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# CONSTRUCTIONS FOR PROJECTIVELY UNIQUE POLYTOPES

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A convex polytope $P$ is projectively unique if every polytope combinatorially isomorphic to $P$ is projectively equivalent to $P$. In this paper are described certain geometric constructions, which are also discussed in terms of Gale diagrams. These constructions are applied to obtain projectively unique polytopes from ones of lower dimension; in particular, they lead to projectively unique polytopes with many vertices.

## 1. Introduction

Let $p$ be a $d$-polytope in $E^d$. Following Grünbaum [1] (as we shall largely do in matters of terminology), we say that $P$ is *projectively unique* if, whenever $P'$ is a polytope combinatorially isomorphic to $P$, there is a projective transformation $\Phi$ of $E^d$ such that $P\Phi = P'$. We shall further assume that $\Phi$ takes each vertex of $P$ into the vertex of $P'$ which corresponds to it under the combinatorial isomorphism. In this respect we follow Perles, who first discussed projectively unique (or "rigid") polytopes in a letter to Grünbaum (who kindly communicated its contents to us). The relaxation of this additional assumption leads to polytopes called "almost rigid" by Perles. However, no examples are known of polytopes which are almost rigid, but not rigid. Indeed, such an example would answer (in the negative) the probably very hard question (compare [1, Exercise 6.5.7]): Does there correspond to every polytope $P$ a combinatorially isomorphic polytope $P'$, such that every combinatorial automorphism of $P'$ is induced by a symmetry of $P'$? In view of this problem, we shall not discuss almost rigid polytopes any further.

Most of Perles' original results are given in [1, §§4.8, 5.5, 6.5 and 11.1]. One other is described by Perles and Shephard [4]; as we shall remark in

Section 6, it is a special case of one of our constructions. These constructions are described geometrically in Section 2, and in terms of Gale diagrams in Section 3; they generalize constructions of Perles and Shephard [3] and may well be of use in other contexts.

Shephard, in an unpublished manuscript which he has kindly let us use, made what is believed to be a complete list of projectively unique 4-polytopes; all these, and the known projectively unique 3-polytopes, can be constructed by our methods. We shall see in Section 7, however, that there are projectively unique polytopes which our constructions do not yield.

## 2. The geometric constructions

The first of our constructions, that of the (free) join, seems to have originated with Sommerville [5]; it was also used by Perles and Shephard [3]. If $P$ and $Q$ are two polytopes, such that

$$\dim \operatorname{conv}(P \cup Q) = \dim P + \dim Q + 1 ,$$

we call the polytope $\operatorname{conv}(P \cup Q)$ the *join* of $P$ and $Q$, and denote it by $P \otimes Q$. In the particular case when $Q$ is a point (0-polytope), $P \otimes Q$ is a pyramid with basis $P$. The subspaces aff $P$ and aff $Q$ are independent (totally skew); thus the union of affinely independent subsets of aff $P$ and aff $Q$ is affinely independent.

The combinatorial structure of the join is easily described (we leave the proof of the proposition to the reader).

**Proposition 2.1.** *The faces of $P \otimes Q$ are precisely the sets of the form $F \otimes G$, where $F$ is a face of $P$ and $G$ is a face of $Q$ (including $F = \emptyset$ or $P$ and $G = \emptyset$ or $Q$).*

We denote by $B(P)$ the group of projective symmetries of the polytope $P$, that is, the group of (necessarily non-singular) projective transformations $\Phi$ of $E^d$ such that $P\Phi = P$. We write $B(P \mid \operatorname{vert} P)$ for the subgroup of $B(P)$ consisting of those projective symmetries of $P$ which leave fixed each vertex of $P$. We then obtain the following useful characterization of the join.

**Theorem 2.2.** *Let $F$ be a $k$-face of a polytope $P$. Then $B(P \mid \text{vert } P)$ is transitive on* relint $F$ *if and only if $F$ is a $k$-simplex, with vertices $x_0, \ldots, x_k$* (say), *and $P$ has faces $P_0, \ldots, P_k$ with $x_i \in P_i$ $(i = 0, \ldots, k)$, such that*

$$P = P_0 \circledvee \ldots \circledvee P_k \, .$$

To prove this, we imbed $E^d$ in $E^{d+1}$ as, say, the coordinate subspace $\xi_{d+1} = 1$. To the projective transformations of $B(P \mid \text{vert } P)$ correspond the linear transformations of $E^{d+1}$ which leave fixed each of the directions through the vertices of $P$. Let $F_1, \ldots, F_r$ be the maximal faces of $P$, each point of which $B(P \mid \text{vert } P)$ leaves fixed. Then, in $E^{d+1}$, each $F_i$ spans a linear subspace $L_i$; the corresponding linear transformations on $L_i$ are the positive scalar multiples of the identity. It is now easy to see that $L_1, \ldots, L_r$ are independent subspaces. From the definition of $F$, each vertex of $F$ lies in a different face $F_i$. Hence, if $F$ is a $k$-face, it is a $k$-simplex, with vertices $x_0, \ldots, x_k$ (say). Rename $F_1, \ldots, F_r$, so that $x_i \in P_i = F_i$ $(i = 1, \ldots, k)$, and let $P_0 = F_{k+1} \circledvee \ldots \circledvee F_r$, so that $x_0 \in P_0$. Since each vertex of $P$ belongs to some face $F_i$, it follows that

$$\begin{aligned} P &= F_1 \circledvee \ldots \circledvee F_r \\ &\phantom{=} P_0 \circledvee \ldots \circledvee P_k \, , \end{aligned}$$

as required. Since the converse is clear, this proves Theorem 2.2.

We now describe our second construction. Let $P$ and $Q$ be polytopes whose affine hulls meet in a single point, which is a relatively interior point of the faces $F$ of $P$ and $G$ of $Q$. We allow $F = P$ or $G = Q$ here. Then the polytope conv$(P \cup Q)$ is called the *subdirect sum* of $P$ and $Q$ *relative to* $F$ and $G$, and is denoted by

$$(P, F) \oplus (Q, G) \, .$$

If $F = P$ and $G = Q$, we have the *direct sum* of Perles and Shephard [3]:

$$P \oplus Q = (P, P) \oplus (Q, Q).$$

The next proposition describes the combinatorial structure of the subdirect sum.

**Proposition 2.3.** *The faces of the subdirect sum* $(P, F) \oplus (Q, G)$ *are precisely those sets of the form either* (a) $F_1 \otimes G_1$, *where* $F_1 \cap F \subset F$ *and* $G_1 \cap G \subset G$, *or* (b) $(F_1, F) \oplus (G_1, G)$, *where* $F_1 \supseteq F$ *and* $G_1 \supseteq G$.

For, the faces are of type (a) or (b) according as the supporting hyperplane determining them misses or contains the common relatively interior point of $F$ and $G$; the details of the proof will be omitted. (Note that $\subset$ means strict inclusion.)

Two particularly important examples of the subdirect sum (apart from the direct sum, which we have already mentioned) are as follows. First, the subdirect sum $(P, F) \oplus (Q, G)$ of $P$ and $Q$ relative to a vertex $F = G$ of each is called a *vertex sum*; $F = G$ is called the *distinguished vertex* of the vertex sum. Second, if $F$ is a vertex of $P$, and $T$ is a simplex, then we say that $(P, F) \oplus (T, T)$ is obtained by *adding the simplex T at the vertex F of P*.

It should be observed that the join can be obtained as a special case of the subdirect sum, and, indeed, as an iterated vertex sum. For, consider $P = P_1 \otimes \ldots \otimes P_r$. Let $x_i$ be any vertex of $P_i$ (for $i = 1, \ldots, r$). Then $T = \mathrm{conv}\{x_1, \ldots, x_r\}$ is an $(r-1)$-simplex, and if we define successively:

$$Q_0 = T, \qquad Q_s = (Q_{s-1}, \{x_s\}) \oplus (P_s, \{x_s\}),$$

for $s = 1, \ldots, r$, then clearly $Q_r = P$.

Our final construction is the dual of the second, and is defined as follows. If $P$ is a polytope, and $F$ a face of $P$, let $P^*$ be a dual of $P$, and $\hat{F}$ its corresponding face. (If we wish, we may assume that $P^*$ is defined to within projective equivalence; see, for example, [2, §2.2, particularly Theorem 14], and compare Theorem 3.3 below.) Let $P$ and $Q$ be polytopes, and let $F$ be a face of $P$ and $G$ a face of $Q$, with $F \neq P$ and $G \neq Q$ (but we allow $F = \emptyset$ or $G = \emptyset$). Then the polytope

$$(P, F) \otimes (Q, G) = ((P^*, \hat{F}) \oplus (Q^*, \hat{G}))^*$$

is called the *subdirect product* of $P$ and $Q$ relative to $F$ and $G$.

As a particular case of the subdirect product, we have the *direct* or *cartesian* product

$$P \otimes Q = (P, \emptyset) \otimes (Q, \emptyset).$$

We shall not give a geometrical description of the subdirect product, since the details are a little complicated. We shall therefore confine ourselves to stating the result which describes its combinatorial structure.

**Proposition 2.4.** *The faces of the subdirect* [illegible]*uct* $(P, F) \otimes (Q, G)$ *are precisely the sets of the form either* (a) $F_1$ [illegible]$_1$, *where* $F_1 \subseteq F$ *and* $G_1 \subseteq G$, *or* (b) $(F_1, F_1 \cap F) \otimes (G_1, G_1 \cap G)$, *where* $F_1 \nsubseteq F$ *and* $G_1 \nsubseteq G$.

## 3. Projectively unique polytopes

We now investigate some of the connexions between our geometric constructions and projectively unique polytopes.

**Theorem 3.1.** *The join* $P \circledvee Q$ *of two polytopes* $P$ *and* $Q$ *is projectively unique if and only if both* $P$ *and* $Q$ *are.*

The proof of this result is a straightforward application of familiar properties of projective transformations; we shall not go into the details. (Compare the proofs of Theorem 2.2, or Theorem 5.1.)

More interesting is the next result, about subdirect sums. It only provides a necessary condition; we must postpone the discussion of sufficient conditions until Section 5.

**Theorem 3.2.** *If* $(P, F) \oplus (Q, G)$ *is projectively unique, then* $P$ *and* $Q$ *are projectively unique, and* $B(P \mid \text{vert } P)$ $(B(Q \mid \text{vert } Q))$ *is transitive on* relint $F$ (relint $G$, *respectively*).

The proof follows immediately on considering the cases where $P$ and $Q$ lie in complementary linear subspaces, with the origin as the common point of relint $F$ and relint $G$.

The polytopes with the property of Theorem 3.2 were described in Theorem 2.2. In view of the remarks there, we see that, if we wish to apply our constructions to projectively unique polytopes, in order to obtain ones of higher dimension, we can confine our attention to three particular cases:

(I) The direct sum of two simplices.
(II) A vertex sum of two polytopes.
(III) The result of adding a simplex at a vertex of a polytope.

In fact, this third case reduces still further; an easy induction argument (compare Theorem 5.3 below) shows that we need only consider:

(III)' The result of adding a segment at a vertex of a polytope.

It would not be at all difficult to prove directly that polytopes obtained by means of (I) or (II) are projectively unique. However, the proofs using Gale diagrams are more straightforward.

Our final remark shows that we need not consider the subdirect product separately (compare [1, Exercise 4.8.30]).

**Theorem 3.3.** *A polytope P is projectively unique if and only if its dual $P^*$ is.*

For, if $P$ is a polytope with $o \in \operatorname{int} P$, then

$$P^* = \{x \in \mathrm{E}^d : \langle x, y\rangle \leqslant 1, \text{ for all } y \in P\}$$

is a dual of $P$. Now, if $P_1$ is a polytope projectively equivalent to $P$, also with $o \in \operatorname{int} P_1$, then $P_1^*$ is projectively equivalent to $P^*$ ([2, Chapter 2, Theorem 14]). The theorem follows at once.

## 4. Gale diagrams

In this section, we shall describe the Gale diagrams of the join and subdirect sum of two polytopes, in terms of the Gale diagrams of the original polytopes. It is sufficiently general, for our purposes, to consider the affine (or Gale) transforms of the sets of their vertices. For background information about Gale diagrams, the reader should consult Grünbaum [1, §5.4] or McMullen and Shephard [2, Chapter 3].

We begin by recalling that, to find the affine transform of an ordered set $X = (x_1, ..., x_n)$ of points in $\mathrm{E}^d$, we write a basis for the linear subspace $A(X)$ of affine dependences $(\alpha_1, ..., \alpha_n)$ $\mathrm{E}^n$ of $X$, satisfying

$$\sum_{i=1}^{n} \alpha_i x_i = o, \qquad \sum_{i=1}^{n} \alpha_i = 0,$$

as the column vectors of a matrix; the rows $\bar{x}_1, ..., \bar{x}_n$ of this matrix (which are in an obvious one to one correspondence with the points of $X$) form our affine transform $\bar{X}$. Clearly, $\bar{X}$ depends upon the choice of basis, but it is determined by $X$ up to linear equivalence. If aff $X = \mathrm{E}^d$, then $\bar{X}$ spans $\mathrm{E}^{n-d-1}$ (linearly, and, indeed, positively).

We first consider the join $P \circledcirc Q$, and write vert $P = (p_1, ..., p_m)$ and vert $Q = (q_1, ..., q_n)$, so that vert$(P \circledcirc Q) = (p_1, ..., p_m, q_1, ..., q_n)$. If dim $P = r$ and dim $Q = s$, then dim$(P \circledcirc Q) = r + s + 1$; thus, if $\{a_1, ..., a_{m-r-1}\}$ and $\{b_1, ..., b_{n-s-1}\}$ form bases of $A$(vert $P$) and $A$(vert $Q$) respectively (where we write $a = (\alpha_1, ..., \alpha_m) \in A(\text{vert } P)$, and so on), and we define

$$a'_k = (a_k, o) \qquad (k = 1, ..., m-r-1),$$
$$b'_l = (o, b_l) \qquad (l = 1, ..., n-s-1)$$

(here, $a' = (\alpha_1, ..., \alpha_m, 0, ..., 0)$, and similarly for $b'$) it is clear that $\{a'_1, ..., a'_{m-r-1}, b'_1, ..., b'_{n-s-1}\}$ is linearly independent, and so is a basis of $A$(vert $(P \circledcirc Q)$). Hence, an affine transform of vert$(P \circledcirc Q)$ consists of

$$\tilde{p}_i = (\bar{p}_i, o) \qquad (i = 1, ..., m),$$
$$\tilde{q}_j = (o, \bar{q}_j) \qquad (j = 1, ..., n),$$

where $(\bar{p}_1, ..., \bar{p}_m)$ and $(\bar{q}_1, ..., \bar{q}_n)$ are affine transforms of vert $P$ and vert $Q$, respectively. That is, we place the affine transforms of vert $P$ and vert $Q$ in complementary linear subspaces.

We now consider the subdirect sum $(P, F) \oplus (Q, G)$. Here there are two cases, according as neither $F$ nor $G$ is a vertex, or as one or both are. Let $x = \sum_{i=1}^m \lambda_i p_i = \sum_{j=1}^n \mu_j q_j$ belonging to relint $F \cap$ relint $G$ be the common point of $P$ and $Q$, where $\lambda_i, \mu_j \geqslant 0$, $\sum_{i=1}^m \lambda_i = 1 = \sum_{j=1}^n \mu_j$, and $\lambda_i = 0 = \mu_j$ if $p_i \notin F$ and $q_j \notin G$; we can also assume $\lambda_i, \mu_j > 0$ if $p_i \in F$ and $q_j \in G$. As before, let $\{a_1, ..., a_{m-r-1}\}$ and $\{b_1, ..., b_{n-s-1}\}$ be bases of $A$(vert $P$) and $A$(vert $Q$), respectively.

In case neither $F$ nor $G$ is a vertex,

$$a'_k = (a_k, o) \qquad (k = 1, ..., m-r-1),$$
$$b'_l = (o, b_l) \qquad (l = 1, ..., n-s-1),$$
$$c = (\lambda_1, ..., \lambda_m, -\mu_1, ..., -\mu_n)$$

form a basis of $A(\text{vert}((P, F) \oplus (Q, G)))$; the last affine dependence expresses the relation involving the common point $x$ mentioned above. Thus, an affine transform of $\text{vert}((P, F) \oplus (Q, G))$ consists of

$$\tilde{p}_i = (\bar{p}_i, o, \lambda_i) \qquad (i = 1, \ldots, m)\,,$$

$$\tilde{q}_j = (o, \bar{q}_j, -\mu_j) \qquad (j = 1, \ldots, n)\,,$$

where $(\bar{p}_1, \ldots, \bar{p}_m)$ and $(\bar{q}_1, \ldots, \bar{q}_n)$ are affine transforms of vert $P$ and vert $Q$, respectively, as before.

In the other case, suppose, without loss of generality, that $x = p_1$ is the vertex of $P$ lying in relint $G$, so that $\lambda_1 = 1$. As will be seen, the case where $G$ is also a vertex is covered by the argument. Our basic affine dependences are obtained from those of vert $P$ and vert $Q$ by everywhere replacing $p_1$ by its expression

$$p_1 = \sum_{j=1}^{n} \mu_j q_j\,.$$

(Remember that $p_1$ is no longer a vertex of $(P, F) \oplus (Q, G)$.) Hence, a basis of $A(\text{vert}((P, F) \oplus (Q, G)))$ consists of (in an obvious notation)

$$a'_k = (\alpha_{2k}, \ldots, \alpha_{mk}, \mu_1 \alpha_{1k}, \ldots, \mu_n \alpha_{1k}) \qquad (k = 1, \ldots, m-r-1),$$

$$b'_l = (0, \ldots, 0, \beta_{1l}, \ldots, \beta_{nl}) \qquad (l = 1, \ldots, n-s-1),$$

and so an affine transform of $\text{vert}((P, F) \oplus (Q, G))$ consists of

$$\tilde{p}_i = (\bar{p}_i, o) \qquad (i = 2, \ldots, m),$$

$$\tilde{q}_j = (\mu_j \bar{p}_1, \bar{q}_j) \qquad (j = 1, \ldots, n)\,.$$

If $G$ is also a vertex, $\{q_1\}$ say, we see that we have symmetry between the $p$'s and $q$'s (since $\mu_1 = 1$).

## 5. Stable Gale diagrams

To projective transformations on polytopes correspond *equivalences* on their Gale diagrams, which are transformations of the form

$$\bar{x}_i \to \nu_i \bar{x}_i \Psi\,,$$

where $\nu_i > 0$ for all $i$, and $\Psi$ is a non-singular linear transformation. Two Gale diagrams related in this way are called *equivalent*. To combinatorially isomorphic polytopes correspond *isomorphic* diagrams, where the (one-to-one) mappings preserve the relationships of the form

$$o \in \text{relint conv } \overline{Y} ,$$

for all subsets $\overline{Y}$ of $\overline{X}$. We say a Gale diagram is *stable* if every isomorphic diagram is equivalent to it; thus a polytope is projectively unique if and only if its Gale diagram is stable.

We have already discussed in Section 3 the connexion between our geometric constructions and projective uniqueness. However, we left open there the question of sufficiency with the subdirect sums. Here we complete that discussion. By the results of Section 3, we can restrict our attention to projectively unique polytopes $P$ and $Q$. We had three cases to consider, and we deal with them in the order given in Section 3.

**Theorem 5.1.** *The direct sum of two simplices is projectively unique.*

For, if the sum is $d$-dimensional, it has $d + 2$ vertices. Its Gale diagram is therefore one-dimensional, and is clearly stable (compare [2, Chapter 3]).

**Theorem 5.2.** *A vertex sum of two projectively unique polytopes is projectively unique.*

For, let the two polytopes $P$ and $Q$ have Gale diagrams $\hat{P}$ and $\hat{Q}$, with $\bar{p} \in \hat{P}$ and $\bar{q} \in \hat{Q}$ corresponding to the distinguished vertex $p = q$. Then the Gale diagram of $K = (P, \{p\}) \oplus (Q, \{q\})$ is

$$\hat{K} = (\hat{P} \setminus \{\bar{p}\}) \cup (\hat{Q} \setminus \{\bar{q}\}) \cup \{\bar{p} + \bar{q}\},$$

where we regard $\hat{P}$ and $\hat{Q}$ as lying in complementary linear subspaces. But any isomorphic diagram $\hat{K}_1$ has a similar decomposition into two subsets lying in complementary subspaces $L$ and $M$, and a single point $\bar{p}_1 + \bar{q}_1$ ($\bar{p}_1 \in L$, $\bar{q}_1 \in M$). Then $\hat{P}_1 = (\hat{K}_1 \cap L) \cup \{\bar{p}_1\}$ and $\hat{Q}_1 = (\hat{K}_1 \cap M) \cup \{\bar{q}_1\}$ are isomorphic to $P$ and $Q$, respectively; since these isomorphisms can be realized by equivalences, it follows that $K_1$ is equivalent to $K$, so that $K$ is stable, as was to be shown.

Note that a similar, but easier argument gives a proof of Theorem 3.1.

Our final result characterizes the remaining cases in which our constructions yield a projectively unique polytope.

**Theorem 5.3.** *Let P be a projectively unique polytope, and let x be a vertex of P. Then adding a segment to P at x yields a projectively unique polytope, except when P is a vertex sum, with x as distinguished vertex.*

The necessity of this condition is obvious from geometric considerations; if $P = (P_1, \{x_1\}) \oplus (P_2, \{x_2\})$ (with $x_1 = x_2 = x$), we can form a polytope combinatorially isomorphic to $(P, \{x\}) \oplus (I, I)$, where $I$ is a segment, by parallel displacement of $P_2$, moving $x_2$ to a new position in $I$ distinct from $x_1$. Clearly, this new polytope will not (in general) be projectively equivalent to $P$. However, it is by no means obvious (and the author was unable to prove) geometrically that the condition is sufficient. So, to prove this, we consider the Gale diagram $\hat{P}$ of $P$. The Gale diagram of $(P, \{x\}) \oplus (I, I)$ is obtained (up to equivalence) by replacing the point $\bar{x}$ of $\hat{P}$ corresponding to $x$ by the repeated pair $(\bar{x}, \bar{x})$. If we can obtain an equivalent Gale diagram of $P$ by replacing $\bar{x}$ by $\bar{x}' \neq \lambda\bar{x}$, and keeping fixed the remaining points of $P$, then the diagram obtained by replacing $\bar{x}$ by the pair $(\bar{x}, \bar{x}')$ is isomorphic, but clearly not equivalent to the original Gale diagram of $(P, \{x\}) \oplus (I, I)$. We thus conclude that the Gale diagram of $(P, \{x\}) \oplus (I, I)$ is stable if and only if $\bar{x}$ is determined (up to positive multiple) by the positions of the remaining points of $\hat{P}$. But clearly $\bar{x}$ is not so determined precisely when these remaining points lie in complementary linear subspaces (recall that $\hat{P}$ is stable, so that the possible positions of $\bar{x}$ can be attained by equivalences). Comparing with the proof of Theorem 5.2, we see at once that $P$ is a vertex sum, with $x$ as distinguished vertex, as we wished to show.

## 6. Projectively unique polytopes with many vertices

Results exactly analogous to those of Section 5 hold for subdirect products of polytopes (by virtue of Theorem 3.3). In particular, if, for $i = 1, ..., k$, $P_i$ is a $d_i$-dimensional prism, whose basis is a $(d_i-1)$-dimensional simplex $F_i$, then the subdirect product

$$P = (P_1, F_1) \otimes \ldots \otimes (P_k, F_k)$$

is projectively unique (it is dual to a repeated vertex sum of bipyramids with simplicial bases). Easy calculations, using Proposition 2.4, show that $P$ has $d_1 + \ldots + d_k + k + 1$ facets and $d_1 \ldots d_k + d_1 + \ldots + d_k$ vertices. It can be proved (though the details are involved) that $P$ is the polytope whose Gale diagram was constructed directly by Perles, who remarked that, by suitable choice of $d_1, \ldots, d_k$, one could then find projectively unique $d$-polytopes with about $3^{d/3}$ vertices. We refer the reader to [4] for further details.

## 7. Remarks

If we apply any of our three constructions to projectively unique rational polytopes, which are (projectively equivalent to) polytopes whose vertices have rational cartesian coordinates, then we obtain rational polytopes. Since Perles has found non-rational projectively unique polytopes (see [1, Theorem 5.5.4]), we conclude (not surprisingly) that our constructions cannot yield all projectively unique polytopes, starting from points as basic building blocks.

In fact, we can even give a rational polytope which (we believe) cannot be so constructed. Following the notation of Perles and Shephard [4], if we regard the points $e_{j_1 \ldots j_k}$ of their Gale diagram as being linearly (rather than affinely) independent, and add to the diagram the further point

$$e_0 = \sum_{j_1, \ldots, j_k} e_{j_1 \ldots j_k},$$

we obtain a new stable Gale diagram, of a rational $(d_1 + \ldots + d_k)$-polytope with $d_1 \ldots d_k + d_1 + \ldots + d_k + 1$ vertices, one more than in the previous example. If $k \geqslant 3$ and $d_i = 3$ for all $i$, this number of vertices seems not to be attainable by means of our constructions.

On the more positive side, the projectively unique 3-polytopes have been characterized as precisely those with at most 9 edges (see [1, Exercise 4.8.30]), and Shephard (private communication) has independently made a list, believed to be complete, of the projectively unique 4-polytopes. All of these polytopes can be constructed by the methods described here.

Most of the results of this paper appear in my Ph. D. Thesis (Birmingham, 1968), written under the supervision of Professor G.C. Shephard. However, the account remained incomplete, until I recently found the characterization of Theorem 5.3. My thanks are due to Professor Shephard for his patient reading and commenting on a previous version of this work, and to Professor B. Grünbaum, without whose prompting this paper would probably never have seen the light of day.

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