Journal of Multivariate Analysis 105 (2012) 412–421

Contents lists available at SciVerse ScienceDirect

Journal of Multivariate Analysis

journal homepage: www.elsevier.com/locate/jmva

# The Schur concavity, Schur multiplicative and harmonic convexities of the second dual form of the Hamy symmetric function with applications

Yu-Ming Chu [a,*], Wei-Feng Xia [b], Xiao-Hui Zhang [a]

[a] *Department of Mathematics, Huzhou Teachers College, Huzhou 313000, PR China*

[b] *School of Teacher Education, Huzhou Teachers College, Huzhou 313000, PR China*

ARTICLE INFO

*Article history:*
Received 16 September 2008
Available online 12 August 2011

*AMS 2010 subject classification:*
primary 05E05
secondary 26B25

*Keywords:*
Hamy symmetric function
Second dual form
Schur concave
Schur multiplicatively convex
Schur harmonic convex

ABSTRACT

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$, the second dual form of the Hamy symmetric function is defined by

$$H_n^{**}(x, r) = H_n^{**}(x_1, x_2, \ldots, x_n; r) = \prod_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left( \sum_{j=1}^{r} x_{i_j} \right)^{\frac{1}{r}},$$

where $r \in \{1, 2, \ldots, n\}$ and $i_1, i_2, \ldots, i_n$ are positive integers.

In this paper, we prove that $H_n^{**}(x, r)$ is Schur concave, and Schur multiplicatively and harmonic convex in $\mathbb{R}^n_+$. Some applications in inequalities and reliability theory are presented.



## 1. Introduction

Throughout this paper, we use $\mathbb{R}^n$ to denote the $n$-dimensional Euclidean space, and $\mathbb{R}^n_+ = \{(x_1, x_2, \ldots, x_n) : x_i > 0, i = 1, 2, \ldots, n\}$. In particular, we use $\mathbb{R}$ to denote $\mathbb{R}^1$.

For $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ and $r \in \{1, 2, \ldots, n\}$, the Hamy symmetric function [21] is defined by

$$H_n(x, r) = H_n(x_1, x_2, \ldots, x_n; r) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \left( \prod_{j=1}^{r} x_{i_j} \right)^{\frac{1}{r}},$$

where $i_1, i_2, \ldots, i_n$ are positive integers.

Corresponding to this is the $r$-th order Hamy mean

$$\sigma_n(x, r) = \sigma_n(x_1, x_2, \ldots, x_n; r) = \frac{1}{\binom{n}{r}} H_n(x, r),$$

where $\binom{n}{r} = \frac{n!}{(n-r)!r!}$. Hara et al. [21] established the following refinement of the classical arithmetic and geometric mean inequalities:

$$G_n(x) = \sigma_n(x, n) \le \sigma_n(x, n-1) \le \cdots \le \sigma_n(x, 2) \le \sigma_n(x, 1) = A_n(x).$$

Here $A_n(x) = \frac{1}{n} \sum_{i=1}^{n} x_i$ and $G_n = (\prod_{i=1}^{n} x_i)^{\frac{1}{n}}$ denote the classical arithmetic and geometric means, respectively.

* Corresponding author.
*E-mail addresses:* chuyuming@hutc.zj.cn (Y.-M. Chu), xwf212@hutc.zj.cn (W.-F. Xia), xhzhang@hutc.zj.cn (X.-H. Zhang).

0047-259X/$ – see front matter © 2011 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmva.2011.08.004

Paper [28] contains some interesting inequalities including the fact that $(\sigma_n(x,r))^{\frac{1}{r}}$ is log-concave. More results can be found in [5]. In [20], the Schur convexity of the Hamy symmetric function $H_n(x,r)$ and its generalization were discussed. In [9], the authors proved that $H_n(x,r)$ is Schur harmonic convex in $\mathbb{R}^n_+$ for $r \in \{1,2,\ldots,n\}$.

In [24], Jiang introduced the first dual form of the Hamy symmetric function as follows:

$$H_n^*(x,r) = H_n^*(x_1,x_2,\ldots,x_n;r) = \prod_{1\le i_1<i_2<\cdots<i_r\le n}\left(\sum_{j=1}^{r} x_{i_j}^{\frac{1}{r}}\right),$$

and proved that $H_n^*(x,r)$ is Schur concave and Schur multiplicatively convex in $\mathbb{R}^n_+$ for $r \in \{1,2,\ldots,n\}$.

The Schur convexity or concavity was introduced by Schur [39] in 1923; it has many applications in analytic inequalities [31,41,47], mathematical programming [16,23,35], probability theory and stochastic processes [22,26,29,40, 44,45], and multivariate statistics [1–4,11–15,25,27,33,34,36,38,43].

Dalal and Fortini [13] proved that

$$P\{X+Y\le c\} \ge P\left\{\sqrt{2}\max(X,Y)\le c\right\} \tag{1.1}$$

for every $c > 0$ if $X$ and $Y$ are nonnegative random variables with joint density $f$ such that $f(\sqrt{x},\sqrt{y})$ is Schur convex, and inequality (1.1) is reversed if "convex" is replaced by "concave". Inequality (1.1) can be used to provide conservative simultaneous confidence intervals for all $a_1\mu_1 + a_2\mu_2$ based on samples from $N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$, all parameters unknown.

In [25], Joe established a sufficient condition for the maximality and minimality of the class $\mathcal{P}(p_1,p_2,\ldots,p_n)$ of possible probability matrices $P = (p_{i_j})_{i\neq j}$ associated with paired comparisons in terms of the Schur convexity on $\mathcal{P}$.

Nappo and Spizzichino [33] researched the law of scaled empirical total time on test (TTT)-plots of exchangeable lifetimes; they gave a monotonicity property in the case of absolutely continuous distributions with Schur concave (or Schur convex) densities.

Recently, the Schur multiplicative convexity was introduced and investigated in the literature [10,17,18,30,42]. But only paper [9] discussed the Schur harmonic convexity.

In this paper, we introduce the second dual form of the Hamy symmetric function as follows:

$$H_n^{**}(x,r) = H_n^{**}(x_1,x_2,\ldots,x_n;r) = \prod_{1\le i_1<i_2<\cdots<i_r\le n}\left(\sum_{j=1}^{r} x_{i_j}\right)^{\frac{1}{r}} \tag{1.2}$$

for $x = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n_+$ and $r \in \{1,2,\ldots,n\}$, where $i_1,i_2,\ldots,i_n$ are positive integers.

The purpose of this paper is to prove that $H_n^{**}(x,r)$ is Schur concave, and Schur multiplicatively and harmonic convex in $\mathbb{R}^n_+$. Some applications in inequalities and reliability theory are presented.

## 2. Definitions and lemmas

For the convenience of the readers, we introduce some definitions and lemmas, which we present in this section.

**Definition 2.1.** Let $E \subseteq \mathbb{R}^n$ be a set; a real-valued function $F : E \to \mathbb{R}$ is said to be Schur convex on $E$ if

$$F(x_1,x_1,\ldots,x_n) \le F(y_1,y_2,\ldots,y_n) \tag{2.1}$$

for each pair of $n$-tuples $x = (x_1,\ldots,x_n)$ and $y = (y_1,\ldots,y_n)$ in $E$, such that $x \prec y$, that is

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \quad k = 1,2,\ldots,n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the $i$th largest component in $x$. $F$ is said to be Schur concave if $-F$ is Schur convex.

**Definition 2.2.** Let $E \subseteq \mathbb{R}^n_+$ be a set; a real-valued function $F : E \to \mathbb{R}_+$ is said to be Schur multiplicatively convex on $E$ if (2.1) holds for each pair of $n$-tuples $x = (x_1,x_2,\ldots,x_n)$ and $y = (y_1,y_2,\ldots,y_n)$ in $E$, such that $\log x = (\log x_1, \log x_2,\ldots,\log x_n) \prec \log y = (\log y_1,\log y_2,\ldots,\log y_n)$. $F$ is said to be Schur multiplicatively concave if $\frac{1}{F}$ is Schur multiplicatively convex.

**Definition 2.3.** Let $E \subseteq \mathbb{R}_+^n$ be a set; a real-valued function $F : E \to \mathbb{R}_+$ is said to be Schur harmonic convex on $E$ if (2.1) holds for each pair of $n$-tuples $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ in $E$, such that $\frac{1}{x} = (\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}) \prec \frac{1}{y} = (\frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_n})$. $F$ is said to be Schur harmonic concave if inequality (2.1) is reversed.

**Lemma 2.1** (*see [31]*). *Suppose that $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is a continuous symmetric function. If $f$ is differentiable in $\mathbb{R}_+^n$, then $f$ is Schur convex in $\mathbb{R}_+^n$ if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \geq 0 \tag{2.2}$$

*for all $i, j = 1, 2, \ldots, n$ with $i \neq j$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$. And $f$ is Schur concave if and only if inequality (2.2) is reversed for all $i, j = 1, 2, \ldots, n$ with $i \neq j$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$. Here $f$ is a symmetric function in $R_+^n$, which means that $f(Px) = f(x)$ for all $x \in \mathbb{R}_+^n$ and any $n \times n$ permutation matrix $P$.*

**Remark 2.1.** Since $f$ is symmetric, the Schur's condition in Lemma 2.1, that is (2.2) can be reduced to

$$(x_1 - x_2)\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right) \geq 0.$$

**Lemma 2.2** (*see [10,18]*). *Suppose that $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is a continuous symmetric function. If $f$ is differentiable in $\mathbb{R}_+^n$, then $f$ is Schur multiplicatively convex in $\mathbb{R}_+^n$ if and only if*

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial f}{\partial x_1} - x_2\frac{\partial f}{\partial x_2}\right) \geq 0$$

*for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$.*

**Lemma 2.3** (*see [9]*). *Suppose that $f : \mathbb{R}_+^n \to \mathbb{R}_+$ is a continuous symmetric function. If $f$ is differentiable in $\mathbb{R}_+^n$, then $f$ is Schur harmonic convex in $\mathbb{R}_+^n$ if and only if*

$$(x_1 - x_2)\left(x_1^2\frac{\partial f}{\partial x_1} - x_2^2\frac{\partial f}{\partial x_2}\right) \geq 0$$

*for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$.*

**Lemma 2.4** (*see [18–20]*). *Suppose that $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geq s$, then*

$$\frac{c - x}{\frac{nc}{s} - 1} = \left(\frac{c - x_1}{\frac{nc}{s} - 1}, \frac{c - x_2}{\frac{nc}{s} - 1}, \ldots, \frac{c - x_n}{\frac{nc}{s} - 1}\right) \prec (x_1, x_2, \ldots, x_n) = x.$$

**Lemma 2.5** (*see [19]*). *Suppose that $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ and $\sum_{i=1}^n x_i = s$. If $c \geq 0$, then*

$$\frac{c + x}{\frac{nc}{s} + 1} = \left(\frac{c + x_1}{\frac{nc}{s} + 1}, \frac{c + x_2}{\frac{nc}{s} + 1}, \ldots, \frac{c + x_n}{\frac{nc}{s} + 1}\right) \prec (x_1, x_2, \ldots, x_n) = x.$$

## 3. Main results

**Theorem 3.1.** *$H_n^{**}(x, r)$ is Schur concave in $\mathbb{R}_+^n$.*

**Proof.** According to Lemma 2.1 and Remark 2.1, we only need to prove that

$$(x_1 - x_2)\left(\frac{\partial H_n^{**}(x, r)}{\partial x_1} - \frac{\partial H_n^{**}(x, r)}{\partial x_2}\right) \leq 0 \tag{3.1}$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ and $r \in \{1, 2, \ldots, n\}$.

We divide the proof into four cases.

*Case* 1. If $r = 1$, then (1.2) leads to

$$H_n^{**}(x, 1) = H_n^{**}(x_1, x_2, \ldots, x_n; 1) = \prod_{i=1}^n x_i. \tag{3.2}$$

It follows from (3.2) that

$$(x_1 - x_2)\left(\frac{\partial H_n^{**}(x,1)}{\partial x_1} - \frac{\partial H_n^{**}(x,1)}{\partial x_2}\right) = -\frac{(x_1-x_2)^2}{x_1x_2}\prod_{i=1}^n x_i \le 0.$$

*Case* 2. If $r = n$, then from (1.2) we clearly see that

$$H_n^{**}(x,n) = H_n^{**}(x_1,x_2,\ldots,x_n;n) = \left(\sum_{i=1}^n x_i\right)^{\frac{1}{n}}. \tag{3.3}$$

From (3.3) we get

$$\frac{\partial H_n^{**}(x,n)}{\partial x_1} = \frac{\partial H_n^{**}(x,n)}{\partial x_2} = \frac{1}{n}\left(\sum_{i=1}^n x_i\right)^{\frac{1}{n}-1} \tag{3.4}$$

and

$$(x_1 - x_2)\left(\frac{\partial H_n^{**}(x,n)}{\partial x_1} - \frac{\partial H_n^{**}(x,n)}{\partial x_2}\right) = 0.$$

*Case* 3. If $n \ge 3$ and $r = 2$, then it follows from (1.2) that

$$\begin{aligned} H_n^{**}(x,2) &= H_n^{**}(x_1,x_2,\ldots,x_n;2) \\ &= H_{n-1}^{**}(x_2,x_3,\ldots,x_n;2)(x_1+x_2)^{\frac{1}{2}}\prod_{j=3}^n (x_1+x_j)^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

From (3.5) we get

$$\frac{\partial H_n^{**}(x,2)}{\partial x_1} = \frac{1}{2}H_n^{**}(x,2)\left(\frac{1}{x_1+x_2} + \sum_{j=3}^n \frac{1}{x_1+x_j}\right). \tag{3.6}$$

Similarly, we have

$$\frac{\partial H_n^{**}(x,2)}{\partial x_2} = \frac{1}{2}H_n^{**}(x,2)\left(\frac{1}{x_1+x_2} + \sum_{j=3}^n \frac{1}{x_2+x_j}\right). \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\begin{aligned} &(x_1 - x_2)\left(\frac{\partial H_n^{**}(x,2)}{\partial x_1} - \frac{\partial H_n^{**}(x,2)}{\partial x_2}\right) \\ &\quad = -\frac{1}{2}(x_1-x_2)^2 H_n^{**}(x,2)\sum_{j=3}^n \frac{1}{(x_1+x_j)(x_2+x_j)} \\ &\quad \le 0. \end{aligned}$$

*Case* 4. If $n \ge 4$ and $3 \le r \le n-1$, then (1.2) leads to

$$\begin{aligned} H_n^{**}(x,r) &= H_n^{**}(x_1,x_2,\ldots,x_n;r) \\ &= H_{n-1}^{**}(x_2,x_3,\ldots,x_n,r)\prod_{3\le i_1<i_2<\cdots<i_{r-1}\le n}\left(x_1+\sum_{j=1}^{r-1}x_{i_j}\right)^{\frac{1}{r}}\prod_{3\le i_1<i_2<\cdots<i_{r-2}\le n}\left(x_1+x_2+\sum_{j=1}^{r-2}x_{i_j}\right)^{\frac{1}{r}}. \end{aligned} \tag{3.8}$$

Making use of (3.8) and differentiating $H_n^{**}(x,r)$ with respect to $x_1$, we get

$$\frac{\partial H_n^{**}(x,r)}{\partial x_1} = \frac{1}{r}H_n^{**}(x,r)\left[\sum_{3\le i_1<i_2\cdots<i_{r-1}\le n}\left(x_1+\sum_{j=1}^{r-1}x_{i_j}\right)^{-1} + \sum_{3\le i_1<i_2\cdots<i_{r-2}\le n}\left(x_1+x_2+\sum_{j=1}^{r-2}x_{i_j}\right)^{-1}\right]. \tag{3.9}$$

Similarly, we have

$$\frac{\partial H_n^{**}(x,r)}{\partial x_2} = \frac{1}{r}H_n^{**}(x,r)\left[\sum_{3\le i_1<i_2\cdots<i_{r-1}\le n}\left(x_2+\sum_{j=1}^{r-1}x_{i_j}\right)^{-1} + \sum_{3\le i_1<i_2\cdots<i_{r-2}\le n}\left(x_1+x_2+\sum_{j=1}^{r-2}x_{i_j}\right)^{-1}\right]. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$(x_1 - x_2)\left(\frac{\partial H_n^{**}(x,r)}{\partial x_1} - \frac{\partial H_n^{**}(x,r)}{\partial x_2}\right)$$
$$= -\frac{1}{r}(x_1 - x_2)^2 H_n^{**}(x,r) \sum_{3\le i_1<i_2\cdots<i_{r-1}\le n}\left[\left(x_1 + \sum_{j=1}^{r-1} x_{i_j}\right)^{-1}\left(x_2 + \sum_{j=1}^{r-1} x_{i_j}\right)^{-1}\right]$$
$$\le 0.$$

Therefore, inequality (3.1) follows from Cases 1–4, and the proof of Theorem 3.1 is completed. □

**Theorem 3.2.** *$H_n^{**}(x,r)$ is Schur multiplicatively convex in $\mathbb{R}_+^n$.*

**Proof.** By Lemma 2.2, we only need to prove that

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial H_n^{**}(x,r)}{\partial x_1} - x_2\frac{\partial H_n^{**}(x,r)}{\partial x_2}\right) \ge 0 \tag{3.11}$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$.

We divide the proof into four cases.

*Case* I. If $r = 1$, then (3.2) leads to

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial H_n^{**}(x,1)}{\partial x_1} - x_2\frac{\partial H_n^{**}(x,1)}{\partial x_2}\right) = 0.$$

*Case* II. If $r = n$, then (3.4) implies that

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial H_n^{**}(x,n)}{\partial x_1} - x_2\frac{\partial H_n^{**}(x,n)}{\partial x_2}\right)$$
$$= \frac{1}{n}(\log x_1 - \log x_2)(x_1 - x_2)\left(\sum_{i=1}^{n} x_i\right)^{\frac{1}{n}-1}$$
$$\ge 0.$$

*Case* III. If $n \ge 3$ and $r = 2$, then it follows from (3.6) and (3.7) that

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial H_n^{**}(x,2)}{\partial x_1} - x_2\frac{\partial H_n^{**}(x,2)}{\partial x_2}\right)$$
$$= \frac{1}{2}H_n^{**}(x,2)(\log x_1 - \log x_2)(x_1 - x_2)\left[\frac{1}{x_1 + x_2} + \sum_{j=3}^{n}\frac{x_j}{(x_1 + x_j)(x_2 + x_j)}\right]$$
$$\ge 0.$$

*Case* IV. If $n \ge 4$ and $3 \le r \le n - 1$, then from (3.9) and (3.10) we have

$$(\log x_1 - \log x_2)\left(x_1\frac{\partial H_n^{**}(x,r)}{\partial x_1} - x_2\frac{\partial H_n^{**}(x,r)}{\partial x_2}\right)$$
$$= \frac{1}{r}H_n^{**}(x,r)\left[\sum_{3\le i_1<i_2<\cdots<i_{r-1}\le n}\frac{\sum_{j=1}^{r-1} x_{i_j}}{\left(x_1 + \sum_{j=1}^{r-1} x_{i_j}\right)\left(x_2 + \sum_{j=1}^{r-1} x_{i_j}\right)}\right.$$
$$\left. + \sum_{3\le i_1<i_2<\cdots<i_{r-2}\le n}\frac{1}{x_1 + x_2 + \sum_{j=1}^{r-2} x_{i_j}}\right](x_1 - x_2)(\log x_1 - \log x_2)$$
$$\ge 0.$$

Therefore, inequality (3.11) follows from Cases I–IV, and the proof of Theorem 3.2 is completed. □

**Theorem 3.3.** *$H_n^{**}(x,r)$ is Schur harmonic convex in $\mathbb{R}_+^n$.*

**Proof.** From Lemma 2.3, we clearly see that we only need to prove that

$$(x_1 - x_2)\left(x_1^2 \frac{\partial H_n^{**}(x,r)}{\partial x_1} - x_2^2 \frac{\partial H_n^{**}(x,r)}{\partial x_2}\right) \ge 0 \tag{3.12}$$

for all $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$.

We divide the proof into four cases.

*Case* A. If $r = 1$, then it follows from (3.2) that

$$(x_1 - x_2)\left(x_1^2 \frac{\partial H_n^{**}(x,1)}{\partial x_1} - x_2^2 \frac{\partial H_n^{**}(x,1)}{\partial x_2}\right) = (x_1 - x_2)^2 \prod_{i=1}^{n} x_i \ge 0.$$

*Case* B. If $r = n$, then (3.4) leads to

$$(x_1 - x_2)\left(x_1^2 \frac{\partial H_n^{**}(x,n)}{\partial x_1} - x_2^2 \frac{\partial H_n^{**}(x,n)}{\partial x_2}\right) = \frac{1}{n}(x_1 - x_2)^2(x_1 + x_2)\left(\sum_{i=1}^{n} x_i\right)^{\frac{1}{n}-1} \ge 0.$$

*Case* C. If $n \ge 3$ and $r = 2$, then (3.6) and (3.7) imply that

$$(x_1 - x_2)\left(x_1^2 \frac{\partial H_n^{**}(x,2)}{\partial x_1} - x_2^2 \frac{\partial H_n^{**}(x,2)}{\partial x_2}\right) = \frac{1}{2} H_n^{**}(x,2)(x_1 - x_2)^2 \left[1 + \sum_{j=3}^{n} \frac{x_1 x_2 + x_j(x_1 + x_2)}{(x_1 + x_j)(x_2 + x_j)}\right] \ge 0.$$

*Case* D. If $n \ge 4$ and $3 \le r \le n-1$, then from (3.9) and (3.10) we get

$$\begin{aligned}(x_1 - x_2)\left(x_1^2 \frac{\partial H_n^{**}(x,r)}{\partial x_1} - x_2^2 \frac{\partial H_n^{**}(x,r)}{\partial x_2}\right) &= (x_1 - x_2)^2 \left[\sum_{3 \le i_1 < i_2 < \cdots < i_{r-1} \le n} \frac{x_1 x_2 + (x_1 + x_2)\sum_{j=1}^{r-1} x_{i_j}}{\left(x_1 + \sum_{j=1}^{r-1} x_{i_j}\right)\left(x_2 + \sum_{j=1}^{r-1} x_{i_j}\right)}\right. \\ &\quad \left. + \sum_{3 \le i_1 < i_2 < \cdots < i_{r-2} \le n} \frac{x_1 + x_2}{x_1 + x_2 + \sum_{j=1}^{r-2} x_{i_j}}\right] \frac{H_n^{**}(x,r)}{r} \\ &\ge 0.\end{aligned}$$

Therefore, inequality (3.12) follows from Cases A–D, and the proof of Theorem 3.3 is completed. □

## 4. Applications in inequalities

In this section, we establish some inequalities by use of Theorems 3.1–3.3 and the theory of majorization.

The following result easily follows from Lemmas 2.4 and 2.5 together with Theorems 3.1–3.3.

**Corollary 4.1.** *Suppose that $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$ with $\sum_{i=1}^{n} x_i = s$. If $c_1 \ge s$, $c_2 \ge 0$ and $r \in \{1, 2, \ldots, n\}$, then*

(i) $H_n^{**}(c_1 - x, r) \ge \left(\frac{nc_1}{s} - 1\right)^{\frac{n!}{r[r!(n-r)!]}} H_n^{**}(x, r)$;

(ii) $H_n^{**}\left(e^{\frac{c_1 - x}{\frac{nc_1}{s} - 1}}, r\right) \le H_n^{**}(e^x, r)$;

(iii) $H_n^{**}\left(\frac{1}{x}, r\right) \ge \left(\frac{nc_1}{s} - 1\right)^{\frac{n!}{r[r!(n-r)!]}} H_n^{**}\left(\frac{1}{c_1 - x}, r\right)$;

(iv) $H_n^{**}(c_2 + x, r) \ge \left(\frac{nc_2}{s} + 1\right)^{\frac{n!}{r[r!(n-r)!]}} H_n^{**}(x, r)$;

(v) $H_n^{**}\left(e^{\frac{c_2 + x}{\frac{nc_2}{s} + 1}}, r\right) \le H_n^{**}(e^x, r)$;

(vi) $H_n^{**}\left(\frac{1}{x}, r\right) \ge \left(\frac{nc_2}{s} + 1\right)^{\frac{n!}{r[r!(n-r)!]}} H_n^{**}\left(\frac{1}{c_2 + x}, r\right)$.

*If we take $c_1 = s = 1$ and $r = 1$ in Corollary 4.1*(i) *and* (iv), *and take $c_2 = s = 1$ and $r = n$ in Corollary 4.1*(iii) *and* (vi), *respectively, then we obtain the following.*

**Corollary 4.2.** *If* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ *with* $\sum_{i=1}^n x_i = 1$, *then*

(i) $\prod_{i=1}^n \left(\frac{1}{x_i} - 1\right) \geq (n-1)^n$;

(ii) $\prod_{i=1}^n \left(\frac{1}{x_i} + 1\right) \geq (n+1)^n$;

(iii) $\frac{\sum_{i=1}^n \frac{1}{x_i}}{\sum_{i=1}^n \frac{1}{1-x_i}} \geq (n-1)$;

(iv) $\frac{\sum_{i=1}^n \frac{1}{x_i}}{\sum_{i=1}^n \frac{1}{1+x_i}} \geq (n+1)$.

**Remark 4.1.** Inequalities in Corollary 4.2(i) and (ii) are the well-known Weierstrass inequalities (see [6, p. 260]).

**Corollary 4.3.** *If* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ *and* $r \in \{1, 2, \ldots, n\}$, *then*

$$\left(\frac{nr}{\sum_{i=1}^n \frac{1}{x_i}}\right)^{\frac{n!}{r[r!(n-r)!]}} \leq H_n^{**}(x, r) \leq \left(\frac{r}{n}\sum_{i=1}^n x_i\right)^{\frac{n!}{r[r!(n-r)!]}}.$$

**Proof.** Corollary 4.3 follows from Theorems 3.1 and 3.3 together with the facts that

$$\left(\frac{\sum_{i=1}^n x_i}{n}, \frac{\sum_{i=1}^n x_i}{n}, \ldots, \frac{\sum_{i=1}^n x_i}{n}\right) \prec (x_1, x_2, \ldots, x_n) = x$$

and

$$\left(\frac{\sum_{i=1}^n \frac{1}{x_i}}{n}, \frac{\sum_{i=1}^n \frac{1}{x_i}}{n}, \ldots, \frac{\sum_{i=1}^n \frac{1}{x_i}}{n}\right) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_n}\right) = \frac{1}{x}. \quad \square$$

**Corollary 4.4.** *If* $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n_+$ *and* $r \in \{1, 2, \ldots, n\}$, *then*

$$H_n^{**}(x, r) \geq \left(r\prod_{i=1}^n x_i^{\frac{1}{n}}\right)^{\frac{n!}{r[r!(n-r)!]}}.$$

**Proof.** Corollary 4.4 follows from Theorem 3.2 and the fact that

$$\log\left(\prod_{i=1}^n x_i^{\frac{1}{n}}, \prod_{i=1}^n x_i^{\frac{1}{n}}, \ldots, \prod_{i=1}^n x_i^{\frac{1}{n}}\right) \prec \log(x_1, x_2, \ldots, x_n) = \log x. \quad \square$$

**Corollary 4.5.** *Suppose that* $A \in M_n(C)(n \geq 2)$ *is a complex matrix,* $\lambda_1, \lambda_2, \ldots, \lambda_n$ *and* $\sigma_1, \sigma_2, \ldots, \sigma_n$ *are the eigenvalues and singular values of* $A$, *respectively. If* $A$ *is a positive definite Hermitian matrix and* $r \in \{1, 2, \ldots, n\}$, *then*

(i) $\prod_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \left(\sum_{j=1}^r \lambda_{i_j}\right)^{\frac{1}{r}} \leq \left(\frac{r}{n} trA\right)^{\frac{n!}{r[r!(n-r)!]}}$;

(ii) $\prod_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \left(\sum_{j=1}^r \lambda_{i_j}^{-1}\right)^{\frac{1}{r}} \geq \left(\frac{nr}{trA}\right)^{\frac{n!}{r[r!(n-r)!]}}$;

(iii) $\prod_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \left(\sum_{j=1}^r \lambda_{i_j}\right)^{\frac{1}{r}} \geq \left[r(\det A)^{\frac{1}{n}}\right]^{\frac{n!}{r[r!(n-r)!]}}$;

(iv) $\prod_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \left(\sum_{j=1}^r \lambda_{i_j}\right)^{\frac{1}{r}} \leq \prod_{1 \leq i_1 < i_2 \cdots < i_r \leq n} \left(\sum_{j=1}^r \sigma_{i_j}\right)^{\frac{1}{r}}$.

**Proof.** From the assumption in Corollary 4.5, we clearly see that

$$(\lambda_1, \lambda_2, \ldots, \lambda_n), \quad (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{R}_+^n,$$

$$\sum_{i=1}^{n} \lambda_i = trA \tag{4.1}$$

and

$$\prod_{i=1}^{n} \lambda_i = \det A. \tag{4.2}$$

On the other hand, a result due to Weyl [46] (see also [31, p. 231]) gives

$$\log(\lambda_1, \lambda_2, \ldots, \lambda_n) \prec \log(\sigma_1, \sigma_2, \ldots, \sigma_n). \tag{4.3}$$

Therefore, Corollary 4.5(i) follows from (4.1) and Theorem 3.1, Corollary 4.5(ii) follows from (4.1) and Theorem 3.3, Corollary 4.5(iii) follows from (4.2) and Theorem 3.2, and Corollary 4.5(iv) follows from (4.3) and Theorem 3.2. □

**Corollary 4.6.** *Let $\mathcal{A} = A_1A_2\cdots A_{n+1}$ be a n-dimensional simplex in $\mathbb{R}^n$, and $P$ be an arbitrary point in the interior of $\mathcal{A}$. If $B_i$ is the intersection point of straight line $A_iP$ and hyperplane $\sum_i = A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}$, $i = 1, 2, \ldots, n+1$ and $r \in \{1, 2, \ldots, n+1\}$, then*

(i) $\prod_{1\leq i_1<i_2\cdots<i_r\leq n+1}\left(\sum_{j=1}^{r}\frac{PB_{i_j}}{A_{i_j}B_{i_j}}\right)^{\frac{1}{r}} \leq \left(\frac{r}{n+1}\right)^{\frac{n!}{r[r!(n-r)!]}}$;

(ii) $\prod_{1\leq i_1<i_2\cdots<i_r\leq n+1}\left(\sum_{j=1}^{r}\frac{PA_{i_j}}{A_{i_j}B_{i_j}}\right)^{\frac{1}{r}} \leq \left(\frac{nr}{n+1}\right)^{\frac{n!}{r[r!(n-r)!]}}$;

(iii) $\prod_{1\leq i_1<i_2\cdots<i_r\leq n+1}\left(\sum_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{PB_{i_j}}\right)^{\frac{1}{r}} \geq [(n+1)r]^{\frac{n!}{r[r!(n-r)!]}}$;

(iv) $\prod_{1\leq i_1<i_2\cdots<i_r\leq n+1}\left(\sum_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{PA_{i_j}}\right)^{\frac{1}{r}} \geq \left(\frac{(n+1)r}{n}\right)^{\frac{n!}{r[r!(n-r)!]}}$.

**Proof.** We clearly see that

$$\sum_{i=1}^{n+1}\frac{PB_i}{A_iB_i} = 1 \tag{4.4}$$

and

$$\sum_{i=1}^{n+1}\frac{PA_i}{A_iB_i} = n. \tag{4.5}$$

Therefore, Corollary 4.6(i) follows from (4.4) and Theorem 3.1, Corollary 4.6(ii) follows from (4.5) and Theorem 3.1, Corollary 4.6(iii) follows from (4.4) and Theorem 3.3, and Corollary 4.6(iv) follows from (4.5) and Theorem 3.3. □

**Remark 4.2.** Mitrinović et al. [32, p. 473–479] established a series of inequalities for $\frac{PA_i}{A_iB_i}$ and $\frac{PB_i}{A_iB_i}$, $i = 1, 2, \ldots, n+1$. Obviously, our inequalities in Corollary 4.6 are different from theirs.

**Corollary 4.7.** *Suppose that $X_1, X_2, \ldots, X_n$ are independent random variables with characteristic functions $\varphi_{X_1}(t), \varphi_{X_2}(t), \ldots, \varphi_{X_n}(t)$, and generating functions $\psi_1(s), \psi_2(s), \ldots, \psi_n(s)$. If $\eta = \sum_{i=1}^{n} X_i$ with characteristic function $\varphi_\eta(t)$ and generating function $\psi_\eta(s)$, then*

$$\prod_{1\leq i_1<i_2<\cdots<i_r\leq n}\left(\sum_{j=1}^{r}\varphi_{X_{i_j}}(t)\right)^{\frac{1}{r}} \geq \left(r\varphi_\eta^{\frac{1}{n}}(t)\right)^{\frac{n!}{r[r!(n-r)!]}}$$

*and*

$$\prod_{1\leq i_1<i_2<\cdots<i_r\leq n}\left(\sum_{j=1}^{r}\psi_{i_j}(s)\right)^{\frac{1}{r}} \geq \left(r\psi_\eta^{\frac{1}{n}}(s)\right)^{\frac{n!}{r[r!(n-r)!]}}$$

*for $r \in \{1, 2, \ldots, n\}$.*

**Proof.** We clearly see that

$$\varphi_\eta(t) = \prod_{i=1}^{n} \varphi_{X_i}(t) \tag{4.6}$$

and

$$\psi_\eta(s) = \prod_{i=1}^{n} \psi_i(s). \tag{4.7}$$

Therefore, Corollary 4.7 follows from (4.6) and (4.7) together with Theorem 3.2. □

**Corollary 4.8.** *Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_n$ be independent random variables, and $\overline{F}_i(t) = P\{X_i > t\}$ and $\overline{G}_i(t) = P\{Y_i > t\}$, $i = 1, 2, \ldots, n$. If $(\overline{F}_1(t), \overline{F}_2(t), \ldots, \overline{F}_n(t)) \prec (\overline{G}_1(t), \overline{G}_2(t), \ldots, \overline{G}_n(t))$, then*

$$H_n^{**}(EX_1, EX_2, \ldots, EX_n; r) \geq H_n^{**}(EY_1, EY_2, \ldots, EY_n; r)$$

*for $r \in \{1, 2, \ldots, n\}$.*

**Proof.** It follows from [31, p. 350] that

$$(EX_1, EX_2, \ldots, EX_n) \prec (EY_1, EY_2, \ldots, EY_n). \tag{4.8}$$

Therefore, Corollary 4.8 follows from Theorem 3.2 and (4.8). □

## 5. Application in reliability

In this section, we discuss an application of the proposed $H_n^{**}(x, r)$ in analysis the reliability of $K$ out of $N$ systems with dependent components based on Copula function.

In reliability theory, a $K$ out of $N$ system is a system with $N$ components, which functions if and only if $K$ or more of the components function. For the system that the $N$ components function independently, Boland and Proschan [3] have obtained a result on Schur-concave and Schur-convex of the reliability function of the system. Unfortunately, for the situation with dependent components, the reliability function of the system is a multivariate cumulative distribution function of all components and its analytic representation is often too complex.

To overcome the issue, the multivariate Copula function [8] can be used to estimate the reliability function of the system by splitting it into two parts, namely, determining the marginal distribution of each component, and determining the dependence structure of them, which specify a meaningful copula function.

Furthermore, as well known, the Sklar's theorem [37] is the most important theorem regarding to copula functions, because it provides a way to analyze the dependence structure of multivariate distributions by an adequate copula function.

In the following, inspired by the work of Bregman and Klüppelberg [7], we will first introduce a special copula function and discuss the relationship between it and our $H_n^{**}(x, r)$ and then obtain a simple result on the reliability of the systems with dependent components.

For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n$, let $y = (y_1, y_2, \ldots, y_n) \triangleq x^{-1} = (x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}) \in \mathbb{R}_+^n$, we define

$$S_n(y_1, \ldots, y_n) = (y_1^{-n} + \cdots + y_n^{-n})^{-1/n},$$

and then for any $r \in \{1, 2, \ldots, n\}$ and $1 \leq i_1 < i_2 < \cdots < i_r \leq n$, we have

$$S_r(y_{i_1}, \ldots, y_{i_r}) = (y_{i_1}^{-r} + \cdots + y_{i_r}^{-r})^{-1/r},$$

where $(y_{i_1}, \ldots, y_{i_r}) \in \mathbb{R}_+^r$ is a subvector of $y$. According to [7], each $S_r(y_{i_1}, \ldots, y_{i_r})$ is a function of Clayton family of $S$-Copula, namely, $S_\theta(\theta > 0)$ with $\theta = r$.

From the definition of Copula function, we can find easily that

$$\left[H_n^{**}(x, r)\right]^{-1} = \prod_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} S_r(y_{i_1}, \ldots, y_{i_r}), \tag{5.1}$$

which is also a Copula function.

Based on Sklar's theorem and (5.1), it is known that there will be a $K$ out of $N$ systems with dependent components and the reliability function of this system can be expressed as

$$R_K(p) = \sum_{r=K}^{N} \binom{N}{r} \left[H_N^{**}(p, r)\right]^{-1}, \tag{5.2}$$

where $p = (p_1, \ldots, p_N)$ is the vector of component reliabilities which depend on the marginal distribution of each component. From (5.1) and (5.2) together with Theorem 3.1, we clearly see that the maximum value of reliability function $R_K(p)$ will be reached when the components have same reliabilities, e.g., the system is consisted of components of same type.

## Acknowledgment

The first author's work was supported by the NSF of China under grant No. 11071069, NSF of Zhejiang Province under grant Nos. Y6100170 and Y7080185, and the Innovation Team Foundation of the Department of Education of Zhejiang Province under Grant No. T200924.

## References

[1] R.E. Barlow, F. Spizzichino, Schur-concave survival functions and survival analysis, J. Comput. Appl. Math. 46 (3) (1993) 437–447.
[2] P.J. Boland, F. Proschan, Schur convexity of the maximum likelihood function for the multivariate hypergeometric and multinomial distributions, Statist. Probab. Lett. 5 (5) (1987) 317–322.
[3] P.J. Boland, F. Proschan, The reliability of $k$ out of $n$ systems, Ann. Probab. 11 (3) (1983) 760–764.
[4] J.V. Bondar, Universal optimality of experimental designs: definitions and a criterion, Canad. J. Statist. 11 (4) (1983) 325–331.
[5] P.S. Bullen, Handbook of Means and their Inequalities, Kluwer Academic Publishers Group, Dordrecht, 2003.
[6] P.S. Bullen, A Dictionary of Inequalities, Longman, Harlow, 1998.
[7] Y. Bregman, C. Klüppelberg, Ruin estimation in multivariate models with Clayton dependence structure, Scand. Actuar. J. 2005 (6) (2005) 462–480.
[8] X.-H. Chen, Y.-Q. Fan, V. Tsyrennikov, Efficient estimation of semiparametric multivariate copula models, J. Amer. Statist. Assoc. 101 (475) (2006) 1228–1240.
[9] Y.-M. Chu, Y.-P. Lv, The Schur harmonic convexity of the Hamy symmetric function and its applications, J. Inequal. Appl. 2009, Article ID 838529, 10 pages.
[10] Y.-M. Chu, X.-M. Zhang, G.-D. Wang, The Schur geometrical convexity of the extended mean values, J. Convex Anal. 15 (4) (2008) 707–718.
[11] A. Cohen, Optimality properties of some tests for homogeneity in contingency tables, Statistics 18 (4) (1987) 591–597.
[12] A. Cohen, H.B. Sackrowitz, Unbiasedness of tests for homogeneity, Ann. Statist. 15 (2) (1987) 805–816.
[13] S.R. Dalal, P. Fortini, An inequality comparing sums and maxima with application to Behrens–Fisher type problem, Ann. Statist. 10 (1) (1982) 297–301.
[14] M.L. Eaton, M.D. Perlman, A monotonicity property of the power functions of some invariant tests for MANOVA, Ann. Statist. 2 (1974) 1022–1028.
[15] N. Ebrahimi, F. Spizzichino, Some results on normalized total time on test and spacings, Statist. Probab. Lett. 36 (3) (1997) 231–243.
[16] E. El-Neweihi, F. Proschan, J. Sethuraman, Optimal allocation of components in parallel-series and series-parallel systems, J. Appl. Probab. 23 (3) (1986) 770–777.
[17] K.-Z. Guan, Some properties of a class of symmetric functions, J. Math. Anal. Appl. 336 (1) (2007) 70–80.
[18] K.-Z. Guan, A class of symmetric functions for multiplicatively convex function, Math. Inequal. Appl. 10 (4) (2007) 745–753.
[19] K.-Z. Guan, Schur-convexity of the complete symmetric function, Math. Inequal. Appl. 9 (4) (2006) 567–576.
[20] K.-Z. Guan, The Hamy symmetric function and its generalization, Math. Inequal. Appl. 9 (4) (2006) 797–805.
[21] T. Hara, M. Uchiyama, S.-E. Takahasi, A refinement of various mean inequalities, J. Inequal. Appl. 2 (4) (1998) 387–395.
[22] F.W. Huffer, L.A. Shepp, On the probability of covering the circle by random arcs, J. Appl. Probab. 24 (2) (1987) 422–429.
[23] F.K. Hwang, U.G. Rothblum, Partition-optimization with Schur convex sum objective functions, SIAM J. Discrete Math. 18 (3) (2004/2005) 512–524.
[24] W.-D. Jiang, Some properties of dual form of the Hamy's symmetric function, J. Math. Inequal. 1 (1) (2007) 117–125.
[25] H. Joe, Majorization, entropy and paired comparisons, Ann. Statist. 16 (2) (1988) 915–925.
[26] S.C. Kochar, R. Korwar, Stochastic orders for spacings of heterogeneous exponential random variables, J. Multivariate Anal. 57 (1) (1996) 69–83.
[27] S.S. Kochar, J. Rojo, Some new results on stochastic comparisons of spacings from heterogeneous exponential distributions, J. Multivariate Anal. 59 (2) (1996) 272–281.
[28] H.-T. Ku, M.-C. Ku, X.-M. Zhang, Inequalities for symmetric means, symmetric harmonic means, and their applications, Bull. Austral. Math. Soc. 56 (3) (1997) 409–420.
[29] H.-J. Li, M. Shaked, Stochastic majorization of stochastically monotone families of random variables, Adv. Appl. Probab. 25 (4) (1993) 895–913.
[30] D.-M. Li, H.-N. Shi, Schur convexity and Schur-geometrically concavity of generalized exponent mean, J. Math. Inequal. 3 (2) (2009) 217–225.
[31] A.W. Marshall, I. Olkin, Inequalities: Theorey of Majorization and its Applications, Academic Press, New York, 1979.
[32] D.S. Mitrinović, J.E. Pečarić, V. Volenec, Recent Advances in Geometric Inequalities, Kluwer Academic Publishers Group, Dordrecht, 1989.
[33] G. Nappo, F. Spizzichino, Ordering properties of the TTT-plot of lifetimes with Schur joint densities, Statist. Probab. Lett. 39 (3) (1998) 195–203.
[34] S.E. Nevius, F. Proschan, J. Sethuraman, Schur functions in statistics II: stochastic majorization, Ann. Statist. 5 (2) (1977) 263–273.
[35] D.H. Park, The reliability of $k$ out of $n$ systems when $k$ components are equally reliable, Statist. Probab. Lett. 7 (3) (1988) 175–178.
[36] F. Proschan, J. Sethuraman, Schur functions in statistics I: the preservation theorem, Ann. Statist. 5 (2) (1977) 256–262.
[37] S.T. Rachev, Handbook of Heavy Tailed Distributions in Finance, Elsevier, Amsterdam, 2003.
[38] S.M. Ross, M. Shahshahani, G. Weiss, On the duration of the problem of the points, J. Amer. Statist. Assoc. 75 (371) (1980) 663–666.
[39] I. Schur, Über eine Klasse Von Mittelbildungen mit anwendungen auf die determinantentheorie, Sitzunsber. Belin. Math. Ges. 22 (1923) 9–20.
[40] J.G. Shanthikumar, Stochastic majorization of random variables with proportional equilibrium rates, Adv. Appl. Probab. 19 (4) (1987) 854–872.
[41] H.-N. Shi, Schur-convex functions related to Hadamard-type inequalities, J. Math. Inequal. 1 (1) (2007) 127–136.
[42] H.-N. Shi, Y.-M Jiang, W.-D. Jiang, Schur-convexity and Schur-geometrically concavity of Gini means, Comput. Math. Appl. 57 (2) (2009) 266–274.
[43] Y.L. Tong, Some majorization inequalities in multivariate statistical analysis, SIAM Rev. 30 (4) (1988) 602–622.
[44] Y.L. Tong, Rectangular and elliptical probability inequalities for Schur-concave random variables, Ann. Statist. 10 (2) (1982) 637–642.
[45] Y.L. Tong, Probability Inequalities in Multivariate Distributions, Academic Press, New York, 1980.
[46] H. Weyl, Inequalities between two kinds of eigenvalues of a linear transformation, Proc. Nat. Acad. Sci. USA 35 (1949) 408–411.
[47] X.-M. Zhang, Schur-convex functions and isoperimetric inequalities, Proc. Amer. Math. Soc. 126 (2) (1998) 461–470.