Doppler rate estimation on coherent sinusoidal pulse train and its Cramer–Rao lower bound

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The Doppler rate estimation on coherent sinusoidal pulse train, which can be applied in the passive emitter location systems, is investigated in this paper. When the pulse repetition interval (PRI) is constant, a DFT-based Doppler rate estimation algorithm is proposed and its performance is briefly analyzed. In the case of non-constant PRI, a least-squares-fitting based Doppler rate estimator (LSFE) is proposed. The mean square error is computed in closed form and the threshold signal-to-noise ratio (SNR) is analyzed. The Cramer–Rao lower bound on Doppler rate estimation is derived whereafter, and is compared to the mean square error of the LSFE. Monte Carlo simulations show that when operating above the threshold SNR, the proposed approach achieves the CRLB. The threshold SNRs in the simulations are basically coincident with the theoretical values.

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1. Introduction

The problem of locating an emitter from passive measurements is encountered in a variety of radar and sonar applications. The location can be performed either by a single sensor [1,2] or in an array of spatially distributed sensors [3–8]. The single sensor solution has the following advantages: (1) there is no need to synchronize and transfer data between sensors; (2) its system configuration is simple; (3) it is easy to implement in real applications. However, due to relatively few measured information, it is more difficult to design corresponding algorithms for positioning, tracking and parameter estimation. Array with multi-sensor methods can be divided into two types: uniform linear array (ULA) and non-uniform linear array (NULA). For ULA, the restrictive condition that the element distance \( d \) should be smaller than one half of the wavelength of the signal \( \lambda \) limits the size of the array.

Consequently, antenna gain is limited and mutual couplings between the antenna elements significantly affect the performance. For NULA, when \( d \) is larger than \( \lambda/2 \), the measured results may be ambiguous and extra ambiguity resolution operation should be performed. Multi-sensor solutions and single sensor solutions have their own merits and drawbacks. The applications where the size of the passive location system is limited can be of major interest in single sensor solution.

There are several methods of position estimating. The standard method is based on bearing measurements at different points along the sensor trajectory, which is called the bearing method (BM) [9–13]. Another method is based on Doppler shift of the emitter frequency due to relative motion between the emitter and the observer. This method is called the frequency method (FM) [10,14]. If the motion locus is known, the position of a stationary emitter can be estimated from several frequency estimation values taken at different points in the sensor trajectory. The third method, which combines the BM and FM, is called the combined method (CM) [1,2] in the sequel. A research group of National University of Defense Technology of China investigates the single observer passive...
location algorithms and technologies in recent 20 years. Based on the conventional bearing method, they add new observed quantities, such as angle-of-arrival (AOA) rate and Doppler rate, and propose a novel method [15,16], which outperforms conventional methods in location and tracking accuracy and speed of convergence. With the continuous measured bearing and frequency parameters, the geolocalization is computed using conventional two-step methods (also known as decentralized methods) or one-step methods (also known as centralized methods). In this paper, we mainly focus on Doppler rate estimation used for geolocalization, detailed post-processing after parameter estimation can be found in [1,15] and other cross-citations of these two papers.

FM and CM methods all need to measure frequency. The accuracy of frequency estimation must be sufficiently high because the Doppler frequency and Doppler rate are usually small. Since the transmitted radar waves are generally pulsed waveform and the scales of pulse duration are commonly microsecond (μs), the estimated accuracy from single pulse can hardly fulfill the requirements of the passive emitter location systems. As we know, accumulation of multiple pulses can improve the performance of frequency estimation. Multi-pulse frequency estimation algorithms can be divided into two types: non-coherent [17] and coherent [18–21]. The non-coherent algorithm consists of averaging the frequency estimates of individual pulses. Its accuracy is inversely proportional to $\sqrt{N_p}$, where $N_p$ denotes the observation time, or the number of pulses in this case. The non-coherent algorithm requires a long observation time before the emitter can be accurately located. When the number of pulses received by receiver is not so much or the emitter only transmits several pulses in one frequency point, the accuracy will be insufficient. This causes problems in practical application.

In order to estimate range, radial velocity, and acceleration of a target accurately, coherent technologies are widely used in modern radar systems. Coherent pulses are portions of a continuous wave and so the phases from pulse to pulse are in phase with the original wave. Parameters estimation on received coherent pulse train has been investigated in previous works. For example, joint estimation of delay, Doppler, and Doppler rate [18], measurement of range, radial velocity, and acceleration [19,20], frequency estimation from short pulses of sinusoidal signals [21], etc.

Since the carrier frequency and initial phase are prior parameters in radar, coherent accumulation can easily be implemented. In passive emitter location, however, these two parameters are unknown commonly. In order to use the coherent information in passive location, some extra processes should be taken. The Doppler rate-of-change (also called Doppler rate) of the signal received from remote emitter can be used for emitter location [15,16]. To this end very accurate frequency estimates are necessary.

Frequency estimation from short coherent pulses of a sinusoidal signal was investigated in [22]. The Cramer–Rao lower bound on differential Doppler frequency estimate was derived in [23], where the threshold SNR was also analyzed. Doppler shifted estimation can then be obtained from the difference of the adjacent Doppler estimations, which belongs to a kind of indirection method. In fact, the Doppler shifted frequency can be extracted directly. Since there is no prior knowledge, the frequency of each pulse should be estimated first, and a coherent accumulation can be performed subsequently. The accumulation results of successive pulse contain the information of Doppler rate, which can be extracted by parameters estimation methods, e.g., maximum likelihood estimation (MLE) [24], discrete polynomial-phase transform (DPT) [25,26], least mean square (LMS) [27], and others [28,29]. When the PRI is constant, MLE, DPT, and Kay estimator [28] can be used. However when the PRI is varying (e.g., stagger, jitter, sliding, etc), least-squares-fitting (LSF) will be a good choice.

In this paper, we investigate the Doppler rate estimation algorithm on coherent sinusoidal pulse train. First, we need to detect the pulses and measure the leading edge and trailing edge of each pulse. The methods proposed in [30,31] can estimate the leading and trailing edges under low SNR condition and can be used in our algorithm. When the PRI is constant, a Doppler rate estimation algorithm based on DPT is proposed, and the performance is briefly analyzed. In the case of non-constant PRI, a LSF based Doppler rate estimator is investigated. Then we derive the Cramer–Rao lower bound on Doppler rate estimation. Thereafter the mean square error of LSF is computed in closed form and compared to the CRLB. The threshold signal-to-noise ratio (SNR) is also analyzed. In Section 4, we extend our algorithm to other forms of coherent pulse train. The coherent LFM (linear frequency modulated) pulse train is taken as an example. Finally Monte Carlo simulations are conducted to compare the performance of the LSF estimator against the CRLB for various signal-to-noise ratios. A typical non-rectangular pulse shape, i.e., Gaussian pulse, is used in simulations to demonstrate the performances for non-rectangular cases.

2. Mathematical model

Consider a stationary emitter with coordinates (0, 0) and a sensor is moving relative to the emitter (Fig. 1). In this case, the delay that signals propagate from the emitter
to the sensor is a function of time
\[ r(t) = s(t)/c \]
where \( c \) is the speed of wave propagation in free space.

In very short time, \( s(t) \) can be approximated by a binomial [18]
\[ s(t) \approx c_0 + v_0 t + a_0 t^2/2 \]
where \( c_0, v_0, a_0 \) are the range, radial velocity, and acceleration of the sensor, respectively, at time \( t = 0 \). The typical value of \( v_0 \) is from 0 m/s to 7.9 km/s and the typical value of \( a_0 \) is from 0 m/s² to 500 m/s². Then the noise-free received signal can be modeled as
\[ r(t) = \sum_p g(t - T_p) A_p \exp[j(2\pi f_B t - \pi a t^2 + \theta)] \]
where \( p \) is the index of pulse; \( f_B, f_r, \) and \( \theta_r \) denote the radio frequency and initial phase respectively; \( A_p \) and \( T_p \) are the amplitude and starting time of the \( p \)-th pulse; \( g(t) \) is the envelope of the baseband transmitted pulse with \( 1/T_{OT} \)
\[ g_{\alpha \Delta}(t) = \left\{ \begin{array}{ll}
1, & \text{for } t = 1/T_{OT} \\[0, & \text{for } t \neq 1/T_{OT}
\end{array} \right. \]
where \( T_{OT} \) is the total pulse “on-time”, i.e., the pulse width. Assume that \( g(t) \) is real, time-limited on the interval \([0, T_{OT}]\) and symmetric with respect to \( T_{OT}/2 \). The duty cycle \( \eta \) is defined as \( \eta = T_{OT}/(T_p - T_{p-1}) \).

Substituting (1) into (3), the signal model becomes
\[ r(t) = \sum_p g(t - T_p) A_p \exp[j(2\pi f_B t - \pi a t^2 + \theta)] \]
where \( f_B = f_r - f_s/v_0/c, f_r = f_s/a_0/c \) denotes Doppler rate, and \( \theta_r = -2\pi f_I T_0 + \theta_0 \).

Consider the frequency measurement device to be a digital receiver of bandwidth \( W \) that records \( N_p \) pulses from the emitter to be located. The local frequency of the receiver is \( f_{\text{LO}} \). Suppose that the receiver samples at or above the Nyquist rate, \( 2W \), and that the noise power within the receiver’s bandwidth is \( \sigma^2 \). Then the pulse train the discrete signal model appears as
\[ r(n) = \sum_p g(n - k_0) A_p \exp[j(2\pi f_B n - \pi a n^2 + \theta)] + w_p(n) \]
where \( \Delta \), \( f_r \) and \( k_0 \) denote the sample interval, intermediate frequency and first sample number of the \( p \)-th pulse respectively, \( w_p(n) \) is an additive white Gaussian noise (AWGN). In passive emitter location scenarios, there is no a priori knowledge of the received signals, therefore, the receiver should detect the intercepted pulses and measure the leading edge and trailing edge of each pulse. In our derivations we assume that all pulses are detected and the leading and trailing edges are estimated without error. The performance will degrade when these two assumptions are not meet, which can be observed from the simulation results in Section 7.

The signal of the \( p \)-th pulse can be written as
\[ r_p(n) = g(n - k_0) A_p \exp[j(2\pi f_B n - \pi a n^2 + \theta)] + w_p(n) \]
\[ r_p(n) \]
the baseband signal can be written as
\[ r_p(n') = g(n' - k_0) A_p \exp[j(2\pi f_B(n' + k_0) - \pi a(n' + k_0)^2 + \theta)] + w_p(n' + k_0) \]
where \( f_e = f_r - f_s, n' = n - k_0 \) \((n' = 1, \ldots, N_p)\) and \( N_p \) denotes the sample number of each pulse. Next we will show that in most cases, the \( \pi a n^2 + \theta \) is very small compared to \( 2\pi f_e - \pi a_k n^2 \). Let \( n' = N_p \) and \( N_p \) denotes the pulse width and its value for pulse radar is usually from milliseconds to microseconds. For a typical scenario, \( N_p = 10 \mu s \), the duty cycle \( \eta = 10\% \), the pulse number \( N_p = 10 \), \( \alpha = 500 \text{ m/s}^2, \Delta = 0.1 \mu s \). Thus \( k_0 = N_p = N_p/\eta = 1 \text{ ms} \). When \( SNR = 0 \text{ dB} \), the CRBL [40] of frequency estimation for a 10 μs sinusoid pulse is 3898.48 Hz, i.e., RMSE \( f_p' \) ≥ 3898.48 Hz. Then the frequency error \( f_e \) satisfies that \( f_e = \text{RMSE}(f_p')/\sqrt{N_p} ≥ 1232.81 \text{ Hz} \). Now we evaluate the terms \( 2(\pi f_e - \pi a_k n^2)N_p \) and \( \pi a n^2 \) and find that \( \Delta(\Delta - \pi a_k n^2)N_p \approx 0.07749 \pi a n^2 \approx 1.57e - 7 \). Thus \( \eta \) can be simplified to
\[ r_p(n') \approx A_p \exp[j(2\pi f_e k_0 - \pi a k_0^2 n^2 + \theta)] + w_p(n' + k_0) \]
Taking the discrete Fourier transform of (8) yields
\[ S_p = A_p G(f - (f_s - a_k n^2)) \exp[j(2\pi f_e k_0 - \pi a k_0^2 n^2 + \theta)] + w_p \]
\[ = A_p G(f - (f_s - a_k n^2)) \exp[j(2\pi f_e k_0 - \pi a k_0^2 n^2 + \theta)] + w_p \]
where \( G(f) \) is the spectrum of \( g(t) \). The mean and variance of \( w_p \) are 0 and \( N_p \sigma^2 \) respectively. Since \( f_s - a_k n = 0 \), the spectral line at \( f = 0 \) is selected. Noting that \( G(f - (f_s - a_k n^2)) \) can be expressed as \( |G(f - (f_s - a_k n^2))| \exp[j(2\pi f_e k_0 + a_k n^2 + \theta)] \), (9) can be rewritten as
\[ S_p = A_p' \exp[j(2\pi f_e k_0 - \pi a k_0^2 n^2 + \theta)] + w_p \]
where \( f_s + f_s = f_e, \theta = \theta + \phi, A_p' = A_p / G(f - (f_s - a_k n^2)) \).

When the pulse repetition interval \( T_r \) \((T_r \text{ is the number of samples within one PRI})\) is constant, \( k_0 = p T_r \), (10) will be
\[ S_p = A_p' \exp[j(2\pi f_e k_0 - \pi a k_0^2 n^2 + \theta)] + w_p \]
where \( f_s + f_s = f_e, \theta = \theta + \phi, A_p' = A_p / G(f - (f_s - a_k n^2)) \).

\( S_p \) can be viewed as a chirp signal and its amplitude, initial frequency, and frequency slope are \( A_p', f_s', \) and \( a_s \) respectively. Doppler rate can be obtained by estimating the chirp slope. Many parameter estimation methods for chirp signals have been proposed in previous works, e.g., Newton iteration algorithm [24], DPT [25, 26], etc. Since the computation load of DPT is low and its variance is only about 7% higher than the CRBL at high SNRs [25], Doppler rate in (11) can be calculated by DPT. In [25], it was showed that the optimal delay for estimating frequency slope with DPT is \( r = N/2 \) for all SNR ≥ 0.1831 dB, where \( N \) is the sample number of the chirp signal. Herein, \( N = N_p T_r \). In this case, the variance of \( \hat{a} \) by DPT can be derived as (see Appendix)
\[ \text{Var}(\hat{a}) \approx \left( 1 + \frac{1}{2N(\text{SNR})} \right) \frac{96}{\pi^2 \Delta^4 T_r^2 N_p^4 \text{SNR}_{\text{int}}} \]
where $SNR_{\text{int}} = N_p N_s$SNR is the integrated SNR ($SNR = A_0^2/\sigma^2$ is the SNR of the input signal, and $A_0$ is the amplitude of the signal). In the case of constant PRI, the estimate of Doppler rate is straightforward. In this paper, we mainly discuss the problem of Doppler rate estimation when the PRI is varying.

3. Doppler rate estimation for non-constant PRI

When the PRI is not a constant, DPT or Newton iteration cannot be used to estimate Doppler rate directly. As can be observed from Eq. (10), the phase of noise-free $S_p$ is a parabola, so the least-squares-fitting or linear regression can be utilized to extract three coefficients of the parabola. Steven A Tretter studied the frequency estimation of a noisy sinusoid by linear regression [27], he pointed out that when $SNR > 1$, say, 8 dB, Eq. (10) can be approximated by

$$S_p \approx A_p \exp \left( (2\pi f_0 k_D - \pi ak_2^2 \Delta^2 + \theta + \varepsilon_p) \right). \quad (13)$$

Since $\bar{f}$ is the average of frequency estimates from $N_p$ pulses, its accuracy is comparatively high, therefore, $f_\epsilon$ is relatively small. On the other hand, $\alpha$ is usually small too, then $\left| (f_\epsilon - ak_2 \Delta) \right| \approx 0$ (according to the typical values in Section 2, $(f_\epsilon - ak_2 \Delta) = 1.18 \Delta - 5 \approx 0$). Notice that $\{G(0)\}$ is the pulse energy of the baseband pulse and related to $N_s$ for rectangular and raised cosine pulse shapes is equal to $N_s$. For these cases, $A_p = A_p \{G(0)\} = A_p \{G(0)\} = N_s A_p$. Thus the SNR of (10) is approximately given by $SNR_p \approx N_s A_p^2 / N_s \sigma^2 = N_s A_p^2 / \sigma^2$, which implies $N_s$ times gain after accumulation. When the input SNR is not too low, say, $SNR = 10 \log (A_p^2 / \sigma^2) \geq 0$ dB, the approximate condition can be met.

Doppler rate can be obtained from the phase of (13)

$$\phi_p = 2\pi \bar{f} k_D - \pi ak_2^2 \Delta^2 + \theta + \varepsilon_p. \quad (14)$$

Unfortunately, the phase extracted from (13) is ‘wrapped’ modulo $2\pi$. If the principal value of arg$[S_p]$ obtained by using an inverse tangent can be unwrapped, the parameter $\alpha$ can be estimated using linear regression techniques. Many phase-unwrapping (PU) algorithms have been proposed in the literature [32–38]. Among them, phase unwrapping for chirp signal is mainly discussed in [37,38]. In this paper, we use the PU algorithm proposed in [38]. After phase unwrapping, least-squares-fitting can be applied to estimate the Doppler rate from $\phi_p$.

Let $k_0 = \theta$, $c_1 = 2\pi \bar{f} \Delta$, $c_2 = -\pi \Delta^2$, then (14) can be rewritten as

$$\phi_p = c_0 + c_1 k_D + c_2 k_D^2 + \varepsilon_p. \quad (15)$$

Let $Q = (\phi_1, \ldots, \phi_N)^T$, $R = (a, b, c)^T$, $e = (\varepsilon_1, \ldots, \varepsilon_N)^T$, and $P = \begin{bmatrix} 1 & k_D & k_D^2 \\ k_0 & k_D & k_D^2 \\ \vdots & \vdots & \vdots \\ 1 & k_{N_s-1} & k_{N_s-1}^2 \end{bmatrix}$, where in superscript denotes transpose operation, then (15) can be rewritten as

$$e = Q - PR. \quad (16)$$

Now we can get the estimate of $R$ by applying the least-squares-fitting to (16) [39]

$$\hat{R} = (P^T P)^{-1} P^T Q. \quad (17)$$

Substitution of $P$ and $Q$ in (17) and expansion yield the estimate of $c_2$ as

$$c_2 = \frac{N_p K_1 - K_0 \sum_{p=0}^{N_p} \phi_p}{N_p K_2 - K_0} \quad (18)$$

where $\bar{K} = (1/N_p) \sum_{p=0}^{N_p} k_D$, $K_0 = \sum_{p=0}^{N_p} k_D - \bar{K}$, $K_1 = \sum_{p=0}^{N_p} (k_D - \bar{K}) \phi_p$, $K_2 = \sum_{p=0}^{N_p} (k_D - \bar{K})^2$. Consequently the estimate of Doppler rate can easily be obtained as

$$\hat{\alpha} = - \frac{c_2}{\pi \Delta^2}. \quad (19)$$

Since least-squares-fitting is performed on unwrapped phase, it is necessary to discuss some key problems about phase unwrapping. When the PRI is constant, the unwrapped phase of (11) is

$$\phi_p = 2\pi \bar{f} pT_\Delta - \pi ak_D^2 T_\Delta^2 + \theta + \varepsilon_p \quad (20)$$

where $\bar{f} = \text{mod}(\bar{f}, T_\Delta)$ (herein mod denotes the modulo operation). When the error of $\bar{f}$ is somewhat large, $\bar{f} T_\Delta$ may be larger than 1. Due to the periodicity of trigonometric function is $2\pi$, the phase obtained by unwrapping operation will be as (20). Although $\bar{f}$ may be not equal to the true value $f$, $\alpha$ can still be accurately extracted. For example, when the SNR is 0 dB and the sample interval is 10 ns, the CRLB for frequency estimation of a sinusoid with 100 samples is 38.985 kHz [40]. If the number of pulses in the pulse train is 60 ($N_p = 60$), the accuracy is about $f' \approx 38.985 / \sqrt{60} \approx 5$ kHz after non-coherent accumulation. When the PRI is equal to 1 ms, $f' T_\Delta \approx 5$. It is clear that phase ambiguity has occurred. However, since we only concern the parameter $\alpha$, as long as the second term in $\phi_p$ satisfies $\pi ak_D^2 T_\Delta^2 \leq 2\pi$, $\alpha$ can be extracted without ambiguity, which can be verified from the simulation in Section 7.

When the PRI is constant, the error of $f'$ in (11) has almost no effect on $\hat{\alpha}$. However, it is different for non-constant PRI. The true noise-free phase of (9) is $2\pi \bar{f} k_D - \pi ak_D^2 \Delta^2 + \theta$, and the phase difference of adjacent pulses is $2\pi \bar{f} (k_D - k_{D-1}) \Delta - \pi a(k_D^2 - k_{D-1}^2) \Delta^2$. $k_D - k_{D-1}$ is changing due to non-constant PRI, then when $p \neq q$, $2\pi \bar{f} (k_D - k_{D-1}) \Delta$ and $2\pi \bar{f} (k_q - k_{q-1}) \Delta$ may be in different cycles and $\alpha$ cannot be estimated correctly so that $f'$ should be known here. For a staggered PRI, the Chinese Remainder Theorem can be utilized to solve phase ambiguity and obtain the true value of $f'$. However, stagger ratio used in radar may be large, say, 31/32/33 [42, 43], in this case the range of resolving phase ambiguity is very small. For other cases of PRI diversity (e.g., jitter, sliding), the Chinese Remainder Theorem will be inapplicable. We give a more general method for the estimation of $f'$ herein with respect to PRI diversity. Calculating the phase difference of two adjacent $S_p$ of (9), say, $S_1$ and $S_2$, we get

$$\begin{align*}
\text{atan}(S_2) - \text{atan}(S_1) &= 2\pi \bar{f} (k_2 - k_1) \Delta - \pi a(k_2^2 - k_1^2) \Delta^2 + \varepsilon_2 - \varepsilon_1 \\
&\approx 2\pi \bar{f} (k_2 - k_1) \Delta + \varepsilon_2 - \varepsilon_1
\end{align*} \quad (21)$$
where $\hat{f} = \text{mod}(f^*, (k_2 - k_1)\Delta)$. The approximation is due to the relation $-\pi a(k_3^2 - k_1^2)\Delta^2 \approx 0$. Then

$$\hat{f} = \frac{-\text{Arg}(S_2) - \text{Arg}(S_1)}{2\pi(k_2 - k_1)\Delta}$$

where Arg denotes the principal value of argument. $f^*$ will be

$$f^* = \hat{f} + \frac{l}{(k_2 - k_1)\Delta}, \quad l \in \mathbb{Z}$$

and $l$ can be determined by

$$\max \left\{ \left\lfloor \frac{N_p - 1}{2} \right\rfloor S_p \exp \left\{ -j2\pi \left[ \hat{f} + \frac{l}{(k_2 - k_1)\Delta} \right] k_p\Delta \right\} \right\}.$$  \hspace{1cm} (24)

Expression (24) is a kind of searching method, similar to maximum likelihood searching for sinusoid frequency estimation. Although $S_p$ is a chirp signal, its bandwidth (BW) is usually small (where $BW = \Delta f T_0 \ll 1/(k_2 - k_1)\Delta$ and $T_0$ is the total observation time.), say, several Hertz, therefore, $l$ can be determined by (24) effectively. Since $l$ is an integer and the main error scope can be determined, the search scope needed will not to be very large. If the maximum likelihood-based frequency estimators are used, the frequency estimation variance of each pulse will be [40]

$$\text{Var}(\hat{f}) = \frac{3\sigma^2}{2\pi^2 A_p^2 \Delta^2 N_p(N_p^2 - 1)}.$$ \hspace{1cm} (25)

The accuracy of non-coherent accumulation on $N_p$ pulses will be [23]

$$\text{Var}(\hat{f}) = \frac{3\sigma^2}{2\pi^2 A_p^2 \Delta^2 N_p N_s(N_p^2 - 1)}.$$ \hspace{1cm} (26)

According to the ‘3σ’ rule, the frequency estimation error lies in the region of values from the true frequency of more than three times $\text{Var}(\hat{f})$. Thus the search scope of $l$ can be determined by

$$-\frac{9\sigma^2}{2\pi^2 A_p^2 \Delta^2 N_p N_s(N_p^2 - 1)} < \hat{f} + \frac{l}{(k_2 - k_1)\Delta} < \frac{9\sigma^2}{2\pi^2 A_p^2 \Delta^2 N_p N_s(N_p^2 - 1)}$$

that is

$$\text{floor}[ -\frac{9\sigma^2}{2\pi^2 A_p^2 N_p N_s(N_p^2 - 1)\Delta} - \hat{f} (k_2 - k_1)\Delta ] \leq l \leq \text{ceil}[ -\frac{9\sigma^2}{2\pi^2 A_p^2 N_p N_s(N_p^2 - 1)\Delta} - \hat{f} (k_2 - k_1)\Delta ]$$

where floor[ ] denotes the round operation which rounds a number to the nearest integer less than or equal to it, and ceiling[ ] denotes the round operation which rounds a number to the nearest integer greater than or equal to it.

Once the $l$ which maximizes (24) is found, the estimate of $f^*$ is given by

$$\hat{f} = \hat{f} + l(k_2 - k_1)\Delta$$

(29)

Multiplied by the complex coefficient $\exp(-j2\pi \hat{f} k_p\Delta)$, (9) is converted to

$$S_p \approx A^* \exp[j(2\pi f^* k_p\Delta - \pi a(k_3^2 - k_1^2)\Delta^2 + \theta')] + w_p$$

(30)

where $f^* = f^* - \hat{f}$. Now $\hat{a}$ can be obtained by phase unwrapping and least-squares-fitting.

4. The Cramer–Rao lower bound of Doppler rate estimation

Since the noise term $w_p(n)$ is an additive Gauss white noise, the probability density function (pdf) of $r(n)$ can be written as

$$f(z; \alpha) = F \cdot \exp\left\{-\frac{1}{\sigma^2} \sum_{p=0}^{N_p-1} \sum_{n=k_p+1}^{k_p+N_s} |r(n) - g(n\Delta - r_p)A_p e^{j\theta_p}|^2 \right\}$$

$$= F \cdot \exp\left\{-\frac{1}{\sigma^2} \sum_{p=0}^{N_p-1} \sum_{n=1}^{N_s} \sum_{k=0}^{k_p+N_s} |r(n\Delta + k\lambda)|^2 - g^*(n\Delta)A_p r(n\Delta + k\lambda)e^{j\theta_p} \right\}$$

$$+ g(n\Delta)g^*(n\Delta)A_p^2 \right\}$$

(31)

where $F$ is a constant which is immaterial in further development; $\Delta = 2\pi f + \Delta - \pi a(k_3^2 - k_1^2)\Delta^2 + \theta$; the pulse time $r_p = k_p\Delta$ and $n = n - k_p$. Let $\lambda = \{r_0, r_1, \ldots, r_N, \lambda_0, \lambda_1, \ldots, \lambda_{N_p-1}, \theta, \alpha \}$ be a vector parameter to be estimated. The samples of $g(t)$, i.e., $g(n\Delta)$, are unknown nuisance parameters and hence ignored for simplicity. The log-likelihood function has the form as

$$l = \ln F - \frac{1}{\sigma^2} \sum_{p=0}^{N_p-1} \sum_{n=1}^{N_s} \sum_{k=0}^{k_p+N_s} |r(n\Delta + k\lambda)|^2 - g^*(n\Delta)A_p r(n\Delta + k\lambda)e^{j\theta_p} - g(n\Delta)g^*(n\Delta)A_p^2 \right\}$$

(32)

where superscript $-1$ denotes inverse matrix. $\mathbf{J}$ is the Fisher information matrix (FIM), which is defined as [41]

$$\mathbf{J} = -\mathbb{E} \left[ \frac{\partial^2}{\partial \lambda^2} \right]$$

(34)

The elements of $\mathbf{J}$ is given by

$$J_{ij} = -\mathbb{E} \left[ \frac{\partial}{\partial \lambda_i} \frac{\partial}{\partial \lambda_j} \right] = -\mathbb{E} \left[ \frac{\partial^2}{\partial \lambda_i \partial \lambda_j} \right]$$

(35)

In preparation for computation of the CRLB, the derivatives of the signal with respect to each unknown parameter are found to be

$$\frac{\partial l}{\partial A_p} = -\frac{1}{\sigma^2} \sum_{n=1}^{N_s} \sum_{k=0}^{k_p+N_s} \left[ -g^*(n\Delta) r(n\Delta + k\lambda)e^{-j\theta_p} - g(n\Delta) r(n\Delta + k\lambda)e^{j\theta_p} + 2g(n\Delta)g^*(n\Delta)A_p \right]$$

(36)
\[
\frac{\partial l}{\partial f_j} = -\frac{1}{\sigma} \sum_{p=0}^{N_p-1} \sum_{n=1}^{N} \left[ -g^*(n' \Delta) a_{p} r(n' + k_p) e^{-i \varphi} \right] \frac{\partial g^*(n' \Delta)}{\partial f_j} \frac{\partial a_{p}}{\partial f_j} \frac{e^{-i \varphi}}{\partial \Delta} - g(n' \Delta) a_p r(n' + k_p) e^{i \varphi} \frac{\partial \Delta}{\partial f_j} \left( \frac{\partial \varphi}{\partial f_j} \frac{e^{i \varphi}}{\partial \Delta} \right)
\]

The derivatives of \( \alpha \) are shown in (B.3) of Appendix B. Notice that for convenience, the first equation of (31) is used for deriving \( \alpha \) and the second equation of (31) is used for calculations of other partial derivatives.

In order to simplify the expressions in the following derivation, we define \( A = \sum_{p=0}^{N_p-1} a_{p}^2 \), \( B = \sum_{p=0}^{N_p-1} k_p a_{p}^2 \), \( C = \sum_{p=0}^{N_p-1} a_{p}^2 \), \( D = \sum_{p=0}^{N_p-1} k_p a_{p}^2 \), \( E = \sum_{p=0}^{N_p-1} k_p a_{p}^2 \), \( L_1 = \sum_{n=1}^{N} n^2 \), \( L_2 = \sum_{n=1}^{N} n^4 \), \( L_3 = \sum_{n=1}^{N} n^2 \), \( L_4 = \sum_{n=1}^{N} n^4 \), and \( L_5 = \sum_{n=1}^{N} n^4 \). Thus the elements of the FIM, \( J \), under the signal plus complex AWGN assumption are given by

\[
J_{\alpha \alpha} = -E \left[ \frac{\partial^2}{\partial \alpha^2} \right] = \frac{2L_1 A_p^2}{\sigma^2} \frac{\Delta^2}{\sigma^2}
\]

and

\[
J_{\alpha \beta} = -E \left[ \frac{\partial^2}{\partial \alpha \partial \beta} \right] = \frac{L_1}{\sigma^2}
\]

and

\[
J_{\beta \beta} = -E \left[ \frac{\partial^2}{\partial \beta^2} \right] = \frac{2}{\sigma^2} L_1 A_p^2
\]

and

\[
J_{\alpha \gamma} = -E \left[ \frac{\partial^2}{\partial \alpha \partial \gamma} \right] = \frac{2 \Delta^2}{\sigma^2} \left[ L_3 A_p + 2L_2 B + L_1 C \right]
\]

and

\[
J_{\alpha \delta} = -E \left[ \frac{\partial^2}{\partial \alpha \partial \delta} \right] = \frac{2 \Delta^3}{\sigma^2} \left[ L_5 A_p + 4L_4 B + 6L_3 C + 4L_2 D + L_1 E \right]
\]

and

\[
J_{\beta \gamma} = J_{\beta \delta} = J_{\gamma \delta} = 0
\]

\[
J_{\alpha \gamma} = J_{\alpha \delta} = J_{\beta \gamma} = J_{\beta \delta} = J_{\gamma \delta} = 0
\]

and

\[
J_{\beta \gamma} = J_{\beta \delta} = J_{\gamma \delta} = 0
\]

and

\[
J_{\gamma \gamma} = J_{\gamma \delta} = J_{\delta \gamma} = J_{\delta \delta} = 0
\]

The matrix, \( J \), is then

\[
J_{11} = \frac{2L_1}{\Delta^2} \frac{A_p^2}{\sigma^2}
\]

\[
J_{12} = \frac{1}{\sigma^2}
\]

\[
J_{22} = \frac{1}{\sigma^2} \left[ \begin{array}{cccc} L_1 & 0 & \cdots & 0 \\ 0 & L_1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & L_1 \end{array} \right]
\]

\[
J_{33} = \frac{2 \Delta^4}{\sigma^2} \left[ \begin{array}{cccc} \frac{4L_1 A_p + 6L_2 B + 4L_3 C}{\Delta^2} & \frac{2L_1 A_p + 6L_2 B + 6L_3 C}{\Delta^2} & \frac{2L_1 A_p + 6L_2 B + 6L_3 C + 2L_1 D}{\Delta^2} \\ \frac{2L_1 A_p + 6L_2 B + 6L_3 C + 2L_1 D}{\Delta^2} & \frac{L_1 A_p + 2L_2 B + L_1 C}{\Delta^2} & \frac{L_1 A_p + 2L_2 B + L_1 C}{\Delta^2} \\ \frac{L_1 A_p + 2L_2 B + L_1 C}{\Delta^2} & \frac{L_1 A_p + 2L_2 B + L_1 C}{\Delta^2} & \frac{L_1 A_p + 2L_2 B + L_1 C}{\Delta^2} \end{array} \right]
\]

From the FIM, we can conclude that there is a correlation between the estimates of most of the parameters, except between the pulse amplitudes and the pulse times.

Inversing the FIM matrix and expanding \( L_1 - L_5 \) yield the CRLB of each parameter respectively as

\[
\text{CRLB}(\alpha) = \frac{\Delta^2}{2N_p \text{SNR}_p}
\]

\[
\text{CRLB}(A_p) = \frac{\sigma^2}{N_p}
\]

\[
\text{CRLB}(f_j) = \frac{3 \sigma^2 [180(AE - C^2) + F_1]}{2 \Delta^2 N_p [2160(ACE + 2BCD - E^2 - AD^2 - C^3) + F_2]}
\]

\[
\text{CRLB}(\theta) = \frac{3 \sigma^2 [720(CE - D^2) + 720(N_a + 1)(BE - CD) + F_3]}{2N_p [2160(ACE + 2BCD - E^2 - AD^2 - C^3) + F_2]}
\]

\[
\text{CRLB}(\alpha) = \frac{9 \sigma^2 [A^2 (N_a^2 - 1) + 12(AE - B^2)]}{2 \Delta^2 N_p [2160(ACE + 2BCD - E^2 - AD^2 - C^3) + F_2]}
\]

where \( \text{SNR}_p = A_p^2 / \sigma^2 \) and

\[
F_1 = 360(N_a + 1)(AD - BC) + 120AC(N_a + 1)(2N_a + 1)
\]

\[
-180B^2(N_a + 1)^2 + 60AB(N_a - 1)(N_a + 1)^2
\]

\[
+ A^2 (16N_a^2 + 30N_a^2 - 5N_a^2 - 30N_a - 11)
\]
\[ F_2 = 180(N_b^2 - 1)(4ABD - A^2E - 3AC^2) + 36(2N_b^4 - 5N_b^2 + 3)A^2C - AB^2 + A^3(N_b^4 - 6N_b^2 + 9N_b^4 - 4) \]  
(57)

\[ F_3 = 120AE(N_b + 1)(2N_b + 1) + 720BD(N_b + 1) - 180C^2(N_b + 1)(N_b + 5) + 120AD(N_b + 1)^2(N_b + 2) - 180(N_b + 1)^2(B^2 + 2BC) + 12AC(N_b + 1)(N_b^2 + 1)(7N_b + 8) + 12AB(N_b - 1)(N_b + 1)^2(2N_b + 1)(N_b + 2) + A^2(N_b - 1)(N_b + 1)^2(3N_b^2 + 3N_b + 2) \]  
(58)

Expanding \( AE - C^2 \) yields

\[ AE - C^2 = \left( \sum_{p=0}^{N_b-1} A_p^2 \right) - \left( \sum_{p=0}^{N_b-1} \frac{1}{C_0} \right)^2 \]  
(59)

Since \( k_p \) is the number of samples within the observation time \( (0, T_p) \), which is a large number, therefore, the value of (59) mainly depends on \( \sum_{p=0}^{N_b-1} C_p k_p^2 \), where \( C_p = \sum_{q=0}^{N_b-1} A_q^2 + \sum_{p=0}^{N_b-1} A_p^2 \). Consequently the magnitude of 180(\( AE - C^2 \)) is \( k_0^4 \), while for \( F_1 \) the magnitude is \( N_b k_0 \). Generally, \( N_b < k_p \), therefore, \( F_1 \) can be ignored. In the same way, the magnitude of 2160(\( ACE + 2BCD - EB^2 - AD^2 - C^2 \)) is \( k_0^4 \), and for \( F_2 \), the magnitude is \( N_b k_0^4 \). The CRLB of \( F_1 \) can then be approximated as

\[ \text{CRLB}(f_1) \approx \frac{\sigma^2(AE - C^2)}{8\pi^2 \Delta N_b (ACE + 2BCD - EB^2 - AD^2 - C^2)} \]  
(60)

and the CRLBs of \( \theta \) and \( \alpha \) can be approximated respectively as

\[ \text{CRLB}(\theta) \approx \frac{\sigma^2[(CE - D^2) + (N_b + 1)(BE - CD)]}{2N_b(ACE + 2BCD - EB^2 - AD^2 - C^2)} \]  
(61)

\[ \text{CRLB}(\alpha) \approx \frac{\sigma^2(AC - B^2)}{4\pi^2 \Delta^4 N_b (ACE + 2BCD - EB^2 - AD^2 - C^2)} \]  
(62)

When the amplitude and PRI are constant, the CRLBs of the estimated parameters are reduced to

\[ \text{CRLB}(f_1) = \frac{3T_0^2(N_b^2 - 1)(2N_b - 1)(N_b + 11) + 30T_0(2N_b - 1)(N_b + 1)^2(N_b + 1)(N_b + 5)}{2\pi^2 \Delta^2 \text{SNR}_{\text{int}} T_0^2(N_b^2 - 1)(N_b^2 - 4) + F_4} \]  
(63)

\[ \text{CRLB}(\theta) = \frac{3T_0^2(N_b - 2)(N_b + 1)(3N_b^2 - 3N_b - 2) + 2(N_b^2 - 4) + F_6}{2\pi^2 \Delta^2 \text{SNR}_{\text{int}} T_0^2(N_b^2 - 1)(N_b^2 - 4) + F_5} \]  
(64)

\[ \text{CRLB}(\alpha) = \frac{90}{\pi^2 \Delta^2 \text{SNR}_{\text{int}} T_0^2(N_b^2 - 1)(N_b^2 - 4) + F_5} \]  
(65)

where

\[ F_4 = 5T_0^2(7N_b^2N_b^2 - 6N_b^2N_b + 6N_bN_b - N_b^2 - N_b^2 - 5) + 30T_0(2N_b - 1)(N_b + 1)^2(N_b + 1) + 16N_b^4 + 30N_b^3 - 5N_b^2 - 30N_b - 11 \]  
(66)

\[ F_5 = 5T_0^2(N_b^2 - 1)(N_b^2 - 1) + N_b^4 - 5N_b^2 + 4 \]  
(67)

\[ F_6 = 6T_0^2(N_b - 2)(2N_b - 1)(N_b + 1)(N_b - 1)^2(N_b + 1) + 14N_b^4 + 15N_b^3 \]  
+ 90N_bN_b^3 + 15N_b^2N_b^2 + 30N_b^2 - 13N_b^2 + 90N_bN_b - 29

Since the number of samples of each pulse is large, for example, for a MTI radar, with the pulse width 0.5 \( \mu \)s, \( N_b \) is 50 when the sampling rate is 100 MHz, and the number of pulses used for coherent accumulation is usually larger than 10, the ratio of the first part and the second part of \( F_4 \) will meet (35T_0^2N_b^2N_b^2/30T_0N_bN_b^2) = (35T_0N_b^2/30N_b) \( \approx 1 \) (where \( T_0/N_b \) is the reciprocal of duty cycle and usually larger than 10). The ratio of the first part and the third part of \( F_4 \) will meet (35T_0^2N_b^2N_b^2/16N_b^2) = (35T_0N_b^2/16N_b^2) \( \approx 1 \) in the same way. In addition, the scope of the pulse repetition frequency (PRF, which is the reciprocal of PRI) is usually from several hundreds of Hertz to several tens of thousands Hertz, with the corresponding PRI from several tens of microseconds to several milliseconds. In such a time interval, the number of samples \( T_r \) is sufficiently large, thus the ratio of the first item of (63) and \( F_4 \) is approximately equal to (16T_0^2N_b^2 + 30T_0N_bN_b^2)/35T_0^2N_b^2 = (16T_0^2N_b^2 + 30T_0N_b)/(35N_b^2) \( \approx 1 \) and \( F_4 \) can be ignored. Analyzed in the same way, \( F_5 \) of the denominator of (63)–(65) can also be ignored. CRLB(\( f_1 \)) is simplified to

\[ \text{CRLB}(f_1) \approx \frac{3(2N_b - 1)(8N_b - 11) + 30N_bN_b - 11(N_b + 1)T_r}{2\pi^2 \Delta^2 \text{SNR}_{\text{int}} T_0^2(N_b^2 - 1)(N_b^2 - 4)} \]  
(69)

\[ \text{CRLB}(\theta) \approx \frac{24(N_b - 1)^2}{2\pi^2 \Delta^2 \text{SNR}_{\text{int}} T_0^2(N_b^2 - 1)(N_b^2 - 4)} \]  
\[ \approx \frac{9(N_b^2 - 1)(N_b^2 - 4)}{2\pi^2 \Delta^2 \text{SNR}_{\text{int}} T_0^2(N_b^2 - 1)(N_b^2 - 4)} \]  
(71)

\[ \text{CRLB}(\alpha) \approx \frac{90}{\pi^2 \Delta^2 \text{SNR}_{\text{int}} T_0^2(N_b^2 - 1)(N_b^2 - 4)} \]  
\[ \approx \frac{90}{\pi^2 \Delta^2 \text{SNR}_{\text{int}} T_0^2(N_b^2 - 1)(N_b^2 - 4)} \]  
(72)

where \( T_0 = N_bT_r, \Delta \) is the total observation time. From (72) we can deduce that the CRLB of Doppler rate estimate is inversely proportional to the fourth power of the total observation time \( T_0 \) and inversely proportional to accumulation SNR, namely \( \text{SNR}_{\text{int}} \). Expanding \( \text{SNR}_{\text{int}} \) as \( N_b \text{SNR} \), we can observe that CRLB(\( \alpha \)) is inversely proportional to the input SNR, pulse width, and fifth power of the number of pulses. In a word, the most important factor is
where \( N_s \) is the frequency slope of LFM, we can see that substituting \( N_s \) and SNR in (73) with \( N_p \) and \( N_s \cdot \text{SNR} \) respectively yields (72).

5. Performance analysis

The mean of \( \hat{\alpha} \) is

\[
E[\hat{\alpha}] = -N_p \sum_{p=0}^{N_s-1} \frac{(k_p - K_0)^2 E[\phi_p]}{\pi \Delta^2 (N_p K_2 - K_0^2)}
\]

(74)

where \( E[\phi_p] = c_0 + c_1 k_p + c_2 k_p^2 \), and \( E[\cdot] \) denotes expectation. The variance of \( \hat{\alpha} \) is

\[
\text{Var}[\hat{\alpha}] = \frac{1}{\pi \Delta^4} \text{E} \left\{ N_p \sum_{p=0}^{N_s-1} \frac{(k_p - K_0)^2 E[\phi_p]}{\pi \Delta^2 (N_p K_2 - K_0^2)} \right\}
\]

(75)

where \( \sigma^2 = \sigma_0^2 / (2 N_s A_p^2) \), \( K_3 = \sum_{p=0}^{N_s-1} (k_p - K_0)^2 \sigma_0^2 \), \( K_4 = \sum_{p=0}^{N_s-1} (k_p - K_0)^4 \sigma_0^2 \).

When the amplitude and PRI are constant, the variance of \( \hat{\alpha} \) reduces to

\[
\text{Var}[\hat{\alpha}] = \frac{N_p \sigma_0^2}{\pi \Delta^4} \left( N_p \sum_{p=0}^{N_s-1} [pT_c - (N_p - 1)T_c / 2 + (p(N_p - 1)T_c / 2)]^2 \right)
\]

\[
= \frac{90 \sigma_0^2}{\pi \Delta^4 N_p A_p^2 (N_p^2 - 5 N_p + 4 N_p T_c^2)} 
\]

\[
= \frac{90 \sigma_0^2}{\pi \Delta^4 5N_s A_p N_p T_c^2}
\]

(76)

The approximation condition can be met when \((5N_s^2 - N_p) / N_p^2 \approx 5 / N_p^2 < 0\).

Notice that (76) is equal to (72), namely, when the SNR is higher than the threshold SNR, the root-mean-square error (RMSE) is the same as CRLB.

Generally, the variance of an estimator increases sharply when the SNR is below some threshold SNR. Next, we analyze threshold SNR of the LSFE. First, the frequency of each pulse is estimated by DFT-based (discrete Fourier transform) algorithms. Experience shows that, in DFT-based estimation, the most important factor affecting the goodness of the estimate is the output signal-to-noise ratio (OSNR). The OSNR is defined as the ratio of the signal energy at the point \( \omega = \omega_0 \) (\( \omega_0 \) is the maximum point of the DFT) to the noise energy at this point. The numerical results in [40] show that the threshold OSNR is about 13–14 dB. Therefore, when the sample number of a sinusoid is \( N_s \), the following relation should be met:

10 \log_{10} N_s A_p^2 / \sigma_0^2 \geq 14.

Then the SNR should meet

\[
\text{SNR} \geq \frac{10^{1.4}}{N_s}.
\]

(77)

(78)

On the other hand, in order to apply the Trettler approximation [27] in (13), the SNR should be higher than 8 dB, namely

\[
\text{SNR}_{\text{T}} \approx 10 \log_{10} \frac{N_s A_p^2 / \sigma_0^2}{\gamma} \geq 8 \text{ dB}
\]

(79)

which means the input SNR should meet

\[
\text{SNR} \geq \frac{10^{0.8}}{N_s}.
\]

(80)

Take the maximum of (78) and (80) yields the threshold SNR as

\[
\text{SNR}_{\text{th}} = \max \left\{ \frac{10^{1.4}}{N_s}, \frac{10^{0.8}}{N_s} \right\} = \frac{10^{1.4}}{N_s} \approx \frac{25}{N_s}
\]

(81)

From (81), we can conclude that the threshold SNR is only dependent on the pulse width but independent of other parameters, such as the duty cycle, pulse number, and observation time. Since there is \( N_s \) times gain after coherent accumulation, the SNR needed for Trettler approximation is lower than the threshold OSNR for DFT-based algorithms, which implies that once the frequency of each pulse can be extracted accurately, the LSFE will work well.

6. Extension to other forms of coherent pulse train

In this paper, we mainly focus on Doppler rate estimation on coherent sinusoidal pulse train, which has no phase/frequency modulation. In fact, the proposed method can be extended to radar signals with phase and/or frequency modulation. In this section, we extend our algorithm to coherent LFM pulse train, which is a typical frequency modulated coherent pulse train widely used in modern radar systems.

Next, we directly give the Doppler rate estimation algorithm for coherent LFM pulse train without further derivation:

1. Estimate the TOA \( \hat{\tau}_p \) and pulse width \( \hat{\tau}_p \) of each pulse.
2. According to the property of a LFM signal, its auto-correlation function \( R(t) \) is a sinusoid and its frequency equals to \( \gamma \), where \( \gamma \) is the sweep rate of the LFM. The LFM pulses can be deinterleaved or recognized from the intercepted pulses.
3. Estimate the frequency \( \hat{f}_p \) and frequency sweep rate \( \hat{\gamma}_p \) using the DFT estimator proposed in [25]. The means of frequency and frequency sweep rate can be obtained respectively, i.e., \( \hat{f} = \sum_{p=0}^{N_p} \hat{f}_p \) and \( \hat{\gamma} = \sum_{p=0}^{N_p} \hat{\gamma}_p \).
4. Generate the local reference signal: \( S_{\text{Ref}}(t) = \exp[j(2\pi f T_0 + \pi \hat{\gamma}_p^2)] \)
5. The received signal is processed by quasi-matched filtering:

\[
y(t) = \int r(t) S_{\text{Ref}}(t - \hat{\tau}_p) \, dt
\]

Here, we use the word “quasi-matched filtering” because \( \hat{f} \), \( \hat{\gamma} \), and \( \hat{\tau}_p \) are estimated.

The output of the quasi-matched filter can be written as

\[
S_p = A_p \exp[j(2\pi f_p T_0 - \pi \hat{\gamma}_p^2 + \theta)] + w_p
\]

Notice that \( S_p \) here is similar to (10). Subsequently, our proposed Doppler rate estimation algorithm can be applied.

\[
\text{SNR} \approx 10 \log_{10} \frac{N_s A_p^2 / \sigma_0^2}{\gamma} \geq 8 \text{ dB}
\]

(79)

which means the input SNR should meet

\[
\text{SNR} \geq \frac{10^{0.8}}{N_s}.
\]

(80)
7. Simulation results

Simulations are conducted to characterize the performance of Doppler rate estimator. One thousand Monte Carlo trials per SNR are used to compute a root-mean-square error of the Doppler rate estimates. The parameters of the transmission pulse train are assumed to be radio frequency (RF) = 10 GHz and pulse width (PW, also called pulse duration time) = 1 μs. The radial acceleration $a_0$ is $3 \text{m/s}^2$, i.e., the Doppler rate is $\alpha = f_c a_0/c = 100 \text{Hz/s}$. The receiver converts the frequency to intermediate frequency (IF) with a frequency down-converter. In the simulations the IF $f_I$ is set to 50 MHz. The sampling rate is 100 MHz, i.e., the sample interval $\Delta$ is 10 ns.

In the case of low PRF, three different pulse trains are used. Their PRIs are {1, 32/31, 33/31} ms, {1, 6/5, 27/25, 31/25} ms, and {1, 62/51, 53/51, 61/51, 58/51} ms, i.e., the stagger ratios are $\tau_1:\tau_2:\tau_3 = 31:32:33$, $\tau_1:\tau_2:\tau_3:\tau_4 = 25:30:27:31$, and $\tau_1:\tau_2:\tau_3:\tau_4:\tau_5 = 51:62:53:61:58$, respectively. The stagger ratios are taken from [42] where they have been discussed in detail. The total pulse number for each train is 60, which means the period number of each pulse train is 20, 15, 12, i.e., the observation time are $(31 + 32 + 33)/31 \approx 61.94 \text{ ms}$, $(25 + 30 + 27 + 31)/25 \approx 67.8 \text{ ms}$, and $(51 + 62 + 53 + 61 + 58)/51 \approx 67.06 \text{ ms}$, respectively. It is assumed that the TOA and PW of each pulse are known and the pulse shape is rectangular. The RMSEs and CRLBs are shown in Figs. 2–4.

In the case of medium PRF, two different pulse trains are simulated. The PRIs are {96, 94, 92, 72, 70, 68, 57, 55, 53} μs and {50, 52, 54, 57, 60, 63, 67, 71} μs, respectively. The stagger ratios here are taken from [43] where the performance of radars using such stagger ratios has been investigated in detail. The total pulse numbers used in simulation are 450 and 480, which means the period numbers are 50 and 60, i.e., the observation time is $(96 + 94 + 92 + 72 + 70 + 68 + 57 + 55 + 53) \approx 32.85 \text{ ms}$ and $(50 + 52 + 54 + 57 + 60 + 63 + 67 + 71) \approx 28.44 \text{ ms}$, respectively. It is assumed that the TOA and PW of each pulse are known and the pulse shape is rectangular. The RMSEs and CRLBs are shown in Figs. 5 and 6.
The performance curves with the triangle sign in Figs. 2–6 are the results of constant amplitude cases. While the circle sign is corresponding to random amplitude cases and the varying slope is 1 dB. The sample number of each pulse is 100 with the pulse width of 1 μs and sampling rate 100 MHz. From (81) we can get the threshold SNR $\text{SNR}_{\text{th}} = 6$ dB. As can be seen from simulation results, in the cases of constant amplitude, the threshold SNR is $\text{SNR}_{\text{th}} = 6$ dB when the PRI is $\{96, 94, 92, 72, 70, 68, 57, 55, 53\}$ μs, which is consistent with the theoretical value. The threshold SNRs for the other experiments are $-5$ dB, close to $-6$ dB. When the SNRs are higher than the threshold SNR, the accuracy are all close to CRLB. When the amplitude is fluctuating randomly, the threshold SNRs and accuracy in Fig. 2–4 are almost the same as the constant amplitude cases. For the latter two cases, however, the threshold SNRs increases for about 2 dB and 1 dB respectively.

Although the number of pulses and samples used in the case of medium PRF is much more than that of low PRF, the accuracy of the medium PRF is slightly lower, compared to the low PRF cases. This is because the accuracy of parameter estimation for coherent pulse train is primarily determined by the total observation time, not the number of pulses and samples, which can be observed from the variance of the estimate (i.e., (75) and (76)) and the CRLB (i.e., (62) and (72)).

Simulations are performed for cases where the TOA is unknown or the pulse shape is non-rectangular. In Fig. 7, we compare the simulated results with the cases of known TOA and rectangular pulse shape. The Gaussian pulse is chosen as the non-rectangular pulse. As observed from the results, the threshold SNRs in TOA unknown cases are higher than that in TOA known case. The threshold SNR for Gaussian pulse is higher than that for rectangular pulse. When the SNR is higher than the threshold SNR, the accuracy for rectangular pulse with unknown TOA is closed to that of the case with known TOA. In the case of Gaussian pulse, the accuracy is about 2–3 dB poorer than that of rectangular pulse.

The performance of Doppler rate estimation for coherent LFM pulse train is shown in Fig. 8. The bandwidth of the LFM pulses is 1 MHz and other parameters are kept the same as the sinusoidal case. The impact of TOA and pulse shape on the performance for LFM is similar to the sinusoid case. The threshold SNRs for LFM are higher than those for sinusoid since one more parameter, i.e., the sweep rate of LFM, needs to be estimated. Notice that in Fig. 8, the CRLB for coherent sinusoidal pulse train is used. When there is no a priori knowledge of TOA and pulse shape, the CRLB for coherent LFM pulse train maybe closed to, but different from that for sinusoidal case. From the derivation of CRLB and performance analysis, we can see that when there is no phase/frequency modulation, the CRLB for Doppler rate is independent from the form of $g(t)$.

8. Conclusion

We investigate the Doppler rate estimation algorithms, which can be applied in the passive emitter location
systems. The DFT-based Doppler rate estimator is briefly discussed with respect to constant PRI. To non-constant PRI, however, the least-squares-fitting is used to extract the Doppler rate from the unwrapping phase of the pulse train after coherent accumulation. The mean square error of the least-squares-fitting based Doppler rate estimator is derived in closed form. Subsequently, the threshold SNR is analyzed. The CRLB for this problem is also computed in an analytic form. Further analysis shows that the variance of the least-squares-fitting based Doppler rate estimator is equal to the CRLB, which verifies that when the noise is Gaussian, the least-squares estimates are equivalent to maximum-likelihood estimates. Simulation shows when operating above the threshold SNR, the proposed approach achieves the CRLB. The threshold SNRs in the simulations are basically identical with the theoretical values.

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**Appendix A. The derivation of the mean-square error of Doppler rate estimate w.r.t constant PRI**

The first-order perturbation analysis is used to derive the mean-square error of Doppler rate estimate with respect to constant PRI. The derivation is in a similar way with the derivation of mean-square error of frequency slope of LFM [25].

The mean and variance of the noise term \( \eta_{\omega} \) are zero and \( N_{p} \sigma^2 \) respectively. Recall that the noise-free signal is given by

\[
S_p = A_0 \exp(j2\pi f_p T_D + \alpha p T_D^2 \Delta^2 + \theta').
\]  
(A.1)

The function \( g_p(\omega) \) is given by

\[
g_p(\omega) = \sum_{p=1}^{N_p-1} S_p + \Delta^{2} \exp(-j \omega p T_D) \Delta.
\]  
(A.2)

The perturbation in \( g_p(\omega) \) due to the additive noise is

\[
\delta g_p(\omega) = \sum_{p=1}^{N_p-1} [S_p + \Delta^{2} \exp(-j \omega p T_D) \Delta]
\]  
(A.3)

The functions \( g_p(\omega) \) and \( \delta g_p(\omega) \) and their derivatives, evaluated at the point of global maximum \( \omega_0 = 2\pi c T_D \Delta \), are given by

\[
g_p(\omega_0) = \Delta^{2} A_0 L(N_p - \tau)
\]  
(A.4)

\[
\frac{\partial g_p(\omega)}{\partial \omega} = -j \omega T_D A_0 L(N_p - \tau)^2 \Delta
\]  
(A.5)

\[
\frac{\partial^2 g_p(\omega)}{\partial \omega^2} = -j \frac{3}{2} T_D^2 A_0^2 L(N_p - \tau)^2 \Delta^2
\]  
(A.6)

\[
\delta g_p(\omega) = \sum_{p=1}^{N_p-1} [S_p + \Delta^{2} \exp(-j \omega p T_D) \Delta]
\]  
(A.7)

The second derivative of the function \( f_{N}(\omega) \) is [25]

\[
\frac{\partial^2 f_{p}(\omega)}{\partial \omega^2} \approx - \frac{1}{6} T_D^2 A_0^2 (N_p - \tau)^4 \Delta^2
\]  
(A.9)

Substitution of (A.4), (A.5), (A.7), and (A.8) in the following equation [25]:

\[
\frac{\partial \delta f_{p}(\omega)}{\partial \omega} = 2 \text{Re} \left\{ g_p(\omega) \frac{\partial \delta g_p^2(\omega)}{\partial \omega} + \frac{\partial g_p(\omega)}{\partial \omega} \frac{\partial \delta g_p^2(\omega)}{\partial \omega} \right\}
\]

yields

\[
\frac{\partial \delta f_{p}(\omega)}{\partial \omega} \approx - 2 \Delta T_D A_0^2 L(N_p - \tau) \text{Im} \left\{ \sum_{p=1}^{N_p-1} [S_p + \Delta^{2} \exp(-j \omega p T_D) \Delta] \right\}
\]

where

\[
\eta = \frac{1}{N_p} \sum_{p=1}^{N_p-1} \left[ \left| S_p + \Delta^{2} \exp(-j \omega p T_D) \Delta \right| \right]
\]

Finally using (A.10) and (A.13), Eq. (91) in [25], and the relation \( \delta \alpha = \delta \omega / (2 \pi \alpha T_D) \), we get

\[
E\left[ \left( \delta \alpha \right)^2 \right] \approx \frac{3}{5} \frac{1}{\text{SNR} \cdot \pi T_D^4 \Delta^2 (N_p - \tau)^2} \left[ \left( 1 + \frac{1}{2 \text{SNR}} \right) - \frac{(N_p - 2 \tau)(N_p - 4 \pi \tau + r^2)}{(N_p - r)^2} \right].
\]  
(A.15)

For any given values of \( \Delta \), \( N_p \) and \( \text{SNR} \), \( E\left[ \left( \delta \alpha \right)^2 \right] \) depends on \( \tau \). For all \( \text{SNR} \approx 0.1831 \), the value \( \tau = N_p / 2 \) is optimal; while for \( \text{SNR} \ll 1 \), the optimal choice is \( \tau = 0.4 N_p \). Since the value \( \tau = 0.4 N_p \) is close to \( N_p / 2 \), \( N_p / 2 \) is chosen for all SNRs. With this choice of \( \tau \), (A.15) takes the form

\[
E\left[ \left( \delta \alpha \right)^2 \right] \approx \left( 1 + \frac{1}{2 \text{SNR}} \right) \frac{96}{\pi^2 \Delta^4 T_D^4 N_p N_p \text{SNR}}.
\]  
(A.16)
As we see, at high SNRs, the mean-square error of the estimated \( \hat{\alpha} \) by the DFT method is only about 7% higher (exactly 16/15) than the CRLB, namely (72).

### Appendix B. The derivation of pulse time related Fisher information matrix elements

Herein we re-write the probability density function (pdf) of \( r(n) \) as

\[
f(z; \alpha) = F \cdot \exp \left\{ -\frac{1}{\sigma^2} \sum_{k_p = 0}^{N_p - 1} \sum_{n = k_p + 1}^{k_p + N_i} \left[ r(n) - g(n\Delta - \tau_p)A_p e^{i\phi} \right]^2 \right\}
\]

(B.1)

where \( \phi = 2\pi f_i n \Delta - \pi n^2 \Delta^2 + \theta \), \( \tau_p = k_p \Delta \). The log-likelihood function has the form as

\[
l = \ln F - \frac{1}{2\sigma^2} \sum_{k_p = 0}^{N_p - 1} \sum_{n = k_p + 1}^{k_p + N_i} \left[ r(n) - g(n\Delta - \tau_p)A_p e^{i\phi} \right]^2 - g(n\Delta - \tau_p)A_p r(n)e^{i\phi} - g^*(n\Delta - \tau_p)A^*_p r(n)e^{-i\phi}
\]

(B.2)

The derivation of pulse time related FIM elements utilizes the following properties: \( (1/T_p) \int_{t_p}^{t_p+T} g(t) \, dt = 1 \): \( g(t) \) is real and symmetric with respect to \( T_p/2 \). The partial derivative of \( \tau_p \) is

\[
\frac{\partial l}{\partial \tau_p} = -\frac{1}{\sigma^2} \left\{ -i[2\pi f_i \Delta - 2\pi\alpha (k_p + N_s)\Delta^2] A_p r^*(n)e^{i\phi} |_{n = k_p + N_s} + i[2\pi f_i \Delta - 2\pi\alpha (k_p + N_s)\Delta^2] A_p r^*(n)e^{i\phi} |_{n = k_p + N_s} + i[2\pi f_i \Delta - 2\pi\alpha (k_p + N_s)\Delta^2] A_p r^*(n)e^{i\phi} |_{n = k_p} \right\}
\]

(B.3)

Then

\[
\frac{\partial^2 l}{\partial \tau_p^2} = -\frac{1}{\sigma^2} \left\{ i[4\pi f_i \Delta^2 - 4\pi\alpha (k_p + N_s)\Delta^2] A_p r^*(n)e^{i\phi} |_{n = k_p + N_s} + i[4\pi f_i \Delta^2 - 4\pi\alpha (k_p + N_s)\Delta^2] A_p r^*(n)e^{i\phi} |_{n = k_p + N_s} + i[4\pi f_i \Delta^2 - 4\pi\alpha (k_p + N_s)\Delta^2] A_p r^*(n)e^{i\phi} |_{n = k_p} \right\}
\]

(B.4)

According to \( (1/T_p) \int_{t_p}^{t_p+T} g^2(t) \, dt = 1 \), \( \sum_{n = k_p + 1}^{k_p + N_i} g^2(n\Delta - \tau_p) = N_s \), then \( J_{\tau_p} \) can be obtained as

\[
J_{\tau_p} = -\frac{\partial^2 l}{\partial \tau_p^2} = \frac{2N_s A^2_p}{\Delta^2 \sigma^2}
\]

(B.5)

The derivation of \( J_{\tau_p} \), \( J_{\tau_p} \), \( J_{\tau_p} \), \( J_{\tau_p} \), \( J_{\tau_p} \), \( J_{\tau_p} \), \( J_{\tau_p} \), and \( J_{\tau_p} \) are shown as follows:

\[
J_{\tau_p} = J_{\tau_p} = -\frac{\partial^2 l}{\partial \tau_p^2} = \frac{1}{\sigma^2} \left\{ -r^2(n)e^{i\phi} |_{n = k_p + N_s} + r^*(n)e^{i\phi} |_{n = k_p + N_s} \right\}
\]

(B.6)

\[
J_{\tau_p} = J_{\tau_p} = -\frac{\partial^2 l}{\partial \tau_p^2} = \frac{1}{\sigma^2} \left\{ -2r(n)e^{i\phi} |_{n = k_p + N_s} - 2r^*(n)e^{i\phi} |_{n = k_p + N_s} \right\}
\]

(B.7)

\[
J_{\tau_p} = J_{\tau_p} = -\frac{\partial^2 l}{\partial \tau_p^2} = \frac{1}{\sigma^2} \left\{ -4r^2(n)e^{i\phi} |_{n = k_p + N_s} - 4r^*(n)e^{i\phi} |_{n = k_p + N_s} \right\}
\]

(B.8)
\[
\frac{1}{2} \left(-i\pi (k_p + N_1)^2 \Delta A_p (e^{i\theta} + e^{i\phi} - e^{i\phi} + e^{i\theta}) - i\pi (k_p + N_1)^2 \Delta A_p (e^{i\theta} + e^{i\phi} - e^{i\phi} + e^{i\theta})\right) = 0
\] (B.9)

References