lambda^q = \bigoplus_{i,j} p_i (e_0 \Lambda^q e_0) p_j$.*

Proof. Indeed, by Proposition 11.2 $e_i \Lambda^q e_0$ is a projective module over $e_0 \Lambda^q e_0$, while by Lemma 11.3 it is a quotient (hence a direct summand) of a free rank 1 module. This implies the result. □

In fact, it is easy to see that if i belongs to the kth leg then the idempotent p_i is a polynomial of the element x_k defined in Appendix A.1 projecting to the direct sum of distance$(i, 0)$ eigenspaces of x_k (note that the condition on q in Proposition 11.2 is equivalent to saying that the elements x_k have distinct eigenvalues).

11.2. Affine quivers

Let us now apply the results of the previous subsection to affine quivers. Assume that Q is of type $\tilde{D}_4$, $\tilde{E}_6$, $\tilde{E}_7$, $\tilde{E}_8$. Let δ be a basic imaginary root. By Lemma 10.1, in this case the algebra $e_0 \Lambda_q e_0$ is isomorphic to the generalized double affine Hecke algebra $H = H(t, q')$ for certain t, q' related to q by a simple transformation (for example, $q' = q^\delta$).

Corollary 11.5.

(i) *The Gelfand–Kirillov dimension of Λ^q is* 2.
(ii) *If the condition on q in Proposition* 11.2 *holds, then the algebra Λ^q is naturally Morita equivalent to H. In this case, $\Lambda_q = \bigoplus_{i,j} p_i H p_j$ for the corresponding t, q'.*
(iii) *Suppose that in the situation of* (ii), *q' is a root of unity such that $(q')^\ell$ has order N, and t is so generic that H is an Azumaya algebra* (*of degree ℓN*; *see Section* 5). *Then Λ^q is an Azumaya algebra of degree hN, where h is the Coxeter number of the corresponding Dynkin diagram.*

Proof. (i) Let us put a filtration on Λ^q, which assigns degree 0 to vertex idempotents and degree 1 to edges. It is easy to see that this filtration extends the length filtration on H defined in Section 5. This together with Lemma 11.3 implies that the algebra Λ^q exhibits the quadratic growth in this filtration, hence the result.

(ii) This follows from Corollary 11.4.

(iii) It is easy to see that for a given k the multiplicities of all eigenvalues of the element $T_k \in H$ in an irreducible representation of H are the same, and equal $\ell N/d_k$. Therefore, the rank of the idempotent p_i in such a representation is equal to N times the ith coordinate δ. The sum of such coordinates is the Coxeter number. Thus the statement follows from (ii). $\square$

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References

[1] J. Alev, M.A. Farinati, T. Lambre, A.L. Solotar, Homologie des invariants d'une algebre de Weyl sous l'action d'un group fini, J. Algebra 232 (2000) 564–577.

[2] N. Bourbaki, Groupes et algèbres de Lie, Chapitres IV–VI, Hermann, Paris, 1968.

[3] J.-L. Brylinski, A differential complex for Poisson manifolds, J. Differential Geom. 28 (1988) 93–114.

[4] I. Cherednik, Double affine Hecke algebras, Knizhnik–Zamolodchikov equations, and Macdonald operators, Int. Math. Res. Not. 1992 (9) (1992) 171–180.

[5] I. Cherednik, V. Ostrik, From double Hecke algebra to Fourier transform, Selecta Math. (N.S.) 9 (2) (2003) 161–249.

[6] W. Crawley-Boevey, On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero, Duke Math. J. 118 (2) (2003) 339–352.

[7] W. Crawley-Boevey, Indecomposable parabolic bundles and the existence of matrices in prescribed conjugacy class closures with product equal to the identity, math.AG/0307246.

[8] W. Crawley-Boevey, M. Holland, Noncommutative deformations of Kleinian singularities, Duke Math. J. 92 (1998) 605–635.

[9] W. Crawley-Boevey, P. Shaw, Multiplicative preprojective algebras, middle convolution and the Deligne–Simpson problem, math.RA/0404186.

[10] M. Demazure, H.C. Pinkham, B. Teissier (Eds.), Séminaire sur les Singularités des Surfaces, Held at the Centre de Mathématiques de l'École Polytechnique, Palaiseau, 1976–1977, Lecture Notes in Math., vol. 777, Springer, Berlin, 1980.

[11] V. Dolgushev, Hochschild cohomology versus De Rham cohomology without formality theorems, math.QA/0405177.

[12] P. Etingof, Cherednik and Hecke algebras of varieties with a finite group action, math.QA/0406499.

[13] P. Etingof, A. Oblomkov, Quantization, orbifold cohomology, and Cherednik algebras, math.QA/0311005.

[14] P. Etingof, E. Rains, New deformations of group algebras of Coxeter groups, math.QA/0409261.

[15] P. Etingof, W.L. Gan, A. Oblomkov, Generalized double affine Hecke algebras of higher rank, math.QA/0504089.

[16] W.L. Gan, V. Ginzburg, Deformed preprojective algebras and symplectic reflection algebras for wreath products, J. Algebra 283 (1) (2005) 350–363.

[17] E. Green, An introduction to noncommutative Gröbner bases, in: Computational Algebra, Fairfax, VA, 1993, in: Lect. Notes Pure Appl. Math., vol. 151, Dekker, New York, 1994, pp. 167–190.

[18] N. Jacobson, PI-algebras. An Introduction, Lecture Notes in Math., vol. 441, Springer-Verlag, Berlin, 1975.

[19] L. Le Bruyn, S.P. Smith, M. Van den Bergh, Central extensions of three-dimensional Artin–Schelter regular algebras, Math. Z. 222 (2) (1996) 171–212.

[20] E. Looijenga, Affine Artin groups and the fundamental groups of some moduli spaces, arXiv: math.AG/9801117.

[21] The Magma website, http://magma.maths.usyd.edu.au.
[22] A. Malkin, V. Ostrik, M. Vybornov, Quiver varieties and Lusztig's algebra, math.RT/0403222.
[23] A. Mellit, Algebras generated by elements with given spectrum and scalar sum and Kleinian singularities, math.RA/0406119.
[24] M. Noumi, J.V. Stokman, Askey–Wilson polynomials: An affine Hecke algebraic approach, math.QA/0001033, 2000.
[25] A. Oblomkov, Double affine Hecke algebras of rank 1 and affine cubic surfaces, Int. Math. Res. Not. 2004 (18) (2004) 877–912, math.RT/0306393.
[26] A. Oblomkov, E. Stoica, Finite dimensional representations of double affine Hecke algebra of rank 1, math.RT/0409256.
[27] K. Okamoto, Studies in the Painlevé equations I. Sixth Painlevé equation PVI, Ann. Mat. Pura Appl. 146 (1987) 337–381.
[28] E. Rains, The Magma code for computations in $H(t,q)$, www.math.mit.edu/~etingof/src.tar.gz.
[29] S. Sahi, Nonsymmetric Koornwinder polynomials and duality, Ann. of Math. (2) 150 (1) (1999) 267–282.
[30] J.T. Stafford, M. Van den Bergh, Noncommutative curves and noncommutative surfaces, Bull. Amer. Math. Soc. (N.S.) 38 (2) (2001) 171–216.
[31] J. Stokman, Koornwinder polynomials and affine Hecke algebras, Int. Math. Res. Not. 19 (2000) 1005–1042.
[32] M. Van den Bergh, A relation between Hochschild homology and cohomology for Gorenstein rings, Proc. Amer. Math. Soc. 126 (1998) 1345–1348.
[33] M. Van den Bergh, Blowing up of non-commutative smooth surfaces, Mem. Amer. Math. Soc. 154 (734) (2001).
[34] M. Van den Bergh, Erratum to: "A relation between Hochschild homology and cohomology for Gorenstein rings", Proc. Amer. Math. Soc. 130 (9) (2002) 2809–2810.