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Density via duality

Jaroslav Nešetřil[a,b,*,1] Claude Tardif[c,2]

[a] *Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 11800 Prague, Czech Republic*

[b] *Institute for Theoretical Computer Science (ITI), Charles University, Malostranské nám. 25, 11800 Prague, Czech Republic*

[c] *Department of Mathematics and Statistics, University of Regina, Regina SK, S4S 0A2 Canada*

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Dedicated to the memory of Walter Deuber

Abstract

We present an unexpected correspondence between homomorphism duality theorems and gaps in the poset of graphs and their homomorphisms. This gives a new proof of the density theorem for undirected graphs and solves the density problem for directed graphs.

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1. Introduction

Given two graphs G_1 and G_2, where $G_i = (V_i, E_i)$, a *homomorphism* of G_1 into G_2 is a mapping $f : V_1 \mapsto V_2$ which preserves all the edges: $[f(x), f(y)] \in E_2$ whenever $[x, y] \in E_1$. We write $G_1 \to G_2$ if there exists a homomorphism from G_1 to G_2, and $G_1 \nrightarrow G_2$ otherwise. The class $\mathscr{G}$ of finite graphs endowed with the relation $\to$ is the "skeleton" of the category of finite graphs. This is essentially an ordered set, modulo the equivalence relation $\sim$, where $G_1 \sim G_2$ means that G_1 is *homomorphically equivalent* to G_2, that is, $G_1 \to G_2$ and $G_2 \to G_1$.

* Corresponding author. Department of Applied Mathematics, Faculty of Mathematics and Physics, Charles University, Malostranské nám. 25, 11800 Prague, Czech Republic.

E-mail addresses: nesetril@kam.ms.mff.cuni.cz (J. Nešetřil), tardif@math.uregina.ca (C. Tardif).

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This ordering of graphs from the point of view of homomorphisms has a fascinating structure. For almost every fixed graph H, the question "does the graph G admits a homomorphism into H" is NP-complete (see [7]), thus the relation $\to$ has an intricate local structure. However, it turns out that $\to$ induces a distributive lattice order on the classes of homomorphically equivalent graphs, so that the global structure is reasonably well behaved. One of the motivations of this paper is the "density theorem" of Welzl [17] which gives a flavour of the relation between these two aspects of graph homomorphisms.

Theorem 1 (Welzl [17]). *Let G,H be two finite graphs such that H is not bipartite and there exists a homomorphism from G to H but none from H to G. Then there exists a graph K such that there exist homomorphisms from G to K and from K to H, but none from H to K or from K to G.*

In other words, the relation $\to$ induces a dense quasiorder on $\mathscr{G}$, with the unique exception occurring at the bottom, between the graphs with no edges and the bipartite graphs. This is therefore a statement on the global structure of the category of graphs. However, the construction of the interjacent graph K must take into account the specific instances of G and H, and respect the conditions $G \to K \to H$ and $H \nrightarrow K \nrightarrow G$. These constraints being NP-hard, it is perhaps not surprising that the first proof found by Welzl was a complicated ad hoc argument.

However, Theorem 1 admits a simple natural proof, as was later found out independently by Perles and Nešetřil (see [12]). The pleasing argument intertwines classical results of graph theory with categorical aspects of graph homomorphisms. The present paper is a brief exploration of the developments made possible by this approach. We will show how the argument transposes to the category of directed graphs, connecting the problem of density with an apparently unrelated topic, that is, homomorphism duality.

2. The disjoint union and the product

The disjoint union is the coproduct in the category of graphs. This looks like a trivial construction. Any graph is the disjoint union of its connected components, hence the typical argument "without loss of generality, we can assume that G is connected ..." shows that in many situations, one can dispose of coproducts without even noticing it. We have $\chi(G \cup H) = \max\{\chi(G), \chi(H)\}$, $\omega(G \cup H) = \max\{\omega(G), \omega(H)\}$ and $\gamma_{\mathrm{odd}}(G \cup H) = \min\{\gamma_{\mathrm{odd}}(G), \gamma_{\mathrm{odd}}(H)\}$, where $\chi, \omega, \gamma_{\mathrm{odd}}$ denote, respectively, the chromatic number, the clique number and the odd girth. These identities are consistent with the categorial properties of the coproduct:

$$G \cup H \to K \quad \text{if and only if } G \to K \quad \text{and} \quad H \to K. \tag{1}$$

Thus the coproduct is a supremum with respect to the relation $\rightarrow$. Since the chromatic number and the clique number are increasing and the odd girth is decreasing with respect to $\rightarrow$, the above identities seem natural, and turn out to be easy to prove.

The product $G \times H$ of G and H has $V(G) \times V(H)$ as its vertex set, and $[(u,v),(u',v')]$ is an edge of $G \times H$ if and only if $[u,u']$ is an edge of G and $[v,v']$ is an edge of H. This is the infimum of G and H with respect to the relation $\rightarrow$:

$$K \rightarrow G \times H \quad \text{if and only if } K \rightarrow G \quad \text{and} \quad K \rightarrow H. \tag{2}$$

The identities $\omega(G \times H) = \min\{\omega(G), \omega(H)\}$ and $\gamma_{\text{odd}}(G \times H) = \max\{\gamma_{\text{odd}}(G), \gamma_{\text{odd}}(H)\}$ follow from this characterisation. However, the chromatic number of a product of graphs is an outstanding problem in graph theory.

Conjecture 1 (Hedetniemi [6]). *For any graphs G and H,*

$$\chi(G \times H) = \min\{\chi(G), \chi(H)\}.$$

Very little progress has been made on this question since it has been formulated by Hedetniemi in 1966. This contrasts with the situation of the disjoint union, where the corresponding identity is trivial.

The density problem may be solved using products and coproducts. Suppose that we are given two graphs G and H such that H is not bipartite, $G \rightarrow H$ and $H \nrightarrow G$. We want to find a graph K such that $G \rightarrow K \rightarrow H$ and $H \nrightarrow K \nrightarrow G$. Nešetřil and Perles proposed a solution of the form

$$K = G \cup (X \times H).$$

For any choice of X, we then have $G \rightarrow K \rightarrow H$. The remaining conditions depend on the parameter X.

It is easy to specify conditions on X which will guarantee $H \nrightarrow G \cup (X \times H)$. Since $H \nrightarrow G$, H necessarily contains a nonbipartite connected component H' such that $H' \nrightarrow G$. We can specify that the odd girth of X should be larger than that of H'. We then have $H' \nrightarrow X$, since a nonbipartite graph cannot be mapped by a homomorphism into a graph with a larger odd girth. Therefore $H' \nrightarrow (X \times H)$ whence $H \nrightarrow G \cup (X \times H) = K$.

The remaining condition, $K \nrightarrow G$, is equivalent to $X \times H \nrightarrow G$. This seems less tractable because of the position of X in a product. One is tempted to divide both sides by H to isolate X:

$$X \times H \nrightarrow G \quad \text{if and only if } X \nrightarrow G \div H. \tag{3}$$

At least, this step would seem natural to an immature mathematical mind, but a mathematician would object that the division of graphs is not defined. However, there does indeed exist a graph $G \div H$ with the property described in (3). It has been used by

Lovász [10] in his work on the cancellation law for relational structures, and it is also a fundamental object in the study of Hedetniemi's conjecture [2, 3]. The vertex set of $G \div H$ is the set of all functions from $V(H)$ to $V(G)$, and two functions f and g are joined by an edge if $[f(u), g(v)] \in E(G)$ for all $[u, v] \in E(H)$. This definition allows a natural correspondence between the homomorphisms from $X \times H$ to G and the homomorphisms from X to $G \div H$. It is customary to use an exponential notation and denote this graph G^H rather $G \div H$, because of the structure of its vertex set. Hence condition (3) is usually written as follows:

$$H \times X \nrightarrow G \quad \text{if and only if } X \nrightarrow G^H.$$

This meets the approval of mathematicians, but immature mathematical minds disapprove. For our part, we will be satisfied with the functional characterisation of Eq. (3) and adopt the notation $G \div H$. The arithmetic is then consistent.

To sum up, the right choice of X must be "small enough" so that $H' \nrightarrow X$ and "large enough" so that $X \nrightarrow G \div H$. The classical result of Erdős guarantees the existence of such a graph:

Theorem 2 (Erdős [4]). *There exist graphs with girth and chromatic number as large as we please.*

Selecting X such that $\gamma(X) > \gamma_{\text{odd}}(H')$ and $\chi(X) > \chi(G \div H)$ we then have $H' \nrightarrow X \nrightarrow G \div H$. Therefore

$$G \rightarrow G \cup (X \times H) \rightarrow H$$

and

$$H \nrightarrow G \cup (X \times H) \nrightarrow G.$$

This proves Welzl's density theorem.

3. Directed graphs

Homomorphisms of directed graphs preserve orientation as well as adjacency. In other aspects, the categorial setting remains essentially the same. It seems reasonable to assume that a directed analogue of the density theorem should hold. In fact, this seems to follow directly from Theorem 1. Let $\vec{G}, \vec{H}$ be directed graphs such that $\vec{G} \rightarrow \vec{H}$ and $\vec{H} \nrightarrow \vec{G}$. Let G, H be the undirected base graphs of $\vec{G}$ and $\vec{H}$, respectively. If there exists a graph K such that $G \rightarrow K \rightarrow H$ and $H \nrightarrow K \nrightarrow G$, then for any orientation $\vec{K}$ of K derived from the orientation $\vec{H}$ of H by a homomorphism, we have $\vec{G} \rightarrow \vec{K} \rightarrow \vec{H}$ and $\vec{H} \nrightarrow \vec{K} \nrightarrow \vec{G}$. Therefore Theorem 1 admits a trivial generalisation to directed graphs.

Now, this "trivial generalisation of the density theorem to directed graphs" does not amount to a "density theorem for directed graphs". The reason is quite subtle.

Theorem 1 guarantees the existence of the interjacent graph K only if H is nonbipartite. An exception occurs when H is bipartite. It is easy to dismiss it as trivial, since there exist homomorphisms between any two bipartite graphs. However, the same does not hold for directed graphs. Indeed, the category of directed graphs alone is as rich as the category of directed graphs. Therefore, further arguments would be needed to complete the proof of a "density theorem for directed graphs".

It is worthwhile to run through the argument of the previous section again in the context of directed graphs. Products, coproducts and powers are defined as before, satisfying Eqs. (1)–(3). Suppose that we are given directed graphs $\vec{G}$ and $\vec{H}$ such that $\vec{G} \to \vec{H}$ and $\vec{H} \nrightarrow \vec{G}$. For simplicity, we will also assume that $\vec{H}$ is connected. If there exists a directed graph $\vec{X}$ such that $\vec{H} \nrightarrow \vec{X} \nrightarrow \vec{G} \div \vec{H}$, then $\vec{G} \cup (\vec{X} \times \vec{H})$ is homomorphically interjacent to $\vec{G}$ and $\vec{H}$, just as in the undirected case. The missing punchline is an analogue of Erdős' Theorem 2; which guarantees the existence of $\vec{X}$. A suitable equivalent would state the "independence" between some feature of the orientation of graphs and their chromatic numbers, just as Theorem 2 states the "independence" between girth and chromatic number.

Now, it is a subject of growing curiosity that many aspects of the orientations of a graph are related to its chromatic number. For instance, by Gallai–Roy theorem [5, 15], the chromatic number of a graph is bounded above by the maximum length of a chain in any of its orientations. Minty's painting lemma [11] provides a similar bound using the ratio of forward edges to backward edges in an acyclic orientation of a graph. There are also relations between the chromatic polynomial of a graph and its acyclic orientations [16], and between the list-chromatic number of a graph and its directed Eulerian subgraphs [1].

Therefore, it seems unlikely that there exist a directed analogue of Theorem 2 that would suit our purposes. Note that the use of such a result would necessarily belittle the role of the particular instances of $\vec{H}$ and $\vec{G} \div \vec{H}$, reducing them to features such as their chromatic number, the structure of cycles, and so on. In contrast, our approach will put a great emphasis on the relationship between $\vec{H}$ and $\vec{G} \div \vec{H}$.

Suppose that there exists no directed graph $\vec{X}$ such that $\vec{H} \nrightarrow \vec{X} \nrightarrow \vec{G} \div \vec{H}$. This is an intriguing situation, as it implies a complementarity between homomorphisms *into* $\vec{G} \div \vec{H}$ and *from* $\vec{H}$. For any directed graph $\vec{X}$, $\vec{X} \nrightarrow \vec{G} \div \vec{H}$ implies $\vec{H} \to \vec{X}$; and $\vec{H} \nrightarrow \vec{X}$ implies $\vec{X} \to \vec{G} \div \vec{H}$. The option $\vec{H} \to \vec{X} \to \vec{G} \div \vec{H}$ can be ruled out as it implies $\vec{H} \to \vec{G} \div \vec{H}$ whence $\vec{H} \times \vec{H} \to \vec{G}$. This is contrary to our hypothesis since $\vec{H} \times \vec{H} \sim \vec{H}$. Hence, for any directed graph $\vec{X}$, we then have

$$\vec{X} \nrightarrow \vec{G} \div \vec{H} \quad \text{if and only if } \vec{H} \to \vec{X}. \tag{4}$$

This suggests the following notion:

Definition 3. A couple $(\vec{A}, \vec{B})$ of directed graphs is called a *duality* if for every directed graph $\vec{X}$, we have

$$\vec{X} \nrightarrow \vec{B} \quad \text{if and only if } \vec{A} \to \vec{X}.$$

Such dualities are the antithesis of our approach to density. If $(\vec{A},\vec{B})$ is a duality, then $\vec{A}\to\vec{A}$ implies $\vec{A}\nrightarrow\vec{B}$ whence $\vec{A}\nrightarrow\vec{B}\times\vec{A}$. For any $\vec{K}$ such that $\vec{K}\to\vec{A}$ and $\vec{A}\nrightarrow\vec{K}$, we then have $\vec{K}\to\vec{B}$ thus $\vec{K}\to\vec{B}\times\vec{A}$. In other words, $\vec{A}$ covers $\vec{B}\times\vec{A}$ in the sense that no directed graph lie strictly between $\vec{B}\times\vec{A}$ and $\vec{B}$. These covers will be called "gaps".

Definition 4. A couple $(\vec{G},\vec{H})$ of directed graphs is called a *gap* if $\vec{G}\to\vec{H}$, $\vec{H}\nrightarrow\vec{G}$ and any directed graph $\vec{F}$ satisfying $\vec{G}\to\vec{F}\to\vec{H}$ satisfies either $\vec{G}\sim\vec{F}$ or $\vec{H}\sim\vec{F}$.

"Density" is therefore the property of having no gaps. We have seen how the presence of gaps is linked to the presence of dualities. Algorithmically, the existence of dualities has great implications (see [8]). For instance, if $(\vec{A},\vec{B})$ is a duality, then the problem of deciding if a directed graph $\vec{X}$ admits a homomorphism into $\vec{B}$ is polynomial. We need only to check whether there exists a homomorphism from $\vec{A}$ to $\vec{X}$ and this can be done in polynomial time since $\vec{A}$ is fixed. The dualities in the category of directed graphs are characterised as follows.

Theorem 5 (Komárek [9] and Nešetřil and Tardif [14]). *Given a directed graph $\vec{A}$, there exists a directed graph $\vec{B}_{\vec{A}}$ such that $(\vec{A},\vec{B}_{\vec{A}})$ is a duality if and only if $\vec{A}$ is homomorphically equivalent to a directed tree.*

This result answers our question. As such, there is no "density theorem for directed graphs", because the exceptions are too numerous. However, this is not a negative result: these exceptions correspond to an interesting algorithmic phenomenon. Therefore, instead of a "density theorem", we get a "correspondence theorem", outlining the correspondence between gaps and dualities.

Theorem 6. *The gaps in the category of directed graphs are the couples $(\vec{G},\vec{H})$ such that there exists a duality $(\vec{A},\vec{B})$ with*

$$\vec{B}\times\vec{A}\to\vec{G}\to\vec{B}\quad\text{and}\quad\vec{H}\sim\vec{G}\cup\vec{A}.$$

Conversely, up to homomorphic equivalence, the dualities are the couples $(\vec{H},\vec{G}\div\vec{H})$ such that $\vec{H}$ is connected and $(\vec{G},\vec{H})$ is a gap.

Proof. The proof is essentially a summary of the arguments presented so far. We have seen that if $(\vec{G},\vec{H})$ is a gap and $\vec{H}$ is connected, then $\vec{H}$ and $\vec{G}\div\vec{H}$ must satisfy condition (4) whence $(\vec{H},\vec{G}\div\vec{H})$ is a duality. Also, if $(\vec{A},\vec{B})$ is a duality, then $(\vec{B}\times\vec{A},\vec{A})$ is a gap. These are the main aspects of the correspondence between gaps and dualities. The only points that remain to be discussed are questions of unicity and connectivity.

It is clear from the definition that up to homomorphic equivalence, one member of a duality $(\vec{A},\vec{B})$ uniquely determines the other member. Also, if $(\vec{A},\vec{B})$ is a duality, then we can assume that $\vec{A}$ is connected, for if $\vec{A}=\vec{A}_1\cup\vec{A}_2$ with $\vec{A}\nrightarrow\vec{A}_1$ and $\vec{A}\nrightarrow\vec{A}_2$, then $\vec{A}_1\to\vec{B}$ and $\vec{A}_2\to\vec{B}$ whence $\vec{A}\to\vec{B}$ which implies $\vec{A}\nrightarrow\vec{A}$.

Therefore, we have a correspondence between all dualities and the gaps whose second member is connected: If $\vec{H}$ is connected and $(\vec{G},\vec{H})$ is a gap, then its corresponding duality is $(\vec{H},\vec{G} \div \vec{H})$, which corresponds to the gap $((\vec{G} \div \vec{H}) \times \vec{H},\vec{H})$. However, $\vec{G} \to \vec{H}$ implies $(\vec{G} \div \vec{H}) \times \vec{H} \sim \vec{G}$. This shows that the correspondence is one-to-one and onto.

The remaining gaps are those where the second member is disconnected. For any duality $(\vec{A},\vec{B})$ and any $\vec{G}$ such that $\vec{B} \times \vec{A} \to \vec{G} \to \vec{B}$, $(\vec{G},\vec{G} \cup \vec{A})$ is a gap. Conversely, for any gap $(\vec{G},\vec{H})$, there exists a connected component $\vec{A}$ of $\vec{H}$ such that $\vec{A} \nrightarrow \vec{G}$. Since $(\vec{G},\vec{H})$ is a gap, there exists no directed graph $\vec{X}$ such that $\vec{G} \cup (\vec{X} \times \vec{A})$ is strictly between $\vec{G}$ and $\vec{H}$. From this follows that $\vec{H} \sim \vec{G} \cup \vec{A}$ and that $(\vec{A},\vec{G} \div \vec{A})$ is a duality, with $(\vec{G} \div \vec{A}) \times \vec{G} \to \vec{G} \to \vec{G} \div \vec{A}$. □

Nešetřil and Pultr [13] have shown that the only duality in the category of undirected graphs is (K_1,K_2). Note that modulo the algebraic machinery presented here, this already proves Welzl's density theorem [17].

References

[1] N. Alon, M. Tarsi, Colorings and orientations of graphs, Combinatorica 12 (1992) 125–134.
[2] D. Duffus, N. Sauer, Lattices arising in categorial investigations of Hedetniemi's conjecture, Discrete Math. 152 (1996) 125–139.
[3] H. El-Zahar, N. Sauer, The chromatic number of the product of two 4-chromatic graphs is 4, Combinatorica 5 (1985) 121–126.
[4] P. Erdős, Graph theory and probability, Canad. J. Math. 11 (1959) 34–38.
[5] T. Gallai, On directed paths and circuits, Theory of Graphs (Proc. Colloq., Tihany, 1966), Academic Press, New York, 1968, 115–118.
[6] S.H. Hedetniemi, Homomorphisms of graphs and automata, University of Michigan, Technical Report 03105-44-T, 1966.
[7] P. Hell, J. Nešetřil, On the complexity of H-colourings, J. Combin. Theory Ser. B 48 (1990) 92–110.
[8] P. Hell, J. Nešetřil, X. Zhu, Duality and polynomial testing of tree homomorphisms, Trans. Amer. Math. Soc. 348 (1996) 1281–1297.
[9] P. Komárek, Good characterizations in the class of oriented graphs, Ph.D. Thesis, Charles University, Prague, 1987 (in Czech).
[10] L. Lovász, Operations with structures, Acta Math. Acad. Sci. Hungar. 18 (1967) 321–328.
[11] G.J. Minty, A theorem on n-coloring the points of a linear graph, Amer. Math. Monthly 69 (1962) 623–624.
[12] J. Nešetřil, The homomorphism structure of classes of graphs, recent trends in combinatorics (Mátraháza, 1995), Combin. Probab. Comput. 8 (1999) 177–184.
[13] J. Nešetřil, A. Pultr, On classes of relations and graphs determined by subobjects and factorobjects, Discrete Math. 22 (1978) 287–300.
[14] J. Nešetřil, C. Tardif, Duality theorems for finite structures (characterizing gaps and good characterizations), J. Combin. Theory Ser B 80 (2000) 80–97.
[15] B. Roy, Nombre chromatique et plus longs chemins d'un graphe, Rev. Francaise Informat. Recherche Opérationelle 1 (1967) 129–132.
[16] R.P. Stanley, Acyclic orientations of graphs, Discrete Math. 5 (1973) 171–178.
[17] E. Welzl, Color-families are dense, J. Theoret. Comput. Sci. 17 (1982) 29–41.
[18] X. Zhu, A survey on Hedetniemi's conjecture, Taiwan. J. Math. 2 (1998) 1–24.