Character kernels and degree ratios in finite groups

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1. Introduction

Let \( K = \ker(\chi) \) be the kernel of an irreducible character \( \chi \) of some finite group \( G \). It is known that in certain extreme situations, \( K \) must be nilpotent. For example, \( K \) is nilpotent if \( \chi(1) \) is the maximum of the degrees of the irreducible characters of \( G \) or if \( K \) is minimal (with respect to containment) in the set of kernels of irreducible characters of \( G \). (We discuss these results of S. Garrison and D. Broline more fully below.) In this paper, we obtain some analogous results for irreducible character kernels that are not extreme in either of these senses.

In general, of course, character kernels need not be nilpotent or even solvable. We obtain control, however, over solvable normal subgroups contained in kernels of irreducible characters. Specifically, given an irreducible character kernel \( K \), we establish an upper bound on the Fitting height of the unique largest solvable normal subgroup \( S(K) \) of \( K \). (Recall that the Fitting height \( h(S) \) of a solvable group \( S \) can be viewed as a measure of how far \( S \) is from being nilpotent. By definition, \( h(S) \) is the smallest integer \( h \) such that \( F_h(S) = S \), where \( F_i(S) \) is the \( i \)th term of the ascending Fitting series of \( S \),
defined by setting $F_0(S) = 1$ and $F_i(S)/F_{i-1}(S) = F(S/F_{i-1}(S))$ for $i > 0$.) Of course, our bound on the Fitting height of the solvable radical $S(K)$ automatically applies to all normal solvable subgroups of the irreducible character kernel $K$.

We generalize the notions of minimal irreducible character kernels and irreducible characters of maximal degree as follows. For integers $m \geq 1$, we say that a subgroup $K \subseteq G$ is an $m$th minimal irreducible character kernel in $G$ if there exist irreducible character kernels $K_i$ in $G$ such that $K = K_1 \supseteq K_2 \supseteq \cdots \supseteq K_m$, where $m$ is as large as possible. Similarly, if we fix a subset $\mathcal{X} \subseteq \text{Irr}(G)$, we say that a character $\chi \in \mathcal{X}$ has $m$th maximal degree in $\mathcal{X}$ if there exist characters $\chi_i \in \mathcal{X}$ such that $\chi(1) = \chi_1(1) < \chi_2(1) < \cdots < \chi_m(1)$, where $m$ is as large as possible. (The increased generality obtained by considering subsets $\mathcal{X}$ of $\text{Irr}(G)$ will be useful later, but for now, we take $\mathcal{X} = \text{Irr}(G)$.) If $m = 1$, we know that $m$th minimal kernels are nilpotent and that characters with $m$th maximal degree in $\text{Irr}(G)$ are nilpotent. In general, we have the following.

**Theorem A.** Let $K = \ker(\chi)$, where $\chi \in \text{Irr}(G)$, and let $S = S(K)$. Assume that either

(a) $\chi$ has $m$th maximal degree in $\text{Irr}(G)$ or
(b) $K$ is an $m$th minimal irreducible character kernel in $G$.

Then $h(S) \leq m$.

In fact, if $\chi$ has second maximal degree in $\text{Irr}(G)$, we will show (via an appeal to the classification of simple groups) that $K = \ker(\chi)$ is solvable, and thus $K$ has Fitting height at most $2$ by Theorem A(a). For integers $m \geq 4$, kernels of $m$th maximal degree characters in $\text{Irr}(G)$ definitely need not be solvable, but we leave unresolved the case $m = 3$. (It seems likely, however, that kernels of characters of third maximal degree in $G$ must be solvable.)

Another natural way to generalize the hypothesis that $\chi \in \text{Irr}(G)$ has maximum degree is as follows. Given a real number $a \geq 1$, we can impose the condition that $\psi(1) \leq a \chi(1)$ for all characters $\psi \in \text{Irr}(G)$. (Note that $\chi$ has maximal degree in $\text{Irr}(G)$ if and only if it satisfies this condition with $a = 1$.)

To state our result in this situation, we define an integer valued function $H(a)$ for real numbers $a \geq 1$. We set $H(a)$ to be the maximum of the Fitting heights of all solvable subgroups of the symmetric group $S_n$, where $n = [a]$, the greatest integer in $a$.

**Theorem B.** Let $\chi \in \text{Irr}(G)$, and write $S = S(K)$, where $K = \ker(\chi)$. Suppose that $a$ is a real number such that $\psi(1) \leq a \chi(1)$ for all characters $\psi \in \text{Irr}(G)$. Then $h(S) \leq H(a) + 1$.

If $a < 2$, then $H(a) = 0$, so the assertion of Theorem B in this case is that $S(K)$ is nilpotent. In fact, if $a < 2$, an appeal to the classification of simple groups can be used to show that $K = \ker(\chi)$ must be solvable, and hence $K$ is nilpotent by Theorem B. (This, of course, is a proper generalization of the fact that kernels of characters of maximal degree in $\text{Irr}(G)$ are nilpotent, which is the case $a = 1$.)

In general, we shall see that $H(a) \leq (3/2) \log_2(a)$, and so Theorem B yields a logarithmic bound on the Fitting height of $S(K)$ in terms of the parameter $a$.

Now write $b(G)$ to denote the largest irreducible character degree of a finite group $G$. It is well known that if $N \triangleleft G$ and $G/N$ is nonabelian, then $b(N) \leq b(G)/2$, and it follows that if $G$ is solvable, then its derived length $dl(G)$ satisfies $dl(G) \leq 1 + 2 \log_2(b(G))$. (We present the easy proof of this fact in Section 6.) Using Theorem B, we can obtain a somewhat analogous bound on the Fitting height $h(G)$ of a nonabelian solvable group $G$ in terms of the ratio $b(G)/c(G)$, where $c(G)$ is the minimum degree of a nonlinear character of $G$. (We refer to $b(G)/c(G)$ as the degree ratio of the nonabelian group $G$, and we write $\text{rat}(G) = b(G)/c(G)$.) We have asserted that $h(G)$ bounded in terms of $\text{rat}(G)$ for nonabelian solvable groups $G$, but actually, much more is true: a similar bound can be obtained for the derived length $dl(G)$.

**Theorem C.** Let $G$ be solvable and nonabelian. If $\text{rat}(G) < 2$ then $dl(G) \leq 3$, and in general, $dl(G) \leq 3 + 4 \log_2(\text{rat}(G))$. 
Next, we present some background and history. The fact that $K = \ker(\chi)$ is nilpotent if $\chi$ has maximum degree in $\text{Irr}(G)$ is due to S. Garrison. Actually, Garrison proved a stronger theorem: not only is $K$ is nilpotent, but also $L$ is nilpotent whenever $L/K$ is an abelian chief factor of $G$. (This result is contained in Theorem 12.19 of [2].) The fact that a minimal irreducible character kernel $K$ is nilpotent is a consequence of a stronger theorem of D. Broline, which asserts that $L$ is nilpotent whenever $K \subseteq L \triangleleft G$ and $L/K$ is nilpotent. (This is Theorem 12.24 of [2].)

Let $K \triangleleft G$. We shall say that $K$ is $E$-nilpotent in $G$ if $K$ is nilpotent and also $L$ is nilpotent whenever $L/K$ is an abelian chief factor of $G$. (Note that there may be no such subgroup $L$, so the requirement that $K$ is nilpotent is not redundant in this definition.) Similarly, we shall say that $K$ is strongly $E$-nilpotent if $L$ is nilpotent whenever $K \subseteq L \triangleleft G$ and $L/K$ is nilpotent. Broline's theorem asserts, therefore, that minimal irreducible character kernels are strongly $E$-nilpotent, and Garrison's theorem asserts that kernels of irreducible characters of maximum degree are $E$-nilpotent. (But note that these kernels are not necessarily strongly $E$-nilpotent. If $G$ is dihedral of order 24, for example, then $b(G) = 2$, and there exists an irreducible character of degree 2 with kernel $K$ of order 3. Then $G/K$ is nilpotent but $G$ is not, and thus $K$ is not strongly $E$-nilpotent.)

We mention that if $K$ is an $E$-nilpotent normal subgroup in $G$ and $N \subseteq K$ with $N \triangleleft G$, then $N$ is also $E$-nilpotent in $G$. Certainly, $N$ is nilpotent, and to check that it is $E$-nilpotent, suppose that $M/N$ is an abelian chief factor of $G$. Then either $M \subseteq K$, in which case it is obviously nilpotent, or else $MK/K$ is $G$-isomorphic to $M/N$, and so it is an abelian chief factor in $G$. In the latter case, $MK/K$ is nilpotent, and it follows that $M$ is nilpotent. Similar reasoning shows that if $K$ is strongly $E$-nilpotent in $G$, then $N$ is also strongly $E$-nilpotent in $G$.

As we shall see, the notion of $E$-nilpotence is crucial in the proof of Theorem A. In the case where $K = \ker(\chi)$ is an $m$th minimal kernel, we shall appeal to the full strength of Broline's theorem, and in the case where $\chi$ has $m$th maximal degree, we shall find ourselves essentially reproving Garrison's theorem.

We should also mention another result related to Garrison's theorem. Let $N \triangleleft G$, and recall that by definition, $\text{Irr}(G|N)$ is the set $\{\chi \in \text{Irr}(G) \mid N \not\subseteq \ker(\chi)\}$. Now choose a character $\chi$ in $\text{Irr}(G|N)$ having maximum possible degree, and let $K = \ker(\chi)$. Theorem 4.1 of [4] asserts that $N \cap K$ is $E$-nilpotent in $G$, but in fact, a stronger result is true: $K$ is actually $E$-nilpotent in this situation. Since this implies that $N \cap K$ is $E$-nilpotent, we see that Theorem 4.1 of [4] is a consequence of the following result.

Theorem D. Let $N \triangleleft G$, and suppose that $\chi$ has maximum degree in $\text{Irr}(G|N)$. Then $\ker(\chi)$ is $E$-nilpotent in $G$.

Of course, Theorem D is essentially a generalization of Garrison's theorem. (To see this, take $N = G$.) We will prove an even more general result by considering characters of large degree in certain "closed" subsets $\mathcal{X}$ of $\text{Irr}(G)$, which we will define. As we shall see, the whole set $\text{Irr}(G)$ is closed, as are all subsets of the form $\text{Irr}(G|N)$ for normal subgroups $N$ of $G$, and thus both Garrison's theorem and Theorem D can be viewed as special cases of this more general result, which appears below as Theorem 3.5. Also, this notion of closed sets of characters yields strong forms of Theorems A(a) and B, and these appear as Theorems 3.6 and 4.1.

I close this introduction with thanks to Alex Moretó. It was he who suggested the line of research pursued in this paper by asking whether or not some version of Theorem A might be true.

2. Nearly minimal kernels

The proof of Theorem A(b) is a fairly straightforward application of Broline's theorem, and we dispose of that first. We begin by restating the result in a stronger form.

(2.1) Theorem. Let $K$ is an $m$th minimal irreducible character kernel in $G$, and write $S = S(K)$. Suppose that $S \subseteq T \triangleleft G$, where $T/S$ is nilpotent. Then $h(T) \leq m$.

Proof. If $m = 1$, then $K$ is a minimal kernel in $G$, and thus $S = K$ is strongly $E$-nilpotent in $G$ by Broline's theorem. It follows that $T$ is nilpotent, and hence $h(T) \leq 1$, as required. We can thus assume that $m \geq 2$, and we proceed by induction on $m$. 

By assumption, there exist irreducible character kernels $K_i$ in $G$ such that $K = K_1 > K_2 > \cdots > K_m$, where $m$ is as large as possible. The longest kernel chain descending from $K_2$ thus has $m - 1$ terms, and so $K_2$ is an $(m - 1)$th minimal kernel in $G$. Now $S \cap K_2 = S(K_2)$, so if $S \subseteq K_2$, then in fact, $S = S(K_2)$, and hence $h(T) \leq m - 1$ by the inductive hypothesis, and there is nothing further to prove.

We can assume, therefore, that $S(K_2) < S$. Since $S$ is solvable, we can choose a subgroup $U \subseteq S$ such that $S(K_2) < U \triangleleft G$ and $U/S(K_2)$ is nilpotent, and hence by the inductive hypothesis, $h(U) \leq m - 1$. Now $U \not\subseteq S(K_2) = K_2 \cap S$, and thus $U \not\subseteq K_2$, and we have $K_2 < U/K_2 \subseteq K$. Since there is no irreducible character kernel of $G$ strictly between $K_2$ and $K$, it follows that $K/UK_2$ is a minimal irreducible character kernel in $G/UK_2$, and thus by Broline’s theorem, $K/UK_2$ is strongly E-nilpotent in $G/UK_2$.

Now $T \cap K$ is a solvable normal subgroup of $K$ containing $S = S(K)$, and thus $T \cap K = S$. Then $TK/K \cong T/(T \cap K) = T/S$, and so $TK/K$ is nilpotent. Since $K/UK_2$ is strongly E-nilpotent, it follows that $TK/UUK_2$ is nilpotent, and hence its subgroup $TK/UUK_2$ is nilpotent, and we deduce that $T/(T \cap UK_2)$ is nilpotent. We argue next that $T \cap UK_2 = U$, and thus $T/U$ is nilpotent, and $h(T) = 1 + h(U) \leq m$, as required. Certainly, $U \subseteq T \cap UK_2$, and so by Dedekind’s lemma, $T \cap UK_2 = U(T \cap K_2)$. But $T \cap K_2$ is a solvable normal subgroup of $K_2$, and thus $T \cap K_2 \leq S(K_2) \subseteq U$, and we have $T \cap UK_2 = U$, as wanted. \qed

3. Nearly maximal degrees

The key to the proofs of the theorem of Garrison in [2] and Theorem 4.1 in [4] is the following lemma, which we need for the proofs of Theorems A(a) and D. To state the result, we recall that the vanishing-off subgroup $V(\chi)$ of a character $\chi \in \text{Irr}(G)$ is the subgroup generated by all elements $g \in G$ such that $\chi(g) \neq 0$. Thus $\chi(x) = 0$ for $x \in G - V(\chi)$, and $V(\chi)$ is the smallest subgroup of $G$ with this property.

(3.1) Lemma. Let $H \subseteq G$, and suppose that $\theta \in \text{Irr}(H)$ has the property that $\chi_H = \theta$ for every irreducible constituent $\chi$ of $\theta^G$. Then $V(\theta) \triangleleft G$.

Proof. This is Lemma 12.17 of [2]. \qed

We also need the following easy observation.

(3.2) Lemma. Let $K \triangleleft G$, and suppose that $K \subseteq \ker(\chi)$, where $\chi \in \text{Irr}(G)$. Let $K \subseteq L \triangleleft G$, where either $L = K$ or $L/K$ is minimal normal in $G/K$. Then $L \subseteq V(\chi)$.

Proof. First, $\ker(\chi) \subseteq V(\chi)$ since if $x \in \ker(\chi)$, then $\chi(x) = \chi(1) \neq 0$. Then $K \subseteq V(\chi) \cap L \subseteq L$, and hence either $V(\chi) \cap L = L$ or $V(\chi) \cap L = K$. If $V(\chi) \cap L = L$, then $L \subseteq V(\chi)$, and there is nothing further to prove. We can suppose, therefore, that $V(\chi) \cap L = K$, and we have

$$|L| \cdot |\chi_L, 1_L| = \sum_{x \in L} \chi(x) = \sum_{x \in V(\chi) \cap L} \chi(x) = \sum_{x \in K} \chi(x) = |K| \chi(1) \neq 0,$$

where the second equality holds since $\chi$ vanishes on $L - V(\chi) \cap L$ and the fourth equality holds because $K \subseteq \ker(\chi)$. In this case, $\chi_L$ has a principal constituent, and thus $L \subseteq \ker(\chi) \subseteq V(\chi)$, as wanted \qed

(3.3) Corollary. Let $KM = G$, where $K \triangleleft G$, and suppose that $\theta \in \text{Irr}(M)$ and $K \cap M \subseteq \ker(\theta)$. Let $K \subseteq L \triangleleft G$, and assume either that $L = K$ or that $L/K$ is a minimal normal subgroup of $G/K$. Then $L \cap M = L \cap V(\theta)$.

Proof. The natural isomorphism $G/K \cong M/(K \cap M)$, carries $L/K$ to $(L \cap M)/(K \cap M)$, and thus $(L \cap M)/(K \cap M)$ is either trivial or is a minimal normal subgroup of $M/(K \cap M)$. Since $K \cap M \subseteq \ker(\theta)$, Lemma 3.2 yields $L \cap M \subseteq V(\theta)$, and thus $L \cap M \subseteq L \cap V(\theta)$. The reverse containment is obvious. \qed
Finally, we need the following elementary group-theoretic fact.

**Lemma (3.4)** Let \( L \triangleleft G \), and suppose that \( L \cap M \triangleleft G \) for every maximal subgroup \( M \) of \( G \). Then \( L \) is nilpotent.

**Proof.** It suffices to show for each prime \( p \) that a Sylow \( p \)-subgroup \( P \) of \( L \) is normal in \( G \). If \( P \) is not normal, we can choose a maximal subgroup \( M \) of \( G \) with \( N_G(P) \subseteq M \), and thus \( P \subseteq L \cap M \). But \( L \cap M \triangleleft G \) by hypothesis, and since \( P \in \text{Syl}_p(L \cap M) \), the Frattini argument yields \( G = (L \cap M)N_G(P) \subseteq M \), which is a contradiction. \( \Box \)

Given a group \( G \), we define an ordering on \( \text{Irr}(G) \) by writing \( \alpha < \beta \) if \( \alpha(1) < \beta(1) \) and \( \ker(\alpha) > \ker(\beta) \), where \( \alpha, \beta \in \text{Irr}(G) \). We say that a subset \( \mathcal{X} \subseteq \text{Irr}(G) \) is closed if \( \beta \in \mathcal{X} \) whenever \( \alpha \in \mathcal{X} \) and \( \alpha < \beta \). Of course, the whole set \( \text{Irr}(G) \) is closed, and it is easy to see that subsets of the form \( \text{Irr}(G)/N \) for \( N \triangleleft G \) are also closed.

The following is a variation on Theorem 12.19 of [2]. It includes Theorem D, as well as Garrison’s original result. It also includes a weak form of Broline’s theorem, since if \( K = \ker(\chi) \) is a minimal normal kernel, then \( \{\chi\} \) is closed. (This result does not include the full strength of Broline’s theorem, however, because it does not establish that \( K \) is strongly E-nilpotent.)

**Theorem (3.5)** Let \( \mathcal{X} \subseteq \text{Irr}(G) \) be a closed subset, and suppose that \( \chi \in \mathcal{X} \) has maximal degree in \( \mathcal{X} \). Then \( K = \ker(\chi) \) is E-nilpotent in \( G \).

**Proof.** Suppose that \( K \leq L \triangleleft G \), where either \( L/K \) is an abelian chief factor of \( G \), or else \( L = K \). We must show that \( L \) is nilpotent, so we consider a maximal subgroup \( M \) of \( G \), and we work to show that \( L \cap M \triangleleft G \). We can, of course, assume that \( L \not\subseteq M \), and thus \( LM = G \). If \( K \subseteq M \), then \( K \subseteq L \cap M \subseteq L \) and thus \( L \cap M \triangleleft L \) since \( L/K \) is abelian. Also, of course, \( L \cap M \triangleleft M \), and thus \( L \cap M \triangleleft LM = G \), as required. We can assume, therefore, that \( K \not\subseteq M \), and thus \( KM = G \). Since \( K = \ker(\chi) \), we conclude that \( \chi_M \) is irreducible, and we write \( \theta = \chi_M \).

Now let \( \psi \) be an irreducible constituent of \( \theta^G \). Then \( \theta \) is a constituent of \( \psi_M \), and thus \( \psi(1) > \theta(1) = \chi(1) \). If \( \psi(1) > \chi(1) \), then \( \psi_M \) is reducible, and thus \( M\ker(\psi) \not\subseteq G \), and it follows that \( \ker(\psi) \subseteq L \). Since \( \theta \) is a constituent of \( \psi_M \), we have \( \ker(\psi) \subseteq \ker(\theta) \subseteq \ker(\chi) \), where the second containment holds since \( \chi_M = \theta \), and it is strict because \( \ker(\chi) \not\subseteq M \). Thus \( \chi < \psi \), so \( \psi \in \mathcal{X} \), and this is a contradiction because \( \chi \) has maximum degree among the characters in \( \mathcal{X} \). We conclude that all irreducible constituents \( \psi \) of \( \theta^G \) satisfy \( \psi_M = \theta \), and hence Lemma 3.1 yields \( V(\theta) \triangleleft G \). Finally, since \( L/K \) is either trivial or is a minimal normal subgroup of \( G/K \), Corollary 3.3 yields \( L \cap M = L \cap V(\theta) \triangleleft G \), as wanted. \( \Box \)

Next, we establish a strong form of Theorem A(a).

**Theorem (3.6)** Let \( \mathcal{X} \subseteq \text{Irr}(G) \) be a closed subset, and let \( \chi \in \mathcal{X} \) have \( m \)th maximal degree in \( \mathcal{X} \). Let \( S = S(K) \), where \( K = \ker(\chi) \), and let \( F = F_{m-1}(S) \) be the \((m - 1)\)th term of the ascending Fitting series of \( S \). Then \( S/F \) is \( E \)-nilpotent in \( G/F \), and \( h(S) \leq m \).

**Proof.** Observe that \( h(F) \leq m - 1 \). When we establish that \( S/F \) is \( E \)-nilpotent in \( G/F \), we will know that \( S/F \) is nilpotent, and it will follow that \( h(S) \leq 1 + h(F) \leq m \), proving the last assertion.

If \( m = 1 \), then \( S = K \) is \( E \)-nilpotent in \( G \) by Theorem 3.5. Since \( F = 1 \) in this case, there is nothing further to prove, and thus we can assume that \( m > 1 \). Assuming now that \( S \subseteq T \triangleleft G \), where either \( T = S \) or \( T/S \) is an abelian chief factor of \( G \), our goal is to show that \( T/F \) is nilpotent. By Lemma 3.4, it suffices to show that \( T \cap M \triangleleft G \) whenever \( M \) is a maximal subgroup of \( G \) that contains \( F \). To do this, we fix a maximal subgroup \( M \supseteq F \), and we begin by reasoning as we did in the proof of Theorem 3.5.

If \( T \subseteq M \), then \( T \cap M = T \triangleleft G \), so we can assume that \( T \not\subseteq M \), and thus \( TM = G \). If \( S \subseteq M \), then since \( T/S \) is abelian, it follows that \( T \cap M \triangleleft T \), and since also \( T \cap M \triangleleft M \) and \( TM = G \), we have \( T \cap M \triangleleft G \), as wanted. We can assume, therefore, that \( S \not\subseteq M \), and thus \( SM = G \), and since \( S \subseteq K = \ker(\chi) \), we see that \( \chi_M \) is irreducible, and we write \( \theta = \chi_M \).
Now let $\psi$ be an irreducible constituent of $\theta^G$, and suppose first that $\psi(1) > \chi(1)$. Then $\psi_M$ is reducible, and thus $\ker(\psi/M) \neq G$, so $\ker(\psi) \subseteq M$, and we have $\ker(\psi) \subseteq \ker(\theta) < \ker(\chi)$. Then $\chi < \psi$, so $\psi \in \mathcal{X}$, and since $\psi(1) > \chi(1)$, it follows that there exists an integer $n < m$ such that $\psi$ has $n$th maximal degree in $\mathcal{X}$. Now write $L = \ker(\psi)$, and observe that $S(L) = S \cap L$ since $L \subseteq K$. The inductive hypothesis guarantees that $(S \cap L)/E$ is E-nilpotent in $G/E$, where $E$ is the $(n−1)$th term of the ascending Fitting series for $S \cap L$.

Now $F \leq S \cap M \subseteq K \cap M = \ker(\theta)$. Since $\theta$ is a constituent of $\psi_M$, it follows that $\psi_F$ has a principal constituent, and thus $F \subseteq \ker(\psi) = L$. Since $S \not\subseteq M$ and $L \subseteq M$, we see that $S \not\subseteq L$, and thus $S \cap L < S$, and we can choose a chief factor $U/(S \cap L)$ of $G$ with $U \subseteq S$. Since $S$ is solvable, $U/(S \cap L)$ is an abelian chief factor, and since $(S \cap L)/E$ is E-nilpotent in $G/E$, we conclude that $U/E$ is nilpotent, and thus $h(U) \leq h(E) + 1 < (n−1) + 1 = n < m$. Then $U \subseteq F_{m−1}(S) = F \subseteq L$, and we have $U \subseteq S \cap L$, which is a contradiction.

It follows that no irreducible constituent of $\theta^G$ has degree exceeding $\chi(1) = \theta(1)$, and thus Lemma 3.1 yields $V(\theta) \neq G$. Now $T/S$ is either trivial or is a minimal normal subgroup of $G/S$, and since $S \cap M \subseteq \ker(\theta)$, Corollary 3.3 yields $T \cap M = T \cap V(\theta) \neq G$, as wanted. $\square$

4. Multiples of the degree

Recall that for real numbers $a \geq 1$, we defined $H(a)$ to be the maximum of the Fitting heights of soluble subgroups of the symmetric group $S_n$, where $n = \lceil a \rceil$. The following includes Theorem B.

(4.1) Theorem. Let $\chi \in \mathcal{X}$, where $\mathcal{X}$ is a closed subset of $\text{Irr}(G)$, and let $a \geq 1$ be a real number such that $\psi(1) \leq a\chi(1)$ for all characters $\psi \in \mathcal{X}$. Let $K = \ker(\chi)$ and write $S = S(K)$. Then $h(S) \leq H(a) + 1$.

Before we begin the proof, we recall that if $S$ is a soluble group and $i \geq 0$ is an integer, there is a unique smallest subgroup $N \triangleleft S$ such that $h(S/N) \leq i$, and we write $E_i(S)$ to denote this characteristic subgroup of $S$. (The subgroups $E_i(S)$ are thus the terms of the descending Fitting series of $S$.)

Proof of Theorem 4.1. Let $m = H(a) + 1$, and write $E = E_m(S)$. Then $E \triangleleft G$, and we work to show that $E \leq F(G)$, the Frattini subgroup. One this is established, it will follow that $F(S/E) = F(S)/E$, and thus for subscripts $i \geq 1$, the terms of the ascending Fitting series of $S$ and of $S/E$ correspond, and hence $F_i(S)/E = F_i(S)/E$. In particular, since $F_m(S/E) = S/E$, it follows that $F_m(S) = S$, and thus $h(S) \leq m$, as wanted. It suffices, therefore, to show that $E \subseteq M$, for every maximal subgroup $M$ of $G$.

Let $M$ be maximal in $G$, and observe that if $S \subseteq M$ then $E \subseteq M$, and there is nothing further to prove. We can assume, therefore, that $S \not\subseteq M$, and we write $D = S \cap M$, so that $D < S$. Let $Y = \text{core}_C(D)$, and note that $Y \subseteq D < S$, and thus we can choose a chief factor $X/Y$ of $G$ with $X \subseteq S$. Then $X \not\subseteq D$, and thus $X \not\subseteq M$, and we have $XM = G$.

Since $S$ is soluble, $X/Y$ is abelian, and thus $Y \subseteq X \cap M < X$. Also $X \cap M < M$, and since $XM = G$, we conclude that $X \cap M = G$, and thus since $X \cap M < X$, we have $X \cap M = Y$.

We argue next that $X/Y$ is self-centralizing in $S/Y$. To see this, let $C = C_S(X/Y)$, and observe that $C \triangleleft G$, and thus $C \cap M \triangleleft M$. Also, $[C \cap M, X] \subseteq [C, X] \subseteq Y \subseteq C \cap M$, and thus $X$ normalizes $C \cap M$, and we have $C \cap M \triangleleft XM = G$. But $Y \subseteq C \cap M \subseteq D$ and $Y = \text{core}_C(D)$, and it follows that $C \cap M = Y$, and thus $C = X(C \cap M) = X$, as wanted.

Now $(1_{[1]}^G)^X = (1_Y)^X$ is a sum of the distinct linear characters of $X/Y$, and it follows that $((1^M)^G)^X = (1)^G$ is multiplicity-free, and hence $(1^G)^G$ and $((1^M)^G)^S$ are also multiplicity-free. Also, $(1^{[1]}_S)^S = (1_D)^S$, and we consider a nonprincipal irreducible constituent $\xi$ of this character. Then $\varphi_X$ is a sum of $\varphi(1)$ distinct nonprincipal linear characters of $X/Y$, and these are permuted by $S$.

Let $\xi$ be an irreducible constituent of $(1^G)^G$ lying over $\varphi$, and note that $[(1^G)^G, \xi] = 1$ since $(1^G)^G$ is multiplicity-free. By the Clifford correspondence, $\xi = \eta^G$ for some character $\eta \in \text{Irr}(T)$, where $T$ is the stabilizer of $\varphi$ in $G$. Also, $\eta_S$ is a multiple of $\varphi$, and $[\eta_S, \varphi] = [\xi_S, \varphi] = 1$, where the second equality holds because $((1^G)^G)^S$ is multiplicity-free. We conclude that $\eta_S = \varphi$.

Now let $\gamma$ be an irreducible constituent of $\chi^G$, and note that $\chi$ is a constituent of $\gamma^G$, and thus $\chi(1) \leq \gamma(1) = |G : T|\gamma(1)$. Also, since $S \subseteq \ker(\chi)$, we have $S \subseteq \ker(\gamma)$, and thus $\gamma \eta \in \text{Irr}(T)$ by...
Gallagher’s theorem (6.17 of [2]). Since \( \gamma \eta \) lies over \( \varphi \) and \( T \) is the stabilizer of \( \varphi \) in \( G \), it follows that the character \( \psi = (\gamma \eta)^G \) is irreducible, and we have
\[
\psi(1) = |G : T| \gamma(1) \eta(1) \geq \chi(1) \eta(1) = \chi(1) \psi(1).
\]

We argue next that \( \varphi(1) \leq a \), so we suppose that \( \varphi(1) > a \). Then \( \psi(1) \geq \chi(1) \varphi(1) > a \chi(1) \), and thus by hypothesis, \( \psi \not\in \mathcal{X} \). But \( \psi(1) > \chi(1) \), and we derive a contradiction by showing that \( \ker(\psi) < \ker(\chi) \), and thus \( \chi \prec \psi \).

Let \( L = \ker(\psi) \), and note that \( L \subseteq \ker(\gamma \eta) \) since \( (\gamma \eta)^G = \psi \), and in particular, \( L \subseteq T \). We argue that \( L \subseteq M \). Otherwise, since \( L \triangleleft G \), we have \( LM = G \), and thus \( T = L(M \cap T) \) by Dedekind’s lemma. Since \( L \subseteq \ker(\gamma \eta) \) and \( \gamma \eta \in \text{Irr}(T) \), it follows that \( (\gamma \eta)^{M \cap T} \) is irreducible, and thus \( \eta_{M \cap T} \) is irreducible. Also, we have
\[
[(\eta_{M \cap T}, 1_{M \cap T})] = [(\eta_{M \cap T})^M, 1_M] = [(\eta^G, 1_M)^G] = [\xi, (1_M)^G] = 1,
\]
and it follows that the irreducible character \( \eta_{M \cap T} \) must be the principal character. Then \( 1 \leq a < \varphi(1) = \eta(1) = 1 \), and this contradiction shows that \( L \subseteq M \), as claimed.

Since \( L \triangleleft G \), we have \( L \subseteq \ker((1_M)^G) \), and we conclude that \( L \subseteq \ker(\xi) \). Since \( \xi = \eta^G \), we have \( L \subseteq \ker(\eta) \), and since also \( L \subseteq \ker(\gamma \eta) \), we deduce that \( L \subseteq \ker(\gamma) \). Again using the fact that \( L \triangleleft G \), we have \( L \subseteq \ker(\gamma^G) \), and thus \( L \subseteq \ker(\chi) \) since \( \chi \) is a constituent of \( \gamma^G \). In fact, \( L < \ker(\chi) \) as wanted, since \( L \subseteq M \). This proves that \( \varphi(1) \leq a \), as claimed.

Since \( \varphi_X \) is a sum of \( \varphi(1) \) distinct linear characters of \( X \) that are permuted by \( S \), there exists a homomorphism from \( S \) into the symmetric group on \( \varphi(1) \) symbols. Let \( N \) be the kernel of this homomorphism, and observe that \( N \) fixes some nonprincipal linear character of \( X/Y \). Also \( S/N \) is a solvable group that is isomorphically embedded in the symmetric group on \( \varphi(1) \) symbols, and since \( \varphi(1) \leq a \), we have \( h(S/N) \leq h(a) = m - 1 \) where the inequality follows by the definition of \( h(a) \). Then \( E_{m-1}(S) \subseteq N \), and hence \( E_{m-1}(S) \) has a nontrivial fixed point in \( \text{Irr}(X/Y) \). But \( G \) acts irreducibly on the abelian group \( \text{Irr}(X/Y) \) since \( X/Y \) is a chief factor of \( G \), and since \( E_{m-1}(S) \triangleleft G \), it follows that \( E_{m-1}(S) \) acts trivially on \( \text{Irr}(X/Y) \). Because \( X/Y \) is abelian, we can conclude that \( E_{m-1}(S) \) also acts trivially on \( X/Y \). However, \( C_S(X/Y) = X \), and thus \( E_{m-1}(S) \subseteq X \), and we conclude that \( h(S/X) \leq m - 1 \). Since \( X/Y \) is abelian, we have \( h(S/Y) \leq m \), and thus \( E = E_{m}(S) \subseteq Y \subseteq M \), as wanted. \( \square \)

(4.2) Corollary. Let \( \mathcal{X} \subseteq \text{Irr}(G) \) be closed, and let \( \chi \in \mathcal{X} \). Assume that \( \psi(1) < 2 \chi(1) \) for all characters \( \psi \in \mathcal{X} \). Then \( S(\ker(\chi)) \) is nilpotent.

Proof. Let \( a \) be the maximum of \( \psi(1)/\chi(1) \) as \( \psi \) runs over \( \mathcal{X} \). Then \( 1 \leq a < 2 \), and by Theorem 4.1, we have \( h(S) \leq h(a) + 1 \), where \( S = S(\ker(\chi)) \). But \( [a] = 1 \), and thus \( h(a) = 0 \), and the result follows. \( \square \)

The following result (which is probably known) makes the bound in Theorem 4.1 somewhat more explicit.

(4.3) Lemma. We have \( h(a) \leq (3/2) \log_2(a) \) for \( a \geq 1 \).

Proof. It is no loss to assume that \( a \) is an integer, and we proceed by induction on \( a \). Clearly, \( h(1) = 0 \), \( h(2) = 1 \), \( h(3) = 2 \) and \( h(4) = 3 \), and since \( 2 < (3/2) \log_2(3) \) and \( 3 = (3/2) \log_2(4) \), the desired inequality holds for \( a \leq 4 \). We can assume, therefore, that \( a \geq 5 \), and we let \( n \) be the greatest integer in \((3/2) \log_2(a), \) so \( n \geq 3 \).

We work to show that if \( S \) is a solvable permutation group of degree \( a \), then \( h(S) \leq n \), or equivalently, that \( E_n(S) = 1 \). If \( S \) is intransitive, then by the inductive hypothesis, \( E_n(S) \) is contained in the kernel of the action of \( S \) on each orbit, and the result follows. We can assume, therefore, that \( S \) is transitive.
If $S$ is imprimitive, suppose that the $a$ points permuted by $S$ are partitioned into $b$ blocks of size $c$, where these blocks are permuted by $S$. Let $u$ and $v$ respectively, be the greatest integers in $(3/2)\log_2(b)$ and $(3/2)\log_2(c)$, and note that $u + v \leq n$ since $bc = a$. Since the set of blocks has cardinality $b < a$, it follows by the inductive hypothesis that $E_u(S)$ is contained in the kernel of the action of $S$ on the set of blocks, and so $E_u(S)$ acts on each block. The blocks have size $c < a$, however, and hence by the inductive hypothesis again, $E_v(E_u(S))$ acts trivially in each block. Then $E_n(S) \subseteq E_v(E_u(S)) = 1$, as wanted.

We can now assume that $S$ is primitive, and thus there is an abelian subgroup $A \triangleleft S$ that is complemented by a point stabilizer $T$. If $S$ is not 2-transitive, then $T$ has an orbit with size at most $a/2$, and it follows by the inductive hypothesis that $E_{n-1}(T)$ acts trivially on that orbit, and thus $E_{n-1}(T)$ centralizes some nonidentity element of $A$. Then $N_A(E_{n-1}(T)) > T$, and since $T$ is maximal in $S$ and $T$ contains no nontrivial normal subgroup of $S$, we conclude that $E_{n-1}(T) = 1$. Then $h(T) \leq n - 1$, and thus $h(S) \leq n$.

Finally, we can assume that $S$ is 2-transitive. By B. Huppert’s classification [1] of solvable 2-transitive permutation groups, it follows that with finitely many exceptions, $h(S) \leq 3$. Since $n \geq 3$, we need only consider Huppert’s exceptions, and all of those have Fitting height at most 4. The smallest degree of one of these exceptions is $3^2 = 9$, and so to complete the proof, it suffices to check that $4 < (3/2)\log_2(9)$. Indeed, this inequality holds since $2^8 = 256 < 729 = 9^3$.

5. The solvability of certain kernels

By Theorem 3.5, we know that the kernel $K$ of an irreducible character $\chi$ of maximal degree in some closed set $\mathcal{X} \subseteq \text{Irr}(G)$ is nilpotent. In Theorems 3.6 and 4.1, the maximality hypothesis was relaxed in two different ways, and in each case, we were able to control the Fitting height of the solvable radical $S(K)$. In this section we show (via an appeal to the classification of simple groups) that if the degree of $\chi$ is nearly maximal in $\mathcal{X}$ (either in the sense of Theorem 3.6 or in the sense of Theorem 4.1) then in fact, $K = \ker(\chi)$ is solvable, and so $K = S(K)$, and the Fitting height of $K$ is under control. In first of the two main results in this section, we replace “maximal degree in $\mathcal{X}$” by “second maximal degree in $\mathcal{X}$”, and in the other, we assume that $\psi(1) < 2\chi(1)$ for all members $\psi \in \mathcal{X}$.

(5.1) Theorem. Let $\mathcal{X} \subseteq \text{Irr}(G)$ be a closed subset, and let $K = \ker(\chi)$, where $\chi$ has second-maximal degree in $\mathcal{X}$. Then $K$ is solvable and $h(K) \leq 2$.

(5.2) Theorem. Let $\mathcal{X} \subseteq \text{Irr}(G)$ be a closed subset, and let $K = \ker(\chi)$, where $\chi \in \mathcal{X}$ and $\psi(1) < 2\chi(1)$ for all characters $\psi \in \mathcal{X}$. Then $K$ is nilpotent.

Of course, once we establish that $K$ is solvable, the upper bound on the Fitting height of $K$ in Theorem 5.1 is immediate via Theorem 3.6. Similarly, if we can establish that $K$ is solvable under the hypotheses of Theorem 5.2, then $K$ is nilpotent by Corollary 4.2. For each of Theorem 5.1 and Theorem 5.2, therefore, it suffices to show that $K$ is solvable.

Suppose that $K = \ker(\chi)$ is not solvable under the hypothesis of Theorem 5.1 or Theorem 5.2, and let $N$ be the final term of the derived series of $K$. Then $N > 1$, so if we choose a chief factor $N/M$ of $G$, we see that $N/M$ is a nonsolvable minimal normal subgroup of $G/M$. Let $\mathcal{X}_0 = \{\psi \in \mathcal{X} | M \subseteq \ker(\psi)\}$, and view $\mathcal{X}_0 \subseteq \text{Irr}(G/M)$. Then $\mathcal{X}_0$ is closed in $\text{Irr}(G/M)$, and of course, $\chi \in \mathcal{X}_0$. Also $\chi$ satisfies the hypothesis of Theorem 5.1 or Theorem 5.2 respectively, with $G/M$ in place of $G$ and $X_0$ in place of $\mathcal{X}$, and of course, $N/M \leq \ker(\chi)$. We can thus assume that $M = 1$, and that $N$ is minimal normal in $G$. To prove Theorem 5.1 or Theorem 5.2, therefore, it suffices to show that under the hypotheses of each of those theorems, the kernel $K$ cannot contain a nonsolvable minimal normal subgroup of $G$.

We need the following result, which, of course, relies on the classification of simple groups and a knowledge of their representation theory.

(5.3) Theorem. Let $S$ be a nonabelian simple group, and view $S$ as a normal subgroup of $G = \text{Aut}(S)$. Then there exists a nonprincipal character $\alpha \in \text{Irr}(S)$ such that $\alpha$ extends to $G$. If $S$ is not of the form $\text{PSL}(2, q)$ where
If $S$ is not of the form $PSL(2, q)$, where $q$ is a power of 3, it follows by Proposition 3.7 of [5] that there exist two nonprincipal irreducible characters of $S$ having different degrees such that both extend to $G$, and there is nothing further to prove. If $S = PSL(2, q)$, where $q$ is a power of 3. Then the Steinberg character $\alpha$ of degree $q$ has the desired property. (See the proof of Proposition 3.7 of [5], or see Proposition 3.8 of that paper.)

Next, we offer a related result for arbitrary nonsolvable minimal normal subgroups.

**Corollary (5.4)** Let $N$ be a nonsolvable minimal normal subgroup of $G$, and write $C = C_G(N)$. Then there exists a character $\xi \in \text{Irr}(G)$ such that $\xi_N$ is irreducible and faithful, and $C \subseteq \ker(\xi)$. Also, if the simple direct factors of $N$ are not of the form $PSL(2, q)$, where $q$ is a power of 3, then there exists a second character $\eta \in \text{Irr}(G)$ with $\eta(1) \neq \xi(1)$, and such that $\eta_N$ is irreducible and faithful, and $C \subseteq \ker(\eta)$.

**Proof.** First, observe that $N \cap C = 1$, and thus $NC/C$ is a nonsolvable minimal normal subgroup of $G/C$ and $NC/C \cong N$. Replacing $G$ by $G/C$, therefore, we can assume that $C = 1$.

We may suppose that $N$ is the direct product of $r$ copies of some nonabelian simple group $S$, where $r \geq 1$. Assume that $\gamma \in \text{Irr}(S)$ is a nonprincipal character that extends to Aut$(S)$, and recall that Theorem 5.3 guarantees the existence of such a character, and it guarantees the existence of two such characters with different degrees if $S$ is not of the form $PSL(2, q)$, where $q$ is a power of 3. It suffices, therefore, to construct a character $\theta \in \text{Irr}(N)$ such that $\theta$ extends to $G$ and $\theta(1) = \gamma(1)^r$. Since $N$ is minimal normal in $G$ and $\theta$ is nonprincipal and invariant in $G$, we see that $\theta$ will automatically be faithful.

Recall that the $r$ simple direct factors of $N$ form a conjugacy class of subgroups of $G$. Fix one of these simple factors $S$, and let $B = C_G(S)$ and $M = N_G(S)$. Then $BS \subseteq M$, and $BS/B$ is the socle of $M/B$, and so we can identify $BS/B$ with $S$ and $M/B$ with a subgroup of Aut$(S)$. Under this identification, $\gamma$ corresponds to a character $\gamma^B \in \text{Irr}(BS/B)$, which we view as lying in $\text{Irr}(BS)$. By assumption, $\gamma$ extends to Aut$(S)$, and it follows that $\gamma^B$ extends to a character $\delta \in \text{Irr}(M)$, where $B \subseteq \ker(\delta)$. Let $\gamma = \delta_N$, and observe that $\gamma^B$ agrees with $\gamma$ on $S$, and that each simple direct factor of $N$ different from $S$ is contained in $\ker(\gamma^B)$. In particular, we see that $\gamma^B \in \text{Irr}(N)$.

Now let $T$ be a set of representatives for the right cosets of $M$ in $G$, and observe that $|T| = r$ and the simple direct factors of $N$ are exactly the groups $S^t$, where $t \in T$. The character $(\gamma^B)^t \in \text{Irr}(N)$ agrees with a nonprincipal irreducible character on $S^t$, and each simple direct factor of $N$ other that $S^t$ is contained in the kernel of $(\gamma^B)^t$. It follows that the character $\theta = \prod_T (\gamma^B)^t$ is irreducible, and its degree is $\gamma(1)^r$, as wanted.

It remains to show that $\theta$ extends to $G$. In fact, the tensor-induced character $\delta \otimes G$ is an extension of $\theta$, as required. (See Section 4 of [3] for a brief exposition of tensor induction.) By Lemma 4.1 of [3], it follows that $(\delta \otimes G)_N = \prod_T (\gamma^B)^t = \theta$, and this completes the proof.

The proof of Theorem 5.2 depends only on the part of Corollary 5.4 that asserts the existence one character with certain properties, and so we present that first. The full strength of 5.4 together with some additional information about the groups $PSL(2, q)$, where $q$ is a power of 3, will be used to prove Theorem 5.1. The following technical lemma will be used for both Theorem 5.1 and Theorem 5.2.

**Lemma (5.5)** Let $X$ be a closed set of irreducible characters of $G$, and let $\chi \in X$. Suppose that $N \subseteq \ker(\chi)$ where $N \triangleleft G$, and let $\xi \in \text{Irr}(G)$ be a nonlinear character such that $\xi_N$ is irreducible and faithful and $C_G(N) \subseteq \ker(\xi)$. Then $\xi \chi \in X$.

**Proof.** Since $\chi$ can be viewed as an irreducible character of $G/N$ and $\xi_N$ is irreducible, it follows by Gallagher's theorem (Corollary 6.17 of [2]) that $\xi \chi \in \text{Irr}(G)$. Also, since $\xi$ is nonlinear, the degree of
\( \xi \chi \) exceeds that of \( \chi \), and thus since \( \mathcal{X} \) is closed, it suffices to show that \( \ker(\xi \chi) < \ker(\chi) \) in order to prove that \( \xi \chi \in \mathcal{X} \), as wanted.

First, \( N > 1 \) since \( \xi_N \) is a nonlinear irreducible character of \( N \). Write \( L = \ker(\xi \chi) \), and observe that \( L \cap N \subseteq L \cap \ker(\chi) \subseteq \ker(\xi) \). Since \( \xi_N \) is faithful, we conclude that \( L \cap N = 1 \), and in particular, \( N \not\subseteq L \). Also, \( L \subseteq C_G(N) \subseteq \ker(\xi) \), where the second containment holds by hypothesis, and it follows that \( L \subseteq \ker(\chi) \). In fact, this containment is proper since \( N \subseteq \ker(\chi) \) but \( N \not\subseteq L \). □

**Proof of Theorem 5.2.** We have \( \chi \in \mathcal{X} \), where \( \mathcal{X} \) is closed, and we are assuming that \( \psi(1) < 2\chi(1) \) for all members \( \psi \in \mathcal{X} \). As we have seen, it suffices to show that \( \ker(\chi) \) contains no nonsolvable minimal normal subgroup of \( G \), so we suppose that \( N \) is such a subgroup, and we derive a contradiction.

By Corollary 5.4, there exists a character \( \xi \in \text{Irr}(G) \) such that \( \xi_N \) is irreducible and faithful and \( C_G(N) \subseteq \ker(\xi) \). Also, since \( N \) is nonsolvable and minimal normal in \( G \), it follows that \( \xi_N \) is nonlinear. Writing \( \psi = \xi \chi \), we see by Lemma 5.5 that \( \psi \in \mathcal{X} \), and since \( \psi(1) = \xi(1)\chi(1) \geq 2\chi(1) \), we have a contradiction. □

As was the case for Theorem 5.2, the main tool in the proof of Theorem 5.1 is Corollary 5.4, but here we need the full strength of that result, which asserts that if \( N \) is a nonabelian minimal normal subgroup of \( G \), then there (usually) are two characters of \( G \) with different degrees such that their restrictions to \( N \) are irreducible and faithful, and their kernels contain \( C_G(N) \). This can fail, however, if \( N \) is a direct product of simple groups isomorphic to \( \text{PSL}(2, q) \), where \( q \) is some power of 3, and in that case we must use an alternative argument.

Our proof in the case that \( N \) is a direct product of copies of \( S = \text{PSL}(2, q) \), where \( q \) is a power of 3, relies on the following facts. Let \( Q \in \text{Syl}_3(S) \), so that \( |Q| = q \). Then \( S \) has exactly \( q + 1 \) Sylow 3-subgroups, and \( S \) acts doubly transitively on the set \( \text{Syl}_3(S) \). Let \( H = N_S(Q) \), and write \( (1_H)^S = 1_S + \sigma \), so that \( \sigma \in \text{Irr}(S) \) and \( \sigma(1) = |S:H| - 1 = q \). (Here, \( \sigma \) is the Steinberg character of \( S \). We can avoid appealing to the general theory of Steinberg characters, however, since we need nothing beyond the facts that we have just stated. We should mention, however, that Steinberg characters play a crucial role in the proof of Proposition 3.7 of [5], and that result underlies our Theorem 5.3.)

**Lemma 5.6.** Let \( S = \text{PSL}(2, q) \), where \( q \) is a power of 3, and let \( N = S_1 \times \cdots \times S_r \), where \( S_i \cong S \) for \( 1 \leq i \leq r \). Let \( Q \in \text{Syl}_3(N) \), and write \( H = N_N(Q) \). Then \((1_H)^N\) has a unique irreducible constituent \( \sigma \) such that \( \sigma(1) = |Q| \). Also, \( \sigma \) is faithful, and it has multiplicity 1 in \((1_H)^N\).

**Proof.** Write \( Q = Q_1 \times \cdots \times Q_r \) and \( H = H_1 \times \cdots \times H_r \), where \( Q_i \in \text{Syl}_3(S_i) \) and \( H_i = N_{S_i}(Q_i) \). It is easy to see that

\[
(1_H)^N = (1_{H_1})^{S_1} \times \cdots \times (1_{H_r})^{S_r} = (1_{S_1} + \sigma_1) \times \cdots \times (1_{S_r} + \sigma_r),
\]

where \( \sigma_i \in \text{Irr}(S_i) \) has degree \( q = |Q_i| \). Expanding this product, we see that \((1_H)^N\) is equal to the sum of \( 2^r \) irreducible characters of \( N \), each of which is a product of some subset of the \( \{\sigma_i\} \), with principal characters as the remaining factors. Exactly one of these \( 2^r \) characters has degree equal to \( |Q| = q^2 \), and the result follows since none of the factors \( S_i \) is contained in \( \ker(\sigma) \), and therefore \( \sigma \) is faithful. □

**Proof of Theorem 5.1.** Assuming that \( \chi \) is of second maximal degree in the closed set \( \mathcal{X} \subseteq \text{Irr}(G) \), we have seen that it suffices to show that \( K = \ker(\chi) \) contains no nonsolvable minimal normal subgroup of \( G \). Supposing that \( N \) is such a minimal normal subgroup, we derive a contradiction.

Now \( N \) is a direct product of copies of some simple group \( S \), and we assume first that \( S \) is not of the form \( \text{PSL}(2, q) \), where \( q \) is a power of 3. By Corollary 5.4, there exist characters \( \xi \) and \( \eta \) of \( G \), where \( \xi \) and \( \eta \) have different degrees, \( C_G(N) \) is contained in the kernel of each of them, and \( \xi_N \) and \( \eta_N \) are faithful and irreducible. By Lemma 5.5, the characters \( \xi \chi \) and \( \eta \chi \) lie in \( \mathcal{X} \). Furthermore,
ξχ and ηχ have different degrees, and each of these degrees exceeds χ(1). This contradicts the assumption that χ has second maximal degree in X.

We can now assume that $S \cong PSL(2, q)$, where q is a power of 3. Let $Q \in \text{Syl}_3(N)$ and $M = N_C(Q)$, and write $H = N \cap M$, so that $H = N_M(Q)$. We have $NM = G$, and thus $|G : M| = |N : H|$ is coprime to $|Q|$. Also, since $N \subseteq \ker(\chi)$, it follows that $\chi_M$ is irreducible, and we write $\theta = \chi_M$.

We know from Lemma 5.6 that $(1_H)^N$ has a unique irreducible constituent $\sigma$ such that $(1) = [Q]$, and furthermore, $[1_H^N, \sigma] = 1$ and $\sigma$ is faithful. Now $(1_H)^N = ((1_M)^G)_N$, so there exists an irreducible constituent $\alpha$ of $(1_M)^G$ that lies over $\sigma$, and $\sigma$ occurs with multiplicity 1 as a constituent of $\sigma_N$. Also, $\sigma_N$ is a constituent (not yet known to be irreducible) of $(1_H)^N$, and thus $\sigma$ is the unique irreducible constituent of $\sigma_N$ having degree $\sigma(1)$. It follows that $\alpha_N$ is a multiple of $\sigma$, and hence $\alpha_N = \sigma$. In particular, $\alpha_N$ is faithful and irreducible.

Now let $C = C_G(N)$, so that $C \subseteq N_C(Q) = M$. Since $C < G$, it follows that $C \subseteq \ker((1_M)^G)$, and in particular, $C \subseteq \ker(\alpha)$. We conclude by Lemma 5.5 that the character $\alpha\chi$ is irreducible, and that $\alpha\chi \in X$. Since $\chi$ has second maximal degree in $X$ and $\alpha(1)\chi(1) = \sigma(1)\chi(1) = |Q|\chi(1)$, it follows that every member of $X$ with degree exceeding $\chi(1)$ has degree $|Q|\chi(1)$.

Now let $\sigma \in \text{Irr}(N)$ and $\sigma(1) = |Q|$, where $Q$ is a Sylow 3-subgroup of $N$. It follows that $\sigma$ vanishes on the nonidentity elements of $Q$, and thus $\sigma Q$ is the regular character of $Q$. In particular, each linear character of the abelian group $Q$ occurs with multiplicity 1 as a constituent of $\sigma Q$. Choose an irreducible constituent $\varphi$ of $\sigma Q$ such that $\varphi$ lies over a nonprincipal linear character $\lambda$ of $Q$. Since $\varphi$ is an irreducible constituent of $\sigma Q = \alpha H$, we can choose an irreducible constituent $\beta$ of $\alpha M$ such that $\beta$ lies over $\varphi$ and thus over $\lambda$. Then

$$1 \leq [\beta_0, \lambda] \leq [\alpha \sigma_0, \lambda] = [\sigma_0, \lambda] = 1,$$

and thus $[\beta_0, \lambda] = 1$.

Since $Q < M$ and $\beta < \text{Irr}(M)$, it follows that the irreducible constituents of $\beta_0 Q$ are $M$-conjugate to $\lambda$, and so they are all nonprincipal, and they all appear with multiplicity 1 in $\beta_0 Q$. It follows that $\beta(1)$ cannot exceed the number of nonprincipal linear characters of $Q$, and we have $\beta(1) < |Q|$. Also, $Q \subseteq H < M$, and hence $\beta_0 H$ is a sum of distinct $M$-conjugates of $\varphi$.

Let $T$ be the stabilizer of $\varphi$ in $M$, and let $\eta \in \text{Irr}(T)$ be the Clifford correspondent of $\beta$ with respect to $\varphi$. Then $\eta^M = \beta$ and $[\eta, \varphi] = [\beta, \varphi] = 1$, and we conclude that $\eta_H = \varphi$. Now let $\gamma$ be an arbitrary irreducible constituent of $\theta T$, and note that since $\theta$ is an irreducible constituent of $\gamma^M$, we have $\chi(1) = \theta(1) \leq \gamma^M(1) = |M : T|\gamma(1)$. Also, since $N \subseteq \ker(\chi)$, we have $H \subseteq \ker(\theta)$, and thus $H \subseteq \ker(\gamma)$. It follows by Gallagher's theorem that $\eta \gamma \in \text{Irr}(T)$, and since this character lies over $\eta_H = \varphi$ and $T$ is the stabilizer of $\varphi$ in $M$, we see that the character $\psi = (\eta \gamma)^M$ is irreducible. Also,

$$\psi(1) = |M : T|\eta(1)\gamma(1) \geq \chi(1)\eta(1) = \chi(1)\varphi(1).$$

We argue next that for each irreducible constituent $\xi$ of $\psi^G$, we have $N \cap \ker(\xi) = 1$ and $\ker(\xi) < \ker(\chi)$. Since $\psi$ is a constituent of $\psi H$ and $\psi$ is nonprincipal, it follows that $H \not\subseteq \ker(\psi)$, and thus $H \not\subseteq \ker(\xi)$. Since $N$ is minimal normal in $G$, we have $N \cap \ker(\xi) = 1$, as claimed. We then have $\ker(\xi) \subseteq N_C(Q) = M$, and since $\psi$ is a constituent of $\xi_M$, it follows that $\ker(\xi) = \ker(\psi) < \ker(\eta \gamma)$, where the second containment holds since $\psi = (\eta \gamma)^M$. Furthermore, since $\ker(\xi) \subseteq M$ and $\ker(\psi) < G$, we have $\ker(\xi) \subseteq \ker((1_M)^G)$, and thus $\ker(\xi) \subseteq \ker(\alpha)$. Now $\beta$ is a constituent of $\alpha M$, and since $\ker(\xi) \subseteq M$, we deduce that $\ker(\xi) \subseteq \ker(\beta) \subseteq \ker(\eta)$, where the second containment holds because $\eta^M = \beta$. We now have $\ker(\xi) \subseteq \ker(\eta \gamma)$ and also $\ker(\xi) \subseteq \ker(\eta)$. It follows that $\ker(\xi) \subseteq \ker(\gamma)$. As $\gamma$ lies under $\theta = \chi_M$, it follows that the restriction of $\chi$ to $\ker(\xi)$ has a principal constituent, and since $\ker(\xi) < G$, we conclude that $\ker(\xi) \subseteq \ker(\chi)$. This containment is strict, as claimed, since $N \subseteq \ker(\chi)$ but $N \not\subseteq \ker(\xi)$.

If $\xi$ is an irreducible constituent of $\psi^G$, then $\xi(1) \geq \psi(1) \geq \chi(1)\varphi(1) \geq \chi(1)$, and we show next that $\xi(1) > \chi(1)$. Otherwise, $\xi(1) = \psi(1) = \chi(1)$, and it follows that $\xi_M = \psi$ and $\varphi(1) = 1$, and thus all constituents of $\xi_M = \psi_H$ are linear. Then $H' \subseteq N \cap \ker(\xi) = 1$, so $H$ is abelian. This is a contradiction, however, since $H = N_M(Q)$ and $N$ does not have a normal 3-complement, and we conclude that $\xi(1) > \chi(1)$. 


We have now shown that for all irreducible constituents $\xi$ of $\psi^G$, we have $\xi(1) > \chi(1)$ and $\ker(\xi) < \ker(\chi)$, and since $\mathcal{A}$ is closed, it follows that $\xi \in \mathcal{A}$ for all such characters $\xi$. We have seen, however, that every member of $\mathcal{A}$ with degree exceeding $\chi(1)$ has degree equal to $|Q|\chi(1)$, and it follows that $|Q|\chi(1)$ divides $\psi^G(1) = |G : M|\psi(1)$. Since $\psi = (\eta\gamma)^M$ and $\beta = \eta^M$, we have $\psi(1) = |M : T|\eta(1)\gamma(1) = \beta(1)\gamma(1)$, and thus $|Q|\chi(1)$ divides $|G : M|\beta(1)\gamma(1)$. This holds for every irreducible constituent $\gamma$ of $\theta_T = \chi_T$, and since $\chi(1)$ is an integer linear combination of the degrees $\gamma(1)$ for the various irreducible constituents $\gamma$ of $\chi_T$, we deduce that $|Q|\chi(1)$ divides $|G : M|\beta(1)\gamma(1)$. Then $|Q|$ divides $|G : M|\beta(1)$, and since $|Q|$ and $|G : M|$ are coprime, we see that $|Q|$ divides $\beta(1)$. This is a contradiction, however, since we established previously that $\beta(1) < |Q|$. □

We have now established that for an arbitrary finite group $G$, the kernels of the characters of second maximal degree in $\text{Irr}(G)$ are always solvable. We have been unable to decide the corresponding question for irreducible characters of third maximal degree, but solvability can definitely fail for kernels of irreducible characters of fourth maximal degree. To see this, consider the group $G = A_5 \rtimes E$, where $A_5$ the alternating group of degree 5 and $E$ is nonabelian of order $7^3$. The set of irreducible character degrees of $G$ is $\{1, 3, 4, 5, 7, 7^2, 7^4, 7^5\}$, and thus a character $\chi \in \text{Irr}(G)$ with $\chi(1) = 7$ has fourth maximal degree. The kernel of such a character, however, is the nonsolvable group $A_5$.

6. Degree ratios

Recall that we have written $b(G)$ to denote the maximum of the degrees of the irreducible characters of an arbitrary finite group $G$, and if $G$ is nonabelian, we defined $c(G)$ to be the minimum of the degrees of the nonlinear irreducible character degrees of $G$. Also, we defined the degree ratio $\text{rat}(G)$ of a nonabelian group $G$ by setting $\text{rat}(G) = b(G)/c(G)$. Our main goal in this section is to control the derived length of a nonabelian solvable group in terms of its degree ratio. In particular, we prove the following, which is part of Theorem C.

(6.1) Theorem. Let $G$ be solvable and nonabelian, and write $r = \text{rat}(G)$. Then $\text{dl}(G) \leq 3 + 4\log_2(r)$.

We begin with a preliminary (and well-known) result.

(6.2) Lemma. Let $N \triangleleft G$, where $G/N$ is not abelian. Then $b(N) \leq b(G)/2$.

Proof. Choose $\theta \in \text{Irr}(N)$ with $\theta(1) = b(N)$. Let $\chi \in \text{Irr}(G)$ lie over $\theta$, and let $m = \chi(1)/\theta(1)$, so that $m$ is an integer. Then $b(G) \geq \chi(1) = m\theta(1) = mb(N)$, so if $m > 1$, there is nothing further to prove. Now assume that $m = 1$, so $\chi_N = \theta$, and thus by Gallagher’s theorem, $\chi \beta$ is irreducible for every character $\beta \in \text{Irr}(G/N)$. Since $G/N$ is nonabelian, we can choose $\beta$ to be nonlinear, and we have $b(G) \geq \chi(1)\beta(1) \geq 2\chi(1) = 2\theta(1) = 2b(N)$. □

Using Lemma 6.2, one can easily obtain an upper bound on the derived length of a solvable group $G$ in terms of $b(G)$. This result is analogous to the upper bound in terms of $\text{rat}(G)$ that we give in Theorem 6.1, and indeed, we use the following in the proof of Theorem 6.1.

(6.3) Corollary. Let $G$ be solvable. Then $\text{dl}(G) \leq 1 + 2\log_2(b)$, where $b = b(G)$.

Proof. If $G$ is abelian, the desired inequality is clear, and if $\text{dl}(G) = 2$, then $b \geq 2$, and again the inequality follows. We may assume, therefore, that $\text{dl}(G) > 2$, and we proceed by induction on $\text{dl}(G)$. Let $N = G''$, so that $\text{dl}(N) = \text{dl}(G) - 2$ and $G/N$ is nonabelian. Then $b(N) \leq b/2$ by Lemma 6.2, and the inductive hypothesis, yields

$$\text{dl}(G) = 2 + \text{dl}(N) \leq 2 + 1 + 2\log_2(b/2) = 1 + 2\log_2(b),$$

as wanted. □
Before we proceed with the proof of Theorem 6.1, we digress to compare Lemma 6.2 with a consequence of Sections 4 and 5. Of course, it was not necessary to assume that \( G \) is solvable in Lemma 6.2, but if we want to avoid a solvability hypothesis in the following analogous result, we must appeal to Theorem 5.2 (and thus indirectly to the classification of simple groups). If we are willing to assume that \( G \) is solvable, however, we can appeal directly to Corollary 4.2.

### (6.4) Corollary

Let \( N \triangleleft G \), where \( N \) is not nilpotent. Then \( b(G/N) \leq b(G)/2 \).

**Proof.** Suppose that \( b(G/N) > b(G)/2 \), and choose \( \chi \in \text{Irr}(G/N) \) such that \( \chi(1) = b(G/N) \). If \( \psi \in \text{Irr}(G) \) is arbitrary, we have \( \psi(1) \leq b(G) < 2b(G/N) = 2\chi(1) \), and thus \( \ker(\chi) \) is nilpotent by Theorem 5.2 (or by Corollary 4.2 if we assume that \( \ker(\chi) \) is solvable). But \( N \subseteq \ker(\chi) \), and \( N \) is not nilpotent, and this is a contradiction. \( \Box \)

Using Corollary 6.4 in place of Lemma 6.2, we can easily obtain an upper bound on the Fitting height of a nonabelian solvable group \( G \) in terms of \( \text{rat}(G) \). For Theorem 6.1, however, we want a bound on the derived length, and not just on the Fitting height, and to obtain that, it seems that Corollary 6.4 is not sufficient, and we will have to use other techniques. Nevertheless, we present the Fitting-height inequality here because it is easy to establish using an argument very similar to that in the proof of Corollary 6.3.

### (6.5) Corollary

Let \( G \) be solvable and nonabelian, and write \( r = \text{rat}(G) \). Then \( h(G) \leq 3 + 2\log_2(2 + h(G/N)) \).

**Proof.** The result is trivial if \( h(G) \leq 3 \), so we assume that \( h(G) > 3 \), and we proceed by induction on \( h(G) \). Write \( b = b(G) \) and \( c = c(G) \), and let \( N = F_2(G) \). Since \( N \) is not nilpotent, we have \( b(G/N) \leq b/2 \) by Corollary 6.2. Also, \( G/N \) is nonabelian and \( c(G/N) \geq c \), and thus \( \text{rat}(G/N) \leq r/2 \). Since \( h(G/N) = h(G) - 2 \), the inductive hypothesis applies in \( G/N \), and we have

\[
h(G) = 2 + h(G/N) \leq 2 + 3 + 2\log_2(r/2) = 3 + 2\log_2(2 + h(G/N)).
\]

as wanted. \( \Box \)

We begin work now toward a proof of Theorem 6.1 by establishing a \( p \)-group version of the result. Recall that if \( G \) is a \( p \)-group, then \( G \) is an M-group, and Taketa’s theorem implies that \( d(G) \leq |\text{cd}(G)| \), where \( \text{cd}(G) \) is the set of degrees of the irreducible characters of \( G \). (See Theorem 5.12 of [2].)

### (6.6) Lemma

Let \( r = \text{rat}(G) \), where \( G \) is a nonabelian \( p \)-group. Then \( d(G) \leq 2 + \log_p(r) \).

**Proof.** Let \( p^a \) and \( p^b \), respectively, be the smallest and largest nonlinear irreducible character degrees of \( G \), and observe that \( |\text{cd}(G)| \leq 1 + (b - a + 1) = 2 + b - a \). Also, \( r = p^{b-a} \), so \( \log_p(r) = b - a \), and by Taketa’s theorem, \( d(G) \leq |\text{cd}(G)| \leq 2 + b - a = 2 + \log_p(r) \), as wanted. \( \Box \)

The following results show that in certain special situations, the degree ratio \( \text{rat}(G) \) is an upper bound for the maximum irreducible character degree \( b(K) \) for a normal subgroup \( K \) of \( G \).

### (6.7) Lemma

Let \( K \triangleleft G \), where \( G/K \) is nonabelian, and assume that each member of \( \text{Irr}(K) \) extends to its stabilizer in \( G \). Then \( b(K) \leq \text{rat}(G) \).

**Proof.** Let \( \psi \in \text{Irr}(K) \) with \( \psi(1) = b(K) \). By hypothesis, \( \psi \) extends to its stabilizer \( T \) in \( G \), and we let \( \hat{\psi} \in \text{Irr}(T) \) be such an extension. Since \( G/K \) is nonabelian, there exists a nonlinear character \( \theta \in \text{Irr}(G/K) \), and we choose an irreducible constituent \( \eta \) of \( \theta_T \). Then \( \eta \) is a constituent of \( \eta^G \), and thus

\[
c(G) \leq \theta(1) \leq \eta^G(1) = \eta(1)|G:T|.
\]
Also, $N \subseteq \ker(\theta)$, so $N \subseteq \ker(\eta)$, and thus by Gallagher's theorem, $\widehat{\varphi} \eta$ is an irreducible character of $T$. Of course, $\widehat{\varphi} \eta$ lies over $\theta$, and hence $\langle \widehat{\varphi} \eta \rangle^G \in \Irr(G)$ by the Clifford correspondence. Then

$$b(G) \geq \langle \widehat{\varphi} \eta \rangle^G(1) = \varphi(1) \eta(1) |G : T|,$$

and we conclude that

$$\text{rat}(G) = \frac{b(G)}{c(G)} \geq \frac{\varphi(1) \eta(1) |G : T|}{\eta(1) |G : T|} = \varphi(1) = b(K),$$

as wanted. \qed

\textbf{(6.8) Lemma.} Let $K \triangleleft G$, where $G/K$ is a Frobenius group having an abelian Frobenius kernel. Then $b(K) \leq \text{rat}(G)$.

\textbf{Proof.} Let $\varphi \in \Irr(K)$ with $\varphi(1) = b(K)$, and let $T$ be the stabilizer of $\varphi$ in $G$. Then $|G : T| \varphi(1)$ divides the degree of every irreducible character of $G$ that lies over $\varphi$, and in particular, $b(G) \geq |G : T| \varphi(1)$.

Let $N/K$ be the Frobenius kernel of $G/K$, and write $S = N \cap T$ and $R = NT$. Then $S \triangleleft N$ since $S \supseteq K$ and $N/K$ is abelian, and it follows that $S \triangleleft R$. Since the action of $G/N$ on $N/K$ is Frobenius, so too is the action of $R/N$ on $N/S$, and we have $|R/N| \leq |N/S|$. Then

$$|G : T| = |G : R||R : T| = |G : R||N : S| \geq |G : R||R : N| = |G : N|,$$

and thus $b(G) \geq |G : T| \varphi(1) \geq |G : N| \varphi(1) = |G : N| b(K)$. Since $|G : N|$ is the degree of a nonlinear irreducible character of $G$, we obtain

$$\text{rat}(G) = \frac{b(G)}{c(G)} \geq \frac{|G : N| b(K)}{|G : N|} = b(K),$$

as wanted. \qed

\textbf{Proof of Theorem 6.1.} First, suppose that there exists $K \triangleleft G$ such that $G/K$ is a nonabelian $p$-group for some prime $p$, and let $f \in \text{cd}(G/K)$ with $f > 1$.

If $G$ has a nonlinear irreducible character of degree $e$ not divisible by $p$, this character restricts irreducibly to $K$, and it follows by Gallagher's theorem that $G$ has an irreducible character of degree $ef$. Then

$$r = \frac{b(G)}{c(G)} \geq \frac{ef}{f} = e,$$

and thus

$$r = \frac{b(G)}{c(G)} \geq \frac{b(G)}{e} \geq \frac{b(G)}{r}.$$

Then $b(G) \leq r^2$, and by Corollary 6.3, we have $\text{dl}(G) \leq 1 + 2 \log_2(r^2) = 1 + 4 \log_2(r)$, and there is nothing further to prove in this case.

We can now assume that $G$ has no nonlinear irreducible character of $p'$-degree, and thus $G$ has a normal $p$-complement $N$ by Thompson's theorem. (See Corollary 12.2 of [2].) Also, $N \subseteq K$, and thus $G/N$ is nonabelian. Furthermore, since $N$ is a Hall subgroup of $G$, every irreducible character of $N$ extends to its stabilizer in $G$, and Lemma 6.7 applies. (See Corollary 6.28 or Corollary 8.16 of [2].) We conclude that $b(N) \leq r$, and thus by Corollary 6.3,

$$\text{dl}(N) \leq 1 + 2 \log_2(b(N)) = 1 + 2 \log_2(r).$$
Also, \( \text{rat}(G/N) \leq \text{rat}(G) = r \), and since \( G/N \) is a \( p \)-group, Lemma 6.6 yields
\[
dl(G) \leq \text{dl}(N) + \text{dl}(G/N) \leq (1 + 2 \log_2(r)) + (2 + \log_2(r)) \leq 3 + 3 \log_2(r),
\]
and we are done in this case.

What remains is the case where no nonabelian homomorphic image of \( G \) is a \( p \)-group. In this situation, there exists \( K \triangleleft G \) such that \( G/K \) is a Frobenius group with abelian kernel \( N/K \), where \( G/N \) is cyclic. (See Lemma 12.3 and the following discussion in [2].) By Lemma 6.8, we have \( b(K) \leq r \), and thus by Corollary 6.3,
\[
dl(G) \leq 2 + \text{dl}(K) \leq 2 + 1 + 2 \log_2(b(K)) \leq 3 + 2 \log_2(r),
\]
and the proof is complete. \( \square \)

If \( G \) is solvable and \( \text{rat}(G) < 2 \), Theorem 6.1 yields \( \text{dl}(G) < 7 \), but actually, we can obtain a much better inequality. The following is the part of Theorem C that we have not yet established.

**Theorem.** Let \( G \) be solvable, and assume that \( \text{rat}(G) < 2 \). Then \( \text{dl}(G) \leq 3 \).

It is easy to see that in some sense, Theorem 6.9 is best possible. If \( G \) is the symmetric group \( S_4 \), then \( \text{rat}(G) = 3/2 \) and \( \text{dl}(G) = 3 \), and so the inequality \( \text{dl}(G) \leq 3 \) cannot be strengthened. Also, if \( G = \text{GL}(2, 3) \), then \( \text{rat}(G) = 2 \) and \( \text{dl}(G) = 4 \), and thus the condition \( \text{rat}(G) < 2 \) cannot be weakened.

**Proof of Theorem 6.9.** Since the degree ratio of every nonabelian homomorphic image of \( G \) is less than 2, we can assume that every proper homomorphic image has derived length at most 3. Assuming that \( \text{dl}(G) \) exceeds 3, therefore, we conclude that \( G^{(3)} \) is the unique minimal normal subgroup of \( G \). Also, since \( G \) has a unique minimal normal subgroup, it follows that \( F(G) \) is a \( p \)-group for some prime \( p \), and thus every nilpotent normal subgroup of \( G \) is a \( p \)-group.

If \( G \) is a \( p \)-group, then all irreducible characters of \( G \) are \( p \)-powers, and the hypothesis implies that \( G \) has at most one nonlinear irreducible character. Then \( |\text{cd}(G)| \leq 2 \), and it is known that in this case, \( \text{dl}(G) \leq 2 \). (See Corollary 12.6 of [2].) We can assume, therefore, that \( G \) is not a \( p \)-group.

Suppose that some homomorphic image of \( G \) is a nonabelian \( q \)-group for some prime \( q \), and choose \( K \triangleleft G \) such that \( G/K \) is a \( q \)-group with derived length 2. Since \( \text{rat}(G) < 2 \), and \( G/K \) is non-abelian, it follows from Corollary 4.2 that \( K \) is nilpotent, and so \( K \) is a \( p \)-group. Also, \( q \neq p \) since \( G \) is not a \( p \)-group, and thus \( K \) is a normal Sylow \( p \)-subgroup of \( G \), and hence every irreducible character of \( K \) extends to its stabilizer in \( G \). By Lemma 6.7, therefore, \( b(K) \leq \text{rat}(G) < 2 \), and thus \( K \) is abelian, and we conclude that \( \text{dl}(G) \leq 3 \), as wanted.

We can assume now that no nonabelian homomorphic image of \( G \) is a \( q \)-group for any prime \( q \), and thus we can choose \( K \triangleleft G \) so that \( G/K \) is a Frobenius group with abelian kernel \( N/K \), where \( G/N \) is cyclic. By Lemma 6.8, we have \( b(K) \leq \text{rat}(G) < 2 \), and thus \( K \) is abelian and \( \text{dl}(G) \leq 3 \). \( \square \)

**References**