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Discrete Mathematics 293 (2005) 251–261
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

The Baillon–Simons theorems

Rueiher Tsaur[1]

Department of Computing, Imperial College, London SW7 2BZ, England

Received 2 July 2003; received in revised form 5 April 2004; accepted 18 August 2004
Available online 23 March 2005

Abstract

In this paper, we give combinatorial proofs of Baillon and Simons' almost fixed point and fixed point theorems for discrete-valued mappings (J. Combin. Theory Ser. A 60 (1992) 147–154).

MSC: 52C07; 05C99

Keywords: Fixed points; Lattices

1. Introduction

The almost fixed point and fixed point theorems of Baillon and Simons, as proposed in [1], may be considered as the "discrete" versions (see Section 6 for further discussions) of, respectively, the Halpern–Bergman and Browder fixed point theorems [5,3]. It is well-known that the Brouwer and Kakutani fixed point theorems have numerous generalizations appearing in various formulations; two typical examples, which we discuss in this section, and also were used to prove Baillon and Simons' results in [1], are the Halpern–Bergman and Browder theorems. The outward and, later, inward sets for any compact convex subset of a topological vector space, were first introduced by Halpern in his Ph.D. Thesis and [5]. Let $\mathbb{E}$ be a topological vector space, and A a compact convex subset of $\mathbb{E}$. For any $x \in A$,

E-mail address: rueiher@aol.com.

[1] Partially supported by EPSRC Grant GR/R02344/01.

0012-365X/$ - see front matter
doi:10.1016/j.disc.2004.08.034

the *inward* and *outward sets* of x, denoted by $I_A(x)$ and $O_A(x)$, respectively, are defined as follows:

- $I_A(x) = \{(1-\alpha)x + \alpha y \in \mathbb{E} \mid y \in A, \alpha \in \mathbb{R}, \alpha \geqslant 0\}$,
- $O_A(x) = \{(1-\alpha)x + \alpha y \in \mathbb{E} \mid y \in A, \alpha \in \mathbb{R}, \alpha \leqslant 0\}$.

A continuous mapping $f : A \to \mathbb{E}$ is said to be *inward* if $f(x) \in I_A(x)$ for all $x \in A$. Similarly, f is said to be *outward* if $f(x) \in O_A(x)$ for all $x \in A$. Halpern and Bergman's theorem can be stated as follows.

Theorem 1.1 ([5, Halpern–Bergman fixed point theorem]). *Let A be a compact convex subset of* $\mathbb{R}^n$ (*resp. locally convex space* $\mathbb{E}$), *and* $f : A \to \mathbb{R}^n$ (*resp.* $f : A \to \mathbb{E}$) *a continuous inward or outward mapping. Then f has a fixed point.*

The construct of the Halpern–Bergman theorem was generalized by Browder to multifunctions [3]. A compact convex-valued upper-semicontinuous multifunction $g : A \to \mathbb{E}$ is said to be *inward* if $g(x) \cap I_A(x) \neq \emptyset$ for all $x \in A$; and g is said to be *outward* if $g(x) \cap O_A(x) \neq \emptyset$ for all $x \in A$.

Theorem 1.2 ([3, Browder fixed point theorem]). *Let A be a compact convex subset of* $\mathbb{R}^n$ (*resp. locally convex space* $\mathbb{E}$), *and* $g : A \to \mathbb{R}^n$ (*resp.* $g : A \to \mathbb{E}$) *a compact convex-valued upper-semicontinuous inward or outward multifunction. Then g has a fixed point.*

The Baillon–Simons almost fixed point theorem and fixed point theorem have certain resemblances with, respectively, the Halpern–Bergman and Browder fixed point theorems. The framework they considered is $\mathbb{Z}^n$, in which the rectangle blocks are considered as compact convex subsets. Since Baillon and Simons used the Halpern–Bergman and Browder theorems in deriving their theorems, it was requested by them [1] to find combinatorial proofs of their results (or of at least one of them). In this paper, we will give combinatorial proofs of both (Baillon and Simons') theorems.

2. The Baillon–Simons almost fixed point and fixed point theorems for discrete-valued mappings

The material presented in this section can be found in [1].

Let $\mathbb{Z}^n$ be the product of n copies of the set $\mathbb{Z}$ of integers which is considered as a lattice group under the pointwise order and algebraic operations. If $x, y \in \mathbb{Z}^n$ such that $x \leqslant y$, we write $[x, y]$ for the *segment*

$$\{z \in \mathbb{Z}^n \mid x \leqslant z \leqslant y\}.$$

Note that some or all components of x, y are allowed to be $-\infty$ or ∞. For any point $\mathbf{d} = (d_1, \ldots, d_n) \in \mathbb{Z}^n$ with $0 \leqslant d_i < \infty$ for all i, let $\mathbf{X}_d = [(0, \ldots, 0), \mathbf{d}]$.

The *directed following* of $x \in \mathbf{X}_d$ in $\mathbf{X}_d$, denoted by F_x, is $[x, x + (1, \ldots, 1)] \cap \mathbf{X}_d$. Let $f : \mathbf{X}_d \to \mathbb{Z}^n$ be any mapping. Then a point $x \in \mathbf{X}_d$ is said to be a *directed following* *almost fixed point* of *f* if

$$\bigwedge f(F_x) \leqslant x \leqslant \bigvee f(F_x).$$

Let $\mathrm{pr}_i : \mathbb{Z}^n \to \mathbb{Z}$ be the *projection* of $\mathbb{Z}^n$ onto its ith factor for all i, $1 \leqslant i \leqslant n$. Then

Theorem 2.1 ([1, Baillon–Simons almost fixed point theorem]). *Let $f : \mathbf{X}_d \to \mathbb{Z}^n$ be any mapping. Then f has a directed following almost fixed point if one of* (1)–(3) *is satisfied*:

(1) *For all $i \in \{1, \ldots, n\}$ and $x \in \mathbf{X}_d$,*

$$(\mathrm{pr}_i(x) = 0 \Rightarrow (\mathrm{pr}_i \circ f)(x) \geqslant 0) \ \&\ (\mathrm{pr}_i(x) = d_i \Rightarrow (\mathrm{pr}_i \circ f)(x) \leqslant d_i).$$

(2) *For all $i \in \{1, \ldots, n\}$ and $x \in \mathbf{X}_d$,*

$$(\mathrm{pr}_i(x) = 0 \Rightarrow (\mathrm{pr}_i \circ f)(x) \leqslant 0) \ \&\ (\mathrm{pr}_i(x) = d_i \Rightarrow (\mathrm{pr}_i \circ f)(x) \geqslant d_i).$$

(3) $f(\mathbf{X}_d) \subseteq \mathbf{X}_d$.

We emphasize that, in Theorem 2.1, *f* is *not* required to be order-preserving.

A *segment multifunction* $f : \mathbf{X}_d \to \mathbb{Z}^n$ is a multifunction which maps each point of $\mathbf{X}_d$ to a segment of $\mathbb{Z}^n$. *f* is *strongly-simplicial* if for any $x, y \in \mathbf{X}_d$, we have

$$(-1, \ldots, -1) \leqslant x - y \leqslant (1, \ldots, 1) \Rightarrow f(x) \cap f(y) \neq \emptyset.$$

Theorem 2.2 ([1, Baillon–Simons fixed point theorem]). *Let $f : \mathbf{X}_d \to \mathbb{Z}^n$ be any strongly-simplicial segment multifunction. Then f has a fixed point, that is, there exists $x \in \mathbf{X}_d$ such that $x \in f(x)$, if one of* (a)–(c) *is satisfied*:

(a) *For all $i \in \{1, \ldots, n\}$ and $x \in \mathbf{X}_d$,*

$$(\mathrm{pr}_i(x) = 0 \Rightarrow \max((\mathrm{pr}_i \circ f)(x)) \geqslant 0) \ \&\ (\mathrm{pr}_i(x) = d_i \Rightarrow \min((\mathrm{pr}_i \circ f)(x)) \leqslant d_i).$$

(b) *For all $i \in \{1, \ldots, n\}$ and $x \in \mathbf{X}_d$,*

$$(\mathrm{pr}_i(x) = 0 \Rightarrow \min((\mathrm{pr}_i \circ f)(x)) \leqslant 0) \ \&\ (\mathrm{pr}_i(x) = d_i \Rightarrow \max((\mathrm{pr}_i \circ f)(x)) \geqslant d_i).$$

(c) $f(\mathbf{X}_d) \subseteq \mathbf{X}_d$.

3. Labeling

In [8], Quilliot developed a lemma ([8, Lemme 2]) to show that every (finite reflexive) Helly graph has the *p*-fixed point property for *p*-graph homomorphisms (*p*: any non-negative integer). Here we generalize Quilliot's lemma (Lemma 3.1) in order to prove Theorems 2.1 and 2.2. The product (*n* times) of *m*-paths P_m is denoted by P_m^n. A maximal clique of P_m^n has 2^n vertices, and is called an *elementary n-cube* of P_m^n. Let *C* be an elementary *n*-cube of P_m^n, and $S \subseteq C$ a subset of *C*. Then *S* is a *filling subset* of *C* if

(i) The cardinality of *S*, $\#S = n + 1$,
(ii) *S* is not contained in any facet of *C*.

Every vertex x of P_m^n may be represented with n coordinates $(x_1, \ldots, x_n)$, all integers between 0 and m. A function $L : V(P_m^n) \to \dot{\mathbf{1}}^n$ from the vertex set of P_m^n to the n-product of $\dot{\mathbf{1}} = \{0, 1\}$, $\dot{\mathbf{1}}^n$, is a *labeling* of P_m^n if

$$L_i(x) = \begin{cases} 0, & x_i = 0, \\ 1, & x_i = m \end{cases}$$

for all i, $1 \leqslant i \leqslant n$, where $x = (x_1, \ldots, x_n)$ and $L(x) = (L_1(x), \ldots, L_n(x))$. We have

Lemma 3.1. *For any given labeling L of P_m^n, there exist elementary n-cubes C, D and filling subsets $S \subseteq C$, $S' \subseteq D$ such that*

(1) $(0, \ldots, 0) \in L(S)$, *and for every coordinate i, $1 \leqslant i \leqslant n$, there exists an individual vertex $x \in S$ such that $L_i(x) = 1$,*
(2) $(1, \ldots, 1) \in L(S')$, *and for every coordinate j, $1 \leqslant j \leqslant n$, there exists an individual vertex $y \in S'$ such that $L_j(y) = 0$.*

Proof. For $i \in \mathcal{N}(=\{0, 1, \ldots, n\})$, let S_i^n denote the $(n-1)$-face of the (closed) n-simplex S^n opposite the point v^i of S^n. The well-known Sperner Lemma applies in the form: let T be a triangulation of S^n with each point of T labeled with an integer in $\mathcal{N}$ such that no point in S_i^n is labeled i. (Such a labeling is called *Sperner* or *admissible*.) Then there is a simplex in T whose points carry all the labels in $\mathcal{N}$ (called a *complete labeled simplex*).

Recall that a graph $G = (V(G), E(G))$ is said to be an n-dimensional *triangulation graph* if there exists a triangulation of S^n with V the 0-face set and E the 1-face set such that $V(G) = V$ and $E(G) = E$. Thus Sperner's lemma for simplicial complexes can be reformulated for triangulation graphs: any admissible (Sperner) labeling of an n-dimensional triangulation graph contains a complete-labeled clique. In the following we shall use Sperner's lemma to prove Lemma 3.1. We say that an induced subgraph A of P_m^n is a k-FACE of P_m^n, $0 \leqslant k \leqslant n$, if (1) A is isomorphic with P_m^k, and (2) there exists $\mathcal{N}' \subseteq \mathcal{N}$, $\#\mathcal{N}' = n - k$, such that for each $x \in V(A)$, we have either $\mathrm{pr}_j(x) = 0$ or $\mathrm{pr}_j(x) = m$ for all $j \in \mathcal{N}'$.

Lemma 3.2. *Let L be a labeling of P_m^n, and A a k-FACE of P_m^n, $0 \leqslant k \leqslant n$. Then for any vertex x of A, we have $L(x) = L(y)$ for some corner vertex (0-FACE) y of A.*

Proof. The FACE A is defined by fixing $n - k$ of the coordinates to be either 0 or m. Thus the labels of all the vertices of A coincide in these $n - k$ coordinate positions. Hence there are only 2^k possible distinct labels associated with the vertices of A. Clearly, however, all of these 2^k labels must occur at the 2^k corner vertices of A. □

Now we are ready to prove Lemma 3.1.

(1) Assume that a labeling $L : V(P_m^n) \to \dot{\mathbf{1}}^n$ is given. Note that P_m^n has 2^n corner vertices, i.e., the vertices whose n coordinates of integers are either 0 or m. For convenience, we denote the corner vertices labeled $(0, \ldots, 0)$ by $\mathbf{0}$, and $(0, \ldots, 1, 0, \ldots, 0)$ by $\mathbf{j}$ when the only 1 occurs in its j-coordinate.

In order to apply Sperner's lemma, we want to regard the (unit) cube as, topologically, a simplex. To do this, we take the n FACETs (i.e., $(n-1)$-FACEs) of the cube incident with

$\mathbf{0} = (0, \ldots, 0)$ as facets of the simplex; the remaining n FACETs of the cube (incident with $(1, \ldots, 1)$) constitute facet number $n + 1$ of the simplex (face opposite $\mathbf{0}$).

Moreover, the 2^n labels need to be replaced, in a consistent manner by "Sperner labels". Consistency just means that vertices having the same label receive the same Sperner label; thus we have a mapping $\lambda : \dot{\mathbf{1}}^n \to \{\mathbf{0}, \ldots, \mathbf{n}\}$. The mapping λ may be chosen arbitrarily, subject only to the condition:

$$(\lambda(x) = \mathbf{0} \Rightarrow x = \mathbf{0}) \ \& \ (\lambda_i(x) = 1 \Rightarrow x_i = 1) \tag{1}$$

for all i, $1 \leqslant i \leqslant n$, where $x = (x_1, \ldots, x_n)$ and $\lambda(x) = (\lambda_1(x), \ldots, \lambda_n(x))$. In other words, λ assigns all but one of the non-zero coordinates (if there are any) to zero.

The final ingredient needed for the application of Sperner's lemma is triangulation (of the "cubical complex" P_m^n). Thus, we choose a spanning triangulation subgraph T of P_m^n which, in accordance with the preceding description, we view as a triangulation graph of S^n, with labeling $\lambda \circ L$.

It is immediate by Lemma 3.2 (and condition (1)) that $\lambda \circ L$ is an admissible labeling of T. Hence there exists a complete-labeled clique S of T. Since T is a spanning triangulation subgraph of P_m^n, therefore there exists an elementary n-cube C of P_m^n such that $S \subseteq C$. Clearly there exists a vertex of S which is labeled $(0, \ldots, 0)$. Furthermore, for any j, $1 \leqslant j \leqslant n$, if $x \in S$ is Sperner labeled $\mathbf{j}$ (for T), then the j-coordinate of label $L(x)$ must be 1.

(2) Prove by simply switching the role of 0 and 1 in the proof (1) above. □

Note that, although we are dealing with cubical structure P_m^n in Lemma 3.1, it is clear that, with similar proofs, we are able to extend above result to cuboidal structures $\bigotimes_{1 \leqslant i \leqslant n} P_{m_i}$.

Corollary 3.3. *If every $d_i > 0$, then for any given labeling L of $\mathbf{X}_d$, there exist F_x, F_y, $x, y \in \mathbf{X}_d$, and filling subsets $\Delta \subseteq F_x$, $\Lambda \subseteq F_y$ such that*

(1) *$(0, \ldots, 0) \in L(\Delta)$, and for every coordinate i, $1 \leqslant i \leqslant n$, there exists an individual point $z \in \Delta$ such that $L_i(z) = 1$,*
(2) *$(1, \ldots, 1) \in L(\Lambda)$, and for every coordinate j, $1 \leqslant j \leqslant n$, there exists an individual point $z \in \Lambda$ such that $L_j(z) = 0$.*

4. Proof of the Baillon–Simons almost fixed point theorem

Proof of Theorem 2.1. Without loss of generality, we may assume that $d_i > 0$ for every i.

(1) For $\mathbf{Y} = [(0, \ldots, 0), (d_1 + 1, \ldots, d_n + 1)]$, we define the function $g : \mathbf{Y} \to \mathbb{Z}^n$ by

$$g(x) = \begin{cases} f(x), & x \in \mathbf{X}_d, \\ f(y), & x \notin \mathbf{X}_d,\ y \in \mathbf{X}_d \text{ s.t. } |\mathrm{pr}_i(x) - \mathrm{pr}_i(y)| \leqslant |\mathrm{pr}_i(x) - \mathrm{pr}_i(z)|,\ \forall z \in \mathbf{X}_d. \end{cases}$$

The extension of g in this way to $\mathbf{Y}$ enables some tedious case analysis to be avoided, later on. Notice that $x \in \mathbf{X}_d$ is a directed following almost fixed point of f if and only if x is a directed following almost fixed point of g (that is, $\bigwedge f(F_x) \leqslant x \leqslant \bigvee f(F_x) \Leftrightarrow \bigwedge g(F_x \cap \mathbf{X}_d) \leqslant x \leqslant \bigvee g(F_x \cap \mathbf{X}_d) \Leftrightarrow \bigwedge g(F_x) \leqslant x \leqslant \bigvee g(F_x)$).

Let us, then, show that g has a directed following almost fixed point in $\mathbf{X}_d$. Define a function $L : \mathbf{Y} \to \{0, 1\}^n$ by

$$L_i(x) = \begin{cases} 0, & \mathrm{pr}_i(x) \leqslant (\mathrm{pr}_i \circ g)(x), \\ 1, & \mathrm{pr}_i(x) > (\mathrm{pr}_i \circ g)(x). \end{cases}$$

Clearly, $L(x) = (L_1(x), \ldots, L_n(x))$ is a labeling. Hence by Corollary 3.3, there exists F_y, $y \in \mathbf{Y}$, and a filling subset $\Delta \subseteq F_y$, such that $(0, \ldots, 0) \in L(\Delta)$, and for every coordinate i, $1 \leqslant i \leqslant n$, there exists an individual point $z \in \Delta$ such that $L_i(z) = 1$. Also note that, since $\Delta \subseteq F_y$ is a filling subset, we must have $y \in \mathbf{X}_d$.

We show that y is a directed following almost fixed point of g. Let z be the point of Δ which is labeled $(0, \ldots, 0)$. Then it is clear that $\mathrm{pr}_i(z) \leqslant (\mathrm{pr}_i \circ g)(z)$ for all i. Thus we have $y \leqslant z \leqslant g(z) \leqslant \bigvee g(F_y)$. Furthermore, for every coordinate i, $1 \leqslant i \leqslant n$, there exists an individual point $w(i) \in \Delta$ such that $L_i(w(i)) = 1$. Therefore we have $(\mathrm{pr}_i \circ g)(w(i)) < \mathrm{pr}_i(w(i))$ for all i; hence $(\mathrm{pr}_i \circ g)(w(i)) \leqslant \mathrm{pr}_i(y)$. Thus we have $\bigwedge g(F_y) \leqslant \bigwedge_i g(w(i)) \leqslant (g_1(w(1)), \ldots, g_n(w(n))) \leqslant (\mathrm{pr}_1(y), \ldots, \mathrm{pr}_n(y)) = y$.

(2) Arguing as in (1) but more simply, we define a function $L' : \mathbf{X}_d \to \{0, 1\}^n$ by

$$L'_i(x) = \begin{cases} 1, & \mathrm{pr}_i(x) \leqslant (\mathrm{pr}_i \circ f)(x) \text{ and } \mathrm{pr}_i(x) \neq 0, \\ 0, & \mathrm{pr}_i(x) > (\mathrm{pr}_i \circ f)(x) \text{ or } \mathrm{pr}_i(x) = 0. \end{cases}$$

It is clear that $L'(x) = (L'_1(x), \ldots, L'_n(x))$ is a labeling. Thus by Corollary 3.3, there exists F_x, $x \in \mathbf{X}_d$, and a filling subset $\Lambda \subseteq F_x$, such that $(1, \ldots, 1) \in L'(\Lambda)$, and for every coordinate i, $1 \leqslant i \leqslant n$, there exists an individual point $y(i) \in \Lambda$ such that $L'_i(y(i)) = 0$.

We claim that x is a directed following almost fixed point of f. Let z be the point of Λ which is labeled $(1, \ldots, 1)$. Then it is clear that $\mathrm{pr}_i(z) \leqslant (\mathrm{pr}_i \circ f)(z)$ for all i. Thus we have $x \leqslant z \leqslant f(z) \leqslant \bigvee f(F_x)$. Let $\Lambda' = \{y(i) \in \Lambda \backslash z \mid (\mathrm{pr}_i \circ f)(y(i)) = \mathrm{pr}_i(y(i)) = 0\}$. If $\Lambda' = \emptyset$, then (in a similar way to the proof of (1)) it is easy to check that $\bigwedge f(F_x) \leqslant x$. If $\Lambda' \neq \emptyset$, then for any $y(i) \in \Lambda'$, we have $\mathrm{pr}_i(x) = \mathrm{pr}_i(y(i)) = (\mathrm{pr}_i \circ f)(y(i)) = 0$. Since $\mathrm{pr}_i(x) \geqslant (\mathrm{pr}_i \circ f)(y(i))$ (in fact, $\mathrm{pr}_i(x) > (\mathrm{pr}_i \circ f)(y(i))$) for all $y(i) \notin \Lambda'$, therefore we also have $\bigwedge f(F_x) \leqslant x$.

(3) This is immediate from (1).

Thus we have completed the proof of Theorem 2.1. □

Remark 4.1. In the proof of Theorem 2.1(2), we may also say that there exists F_x, $x \in \mathbf{X}_d$, and a filling subset $\Lambda \subseteq F_x$, such that there exists $z \in \Lambda$ with $L'(z) = (0, \ldots, 0)$, and for every coordinate i, $1 \leqslant i \leqslant n$, there exists an individual point $y(i) \in \Lambda$ such that $L'_i(y(i)) = 1$. Then it is clear that, for all i, $1 \leqslant i \leqslant n$, we have either $\mathrm{pr}_i(z) > (\mathrm{pr}_i \circ f)(z)$ or $\mathrm{pr}_i(z) = (\mathrm{pr}_i \circ f)(z) = 0$. In the former case, we would have $\mathrm{pr}_i(x) \geqslant (\mathrm{pr}_i \circ f)(z)$, and in the latter case, we would have $\mathrm{pr}_i(x) = \mathrm{pr}_i(z) = (\mathrm{pr}_i \circ f)(z)$. Therefore we must have $\bigwedge f(F_x) \leqslant x$. To show $x \leqslant \bigvee f(F_x)$, it is clear that, for all i, $1 \leqslant i \leqslant n$, we have $\mathrm{pr}_i(x) \leqslant \mathrm{pr}_i(y(i)) \leqslant (\mathrm{pr}_i \circ f)(y(i))$; thus $x \leqslant \bigvee f(F_x)$.

5. Proof of the Baillon–Simons fixed point theorem

We recall that a *two-way infinite path*, denoted by $\mathbb{P}$, is an infinite tree in which each point has exactly two adjacent points [4].

Lemma 5.1. *Let $f : \mathbf{X}_d \to \mathbb{Z}^n$ be any strongly-simplicial segment multifunction, and x a point of $\mathbf{X}_d$. Then for any non-empty subset A of F_x, $\bigcap_{a\in A} f(a) \neq \emptyset$.*

Proof. The segments of $\mathbb{Z}^n$ are exactly the intersections of balls of an (n times) product of two-way infinite paths $\mathbb{P}$; and the finite product of two-way infinite paths is strongly Helly (i.e., any infinite family of pairwise non-disjoint balls has a non-empty intersection). See [7 or 6]. □

Lemma 5.2. *Suppose that Theorem* 2.2 *holds for bounded multifunctions. Then it holds unrestrictedly.*

Proof. Given any finite positive integer $m \geqslant 1$, let $\mathbf{Y} = \{x \in \mathbb{Z}^n \mid \|x-y\|_\infty \equiv \max_{1\leqslant i\leqslant n} |\mathrm{pr}_i(x) - \mathrm{pr}_i(y)| \leqslant m$, for some $y \in \mathbf{X}_d\}$ ($\|\cdot\|_\infty$ is the ℓ_∞-norm). Suppose f is unbounded. We define the multifunction $g : \mathbf{X}_d \to \mathbb{Z}^n$, $x \mapsto g_1(x) \times \cdots \times g_n(x)$ by

$$g_i(x) = \begin{cases} (\mathrm{pr}_i \circ f)(x) \cap \mathrm{pr}_i(\mathbf{Y}) & \text{if } (\mathrm{pr}_i \circ f)(x) \cap \mathrm{pr}_i(\mathbf{Y}) \neq \emptyset, \\ -m & \text{if } z < -m \text{ for all } z \in (\mathrm{pr}_i \circ f)(x), \\ d_i + m & \text{if } z > d_i + m \text{ for all } z \in (\mathrm{pr}_i \circ f)(x). \end{cases}$$

Note that g is well-defined, since f is a segment multifunction from $\mathbf{X}_d$ to $\mathbb{Z}^n$.

It is easy to check that g satisfies the hypotheses (a–c) of Theorem 2.2 if f does. We show that g is a strongly-simplicial multifunction whose images are segments of $\mathbf{Y}$. Let x be any point of $\mathbf{X}_d$. If $f(x) \cap \mathbf{Y} \neq \emptyset$, then since $f(x)$ is a segment of $\mathbb{Z}^n$, it is clear that $g(x) = f(x) \cap \mathbf{Y}$ is a segment of $\mathbf{Y}$. Now, suppose $f(x) \cap \mathbf{Y} = \emptyset$. Without loss of generality, we may assume that $f(x) = [(s_1, \ldots, s_n), (t_1, \ldots, t_n)]$ (where s_j, t_k may be ∞). Clearly, there exists an index subset $M \subseteq \{1, \ldots, n\}$ such that, for any $i \in M$, we have either $s_i, t_i < -m$ or $s_i, t_i > d_i + m$. By the definition of g, it is easy to check that, in the former case we have $g_i(x) = -m$; and in the latter case we have $g_i(x) = d_i + m$. Also note that, for any i such that $i \notin M$, we have $g_i(x) = (\mathrm{pr}_i \circ f)(x) \cap \mathrm{pr}_i(\mathbf{Y}) \neq \emptyset$. Hence we have $g(x) = g_1(x) \times \cdots \times g_n(x)$ is a segment of $\mathbf{Y}$ (in fact, $g(x)$ is contained in one side of the “boundary” of $\mathbf{Y}$). To show that g is strongly-simplicial: by Lemma 5.1, it is clear that for any F_x, $x \in \mathbf{X}_d$, we have $\bigcap_{a\in F_x} g(a) \supseteq g(\bigcap_{a\in F_x} f(a)) \neq \emptyset$ (since $g(\bigcap_{a\in F_x} f(a)) \subseteq g(b)$ for all $b \in F_x$ and $\bigcap_{a\in F_x} f(a) \neq \emptyset$). So g must be strongly-simplicial.

Finally, it is clear that $g(x) \cap \mathbf{X}_d = f(x) \cap \mathbf{X}_d$ for all $x \in \mathbf{X}_d$. Therefore f satisfies Theorem 2.2 if g does. □

We are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Without loss of generality, we may assume that $d_i > 0$ for every i.

(a) From Lemma 5.2, without loss of generality, we may assume that, for any $x \in \mathbf{X}_d$, $f(x)$ is bounded in $\mathbb{Z}^n$.

Let $m_i = \max\{0, u_i\}$, where $u_i = \max(\bigcup_{x,\mathrm{pr}_i(x)=d_i}(\mathrm{pr}_i \circ f)(x)) - d_i + 1$. Let $\mathbf{Y} = [(0, \ldots, 0), (d_1+m_1, \ldots, d_n+m_n)]$. It is easy to check that $\mathbf{Y} = \bigcap_{1\leqslant i\leqslant n}[(0, \ldots, 0), (\infty, \ldots, d_i+m_i, \ldots, \infty)]$. Let us define the multifunction $g : \mathbf{Y} \to \mathbb{Z}^n$ by

$$g(x) = \begin{cases} f(x), & x \in \mathbf{X}_d, \\ f(y), & x \notin \mathbf{X}_d,\ y \in \mathbf{X}_d \text{ s.t. } |\mathrm{pr}_i(x) - \mathrm{pr}_i(y)| \leqslant |\mathrm{pr}_i(x) - \mathrm{pr}_i(z)|,\ \forall z \in \mathbf{X}_d. \end{cases}$$

We claim that, if g has a fixed point in $\mathbf{Y}$, then f has a fixed point in $\mathbf{X}_d$. To show this, suppose $x \in \mathbf{Y}$ is a fixed point of g. It is trivial that x is a fixed point of f if $x \in \mathbf{X}_d$. Hence we assume that $x \notin \mathbf{X}_d$, and we would like to show that the point $y \in \mathbf{X}_d$, $|\mathrm{pr}_i(x)-\mathrm{pr}_i(y)| \leqslant |\mathrm{pr}_i(x)-\mathrm{pr}_i(z)|$, $\forall z \in \mathbf{X}_d$, is a fixed point of f. Note that for any i, $1\leqslant i\leqslant n$, we have, respectively, $\mathrm{pr}_i(y) = \mathrm{pr}_i(x)$ if $0 < \mathrm{pr}_i(x) \leqslant d_i$, and $\mathrm{pr}_i(y) = d_i$ if $\mathrm{pr}_i(x) > d_i$. Also note that, since g (resp. f) is a segment multifunction, we have $a \in g(b)$ (resp. $a \in f(b)$) if and only if $\mathrm{pr}_i(a) \in (\mathrm{pr}_i \circ g)(b)$ (resp. $\mathrm{pr}_i(a) \in (\mathrm{pr}_i \circ f)(b)$) for all i, $1\leqslant i\leqslant n$. Now, since x is a fixed point of g, we have $\mathrm{pr}_i(x) \in (\mathrm{pr}_i \circ g)(x)$ for all i, $1\leqslant i\leqslant n$. From the definition of g, we have $g(x) = f(y)$. Thus $\mathrm{pr}_i(x) \in (\mathrm{pr}_i \circ f)(y)$ for all i, $1\leqslant i\leqslant n$. Hence for all i, $1\leqslant i\leqslant n$, we have:

- If $\mathrm{pr}_i(y) = \mathrm{pr}_i(x)$: Clearly $\mathrm{pr}_i(y) \in (\mathrm{pr}_i \circ f)(y)$;
- If $\mathrm{pr}_i(y) = d_i$ (where $\mathrm{pr}_i(x) > d_i$): By the inward definition of f, we have $\min((\mathrm{pr}_i \circ f)(y)) \leqslant d_i$. Since $f(y)$ is a segment, therefore we have $\mathrm{pr}_i(y) \in [\min((\mathrm{pr}_i \circ f)(y)), \mathrm{pr}_i(x)]_{\mathbb{Z}} \subseteq (\mathrm{pr}_i \circ f)(y)$.

Therefore y is a fixed point of f.
Next, we show that g indeed has a fixed point in $\mathbf{Y}$. Define a function $Q : \mathbf{Y} \to \{0, 1\}^n$ by

$$Q_i(x) = \begin{cases} 0 & \text{if } \mathrm{pr}_i(x) \leqslant y \text{ for some } y \in (\mathrm{pr}_i \circ g)(x), \\ 1 & \text{if } \mathrm{pr}_i(x) > y \text{ for every } y \in (\mathrm{pr}_i \circ g)(x). \end{cases}$$

It is clear that, from the definition of g, $Q(x) = (Q_1(x), \ldots, Q_n(x))$ is a labeling. Hence by Corollary 3.3, there exists a filling subset $\Delta \subseteq F_x$ for some $x \in \mathbf{Y}$, such that

- there exists a unique point $z \in \Delta$ such that $Q(z) = (0, \ldots, 0)$, and
- for every coordinate i, $1\leqslant i\leqslant n$, there exists an individual point, denoted by $z(i)$, $z(i) \in \Delta$, such that $Q_i(z(i)) = 1$.

We claim that z is a fixed point of g; hence based on our previous discussion, f has a fixed point. Suppose z is not a fixed point of g. Then there exists j, $1\leqslant j\leqslant n$, such that $\mathrm{pr}_j(z) \cap (\mathrm{pr}_j \circ g)(z) = \emptyset$. If $\mathrm{pr}_j(z) > \mathrm{pr}_j(w)$ for all $w \in g(z)$, then clearly, the j-coordinate of z would be labeled 1: contradiction. If $\mathrm{pr}_j(z) < \mathrm{pr}_j(w)$ for all $w \in g(z)$, then for the point $z(j) \in \Delta$ whose j-coordinate is labeled 1, we would have $g(z) \cap g(z(j)) = \emptyset$: contradiction.

(b) We may give a proof along lines parallel to the proof of (a) above. However, it is simpler to reduce the assertion of (b) directly to that of (a).

To do this, we define the multifunction $g : \mathbf{X}_d \to \mathbb{Z}^n$ by

$$g(x) = 2x - f(x) \quad \forall x \in \mathbf{X}_d.$$

It is clear that, for any $x \in \mathbf{X}_d$, $g(x)$ is a segment of $\mathbb{Z}^n$. Furthermore, by Lemma 5.1, it is easy to check that g is strongly-simplicial. For any point $x \in \mathbf{X}_d$ such that $\mathrm{pr}_i(x) = 0$, clearly $(\mathrm{pr}_i \circ g)(x) = 2\mathrm{pr}_i(x) - (\mathrm{pr}_i \circ f)(x) = -(\mathrm{pr}_i \circ g)(x)$. Hence, from the definition of f ($\min((\mathrm{pr}_i \circ f)(x)) \leqslant 0$), we have $\max((\mathrm{pr}_i \circ g)(x)) = \max(-(\mathrm{pr}_i \circ f)(x)) \geqslant 0$. Similarly, for any point $x \in \mathbf{X}_d$ such that $\mathrm{pr}_i(x) = d_i$, we have $(\mathrm{pr}_i \circ g)(x) = 2\mathrm{pr}_i(x) - (\mathrm{pr}_i \circ f)(x) = 2d_i - (\mathrm{pr}_i \circ f)(x)$. Since $\max((\mathrm{pr}_i \circ f)(x)) \geqslant d_i$, we have $\min((\mathrm{pr}_i \circ g)(x)) \leqslant 2d_i - (\mathrm{pr}_i \circ f)(x) \leqslant 2d_i - d_i = d_i$. Therefore g satisfies the conditions of Theorem 2.2(a), and hence g has a fixed point, say the point $z \in \mathbf{X}_d$. Then we have $z \in g(z) \Rightarrow z \in 2z - f(z) \Rightarrow z \in f(z)$.

(c) This is immediate from (a).

Thus we have completed the proof of Theorem 2.2. □

Remark 5.3. The idea of the proof of Theorem 2.2(b) comes from [3].

6. Application to topology

Since $\mathbb{Z}^n \subset \mathbb{R}^n$, we may define a mapping $\upuparrows : \mathbb{R}^n \to \mathbb{Z}^n$ by

$$\upuparrows(x) = \min((\uparrow_{\mathbb{R}^n} x) \cap \mathbb{Z}^n), \forall x \in \mathbb{R}^n,$$

where $\uparrow_{\mathbb{R}^n} x = \{y \in \mathbb{R}^n \mid x \leqslant y\}$. It is clear that $\upuparrows$ is well-defined.

The following theorem (Theorem 6.1) is a generalization of Theorem 2 in [1], where Baillon and Simons showed that Theorem 2.1(3) implies the Brouwer fixed point theorem.

Theorem 6.1. *Theorem* 2.1(1) *and* (2) *imply the Halpern–Bergman fixed point theorem for compact convex subsets of* $\mathbb{R}^n$.

Proof. It is enough to show the case for continuous inward mappings, since the case for a continuous outward mapping then follows by a dual argument (see the proof of Theorem 4.3(1) in [5]).

Let I^n be the unit n-cube in $\mathbb{R}^n$. Firstly, we prove it for the n-cube sI^n for any $s \in \mathbb{R}, 0 < s < \infty$. Let $f : sI^n \to \mathbb{R}^n$ be any continuous inward mapping from sI^n to $\mathbb{R}^n$. For any $k \in \mathbb{N}, k \geqslant 1$, we define the mapping $f_k : [(0, \ldots, 0), (k, \ldots, k)] \to \mathbb{Z}^n$ by

$$f_k(x) = \upuparrows\left(\frac{k}{s}\, f\left(\frac{s}{k}\, x\right)\right),$$

for all $x \in [(0, \ldots, 0), (k, \ldots, k)]$. It is clear that f_k satisfies the condition of Theorem 2.1(1) if f is a continuous inward mapping. By Theorem 2.1, f_k must have a directed following almost fixed point, say $y \in [(0, \ldots, 0), (k, \ldots, k)]$.

Since y is a directed following almost fixed point of f_k, i.e., $\bigwedge_{\mathbb{Z}^n} f_k(F_y) \leqslant y \leqslant \bigvee_{\mathbb{Z}^n} f_k(F_y)$, therefore

$$\left[\frac{s\bigwedge_{\mathbb{Z}^n} f_k(F_y)}{k} \leqslant \frac{sy}{k} \leqslant \frac{s\bigvee_{\mathbb{Z}^n} f_k(F_y)}{k}\right]$$
$$\Rightarrow \left[\frac{s\bigwedge_{\mathbb{Z}^n} \upuparrows(\frac{k}{s} f(\frac{s}{k} F_y))}{k} \leqslant \frac{sy}{k} \leqslant \frac{s\bigvee_{\mathbb{Z}^n} \upuparrows(\frac{k}{s} f(\frac{s}{k} F_y))}{k}\right]$$
$$\Rightarrow \left[\frac{s\bigwedge_{\mathbb{R}^n} (\frac{k}{s} f(\frac{s}{k} F_y))}{k} \leqslant \frac{sy}{k} \leqslant \frac{s\bigvee_{\mathbb{R}^n} (\frac{k}{s} f(\frac{s}{k} F_y) + 1)}{k}\right]$$
$$\Rightarrow \left[\bigvee_{\mathbb{R}^n} f\left(\frac{s}{k} F_y\right) \leqslant \frac{sy}{k} \leqslant \bigvee_{\mathbb{R}^n} \left(f\left(\frac{s}{k} F_y\right) + \frac{s}{k}\right) = \bigvee_{\mathbb{R}^n} f\left(\frac{s}{k} F_y\right) + \frac{s}{k}\right].$$

Let $z_k = (sy)/k$: clearly $z_k \in sI^n$. Then it is easy to check that we have $\bigwedge_{\mathbb{R}^n} f(D_{z_k}) \leqslant z_k \leqslant \bigvee_{\mathbb{R}^n} f(D_{z_k}) + s/k$, where $D_{z_k} = \{b \in sI^n \mid z_k \leqslant b \leqslant z_k + s/k\}$. Since the n-cube sI^n is compact and s is a constant, thus let $k \to \infty$ and then by the uniform continuity of f, f has a fixed point.

Now let A be any compact convex subset of $\mathbb{R}^n$, without loss of generality, we may assume that for any $x \in A$, we have $\mathrm{pr}_i(x) > 0$ for each i, $1 \leqslant i \leqslant n$. We claim that any continuous inward mapping $g : A \to \mathbb{R}^n$ from A to $\mathbb{R}^n$ has a fixed point. Since A and $g(A)$ are bounded subsets in $\mathbb{R}^n$, for s sufficiently large, we have the n-cube sI^n properly contains $A \cup g(A)$. Since A is a compact convex subset of $\mathbb{R}^n$ (see for example [2]), for any point $x \in sI^n$, there exists a unique point $y \in A$ such that $d_{\mathbb{R}^n}(x, y) \leqslant d_{\mathbb{R}^n}(x, z)$ for all $z \in A$, and furthermore, the mapping $r : sI^n \to A$ defined by $r(x) = y$ is continuous. Then it is clear that r satisfies the following conditions (since sI^n is a compact convex subset of $\mathbb{R}^n$ containing A as a subset):

- r maps each point of the boundary of sI^n, denoted by $\mathscr{B}(sI^n)$, to a point of $\mathscr{B}(A)$, the boundary of A,
- for any $x \in sI^n \setminus A$, r maps the set of points $(1-\alpha)x + \alpha r(x)$, $0 \leqslant \alpha \leqslant 1$, to $\{r(x)\} \subseteq \mathscr{B}(A)$,
- for any $x \in A$, $r(x) = x$.

Note that $r : sI^n \to A$ is indeed a retraction from sI^n to A.

Let $h = g \circ r$. Clearly, $h : sI^n \to \mathbb{R}^n$ is a continuous inward mapping from sI^n to $\mathbb{R}^n$ (in fact, we have $h(sI^n) \subseteq f(A) \subseteq sI^n$). From above, h has a fixed point, say $w \in f(A)$. We show that $w \in A$, and hence $g(w) = (g \circ r)(w) = h(w) = w$; w is also a fixed point of g. Suppose not, that is, we have $w \notin A$. Then we have $w \in f(A) \setminus A$, hence $g(r(w)) = (g \circ r)(w) = h(w) = w$ (where $r(w) \neq w$). Thus (by the property of inward continuity of g) there exists a point $z \in A$, $z \neq r(w)$ such that $w \in \{(1-\alpha)r(w) + \alpha z \in \mathbb{R}^n \mid \alpha \in \mathbb{R}, \alpha \geqslant 0\} \subseteq I_A(r(w))$. So we have $d_{\mathbb{R}^n}(w, r(w)) > d_{\mathbb{R}^n}(w, z)$: contradiction. □

Hence by Theorem 6.1, we may call Theorem 2.1 the discrete Halpern–Bergman fixed point theorem. On the other hand, since in general Euclidean convex subsets of $\mathbb{R}^n$ cannot be“approximated” by rectangle blocks (segments) of $\mathbb{R}^n$, it is *impossible* to “approximate” the Browder fixed point theorem by Theorem 2.2.

A possible discrete Browder fixed point theorem should have a formulation similar to those given in [9, Theorem 5.4]. Indeed, it is not difficult to show that it can be done by approaches similar to those which we proposed in [9]. However, as already mentioned in [9], suitable combinatorial proofs for these (discrete Browder and Kakutani fixed point) theorems are still unavailable.

Acknowlegdment

The author would like to thank Graham Brightwell of London School of Economics and Michael B. Smyth of Imperial College for many valuable suggestions which led to substantial improvements. He also wishes to thank the referees for their valuable comments, which resulted in the improved presentation of the paper.

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