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Convergent series solution of nonlinear equations

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Abstract: The author's decomposition method [1] provides a new, efficient computational procedure for solving large classes of nonlinear (and/or stochastic) equations. These include differential equations containing polynomial, exponential, and trigonometric terms, negative or irrational powers, and product nonlinearities [2]. Also included are partial differential equations [3], delay-differential equations [4], algebraic equations [5], and matrix equations [6] which describe physical systems. Essentially the method provides a systematic computational procedure for equations containing any nonlinear terms of physical significance. The procedure depends on calculation of the author's A_n, a finite set of polynomials [1,13] in terms of which the nonlinearities can be expressed. This paper shows important properties of the A_n which ensure an accurate and computable convergent solution by the author's decomposition method [1]. Since the nonlinearities and/or stochasticity which can be handled are quite general, the results are potentially extremely useful for applications and make a number of common approximations such as linearization, unnecessary.

Keywords: nonlinear differential equations, decomposition method, nonlinear operators convergence.

1. Introduction

The inversion of linear and nonlinear operators which may be stochastic, or deterministic as a special case, can be carried out using the decomposition method and the A_n polynomials [1]. This makes practical a new approach to dynamical system problems in many areas of scientific research which involve nonlinear and/or stochastic behavior. These can include nuclear reactors [7], lasers, control systems [8], physiological systems such as the cardiovascular system, or the national economy [9] and a variety of others [10]. For many physical systems, strongly nonlinear behavior and even violent fluctuations can be essential features. Turbulence, for example, is such a case. Modeling of such problems leads to nonlinear stochastic (differential or partial differential) operator equations. The prevalent generally used methods for solving these equations have been unsatisfactory in many instances since linearization, perturbation, and averaging procedures can create serious discrepancy between the consequent mathematical solution and the physical model.

2. Preliminary discussion—calculation of the A_n

Let N be a nonlinear (nondifferential) operator such that Ny is a function $f(y)$ such as y^2, y^5,

e^y, $\sin y$, etc. We can also consider operators such that we get $f(y, y')$, $f(y, y', y'')$, etc., e.

$$Ny = \frac{d^0}{dt^0} y \frac{d^1}{dt^1} y = yy' = f(y, y'), \qquad Ny = \left(\frac{d}{dt} y\right)^2 = y'^2 = f(y'),$$

etc., and these further cases have been previously discussed by Adomian and Rach [11], a limit ourselves here to functions $f(y)$.

An operator equation of the form $Fy = x$ where $x = x(t)$ and F is an algebraic or differ operator such that $Fy = Ly + Ny$ where Ly is a linear term and Ny is a nonlinear term $f($ must be invertible although if it is not, it is again decomposed so we can begin with an inve operator. Since we deal with physical systems, inputs are bounded. [1] The equation is solv decomposition writing $y = F^{-1}x = \sum_{n=0}^{\infty} \lambda^n F_n^{-1} x$ where the F_n^{-1} are to be determined and parameters for grouping terms (not a perturbation parameter). [2] Equivalently, $y = \sum_{n=0}^{\infty} \lambda^n y$, nonlinear term Ny is represented by $\sum_{n=0}^{\infty} \lambda^n A_n$ where the A_n are Adomian's previously d polynomials [1]. These are defined such that each A_n depends on $y_0, y_1, \ldots, y_n$ only w calculated from the linear part of the equation and use of initial or boundary conditions. each y_{n+1} is calculable since it depends only on the $y_0, y_1, \ldots, y_n$. For the simple non operators above, the A_n have been given as

$$A_n = \frac{1}{n!} \frac{d^n}{d\lambda^n} f(y(\lambda)) \bigg|_{\lambda=0} = \frac{1}{n!} D^n f \bigg|_{\lambda=0},$$

suppressing arguments. With this $y = L^{-1}x - L^{-1}Ny$ becomes $y = y_0 - L^{-1}\sum_{n=0}^{\infty} A_n$ henc $\sum_{n=0}^{\infty} y_n$ is calculable.

These implicit differentiations are cumbersome and seeking explicit differentiations [11 can write $D = d/d\lambda = (dy/d\lambda)(d/dy)$. Each $D^n f$ is evaluated at $\lambda = 0$ and divided by $n!$. relations are given by

$$\frac{d^n}{d\lambda^n} y(\lambda) = n! y_n, \qquad \frac{d^n}{dy^n} f(y(\lambda)) \bigg|_{\lambda=0} = \frac{d^n f}{dy_n}(y_0) = h_n(y_0).$$

The $D^n f$ for $n > 0$ can be written as a sum from $\nu = 1$ to n of terms $d^\nu f/dy^\nu$ with coeffi which are polynomials in the $d^\nu y/d\lambda^\nu$. Thus

$$D^1 f = \frac{df}{dy} \frac{dy}{d\lambda},$$

$$D^2 f = \frac{d^2 f}{dy^2} \left(\frac{dy}{d\lambda}\right)^2 + \frac{df}{dy} \frac{d^2 y}{d\lambda^2},$$

$$D^3 f = \frac{d^3 f}{dy^3} \left(\frac{dy}{d\lambda}\right)^3 + 3 \frac{d^2 f}{dy^2} \frac{dy}{d\lambda} \frac{d^2 y}{d\lambda^2} + \frac{df}{dy} \frac{d^3 y}{d\lambda^3},$$

$$\vdots$$

Denote the nth coefficient for the nth derivative $D^n f$ by $c(\nu, n)$ and write

$$D^n f = \sum_{\nu=1}^{n} c(\nu, n) F(\nu),$$

e.g.,

$$D^3 f = c(1, 3) F(1) + c(2, 3) F(2) + c(3, 3) F(3).$$

[1] We assume $x(t) - Ny$ is continuous with respect to t, y (in a convex region) and $\partial Ny/\partial y$ exists and is bounc

[2] The λ's are later set equal to 1 but are a convenience in some problems [1].

All $c(i, j)$ can be generated from the recurrence rule (for $1 \leqslant i, j \leqslant n$)

$$c(i,j)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\{c(i,j-1)\}+\frac{\mathrm{d}y}{\mathrm{d}\lambda}\{c(i-1,j-1)\},$$

letting $c(0, 0) = 1$ and $c(1, 0) = 0$. As a convenience define the explicit derivatives

$$\psi(i,j)=\left(\frac{\mathrm{d}^i y}{\mathrm{d}\lambda^i}\right)^j, \qquad F(i)=\frac{\mathrm{d}^i f}{\mathrm{d}y^i}.$$

Now

$$c(0,0)=\psi(1,0)=1,$$

$$c(1,1)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\{c(1,0)\}+\psi(1,1)\{c(0,0)\}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\{0\}+\psi(1,1)\{1\}=\psi(1,1).$$

Observing $c(0, j) = 0$ when $j > 0$,

$$c(2,2)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\{c(2,1)\}+\psi(1,1)\{c(1,1)\}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\{0\}+\psi(1,1)\psi(1,1)=\psi(1,2),$$

$$c(1,2)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\{c(1,1)\}+\psi(1,1)\{c(0,1)\}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\psi(1,1)+\psi(1,1)\{0\}=\psi(2,1),$$

$$c(3,3)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\{c(3,2)\}+\psi(1,1)\{c(2,2)\}=\psi(1,1)\psi(1,2)=\psi(1,3),$$

$$\begin{aligned}c(2,3)&=\frac{\mathrm{d}}{\mathrm{d}\lambda}\{c(2,2)\}+\psi(1,1)\{c(1,2)\}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\psi(1,2)+\psi(1,1)\psi(2,1)\\&=2\psi(1,1)\psi(2,1)+\psi(1,1)\psi(2,1)=3\psi(1,1)\psi(2,1),\end{aligned}$$

$$c(1,3)=\frac{\mathrm{d}}{\mathrm{d}\lambda}\{c(1,2)\}=\frac{\mathrm{d}}{\mathrm{d}\lambda}\psi(2,1)=\psi(3,1)$$

$$\vdots$$

To calculate any $\mathrm{D}^n f$ multiply the $c(\nu, n)$ by $F(\nu)$ for the range from 1 to n and sum. Then the A_n are given by

$$A_n=\frac{1}{n!}\mathrm{D}^n F|_{\lambda=0}.$$

Thus

$$A_0=\frac{1}{0!}\mathrm{D}^0 f|_{\lambda=0}=f(y_0)=h_0(y_0),$$

$$A_1=\frac{1}{1!}\mathrm{D}^1 f|_{\lambda=0}=c(1,1)h_1(y_0)=\psi(1,1)h_1(y_0)=y_1h_1(y_0),$$

$$\begin{aligned}A_2&=\frac{1}{2!}\mathrm{D}^2 f|_{\lambda=0}\\&=\tfrac{1}{2}\{c(1,2)h_1+c(2,2)h_2\}=\tfrac{1}{2}\{\psi(2,1)h_1+\psi(1,2)h_2\}\\&=\tfrac{1}{2}\{2y_2h_1+y_1^2h_2\}=y_2h_1(y_0)+\tfrac{1}{2}y_1^2h_2(y_0),\end{aligned}$$

$$A_3 = \frac{1}{3!}\{c(1,3)h_1 + c(2,3)h_2 + c(3,3)h_3\}$$
$$= \frac{1}{3!}\{\psi(3,1)h_1 + 3\psi(1,1)\psi(2,1)h_2 + \psi(1,3)h_3\}$$
$$= \frac{1}{3!}\{(3!)y_3h_1 + 3!y_1y_2h_2 + y_1^3h_3\}$$
$$= \tfrac{1}{6}\{6y_3h_1 + 6y_1y_2h_2 + y_1^3h_3\} = y_3h_1 + y_1y_2h_2 + \tfrac{1}{6}y_1^3h_3,$$
$$\vdots$$

If $f(y) = y$, $h_0 = y$, $h_1 = 1$, and $h_i = 0$ for $i \geqslant 2$ consequently $A_0 = y_0$, $A_1 = y_1$, $A_2 = y_2, \ldots, A_n = y_n$ or since $y = \sum_{i=0}^{\infty}\lambda^i y_i$ and $n!y_n = \mathrm{d}^n y/\mathrm{d}\lambda^n|_{\lambda=0}$

$$A_n = \frac{1}{n!}\mathrm{D}^n\{f(y)\}\bigg|_{\lambda=0} = \frac{1}{n!}\mathrm{D}^n y\bigg|_{\lambda=0} = \frac{1}{n!}\,\frac{\mathrm{d}^n y}{\mathrm{d}\lambda^n}\bigg|_{\lambda=0},$$
$$A_n = \frac{1}{n!}\{n!y_n\} = y_n.$$

Thus $y_{n+1} = y_{n+1}(y_n)$ so in the linear case, y_{n+1} depends only on the term preceding it as first demonstrated by the author.

We can now write in general

$$A_0 = h_0(y_0), \qquad A_1 = h_1(y_0)y_1, \qquad A_2 = h_1(y_0)y_2 + h_2(y_0)\frac{1}{2!}y_1^2,$$
$$A_3 = h_1(y_0)y_3 + h_2(y_0)y_1y_2 + h_3(y_0)\frac{1}{3!}y_1^3,$$
$$A_4 = h_1(y_0)y_4 + h_2(y_0)\left[\frac{1}{2!}y_2^2 + y_1y_3\right] + h_3(y_0)\frac{1}{2!}y_1^2y_2 + h_4(y_0)\frac{1}{4!}y_1^4,$$
$$A_5 = h_1(y_0)y_5 + h_2(y_0)[y_2y_3 + y_1y_4] + h_3(y_0)\left[y_1\frac{1}{2!}y_2^2 + \frac{1}{2!}y_1^2y_3\right]$$
$$+ h_4(y_0)\frac{1}{3!}y_1^3y_2 + h_5(y_0)\frac{1}{5!}y_1^5,$$
$$A_6 = h_1(y_0)h_6 + h_2(y_0)\left[\frac{1}{2!}y_3^2 + y_2y_4 + y_1y_5\right] + h_3(y_0)\left[\frac{1}{3!}y_2^3 + y_1y_2y_3 + \frac{1}{2!}y_1^2y_4\right]$$
$$+ h_4(y_0)\left[\frac{1}{2!}y_1^2\frac{1}{2!}y_2^2 + \frac{1}{3!}y_1^3y_3\right] + h_5(y_0)\frac{1}{4!}y_1^4y_2 + h_6(y_0)\frac{1}{6!}y_1^6.$$
$$\vdots$$

3. Convergence

Consider the (deterministic) nonlinear equation $Ly + Ny = x$. Then $Ly = x - Ny$ or $y = L^{-1}x - L^{-1}Ny = L^{-1}x - L^{-1}\sum_{n=0}^{\infty}A_n = \sum_{n=0}^{\infty}y_n$ where

$$A_0 = f(y_0) = h_0(y_0), \qquad A_1 = c(1,1)h_1 = y_1h_1(y_0),$$
$$A_2 = c(1,2)h_1 + c(2,2)h_2 = y_2h_1(y_0) + \frac{1}{2!}y_1^2h_2(y_0),$$
$$\vdots$$

which are finite polynomials in y_0 since $y_1, y_2, \ldots$ can be given in terms of y_0.

$$y_0 = L^{-1}x = \int_0^t l(t, \tau)x(\tau)\,\mathrm{d}\tau,$$

assume $|x(t)| < M_1$ for $t \in T$ and $|l(t, \tau)| < M_2$ for $t, \tau \in T$. Hence

$$|y_0| = M_2 M_1 \int_0^t \mathrm{d}\tau = M_2 M_1 t,$$

$$y_1 = -L^{-1}A_0 = -\int_0^t l(t, \tau)h_0(y_0)\,\mathrm{d}\tau,$$

$$y_2 = -L^{-1}A_1 = -\int_0^t l(t, \tau)y_1 h_1(y_0)\,\mathrm{d}\tau,$$

$$\vdots$$

We will consider y^m type nonlinearities and for simplicity y^2 specifically. Now

$$y_1 = -\int_0^t l(t, \tau)y_0^2\,\mathrm{d}\tau, \qquad |y_1| = -M_2\int_0^t |L^{-1}xL^{-1}x|\,\mathrm{d}\tau = -M_2^3 M_1^2 \frac{t^3}{3!},$$

$$y_2 = -L^{-1}(2y_0y_1) = -2L^{-1}L^{-1}xL^{-1}L^{-1}xL^{-1}x, \qquad |y_2| = 2M_2^5 M_1^3 \frac{t^5}{5!}.$$

We can absorb the $1/n!$ into the coefficients and write

$$A_n = \sum_{\nu=1}^{n} C(\nu, n)h_\nu(y_0).$$

Then, for example

$$A_3 = C(1, 3)h_1 + C(2, 3)h_2 + C(3, 3)h_3 \quad \text{or}$$

$$A_5 = C(1, 5)h_1 + C(2, 5)h_2 + \cdots + C(5, 5)h_5$$

etc. The n-term approximation ϕ_n is given by

$$\phi_n = y_0 - L^{-1}\sum_{i=0}^{n-1} A_i, \qquad \phi_{n+1} = y_0 - L^{-1}\sum_{i=0}^{n} A_i = y_0 - L^{-1}\sum_{i=0}^{n-1} - L^{-1}A_n,$$

$$|\phi_{n+1} - \phi_n| = -|L^{-1}A_n| = -|y_{n+1}| \to 0.$$

Examination of the components y_n term by term shows convergent terms for any term y^m. Let's consider $m = 2$ as an example. Let $l(t, \tau)$ be bounded by M_l, $x(t)$ bounded by M_x (n the almost-everywhere sense if x is stochastic). The first term of $y = \sum_{n=0}^{\infty} y_n$ is $y_0 = L^{-1}x = \int_0^t l(t, \tau)x(\tau)\mathrm{d}\tau$ which is $M_l M_x t$. The second is $y_1 = L^{-1}y_0^2 = L^{-1}L^{-1}xL^{-1}x = M_x^2 M_l^3 t^3/3!$. Multiply and divide the second term by M_x to write $(M_x^3 M_l^3 t^3/3!)/M_x$. The third term is $2M_x^3 M_l^5 t^5/5!$ which can be written $(M_x^5 M_l^5 t^5/5!)/(2/M_x^2)$. The next term is $M_x^4 M_l^7 t^7/7! + 4M_x^4 M_l^7 t^7/7!$ or $\{(M_x^7 M_l^7/7!)/M_x^3\} + \{(M_x^7 M_l^7 t^7/7!)/(4/M_x^3)\}$. Each term is in the form $(M_x^n M_l^n t^n/n!)$ divided by increasing powers of M_x, i.e., the series is bounded by $\exp(M_x M_l t)/M_x^\alpha$ where α is a positive exponent. The series converges for all finite t. For y_n we have $\{(M_x M_l t)^{2n+1}/(2n+1)!\}/M_x^n$ and $y_n \to 0$ as $n \to \infty$ for any finite t. A series behaving like $t^n/n!$ would have the $(n+1)$th term divided by the nth term equal to $(t^{n+1}/(n+1)!)/(n!/t^n) = t/n$ and approach zero for finite t as $n \to \infty$. This series converges much faster as illustrated by many examples often yielding stable accurate results in a few terms to a dozen terms.

It has been shown that nonlinear terms can be expanded in the A_n polynomials which approach zero for high n and consequently the decomposition method yields an accurate solution. More complete discussions appear in [10,11,12].

The following (anharmonic oscillator) example clearly shows the expected behavior:

$$\frac{d^2\theta}{dt^2} + k^2 \sin\theta = 0, \quad \theta(0) = \gamma = \text{constant}, \quad \theta'(0) = 0.$$

The solution is

$$\theta(t) = \gamma - \left|\frac{(kt)^2}{2!}\right| \sin\gamma + \left|\frac{(kt)^4}{4!}\right| \sin\gamma\cos\gamma - \left|\frac{(kt)^6}{6!}\right| \cdot |\sin\gamma\cos^2\gamma - 3\sin^3\gamma| + \cdots.$$

If γ is small this reduces to

$$\theta(t) = \gamma\cos kt,$$

i.e., the result for the linear harmonic oscillator.

To verify solutions with $\theta_n = \Sigma_{i=0}^{n-1} y_i$ substitute into $Ly + Ny = x(t)$

$$L\phi_{n+1} + \sum_{i=0}^{n-1} A_i = x(t)$$

thus approximating the derivative term to y_1, for example, by writing $\phi_2 = y_0 + y_1$, the ΣA_i would include only A_0 since $y_1 = -L^{-1}A_0$ and depends only on A_0.

By solving the equation $Ly + Ny = x$ using the approximation $\phi_n = \Sigma_{i=0}^{n-1} y_i$ for y, we have $\phi_n = L^{-1}x - L^{-1}\Sigma A_n$ but $\lim_{n\to\infty}\phi_n = y$ and $\lim_{n\to\infty}\Sigma A_n = Ny$, so the equation is satisfied. Existence and uniqueness are guaranteed by the previously stated [1] continuity and boundedness requirements which are reasonable for physical problems.

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