Local moduli of holomorphic bundles

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We study moduli of holomorphic vector bundles on non-compact varieties. We discuss filtrability and algebraicity of bundles and calculate dimensions of local moduli. As particularly interesting examples, we describe numerical invariants of bundles on some local Calabi–Yau threefolds.

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1. Introduction

This paper is the second part of a long-term project to study bundles on threefolds, their moduli and how they change under birational transformations of the base. An essential component within this plan is the study of the local situation, that is, holomorphic bundles over open threefolds such as a neighbourhood of a reduced local complete intersection Z inside a smooth threefold W. The case when Z is a ruled surface was studied in [4]. In this paper we focus on the case when Z is a curve.

For the local situation, one has to make some delicate choices: algebraic versus holomorphic bundles; a small analytic neighbourhood versus a Zariski neighbourhood versus a formal neighbourhood of Z in W. Our choice for this paper is the nicest possible situation, namely when the conormal bundle of Z inside W is ample. By "nice" we mean that in this case the aforementioned choices all amount to the same results (Theorem 3.2 together with the formal principle 4.3); moreover, filtrability implies that we can study moduli in a very concrete fashion: by considering Ext-spaces modulo bundle isomorphisms, which immediately produce quotient stacks, but *a priori* no moduli spaces. In fact, given the absence of a notion of stability for bundles on non-compact manifolds, there is no preferred way to obtain moduli spaces out of these quotients, and we are led to study stratifications via dimensions and numerical invariants.

Here we discuss dimension of local deformation spaces and provide concrete examples of bundles on some Calabi–Yau threefolds. The threefold situation contrasts totally with the two-dimensional situation, where the components of the local holomorphic Euler characteristic provided the required stratifications of the local moduli (cf. [3, Theorem 4.1] and [5, Theorem 4.15]). Our examples in this paper show that the local holomorphic Euler characteristic is not a satisfactory invariant for the case of a curve inside a threefold. In particular, the invariant $w(E) = h^0\left((\pi_*E)^{\vee\vee}/\pi_*E\right)$, which provided meaningful geometric information in the case of an exceptional curve Z in a surface (where π is the contraction of Z), vanishes identically for exceptional curves in n-folds for $n \geq 3$ (Lemma 5.2). We introduce the notion of partial invariants and tabulate some examples.

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2. Results

Let W be a connected complex manifold (or smooth algebraic variety) and Z a curve contained in W that is reduced, connected and locally a complete intersection. Let $\widehat{Z}$ denote the formal completion of Z in W. Ampleness of the conormal bundle (see Definition 3.1) has a strong influence on the behaviour of bundles on $\widehat{Z}$. We show:

Theorem 3.2. *If the conormal bundle $N^*_{Z,W}$ is ample, then every vector bundle on $\widehat{Z}$ is filtrable. If in addition Z is smooth, then every holomorphic bundle on $\widehat{Z}$ is algebraic.*

It then follows from Peternell's Existence Theorem (Theorem 4.3) that the same results hold true for bundles on an analytic neighbourhood of Z in W. Alternatively, one can work with the more concrete cases where W is the total space of a vector bundle over Z; we consider such spaces with $Z \cong \mathbb{P}^1$ in Sections 6 and 7. It turns out that when $Z \cong \mathbb{P}^1$ and $N^*_{Z,W}$ is ample, then the dimension $\gamma(E)$ of the local deformation space of a bundle E on $\widehat{Z}$ depends *only* on the restriction of E to Z. We show:

Theorem 6.5. *Let $W = \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n}\right)$ and Z the zero section. Let E be a vector bundle on W (or on $\widehat{Z}$) such that $E|_Z$ has splitting type $a_1 \geq \cdots \geq a_r$. Then the dimension of the local deformation space at E is*

$$\gamma(E) = \sum_{a_i - a_j > 1} \sum_{t=0}^{a_i - a_j - 2} (a_i - a_j - 1 - t) \cdot \binom{t + n - 1}{t}.$$

However, for curves other than $\mathbb{P}^1$ a similar statement is very far from the truth, that is, the restriction to Z does not determine the dimension of the deformation space. So the question of local moduli becomes more complicated as well as more interesting.

In Section 8 we consider the case of chains of transversally intersecting $\mathbb{P}^1$'s. We give some numerical conditions on a bundle E under which the dimension of the deformation space depends only on the splitting type (Propositions 8.8 and 8.10); but these are very restrictive conditions relating the splitting type of E and the degree of the conormal bundle, seldom satisfied, but nevertheless also satisfied by certain non-split bundles.

For the case of curves of positive genus, the situation is naturally more complicated. Dimension of the local deformation space of a split bundle is given by an exact formula (Theorem 9.4), but in full generality, even for bundles whose filtrations have fixed degrees the dimension of the local moduli still varies. Note that for a curve Z of positive genus, the isomorphism type of the normal bundle $N_{Z,W}$ is no longer determined by the degree of a filtration, and the dimension of the local deformation space $\gamma(E)$ of a bundle E on $\widehat{Z}$ depends both on $F = E|_Z$ and $N_{Z,W}$. Yet not all hope is lost, and it is still possible to give an upper bound for $\gamma(E)$, at least in the case when $N_{Z,W}$ is a general element of the moduli space $M(Z; r, d)$ of stable vector bundles on Z of rank r and degree d. For $E|_Z = F$ we write

$$\gamma(F, N) := \sum_{t \geq 0} \gamma(F, N, t), \quad \text{where } \gamma(F, N, t) := h^1\left(Z; \mathcal{E}nd(F) \otimes S^t(N^*)\right),$$

and show:

Theorem 9.5. *Let Z be a curve of genus $g \geq 2$. Fix a rank-r vector bundle F on Z and any increasing filtration $\{F_i\}_{i=0}^r$ of F such that $F_r = F$, $F_0 = \{0\}$ and each F_i/F_{i-1} is a line bundle. Set $a_i := \deg(F_i/F_{i-1})$, $1 \leq i \leq r$. Assume that $N_{Z,W}$ is a general element of $M(Z; n - 1, d)$. Then for any bundle E on $\widehat{Z}$ with $E|_Z = F$,*

$$\gamma(F, N_{Z,W}, t) \leq \sum_{i=1}^{r} \sum_{j=1}^{r} \max\left\{0,\ t\left(d + 2g - 2 + a_i - a_j\right)/r + 1 - g\right\} \cdot \binom{t + r - 1}{t}$$

for all $t \geq 1$.

We also point out a rigidity behaviour when $g = 1$ (Proposition 9.9); namely, when Z has genus 1, if both $E|_Z$ and $N_{Z,W}$ are semi-stable, then $\gamma(E|_Z, N_{Z,W}, t) = 0$ for all $t > 0$.

In Section 7 we discuss concrete examples of Calabi–Yau threefolds and provide a table of numerical invariants.

3. Filtrability and Algebraicity

Let W be a connected, complex manifold (or smooth algebraic variety) and $Z \subset W$ reduced, connected and locally a complete intersection in W. Let $\mathcal{I}$ be the ideal sheaf of Z, then the conormal sheaf $\mathcal{I}_{Z,W}/\mathcal{I}^2_{Z,W}$ is locally free, and we write $N_{Z,W}$ for its dual. We will assume that the conormal bundle $N^*_{Z,W}$ is ample. For all integers $m \geq 0$ let $Z^{(m)}$ denote the mth infinitesimal neighbourhood of Z in W, i.e. the closed analytic subspace of W with $\mathcal{I}_Z^{m+1}$ as its ideal sheaf. Thus $Z^{(0)} = Z$. Let $\widehat{Z} := \varprojlim_m Z^{(m)}$ denote the formal completion of Z in W.

In this section we assume that Z is a curve and that the conormal bundle $N^*_{Z,W}$ is ample. We recall the definition of ampleness for a general vector bundle (a.k.a. locally free sheaf of finite rank) according to [9, p. 321]:

Definition 3.1. Let $(X, \mathcal{O}_X)$ be a complex space and $\mathcal{E}$ a locally free $\mathcal{O}_X$-module of finite rank (i.e. the sheaf of sections of a holomorphic vector bundle E). Then $\mathcal{E}$ (or E) is called *ample* if for every coherent sheaf $\mathcal{F}$ on X there exists an integer n such that $\mathcal{G} := \mathcal{F} \otimes_{\mathcal{O}_X} S^n(\mathcal{E})$ is generated as an $\mathcal{O}_X$-module by its global sections.

[Recall that this means that there exist sections $s_1, \ldots, s_p \in \Gamma(X, \mathcal{G})$ such that for every open $U \subseteq X$, $\mathcal{G}(U)$ is generated by $\{s_1|_U, \ldots, s_p|_U\}$ as an $\mathcal{O}_X(U)$-module. This is equivalent by [9, Proposition 2.1] to every stalk $\mathcal{G}_x$ being generated by $\{s_{1,x}, \ldots, s_{p,x}\}$ as an $\mathcal{O}_{X,x}$-module.]

We will also use the term *spanned* for "generated by global sections", which is equivalent on complex spaces.[1]

The aim of this section is to prove:

Theorem 3.2. *If the conormal bundle $N^*_{Z,W}$ is ample, then every vector bundle on $\widehat{Z}$ is filtrable. If in addition Z is smooth, then every holomorphic bundle on $\widehat{Z}$ is algebraic.*

Lemma 3.3. *Assume that $Z \subset W$ is a smooth curve and that $N^*_{Z,W}$ is ample. Then there exists $H \in \operatorname{Pic} \widehat{Z}$ such that $H|_{Z^{(n)}}$ is ample for all n.*

Proof. For each $m > n \geq 0$ the restriction map $\rho_{m,n} \colon \operatorname{Pic} Z^{(m)} \to \operatorname{Pic} Z^{(n)}$ is surjective because $H^2\left(\widehat{Z}; S^i(N^*_{Z,W})\right) = 0$ for all i; the result follows from the definition of $\operatorname{Pic} \widehat{Z}$. $\square$

Lemma 3.4 (Filtrability). *Assume that $Z \subset W$ is a smooth curve and that $N^*_{Z,W}$ is ample. Fix any integer $r \geq 2$ and any rank-r vector bundle E on $\widehat{Z}$. Then there exists an increasing filtration $E_1 \subset \cdots \subset E_{r-1} \subset E_r := E$ of E by subbundles such that $E_1 \in \operatorname{Pic} \widehat{Z}$ and $E_i/E_{i-1} \in \operatorname{Pic} \widehat{Z}$ for all $2 \leq i \leq r$.*

Proof. Fix $H \in \operatorname{Pic} \widehat{Z}$ as in Lemma 3.3. Since $N^*_{Z,W}$ is ample, there is an integer m such that $S^n(N^*_{Z,W})$ is spanned (i.e. generated by global sections) for all integers $n \geq m$. Since $H|_{Z^{(m)}}$ is ample, there is an integer $t \geq 0$ such that $H^1\left(Z^{(m)}; E(tH)|_{Z^{(m)}}\right) = 0$ and $E(tH)|_{Z^{(m)}}$ is spanned. Since $\dim Z^{(m)} = 1$, $h^2\left(Z^{(m)}; \mathcal{F}\right) = 0$ for every coherent sheaf $\mathcal{F}$ on $Z^{(m)}$ and since $S^n(N^*_{Z,W})$ is spanned for all integers $n \geq m$, we get

$$H^1\left(Z^{(m)}; \left(E(tH)|_{Z^{(m)}}\right) \otimes S^n(N^*_{Z,W})\right) = 0 \quad \text{for all } n \geq m$$

and that $\left(E(tH)|_{Z^{(m)}}\right) \otimes S^n(N^*_{Z,W})$ is spanned for all $n \geq m$. Using the exact sequences

$$0 \longrightarrow S^n(N^*_{Z,W}) \longrightarrow \mathcal{O}_Z^{(n)} \longrightarrow \mathcal{O}_Z^{(n-1)} \longrightarrow 0,$$

we get that $E(tH)|_{Z^{(n)}}$ is spanned for all $n \geq m$ and also that the restriction map

$$\eta \colon H^0\left(\widehat{Z}; E(tH)\right) \to H^0\left(Z^{(m)}; E(tH)|_{Z^{(m)}}\right)$$

is surjective. Since $\dim Z^{(m)} = 1$ and $r \geq 2$, there is a nowhere vanishing section

$$\sigma \in H^0\left(Z^{(m)}; E(tH)|_{Z^{(m)}}\right),$$

i.e. an inclusion $(H^{\otimes t})^\vee|_{Z^{(m)}} \to E|_{Z^{(m)}}$ with locally free cokernel. Take $\alpha \in H^0\left(\widehat{Z}; E(tH)\right)$ such that $\eta(\alpha) = \sigma$. Since Z is the support of both $\widehat{Z}$ and $Z^{(m)}$, η induces an inclusion $j \colon (H^{\otimes t})^\vee \to E$ with locally free cokernel. Set $E_1 := j\left((H^{\otimes t})^\vee\right)$. If $r = 2$, then we are done. If $r \geq 3$ we apply induction on r to the rank-$(r-1)$ vector bundle E/E_1. $\square$

Theorem 3.5 (Algebraicity). *Assume that $Z \subset W$ is a smooth curve and that $N^*_{Z,W}$ is ample. Then holomorphic vector bundles on $\widehat{Z}$ are algebraic.*

Proof. By Lemma 3.4, vector bundles on $\widehat{Z}$ are extensions

$$0 \longrightarrow L_{a_r} \longrightarrow E \longrightarrow F_{r-1} \longrightarrow 0$$

and are therefore classified by $\operatorname{Ext}^1\left(F_{r-1}, L_{a_r}\right) = H^1\left(F_{r-1} \otimes L_{a_r}^{-1}\right)$, which is finite-dimensional. Furthermore, each such extension (and hence each such E) is uniquely determined from some finite, infinitesimal neighbourhood $X^{(n)}$, an algebraic object, and consequently E is algebraic. $\square$

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[1] The notion of a *spanned* vector bundle is more general on spaces over non-algebraically closed fields.

4. Local deformation space

Let W be a connected, complex manifold (or smooth algebraic variety) and $X \subset W$ reduced, connected and locally a complete intersection in W. We recall some general definitions and properties of deformation spaces.

Definition 4.1 (*Deformation of a Vector Bundle, [16, p. 113]*). Let X be complex space and V_0 a holomorphic vector bundle on X. A *deformation* of V_0 is a pair (S, V) consisting of a germ (S, s_0) of a complex space and a vector bundle V on $X \times S$ together with an isomorphism $V_0 \cong V|_{X \times \{s_0\}}$.

Let (S', V') be another deformation of V_0. A *morphism* $(S', V') \to (S, V)$ is a pair (α, f) consisting of a morphism $\alpha \colon S' \to S$ of germs of spaces and a morphism of vector bundles $f \colon V' \to V$ over α (i.e. a holomorphic map which maps fibrewise and is linear on fibres), which respects the isomorphisms of the central fibres.

A deformation (S, V) of V_0 is called

1. *complete* if for every deformation (S', V') of V_0 there exists a morphism

   $$(\alpha, f) \colon (S', V') \to (S, V),$$

2. *effective* if the following is true:

   If (S', V') is another deformation of V_0 and if $(\alpha, f), (\beta, g) \colon (S', V') \to (S, V)$ are morphisms of deformations, and if $T(\alpha), T(\beta) \colon T(S') \to T(S)$ are the corresponding tangential maps, then $T(\alpha) = T(\beta)$,

3. *semi-universal* if (S, V) is complete and effective.

In this paper we calculate the minimal dimension of a semi-universal deformation for bundles on $\widehat{X}$ in the case when $N_{X,W}^*$ is ample. The existence of a finite-dimensional, semi-universal deformation in this situation is given by results of Peternell:

Remark 4.2. If the conormal sheaf $N_{X,W}^*$ is ample, the deformation space of a bundle on $\widehat{X}$ is finite-dimensional:

Fix an integer $m \geq 0$, a vector bundle E_m on $X^{(m)}$ and set $E_0 := E_m|_X$. If

$$h^2 \left(X; \mathcal{E}nd\, E_0 \otimes S^m(N_{X,W}^*) \right) = 0,$$

then there exists a vector bundle E_{m+1} on $X^{(m+1)}$ such that $E_{m+1}|_{X^{(m)}} \cong E_m$ [16, Satz 1].

Now let F be a vector bundle over X such that $h^2 \left(X; \mathcal{E}nd\, F \otimes S^t(N_{X,W}^*) \right) = 0$ for all $t > 0$. If $N_{X,W}^*$ is ample, then $h^1 \left(X; \mathcal{E}nd\, F \otimes S^t(N_{X,W}^*) \right) = 0$ for $t \gg 0$, and hence

$$\gamma = \sum_{t \geq 0} h^1 \left(X; \mathcal{E}nd\, F \otimes S^t(N_{X,W}^*) \right) < +\infty.$$

Then there exists a vector bundle G on $\widehat{X}$ such that $G|_X \cong F$, and for a fixed such G the *deformation space* of G is isomorphic to $\mathbb{C}^\gamma$ ([17, Satz 2], and first Bemerkung at p. 115, and see also [7, Theorem 10.3.16]). There is a vector bundle A on an analytic neighbourhood U of X in W such that $A|_{\widehat{X}} = G$, and hence $A|_X \cong F$ [17, Satz 3].

We will work with objects on infinitesimal neighbourhoods $X^{(m)}$ or on the formal completion $\widehat{X}$ of $X \subset W$. However, in general the goal is to obtain statements about an actual neighbourhood of X, i.e. a tubular neighbourhood. Extending objects that are defined on the formal neighbourhood to a tubular neighbourhood is done via the *Formal Principle*.

We restrict our attention to the case when X is an exceptional set, i.e. when W is the resolution $\sigma \colon W \to W'$ of some (possibly singular) space W', $\sigma(X) = x \in W'$ and $W \setminus X \cong W' \setminus \{x\}$ via σ.

Theorem 4.3 (*Existence, [16]*). Let W be a complex space and let X be an exceptional subspace. Let $\mathcal{E}$ be a locally free sheaf on the formal completion $\widehat{X}$ of X. Then there is a locally free sheaf $\mathcal{F}$ on a neighbourhood $U \supset X$ in W such that $\mathcal{F}|_{\widehat{X}} \cong \mathcal{E}$. Moreover, $\mathcal{F}$ is uniquely determined by the germ of the embedding $X \hookrightarrow W$. $\quad\square$

5. Local holomorphic Euler characteristic

Suppose that Z is an exceptional curve in W, so that Z can be contracted to a point, and let W' be the space obtained from W by contracting Z to a (singular) point x, and let $\pi \colon W \to W'$ be the contraction. Following Blache [6, 3.9], we define the *local holomorphic Euler characteristic* of reflexive sheaves near Z.

Definition 5.1. The *local holomorphic Euler characteristic* of a reflexive sheaf $\mathcal{F}$ near $Z \subset W$ is

$$\chi(Z, \mathcal{F}) := \chi(x, \pi_* \mathcal{F}) := h^0 \left(W'; (\pi_* \mathcal{F})^{\vee\vee} / \pi_* \mathcal{F} \right) + \sum_{i=1}^{r-1} (-1)^{i-1} h^0 \left(W'; R^i \pi_* \mathcal{F} \right), \tag{5.1}$$

where $r = \operatorname{rk} \mathcal{F}$.

Since in our case W is of cohomological dimension 1, the higher direct images $R^i \pi_* \mathcal{F}$ in Eq. (5.1) are all zero for $i > 1$.

Lemma 5.2. *Let Z be a curve of codimension $n \geq 2$ in W. Then for any reflexive sheaf $\mathcal{F}$ on W we have*

$$h^0 \left(W'; (\pi_* \mathcal{F})^{\vee\vee} / \pi_* \mathcal{F} \right) = 0.$$

Proof. The map π contracts Z to the point x, whereas the restriction $\pi|_{W \setminus Z} \colon W \setminus Z \to W' \setminus \{x\}$ is an isomorphism. Let $\widetilde{U} \subseteq W$ be an open set containing Z and let $U = \pi(\widetilde{U})$. For a section $\sigma \in \Gamma(U \setminus \{x\}, \pi_* \mathcal{F})$ there is thus a corresponding section $\widetilde{\sigma} \in \Gamma(\widetilde{U} \setminus Z, \mathcal{F})$ such that $\pi_* \widetilde{\sigma} = \sigma$.

However, Z is of codimension $n > 1$ in W, so $\widetilde{\sigma}$ can be extended uniquely (by Hartogs' Theorem) to a section $\widetilde{\tau} \in \Gamma(\widetilde{U}, \mathcal{F})$ such that $\widetilde{\tau}|_{\widetilde{U} \setminus Z} = \widetilde{\sigma}$, and thus the image $\tau := \pi_* \widetilde{\tau}$ is a unique extension of σ over the singular point, i.e. $\tau|_{U \setminus \{x\}} = \sigma$. It follows that $\Gamma(U \setminus \{x\}, \pi_* \mathcal{F}) \cong \Gamma(U, \pi_* \mathcal{F})$ for all open sets $U \subseteq W'$, making $\pi_* \mathcal{F}$ a *normal* sheaf in the sense of Barth, and by [10, Proposition 1.6] $\pi_* \mathcal{F}$ is reflexive; i.e. $(\pi_* \mathcal{F})^{\vee\vee} \cong \pi_* \mathcal{F}$. $\square$

Remark 5.3. Note that this result is in strong contrast with the case of surfaces ($n = 1$), for which $h^0 \left(W'; (\pi_* \mathcal{F})^{\vee\vee} / \pi_* \mathcal{F} \right)$ attains a wide variety of values (compare with [5, Theorem 2.16]).

Corollary 5.4. *If the codimension of the curve Z in W is at least 2, then for any reflexive sheaf $\mathcal{F}$ on W, we have $\chi(Z, \mathcal{F}) = h^0 \left(W'; R^1 \pi_* \mathcal{F} \right)$.* $\square$

6. Total space of $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n}$

Let W be an $(n+1)$-dimensional complex manifold, $Z \subset W$ with $Z \cong \mathbb{P}^1$ and $N_{Z,W}$ isomorphic to the direct sum of n line bundles of degree -1. In fact, we can assume that $W = \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n} \right)$. We denote by $\widetilde{Z}$ the inverse limit of all complex manifolds W' for which W' is an open neighbourhood of Z in W and by $\widehat{Z} = \varprojlim_m Z^{(m)}$ the formal completion of Z in W.

Lemma 6.1. $\mathrm{Pic}\, \widetilde{Z} \equiv \mathrm{Pic}\, \widehat{Z} \cong \mathbb{Z}$. *More precisely, every line bundle L on $\widetilde{Z}$ or $\widehat{Z}$ is uniquely determined by the integer $\deg(L|_Z)$.*

Proof. To check the result for $\widehat{Z}$, it is sufficient to show that for all integers $i \geq 0$ the restriction map $\rho_i \colon \mathrm{Pic}\, Z^{(i+1)} \to \mathrm{Pic}\, Z^{(i)}$ is bijective. The map ρ_i is injective (resp. surjective) because $H^1 \left(Z; S^i(N_{Z,W}^*) \right) = 0$ (resp. $H^2 \left(Z; S^i(N_{Z,W}^*) \right) = 0$). The restriction map $\mathrm{Pic}\, \widetilde{Z} \to \mathrm{Pic}\, \widehat{Z}$ is bijective by Grothendieck's existence theorem [8, 5.1.4]. $\square$

Notation: Let L_k denote the only line bundle on $\widetilde{Z}$ or $\widehat{Z}$ such that $\deg(L|_Z) = k$. Also, for any $\mathcal{O}_W$-module $\mathcal{F}$, we write

$$\mathcal{F}^{(n)} := \mathcal{F} \otimes_{\mathcal{O}_W} \left(\mathcal{O}_W / \mathcal{I}^{n+1} \right),$$

where $\mathcal{I}$ is the ideal sheaf of $Z \subset W$. We can think of $\mathcal{F}^{(n)}$ as the sheaf of sections of $\mathcal{F}$ on the nth infinitesimal neighbourhood of Z.

Lemma 6.2. $H^1 \left(\widetilde{Z}; L_k \right) = H^1 \left(\widehat{Z}; L_k \right) = 0$ *for all $k \geq -1$.*

Proof. Direct calculation. $\square$

Proposition 6.3. *Let E be a vector bundle on $\widetilde{Z}$ or $\widehat{Z}$ such that $E|_Z$ has splitting type $a_1 \geq \cdots \geq a_r$ with $a_r \geq a_1 - 1$. Then $E \equiv \bigoplus_{i=1}^r L_{a_i}$.*

Proof. Use Lemmas 3.4 and 6.2. $\square$

Remark 6.4. By Remark 4.2 the dimension of the local deformation space of any vector bundle E on $\widehat{Z}$ is

$$\gamma = \sum_{t \geq 0} h^1 \left(Z; \mathcal{E}nd(E|_Z) \otimes S^t(N_{Z,W}^*) \right) < +\infty.$$

Now $a_1 \geq \cdots \geq a_r$ denote the splitting type of $E|_Z$. Since here $N_{Z,W}^* \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus n}$, we have

$$\gamma(E) = \sum_{t \geq 0} \binom{t + n - 1}{t} h^1 \left(\mathbb{P}^1; \mathcal{E}nd(E|_Z)(t) \right).$$

Thus the dimension of the deformation space of E depends only on n and the splitting type of $E|_Z$. By Grothendieck's existence theorem the same is true with $\widetilde{Z}$ instead of $\widehat{Z}$.

We calculate $\gamma(E)$:

Theorem 6.5 (*Dimension of Local Deformation Space*). *Let E be a vector bundle on $\widetilde{Z}$ or $\widehat{Z}$ such that $E|_Z$ has splitting type $a_1 \geq \cdots \geq a_r$. Then the dimension of the local deformation space at E is*

$$\gamma(E) = \sum_{a_i - a_j > 1} \sum_{t=0}^{a_i - a_j - 2} (a_i - a_j - 1 - t) \cdot \binom{t + n - 1}{t}.$$

Proof. According to Remark 6.4, the dimension $\gamma(E)$ does not depend on the choice of extension E, so we can assume $E = L_{a_1} \oplus \cdots \oplus L_{a_r}$. In this case,

$$\mathscr{E}nd\, E = \bigoplus_{1 \leq i,j \leq r} L_{a_j - a_i},$$

and consequently

$$h^1\left(Z; \mathscr{E}nd\, E \otimes S^t(N^*_{Z,W})\right) = \sum_{1 \leq i,j \leq r} h^1\left(\mathbb{P}^1; L_{a_i - a_j} \otimes S^t(N^*_{Z,W})\right).$$

If $a_j - a_i \geq -1$, then $h^1\left(\mathbb{P}^1; L_{a_j - a_i}\right) = 0$, so we have

$$\gamma(E) = \sum_{t \geq 0} \sum_{a_i - a_j > 1} h^1\left(\mathbb{P}^1; L_{a_j - a_i} \otimes S^t(N^*_{Z,W})\right).$$

We perform the computations on $W = \text{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus n}\right)$ using the standard coordinate charts $(z, u_1, \ldots, u_n) \mapsto (z^{-1}, zu_1, \ldots, zu_n)$. In these charts, the bundle $L_{a_j - a_i}$ has transition function $z^{a_i - a_j}$, and consequently a non-trivial 1-cocycle σ on the tth infinitesimal neighbourhood has an expression of the form:

$$\sigma = \sum_{i_1 + \cdots + i_n = t} \sum_{k = a_j - a_i + 1 + t}^{-1} \sigma_{i_1 \ldots i_n k}\, z^k u_1^{i_1} \cdots u_n^{i_n},$$

and non-zero terms occur only when $t \leq a_i - a_j - 2$. Hence, for each pair $a_i - a_j > 1$ there are $\sum_{t=0}^{a_i - a_j - 2}(a_i - a_j - 1 - t) \cdot \binom{t + n - 1}{t}$ terms. $\square$

7. Local Calabi–Yau threefolds

In this section we specialise to the case of local Calabi–Yau threefolds. We provide only simple examples to give some idea of the behaviour of the local holomorphic Euler characteristic (χ for short). The general *Macaulay* 2 algorithm to compute numerical invariants for bundles on threefolds will appear in the companion paper [12]. Here we tabulate some invariants for the local threefolds

$$W_i := \text{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-i) \oplus \mathcal{O}_{\mathbb{P}^1}(i-2)\right).$$

The cases $i = 1, 2, 3$ present quite different behaviour and are particularly interesting from the point of view of birational geometry, see [11]. Recall from Eq. (5.1) that the local holomorphic Euler characteristic of a rank-2 bundle is by definition $\chi(Z, \mathcal{F}) := w + h$, where $w := h^0\left(W'; (\pi_*\mathcal{F})^{\vee\vee}/\pi_*\mathcal{F}\right)$ and $h := h^0\left(W'; R^1\pi_*\mathcal{F}\right)$. However, by Lemma 5.2, on threefolds we have $w \equiv 0$; and even worse, for $i > 1$ bundles on W_i have $\chi = \infty$. To remedy this situation, we introduce some partial invariants, obtained by restriction to a surface.

Definition 7.1. Let $D_i \cong \text{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-i)\right)$, considered as the hypersurface $\text{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-i) \oplus 0\right)$ of W_i. We define the *partial invariants*:

$$h'(\mathcal{F}) := h\left(\mathcal{F}|_{D_i}\right),$$
$$w'(\mathcal{F}) := w\left(\mathcal{F}|_{D_i}\right).$$

Some comments about the W_i for $i = 1, 2, 3$:

1. $W_1 := \text{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)\right)$

   Note that W_1 is just the specialisation to $n = 2$ of the spaces in the previous section; and appears in the well-known diagram of the basic flop: Let X be the cone over the ordinary double point defined by the equation $xy - zw = 0$ on $\mathbb{C}^4$. The basic flop is described by the diagram:

Table 1
Partial invariants of some bundles on W_1, W_2 and W_3 (see Definition 7.1)

	W_1			W_2			W_3		
	χ	h'	w'	χ	h'	w'	χ	h'	w'
$j = 0$, any p	0	0	0	0	0	0	∞	0	0
$j = 1$, any p	0	0	1	0	0	0	∞	0	0
$j \geq 2$, $p = 0$	$f_0(j)$	$f_1(j)$	$g_1(j)$	∞	$f_2(j)$	$g_2(j)$	∞	$f_3(j)$	$g_3(j)$
$j = 3$, $p = u_1$	3	2	1	∞	2	1	∞	2	1
$j = 4$, $p = zu_1$	6	3	1	∞	3	0	∞	3	0
$j = 4$, $p = z^3 u_1$	7	4	6	∞	3	2	∞	3	1
$j = 4$, $p = z^3 u_1^2$	9	5	6	∞	4	2	∞	3	2
$j = 5$, $p = zu_1$	10	4	1	∞	4	0	∞	4	0
$j = 5$, $p = z^3 u_1$	11	5	6	∞	4	2	∞	4	1
$j = 5$, $p = z^3 u_1^2$	16	7	6	∞	6	2	∞	5	2

where $\widetilde{X} := \widetilde{X}_{x,y,z,w}$ is the blow-up of X at the vertex $x = y = z = w = 0$, $X^- := \widetilde{X}_{x,z}$ is the blow-up of X along $x = z = 0$ and $X^+ := \widetilde{X}_{y,w}$ is the blow-up of X along $y = w = 0$. The *basic flop* is the transformation from X^- to X^+.

By Theorem 3.5 holomorphic bundles on W_1 are algebraic, and this has the nice consequence that χ is finite over W_1.

2. $W_2 := \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(0)\right)$

Note that $W_2 \cong Z_2 \times \mathbb{C}$, and we can contract the zero section on the first factor, thus obtaining a singular family $X_2 \times \mathbb{C}$, where X_2 is the surface containing an ordinary double-point singularity defined by $xy - z^2 = 0$ in $\mathbb{C}^3$. Holomorphic bundles on W_2 have infinite χ, but h' and w' take finite values on W_2.

Note that in contrast to W_1, there are strictly holomorphic (not algebraic) bundles on W_2, although every rank-2 bundle on W_2 is still an extension of line bundles.

3. $W_3 := \mathrm{Tot}\left(\mathcal{O}_{\mathbb{P}^1}(-3) \oplus \mathcal{O}_{\mathbb{P}^1}(+1)\right)$

Here not even a partial contraction of the zero section is possible, but nevertheless we can still calculate partial h' and w'.

Again, on W_3 there are strictly holomorphic (not algebraic) bundles, and moreover, there are (many) rank-2 bundles which are not extensions of line bundles.

Given a rank-2 bundle E on W_i with vanishing first Chern class, its restriction to the zero section Z determines an integer $j \geq 0$ called the *splitting type* of E, such that $E|_Z \cong \mathcal{O}_{\mathbb{P}^1}(j) \oplus \mathcal{O}_{\mathbb{P}^1}(-j)$. We consider bundles E that are extensions of line bundles

$$0 \longrightarrow \mathcal{O}(-j) \longrightarrow E \longrightarrow \mathcal{O}(j) \longrightarrow 0.$$

Such a bundle is uniquely determined by j and an extension class p. We tabulate the values of χ, h', w' in Table 1, in which the formulae for the split bundles are as follows:

$$f_0(j) = \left(j^3 - j\right)/6 \qquad\qquad g_1(j) = \left(j^2 + j\right)/2$$

$$f_1(j) = \left(j^2 - j\right)/2 \qquad\qquad g_2(j) = \left\lfloor \frac{j}{2} \right\rfloor j - \left\lfloor \frac{j}{2} \right\rfloor^2$$

$$f_2(j) = \left\lfloor \frac{j}{2} \right\rfloor \left(j - \left\lfloor \frac{j}{2} \right\rfloor \right) \qquad\qquad g_3(j) = \frac{1}{2} \left\lfloor \frac{j}{3} \right\rfloor \left(2j - 1 - 3 \left\lfloor \frac{j}{3} \right\rfloor \right)$$

$$f_3(j) = \frac{1}{2} \left\lfloor \frac{j+1}{3} \right\rfloor \left(2j + 1 - 3 \left\lfloor \frac{j+1}{3} \right\rfloor \right).$$

8. Neighbourhoods of $\mathbb{P}^1$-chains

Fix an integer $a > 0$. Since the case $a = 1$ was studied in Section 6, we will often silently assume $a \geq 2$. Let Y_a be the nodal and connected projective curve with a irreducible components, say $T_1, \ldots, T_a$, such that $T_i \cong \mathbb{P}^1$ for all i and $T_i \cap T_j = \varnothing$ if $|i - j| \geq 2$, and $\#(T_j \cap T_{j+1}) = 1$ for all $j = 1, \ldots, a - 1$. Notice that the plurigenus is $p_a(Y_a) = 0$. Let W be an n-dimensional complex manifold (or smooth algebraic variety) and $Z_a \subset W$ a compact complex subspace such that $Z_a \cong Y_a$. Abusing notation, we will also call T_i the irreducible components of Z_a. Since Z_a is a local complete intersection in W, its conormal sheaf $N^*_{Z_a, W}$ is locally free.

Every vector bundle on Y_a is isomorphic to a direct sum of line bundles. Hence set $N^*_{Z_a, W} \cong L_1 \oplus \cdots \oplus L_{n-1}$ with $L_j \in \mathrm{Pic}\, Z_a$. For each $1 \leq j \leq n - 1$ and each $1 \leq i \leq a$ set

$$b_{i,j} := \deg(L_j|_{T_i})$$

and set

$$b_i := \min_{1 \leq j \leq n-1} b_{i,j}.$$

The first set of integers may depend upon a choice of the ordering $L_1, \ldots, L_{n-1}$ of the indecomposable factors of $N^*_{Z_a, W}$. The bundle $N^*_{Z_a, W}$ is ample if and only if each L_j is ample, i.e. if and only if each $L_j|_{T_i}$ is ample, i.e. if and only if $b_{i,j} > 0$ for all $1 \le j \le n-1$ and all $1 \le i \le a$. Notice that the case $b_{i,j} = 1$ for all i, j is the natural generalisation of Section 6. From now on we will assume the ampleness of $N^*_{Z_a, W}$.

Remark 8.1. We have $\operatorname{Pic} Z_a \cong \mathbb{Z}^{\oplus a}$. More precisely, for every ordered set $(d_1, \ldots, d_a)$ of a integers there is (up to isomorphisms) a unique line bundle L on Z_a such that $\deg(L|_{T_i}) = d_i$ for all i. Then L is ample if and only if it is very ample if and only if $d_i > 0$ for all i, and L is spanned if and only if $d_i \ge 0$ for all i. Furthermore, if L is spanned, then $h^1(Z_a; L) = 0$ and hence $h^0(Z_a; L) = 1 + \sum_{i=1}^a d_i$ (Riemann–Roch).

Set $\mathbf{deg}(L) := \big(\deg(L|_{T_1}), \ldots, \deg(L|_{T_a})\big) \in \mathbb{Z}^{\oplus a}$.

Remark 8.2. Since $\dim Z_a = 1, h^2(Z_a; \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on Z_a. We have $h^1(Z_a; L) = 0$ for every line bundle on Z_a such that $\deg(L|_{T_j}) \ge 0$ for at least $a-1$ indices $j \in \{1, \ldots, a\}$ and $\deg(L|_{T_i}) \ge -1$ for all i (use $a-1$ Mayer–Vietoris exact sequences and the cohomology of line bundles on $\mathbb{P}^1$).

Lemma 8.3. *Pic* $\widetilde{Z_a} \equiv$ *Pic* $\widehat{Z_a} \cong \mathbb{Z}^{\oplus a}$*. More precisely, every line bundle L on $\widetilde{Z_a}$ or $\widehat{Z_a}$ is uniquely determined by the ordered set of a integers* $\mathbf{deg}(L|_{Z_a})$*.

Proof. To check the result for $\widehat{Z_a}$ it is sufficient to show that for all integers $i \ge 0$ the restriction map $\rho_i \colon \operatorname{Pic} Z_a^{(i+1)} \to \operatorname{Pic} Z_a^{(i)}$ is bijective. The map ρ_i is injective (resp. surjective), because $H^1(Z_a; S^i(N^*_{Z_a, W})) = 0$ (resp. $H^2(Z_a; S^i(N^*_{Z_a, W})) = 0$) (Remark 8.2). The restriction map $\operatorname{Pic} \widetilde{Z_a} \to \operatorname{Pic} \widehat{Z_a}$ is bijective by Grothendieck's existence theorem [8, 5.1.4]. $\quad\square$

Notation: For each rank-r vector bundle F on Z_a let $a_{i,1}(F) \ge \cdots \ge a_{i,r}(F)$ be the splitting type of $F|_{T_i}$. Set $\epsilon_i(F) := a_{i,1} - a_{1,r}$, $1 \le i \le a$. For each rank-r vector bundle E on $\widehat{Z_a}$ set $a_{i,h}(E) := a_{i,h}(E|_{Z_a})$, $1 \le i \le a$, $1 \le h \le r$, and $\epsilon_i(E) := \epsilon_i(E|_{Z_a}) = a_{i,1}(E) - a_{i,r}(E)$, $1 \le i \le a$.

Notation: For every integer i such that $1 \le i \le a$ set $T[i] := T_1 \cup T_i$. For each $1 \le i \le n-1$ set $P_i := T_i \cap T_{i+1}$. Set $T[0] := \varnothing$.

Lemma 8.4. *Let F be a rank-r vector bundle on Z_a such that $a_{i,r} \ge -1$ for all i and $a_{i,r} \ge 0$ for at least $a-1$ indices. Then $h^1(Z_a; F) = 0$. If $a_{i,r} \ge 0$ for all i, then F is spanned.*

Proof. Notice that $a_{i,r} \ge -1$ if and only if $h^1(T_i; F|_{T_i}) = 0$ and that $a_{i,r} \ge 0$ if and only if $F|_{T_i}$ is spanned. In particular $h^1(T_1; F|_{T_1}) = 0$. Hence we may assume $a \ge 2$. Fix an integer $1 \le i < a$ and assume $h^1(T[i]; F|_{T[i]}) = 0$. To get the first assertion of the Lemma by induction on i it is sufficient to prove $h^1(T[i+1]; F|_{T[i+1]}) = 0$. To get the second assertion we also need to prove that if $F|_{T[i]}$ and $F|_{T_i}$ are spanned, then $F|_{T[i+1]}$ is spanned. Consider the Mayer–Vietoris exact sequence

$$0 \longrightarrow F|_{T[i+1]} \longrightarrow F|_{T[i]} \oplus F|_{T_{i+1}} \longrightarrow F|_{\{P_i\}} \longrightarrow 0. \tag{8.1}$$

By assumption, $h^1(T[i]; F|_{T[i]}) = h^1(T_{i+1}; F|_{T_{i+1}}) = 0$. If $a_{i+1} \ge 0$, then $F|_{T_i}$ is spanned, and hence the restriction map

$$\eta \colon H^0(T[i]; F|_{T[i]}) \oplus H^0(T_{i+1}; F|_{T_{i+1}}) \longrightarrow F|_{\{P_i\}}$$

is surjective. The cohomology exact sequence of (8.1) gives $h^1(T[i+1]; F|_{T[i+1]}) = 0$. If $a_{i+1,r} = -1$, then the inductive assumption gives that $F|_{T[i]}$ is spanned. Hence the first component of η is surjective, and consequently so is η, and again we obtain $h^1(T[i+1]; F|_{T[i+1]}) = 0$. Similarly, fixing any $Q \in T[i+1]$ we see that if both $F|_{T[i]}$ and $F|_{T_{i+1}}$ are spanned, then $F|_{T[i+1]}$ is spanned, concluding the inductive proof. $\quad\square$

Lemma 8.5. *Let E be a rank-r vector bundle on $\widehat{Z_a}$. Assume that $b_i \ge \epsilon_i(E) - 1$ for all $1 \le i \le a$, and assume the existence of $b \in \{1, \ldots, a\}$ such that $b_i \ge \epsilon_i(E)$ for all $i \in \{1, \ldots, a\} \setminus \{b\}$ and all $1 \le j \le n-1$. Then the natural restriction map $\rho_E \colon H^1(\widehat{Z_a}; E) \to H^1(Z_a; E|_{Z_a})$ is bijective.*

Proof. The surjectivity of ρ_E comes just from Remark 4.2, it uses only that $\dim Z_a = 1$ and that $X = Z_a$ is reduced and a local complete intersection in W. To check the injectivity of ρ_E it is sufficient to prove $H^1(Z_a; S^m(N^*_{Z_a, W}) \otimes (E|_{Z_a})) = 0$ for all $m \ge 1$. Fix an integer $m \ge 1$ and set $F := S^m(N^*_{Z_a, W}) \otimes (E|_{Z_a})$ and $t := \binom{n+m-1}{n-1} r$. Notice that $t = \operatorname{rk} F$. Notice that $a_{i,t}(F) = mb_i + a_{i,r}$. Since $m \ge 1$, our assumptions give $a_{i,t}(F) \ge -1$ for all i and $a_{i,t}(F) \ge 0$ for at least $a-1$ indices i. Apply Lemma 8.4. $\quad\square$

Proposition 8.6. *Fix any rank-r vector bundle E on $\widehat{Z_a}$ and a decomposition $E|_{Z_a} \cong L_1 \oplus \cdots \oplus L_r$ with $L_h \in \operatorname{Pic} Z_a$, $1 \le h \le r$. Let $\widehat{L_h}$ denote the only line bundle on $\widehat{Z_a}$ such that $\widehat{L_h}|_{Z_a} \cong L_h$ (Lemma 8.3). Assume $b_i \ge 2\epsilon_i(E) - 1$ for all i and $b_i \ge 2\epsilon_i(E)$ for at least $a-1$ indices i. Then $E \cong \widehat{L_1} \oplus \cdots \oplus \widehat{L_r}$.*

Proof. We fix an isomorphism $\alpha_0 : E \to L_1 \oplus \cdots \oplus L_n$. Set $A := \mathcal{H}om\,(E, \widehat{L}_1 \oplus \cdots \oplus \widehat{L}_r)$. It is sufficient to show the existence of $\alpha \in H^0\left(\widehat{Z_a}; A\right)$ which induces an isomorphism $E \to \widehat{L}_1 \oplus \cdots \oplus \widehat{L}_r$, or, using Nakayama's Lemma, to prove the existence $\alpha \in H^0\left(\widehat{Z_a}; A\right)$ such that $\alpha|_{Z_a} = \alpha_0$. Equivalently, it is sufficient to show the existence of

$$\left\{\alpha_m \in H^0\left(Z_a^{(m)}; A|_{Z_a^{(m)}}\right)\right\}_{m \geq 1} \quad \text{such that } \alpha_m|_{Z^{(m-1)}} = \alpha_{m-1}.$$

Fix an integer $m \geq 1$ and assume that $\alpha_0, \ldots, \alpha_{m-1}$ have been constructed as above. The obstruction to the existence of α_m is an element of $H^1\left(Z_a; S^m(N^*_{Z_a, W}) \otimes (A|_{Z_a})\right)$. Notice that $\epsilon_i(A) = 2\epsilon_i(E)$ for all $1 \leq i \leq a$, and now apply Lemma 8.5. $\quad\square$

Remark 8.7. Let E be a vector bundle on $\widehat{Z_a}$. Set $F := E|_{Z_a}$. By Remark 4.2 the cohomology group $H^1\left(\widehat{Z_a}; \mathcal{E}nd\,E\right)$ is finite-dimensional, and $H^1\left(\widehat{Z_a}; \mathcal{E}nd\,E\right)$ (resp. $H^1\left(Z_a; \mathcal{E}nd\,F\right)$) is the tangent space to the deformation space of E (resp. F). Remark 4.2 gives the surjectivity of the restriction map

$$\rho_{\mathcal{E}nd\,E} : H^1\left(\widehat{Z_a}; \mathcal{E}nd\,E\right) \to H^1\left(Z_a; \mathcal{E}nd\,F\right).$$

In Proposition 8.8 we will give a sufficient condition for the injectivity of the map $\rho_{\mathcal{E}nd\,E}$. Then in Proposition 8.10 we will give other cases in which we are able to "partially control" $\ker(\rho_{\mathcal{E}nd\,E})$.

The proof of Proposition 8.6 gives the following result.

Proposition 8.8. *Fix a rank-r vector bundle E on $\widehat{Z_a}$. Assume that $b_i \geq 2\epsilon_i(E) - 1$ for all i and assume $b_i \geq 2\epsilon_i(E)$ for at least $a - 1$ indices i. Then the restriction map $\rho_{\mathcal{E}nd\,E} : H^1\left(\widehat{Z_a}; \mathcal{E}nd\,E\right) \to H^1\left(Z_a; \mathcal{E}nd\,(E|_{Z_a})\right)$ is bijective.* $\quad\square$

Remark 8.9. Fix an integer $m \geq 0$ and a vector bundle A on $\widehat{Z_a}$. Let

$$\rho_{A,m} : H^1\left(\widehat{Z_a}; A\right) \to H^1\left(Z_a^{(m)}; A|_{Z_a^{(m)}}\right).$$

Remark 4.2 gives the surjectivity of $\rho_{A,m}$ and that the map $\rho_{A,m}$ is injective if

$$h^1\left(Z_a; S^t(N^*_{Z,W}) \otimes A\right) = 0 \quad \text{for all } t \geq m+1.$$

Since $b_i\left(S^t(N^*_{Z,W})\right) = tb_i$, the proof of Lemma 8.5 gives that $\rho_{A,m}$ is injective if $(m+1)b_i \geq \epsilon_i(A) - 1$ for all i and $(m+1)b_i \geq \epsilon_i(A)$ for at least $a - 1$ indices i. Now we fix a vector bundle E on $\widehat{Z_a}$ and take $A := \mathcal{E}nd\,(E|_{Z_a})$. Since $\epsilon_i(\mathcal{E}nd\,E) = 2\epsilon_i(E)$, we get the following result.

Proposition 8.10. *Fix a rank-r vector bundle E on $\widehat{Z_a}$ and an integer $m \geq 0$. If $(m+1)b_i \geq 2\epsilon_i(E) - 1$ for all i and $(m+1)b_i \geq 2\epsilon_i(E)$ for at least $a - 1$ indices i, then the restriction map*

$$\rho_{\mathcal{E}nd\,E,m} : H^1\left(\widehat{Z_a}; \mathcal{E}nd\,E\right) \to H^1\left(Z_a^{(m)}; \mathcal{E}nd\,(E|_{Z_a^{(m)}})\right)$$

is bijective. $\quad\square$

9. Neighbourhoods of positive-genus curves

Let W be an n-dimensional complex manifold or an n-dimensional smooth algebraic variety and $Z \subset W$ a closed submanifold which is a smooth and connected curve of genus $g > 0$. *We assume that the conormal bundle $N^*_{Z,W}$ is ample.*

Remark 9.1. Let F be a semi-stable vector bundle on Z. Since we work in characteristic zero, $\mathcal{E}nd\,F \cong \mathcal{O}_Z \oplus \text{ad}\,F$, and ad F is a semi-stable vector bundle on Z of degree zero [14, Theorem 2.5 or 2.6].

Let $M(Z; r, d)$ denote the moduli space of all stable vector bundles on Z with rank r and degree d. If $g \geq 2$, then $M(Z; r, d)$ is a non-empty irreducible variety of dimension $r^2(g - 1) + 1$.

Remark 9.2. Set $d := \deg N_{Z,W}$. Since $\dim Z = 1$, $h^2(Z; \mathcal{F}) = 0$ for every coherent sheaf $\mathcal{F}$ on Z. We also recall that $h^0(Z; A) = 0$ if A is a semi-stable (resp. stable) vector bundle and $\deg A < 0$ (resp. $\deg A \leq 0$). Thus the restriction map $\text{Pic}\,\widehat{Z} \to \text{Pic}\,Z$ is surjective. The restriction map $\text{Pic}\,\widehat{Z} \to \text{Pic}\,Z$ is injective if $h^1\left(Z; S^t(N^*_{Z,W})\right) = 0$ for all integers $t \geq 1$, i.e. if

$$\alpha(N_{Z,W}, t) := h^0\left(Z; S^t(N_{Z,W}) \otimes \omega_Z\right) = 0$$

for all integers $t \geq 1$.

Now we will give some conditions which ensure that $\alpha(N_{Z,W}, t) = 0$: Let $\mu_+(N_{Z,W})$ denote the maximal slope of a non-zero subsheaf of $N_{Z,W}$. The ampleness of $N^*_{Z,W}$ implies $\mu_+(N_{Z,W}) < 0$. Since the tensor product of semi-stable vector bundles is semi-stable [14, Theorem 2.6], we get $\mu_+(S^t(N_{Z,W})) = t\mu_+(N_{Z,W})$. Hence $\alpha(N_{Z,W}, t) = 0$ if $N_{Z,W}$ is semi-stable and $d < (n-1)(2-2g)$ or $N_{Z,W}$ is stable and $d \leq (n-1)(2-2g)$. If $N_{Z,W}$ is a general element of $M(Z; n-1, d)$, then $\alpha(N_{Z,W}, t) = 0$ for all $t \geq 1$ if and only if $d \leq (n-1)(1-g)$.

For any vector bundle F on Z and every integer $t \geq 0$ set

$$\gamma(F, N, t) := h^1\left(Z; \mathcal{E}nd\, F \otimes S^t(N^*)\right)$$

and

$$\gamma(F, N) := \sum_{t \geq 0} \gamma(F, N, t).$$

The ampleness of $N_{Z,W}^*$ implies $\gamma(F, N_{Z,W}) < +\infty$. Furthermore, there is a vector bundle E on $\widehat{Z}$ such that $E|_Z \cong F$, the deformation space of any such E is smooth and $\gamma(F, N_{Z,W})$ is the dimension of the deformation space of any such E. Hence our main aim will be the computation of the integers $\gamma(F, N, t)$ for some N and/or some F.

Remark 9.3. Set $r := \mathrm{rk}\, F$. We have

$$\gamma(F, N, 0) = h^1(Z; \mathcal{E}nd\, F) = r^2(g - 1) + h^0(Z; \mathcal{E}nd\, F),$$

where $h^0(Z; \mathcal{E}nd\, F) \geq 1$, with equality if F is stable. For an arbitrary F we cannot say much about the integer $h^0(Z; \mathcal{E}nd\, F)$ and hence about the integer $\gamma(F, N, 0)$.

As an example, assume $r \geq 2$ and $F \cong \bigoplus_{i=1}^r L_i$, where the L_i are line bundles. Thus

$$h^0(Z; \mathcal{E}nd\, F) = \sum_{i=1}^r \sum_{j=1}^r h^0\left(Z; L_i \otimes L_j^*\right).$$

Hence $h^0(Z; \mathcal{E}nd\, F) \geq r$, with equality if and only if $h^0\left(Z; L_i \otimes L_j^*\right) = 0$ for all $i \neq j$. For all integers r, a, x there exists a decomposable F as above with rank r, degree a and $h^0(Z; \mathcal{E}nd\, F) \geq x$.

Theorem 9.4. *Assume $g \geq 2$ and fix $F \cong \bigoplus_{i=1}^r L_i$, where each L_i is a line bundle of degree a_i. Assume that N is a general element of $M(Z; n - 1, d)$. Then*

$$\gamma(F, N, t) = \sum_{i=1}^r \sum_{j=1}^r \max\left\{0,\ t\left(d + 2g - 2 + a_i - a_j\right)/r + 1 - g\right\} \cdot \binom{t + r - 1}{t} \tag{9.1}$$

for all $t \geq 1$.

Proof. Note that $\mathcal{E}nd\, F \cong \bigoplus_{i=1}^r \bigoplus_{j=1}^r L_i \otimes L_j^*$, that $L_i \otimes L_j^*$ has degree $a_i - a_j$ if $i \neq j$ and that $L_i \otimes L_j^* \cong \mathcal{O}_Z$ if $i = j$. For all integers m, t such that $t \geq 0$ set

$$u_{t,m} := \max\{0,\ t(d + 2g - 2 + m)/r + 1 - g\} \cdot \binom{t + r - 1}{t}.$$

Fix any $M \in \mathrm{Pic}\, Z$ and take a general $N \in M(Z; n - 1, d)$. By [2, Theorem 1], we have $h^1\left(Z; S^t(N^*) \otimes M\right) = h^0\left(Z; S^t(N) \otimes M^* \otimes \omega_Z\right) = u_{t,m}$. Notice that in the decomposition of $\mathcal{E}nd\, F$ only finitely many line bundles appear. Hence $\gamma(F, N, t) = \sum_{i=1}^r \sum_{j=1}^r u_{t,a_i - a_j}$. □

We stress that in the statement of Theorem 9.4 we first choose the line bundles L_i and then take a general N. If we fix a general N, then for all choices of $L_i \in \mathrm{Pic}^{a_i}(Z)$ we only claim the inequality

$$\gamma(F, N, t) \leq \sum_{i=1}^r \sum_{j=1}^r \max\left\{0,\ t\left(d + 2g - 2 + a_i - a_j\right)/r + 1 - g\right\} \cdot \binom{t + r - 1}{t}$$

(just use semi-continuity and the proof of Theorem 9.4 just given), but it is easy to get examples (say, with $t = 1$) in which this inequality is strict.

Theorem 9.5 (*Dimension of Local Moduli*). *Assume $g \geq 2$. Fix a rank-r vector bundle F on Z and any increasing filtration $\{F_i\}_{i=0}^r$ of F such that $F_r = F$, $F_0 = \{0\}$ and each F_i/F_{i-1} is a line bundle. Set $a_i := \deg(F_i/F_{i-1})$, $1 \leq i \leq r$. Assume that N is a general element of $M(Z; n - 1, d)$. Then*

$$\gamma(F, N, t) \leq \sum_{i=1}^r \sum_{j=1}^r \max\left\{0,\ t\left(d + 2g - 2 + a_i - a_j\right)/r + 1 - g\right\} \cdot \binom{t + r - 1}{t} \tag{9.2}$$

for all $t \geq 1$.

Proof. If $r = 1$, then $F = F_1$ and we may apply Theorem 9.4. Assume $r \geq 2$. By Theorem 9.4 it is sufficient to prove the existence of an integral smooth affine curve T, $0 \in T$, and a flat family $\{F_\lambda\}_{\lambda \in T}$ such that $F_0 \cong \bigoplus_{i=1}^{r} F_i / F_{i-1}$ and $F_\lambda \cong F$ for all $\lambda \in T \setminus \{0\}$. Notice that $F_1 = F_1 / F_0$ is a line bundle. Call ϵ the extension

$$0 \longrightarrow F_1 \longrightarrow F \longrightarrow F/F_1 \to 0. \tag{9.3}$$

Set $T := \mathbb{C}$. For every $\lambda \in \mathbb{C} \setminus \{0\}$, the extension $\lambda \epsilon$ has as its middle sheaf space F_λ a vector bundle isomorphic to F. The zero-extension of F/F_1 by F_1 has $F_1 \oplus F/F_1$ as its middle sheaf. Then use induction on r. $\quad\square$

Corollary 9.6. *Assume $g \geq 2$. Fix any integer c and assume that (N, F) is general in $M(Z; n - 1, d) \times M(Z; r, cr)$. Then*

$$\gamma(F, N, t) \leq r^2 \cdot \binom{t + r - 1}{t} \cdot \max \{0, \ t\,(d + 2g - 2)\,/r + 1 - g\}.$$

Proof. Fix any $R \in \mathrm{Pic}^{-c}(Z)$. Since $\mathcal{E}nd\, F \cong \mathcal{E}nd\,(F \otimes R)$ and $(N, F \otimes R)$ is general in $M(Z; n - 1, d) \times M(Z; r, 0)$, it is sufficient to do the case $c = 0$. Apply Theorem 9.5 to $\mathcal{O}_Z^{\oplus r}$ taking $F_i := \mathcal{O}_Z^{\oplus i}$ for all $1 \leq i \leq r$. Hence $a_i = 0$ for all i. Every vector bundle on Z is a flat limit of a family of stable vector bundles with the same degree and the same rank [15, Proposition 2.6]. Apply this observation to $\mathcal{O}_Z^{\oplus r}$ and use the semi-continuity theorem for cohomology groups. $\quad\square$

Corollary 9.7. *Assume $g \geq 2$. Fix any integer d and assume that (N, F) is general in $M(Z; n-1, d) \times M(Z; r, a)$. Write $a = rx + y$ with $x, y \in \mathbb{Z}$ and $0 \leq y < r$. Set $c := \min \{y, r - y\}$. Then $\gamma(F, N, t)$, $t \geq 1$, satisfies the inequality (9.2) with $a_i := 0$ for $1 \leq i \leq r - 1$ and $a_r := c$.*

Proof. First assume $c = y$, i.e. $0 \leq y \leq \lfloor r/2 \rfloor$. Fix any $R \in \mathrm{Pic}^{-x}(Z)$ and any $M \in \mathrm{Pic}^y(Z)$. Note that $\mathcal{E}nd\, F \cong \mathcal{E}nd\,(F \otimes R)$ and $(N, F \otimes R)$ is general in $M(Z; n - 1, d) \times M(Z; r, y)$, and every vector bundle on Z is a flat limit of a family of stable vector bundles with the same degree and the same rank [15, Proposition 2.6]. Apply this observation to $M \oplus \mathcal{O}_Z^{\oplus(r-1)}$ and use the semi-continuity theorem for cohomology groups.

Now assume $c = r - y$. Notice that $\mathcal{E}nd\, F \cong \mathcal{E}nd\,(F^*)$ and that $-a \equiv c \,(\mathrm{mod}\ r)$. Apply the first part of the proof to the vector bundle F^*. $\quad\square$

Remark 9.8. For any rank-$(r \geq 2)$ vector bundle F there are many filtrations as in the statement of Theorem 9.5 with very different numerical invariants. However, when $r = 2$, there is always a numerically better filtration, as we will soon see. Let F be a rank-r vector bundle. First assume that F is not semi-stable. Then F fits in an extension of line bundles

$$0 \longrightarrow L_1 \longrightarrow F \longrightarrow L_2 \longrightarrow 0$$

with $\deg L_1 > \deg L_2$. Furthermore, this extension is unique and minimises the integer $|\deg L_1 - \deg L_2|$.

Note that the integer $\deg(L_1) + \deg(L_2) = \deg(F)$ is fixed. To get the best estimate from inequality (9.2), we need to find an extension (9.3) which minimises the integer $|\deg L_1 - \deg L_2|$. If F is semi-stable but not stable, then we may find an extension (Theorem 9.4) with $\deg L_1 = \deg L_2$. Now assume that F is stable. There is a unique integer $s(F)$ such that $s(F) \equiv \deg F \,(\mathrm{mod}\ 2)$, $0 < s(F) \leq g$, and there is an extension (9.3) with $\deg L_2 = \deg L_1 + s(F)$ [13, Proposition 3.1]. Such an extension is not always unique (see e.g. [13, Corollary 4.6 and Theorem 5.1]), but any of those minimises the integer $|\deg L_1 - \deg L_2|$.

Proposition 9.9. *Assume that $g = 1$ and that both E and $N_{Z,W}$ are semi-stable and write $F := E|_Z$. Then $\gamma(F, N_{Z,W}, t) = 0$ for all $t \geq 1$, and consequently the deformation space of E coincides with the deformation space of $E|_Z$.*

Proof. By Serre duality we need to prove that for $t \geq 1$

$$h^0 \left(Z; S^t(N_{Z,W}) \otimes \mathcal{E}nd\,(E|_Z)\right) = 0.$$

Since $N_{Z,W}^*$ is ample, we have $\deg \left(S^t(N_{Z,W}) \otimes \mathcal{E}nd\,(E|_Z)\right) < 0$. By Atiyah's classification of vector bundles on elliptic curves [1] or by [14, Theorem 2.6], $S^t(N_{Z,W}) \otimes \mathcal{E}nd\,(E|_Z)$ is semi-stable. Hence $h^0 \left(Z; S^t(N_{Z,W}) \otimes \mathcal{E}nd\,(E|_Z)\right) = 0$ [1]. $\quad\square$

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