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Journal of Functional Analysis 255 (2008) 1667–1691

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

Slanted matrices, Banach frames, and sampling

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Received 10 August 2007; accepted 12 June 2008

Available online 22 July 2008

Communicated by N. Kalton

Abstract

In this paper we present a rare combination of abstract results on the spectral properties of slanted matrices and some of their very specific applications to frame theory and sampling problems. We show that for a large class of slanted matrices boundedness below of the corresponding operator in ℓ^p for some p implies boundedness below in ℓ^p for all p. We use the established result to enrich our understanding of Banach frames and obtain new results for irregular sampling problems. We also present a version of a non-commutative Wiener's lemma for slanted matrices.

Keywords: Slanted matrices; Boundedness below; Banach frames; Irregular sampling; Non-uniform sampling

1. Introduction

In this paper we study certain properties of so-called slanted matrices, which occur naturally in different fields of pure and applied analysis. A matrix is slanted if it has a decay property such that the coefficients vanish away from a diagonal, which is not necessarily the main diagonal; ideally, non-zero coefficients of such a matrix are contained between two parallel slanted lines. Potential applications of the theory of slanted matrices range through wavelet theory and signal

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[1] The first author was supported in part by NSF grants DM-0504788.

[2] The second author is supported in part by RFBR grant 07-01-00131.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2008.06.024

processing [17–19,24,33], frame and sampling theory [1–3,8,9,30], differential equations [13, 14,16], and even topology of manifolds [43]. Here we especially emphasize the use of slanted matrices in frame theory and related fields.

We begin with a few explicit examples illustrating the appearance of slanted matrices. The reader, of course, should keep in mind that the standard case of banded matrices is a particular case of slanted banded matrices. Below are less trivial examples and the first of them concerns sampling in shift invariant spaces.

Example 1.1. It is well known that the Paley–Wiener space $PW_{1/2} = \{f \in L^2(\mathbb{R})\colon \operatorname{supp} \hat{f} \subset [-1/2, 1/2]\}$ can also be described as

$$PW_{1/2} = \left\{ f \in L^2(\mathbb{R})\colon\ f = \sum_{k\in\mathbb{Z}} c_k \phi(\cdot - k),\ c \in \ell^2(\mathbb{Z}) \right\}, \tag{1.1}$$

where $\phi(x) = \frac{\sin \pi(x-k)}{\pi(x-k)}$ and the series converges in $L^2(\mathbb{R})$ (see e.g., [3]). Because of this equivalent description of $PW_{1/2}$, the problem of reconstructing a function $f \in PW_{1/2}$ from the sequence of its integer samples, $\{f(i)\}_{i\in\mathbb{Z}}$, is equivalent to finding the coefficients $c \in \ell^2$ such that $\{f(i)\} = Ac$ where $A = (a_{i,j})$ is the matrix with entries $a_{i,j} = \phi(i - j)$. It is immediate, however, that $A = I$ is the identity matrix and, therefore,

$$f = \sum_{k\in\mathbb{Z}} f(k)\phi(\cdot - k).$$

If, instead, we sample a function $f \in PW_{1/2}$ on $\frac{1}{2}\mathbb{Z}$, then we obtain the equation $\{f(\frac{i}{2})\} = Ac$. In this case, the sampling matrix A is defined by $a_{i,j} = \phi(\frac{i}{2} - j)$ and is no longer diagonal—it has constant values on slanted lines with slopes 1/2, for instance, $a_{2j,j} = 1$. If $\phi = \frac{\sin \pi(x-k)}{\pi(x-k)}$ in (1.1) is replaced by a function ψ supported on $[-\frac{M}{2}, \frac{M}{2}]$, then the matrix $A = (a_{i,j})$ is zero outside the slanted band $|j - i/2| \leqslant M$. Clearly, this matrix is not banded in the classical sense. If we move to the realm of irregular sampling [3], the sampling matrix will be given by $a_{i,j} = \phi(x_i - j)$, where x_i, $i \in \mathbb{Z}$, are the sampling points. In this case, we no longer have constant values on slanted lines, but the slanted structure is still preserved if we have the same number of sampling points per period. An important fact [3] is that any function can be reconstructed from its samples at x_i, $i \in \mathbb{Z}$, if and only if the sampling matrix is bounded below and above. The main emphasis of this paper is to study this particular property of abstract slanted matrices.

The next example deals with frames in Hilbert spaces. Meaningful extension of the notion of frames to Banach spaces is a non-trivial problem which provided some inspiration for our abstract results. In the example below we only give a brief introduction and defer precise definitions to Section 3.

Example 1.2. Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\varphi_n \in \mathcal{H}$, $n \in \mathbb{Z}$, is a *frame* for $\mathcal{H}$ if for some $0 < a \leqslant b < \infty$

$$a\|f\|^2 \leqslant \sum_{n\in\mathbb{Z}} \left|\langle f, \varphi_n\rangle\right|^2 \leqslant b\|f\|^2 \tag{1.2}$$

for all $f \in \mathcal{H}$. The operator $T : \mathcal{H} \to \ell^2$, $Tf = \{\langle f, \varphi_n\rangle\}_{n\in\mathbb{Z}}$, $f \in \mathcal{H}$, is called an *analysis operator*. It is an easy exercise to show that a sequence $\varphi_n \in \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if its analysis operator has a left inverse. The adjoint of the analysis operator, $T^* : \ell^2 \to \mathcal{H}$, is given by $T^*c = \sum_{n\in\mathbb{Z}} c_n\varphi_n$, $c = (c_n) \in \ell^2$. The *frame operator* is $T^*T : \mathcal{H} \to \mathcal{H}$, $T^*Tf = \sum_{n\in\mathbb{Z}}\langle f, \varphi_n\rangle\varphi_n$, $f \in \mathcal{H}$.

Traditionally (see [20,23,30] and references therein), the frame properties are studied via the spectral properties of the frame operator. In this paper we show that some work can be done already at the level of the analysis operator. This makes extensions to Banach spaces easier since the analysis operator is more amenable to such. Connection with slanted matrices is readily illustrated if we consider a frame in $\ell^2(\mathbb{Z})$ which consists of two copies of an orthonormal basis. Then a section of the matrix of the analysis operator with respect to that basis looks like

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0. \end{matrix}$$

Clearly, the slant of the matrix may serve as a natural measure of redundancy of a frame.

The following example illustrates the role of slanted matrices in wavelet theory.

Example 1.3. In signal processing and communication, a sequence s (a discrete signal) is often split into a finite set of *compressed sequences* $\{s_1, \ldots, s_r\}$ from which the original sequence s can be reconstructed or approximated. The compression is often performed with *filter banks* [24, 33] using the cascade algorithm. One way to introduce filters, in the simplest case, is to use the two-scale equation of the multiresolution analysis (MRA):

$$\varphi(x) = \sum_{n\in\mathbb{Z}} a_n\varphi(2x - n),$$

where $\varphi \in L^2(\mathbb{R})$ is the so-called *scaling function*. The filter coefficients a_n, $n \in \mathbb{Z}$, in the above equation are the Fourier coefficients of the *low-pass filter* $m_0 \in L^2(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, which is a periodic function given by

$$m_0(\xi) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi in\xi}, \quad \xi \in \mathbb{R}.$$

It is clear that the two-scale equation has the following equivalent form in the Fourier domain:

$$\hat{\varphi}(2\xi) = m_0(\xi)\hat{\varphi}(\xi), \quad \xi \in \mathbb{R}.$$

An important role in the MRA theory is played by the *periodization* $\sigma_\varphi \in L^\infty(\mathbb{T})$ of the scaling function φ, which is defined by

$$\sigma_\varphi(\xi) = [\hat{\varphi}, \hat{\varphi}](\xi) = \sum_{n\in\mathbb{Z}} \left|\hat{\varphi}(\xi + n)\right|^2.$$

It is a standard fact (see, e.g., [6, Lemma 2.11]) that this periodization satisfies

$$\sigma_\varphi(\xi/2) = \left|m_0(\xi)\right|^2 \sigma_\varphi(\xi) + \left|m_0(\xi + 1/2)\right|^2 \sigma_\varphi(\xi + 1/2).$$

In fact, σ_φ is the Perron–Frobenius eigen-vector of the *transfer operator* R_{m_0} which acts on different spaces of periodic functions via

$$(R_{m_0} f)(\xi) = \left|m_0(\xi)\right|^2 f(\xi) + \left|m_0(\xi + 1/2)\right|^2 f(\xi + 1/2). \tag{1.3}$$

In [18] there is a detailed account of the relation between the spectral properties of the transfer operator on different function spaces and the properties of the corresponding MRA filters, scaling functions, and wavelets. Here we will just recall that the convergence rate of the above mentioned cascade algorithm is controlled by the second biggest eigenvalue of R_{m_0}. The reason we use the transfer operator as an example is because of its matrix with respect to the Fourier basis in $L^2(\mathbb{T})$. Following [18, Section 3.2], we let

$$c_n = \sum_{k\in\mathbb{Z}} \bar{a}_k a_{n+k}.$$

Then the Fourier coefficients of $R_{m_0} f$ and f are related via

$$(R_{m_0} f)_n = \sum_{k\in\mathbb{Z}} c_{2n-k} f_k,$$

and, hence, this is, indeed, a slanted matrix. In particular, if

$$m_0(\xi) = a_0 + a_1 e^{2\pi i\xi} + a_2 e^{2\pi i 2\xi} + a_3 e^{2\pi i 3\xi},$$

a section of this matrix looks like

$$\begin{matrix}
c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 & c_2 & c_1 & c_0 & c_{-1} & c_{-2} & c_{-3}.
\end{matrix}$$

Due to the special Laurent-type structure of this matrix there has been a lot of results on the spectral properties of such matrices (see [18] and references therein). Since we are interested in more general slanted matrices, we cannot use most of those results. Observe also, that this matrix has the "opposite" slant compared to the matrices in the previous examples. In fact, as shown in Lemma 2.2 below such matrices cannot be bounded below and, therefore, are less relevant to this paper.

The rest of this paper is organized as follows. Section 2 is devoted to abstract results. We give precise definitions of different classes of slanted matrices in Section 2.1 and study some of their basic properties in 2.2. In Section 2.3 we state and prove one of our main theorems. Specifically, slanted matrices with some decay, viewed as operators on $\ell^p(\mathbb{Z}^d, (X_n))$ spaces (where

X_n is a Banach space), are either universally bounded below for all $p \in [1, \infty]$, or do not have this property for any $p \in [1, \infty]$. In Section 2.4 we use this theorem to obtain a version of Wiener's Tauberian lemma and a result on subspace complementation in Banach spaces. Section 3 is devoted to some applications of the results of Section 2. Specifically, in Section 3.1, the reconstruction formulae for Hilbert frames are extended to Banach frames under certain localization conditions related to slanted matrices. Gabor systems having this localization property are then presented as an example. Section 3.2 exhibits an application of slanted matrices to sampling theory.

2. Abstract results

2.1. Slanted matrices: definitions

We prefer to give a straightforward definition of slanted matrices in the relatively simple case that arises in applications presented in this paper, mainly in connection with sampling theory. For that reason, we restrict our attention to the group $\mathbb{Z}^d$, $d \in \mathbb{N}$, and leave the case of more general locally compact Abelian groups for future research in the spirit of [11,12,15]. We believe, also, that some of the results below may be extended to matrices indexed by discrete metric spaces.

For each $n \in \mathbb{Z}^d$ we let X_n and Y_n be (complex) Banach spaces and $\mathfrak{L}^p = \ell^p(\mathbb{Z}^d, (X_n))$ be the Banach space of sequences $x = (x_n)_{n \in \mathbb{Z}^d}$, $x_n \in X_n$, with the norm $\|x\|_p = (\sum_{n \in \mathbb{Z}^d} \|x_n\|_{X_n}^p)^{1/p}$ when $p \in [1, \infty)$ and $\|x\|_\infty = \sup_{n \in \mathbb{Z}^d} \|x_n\|_{X_n}$. By $\mathfrak{c}_0 = \mathfrak{c}_0(\mathbb{Z}^d, (X_n))$ we denote the subspace of $\mathfrak{L}^\infty$ of sequences vanishing at infinity, that is $\lim_{|n| \to \infty} \|x_n\| = 0$, where $|n| = \max_{1 \leqslant k \leqslant d} |n_k|$, $n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}^d$. We will use this multi-index notation throughout the paper. Note that when $X_n = \mathbb{C}$ for all n, then $\mathfrak{L}^p$ is the standard space of complex-valued sequences $\ell^p(\mathbb{Z}^d)$.

Let $a_{mn} : X_n \to Y_m$ be bounded linear operators. The symbol $\mathbb{A}$ will denote the operator matrix (a_{mn}), $m, n \in \mathbb{Z}^d$. In this paper, we are interested only in those matrices that give rise to bounded linear operators that map $\mathfrak{L}^p$ into $\mathfrak{L}^p$ for all $p \in [1, \infty]$ and $\mathfrak{c}_0$ into $\mathfrak{c}_0$. We let $\|\mathbb{A}\|_p$ be the operator norm of $\mathbb{A}$ in $\mathfrak{L}^p = \ell^p(\mathbb{Z}^d, (Y_n))$ and $\|\mathbb{A}\|_{\sup} = \sup_{m,n \in \mathbb{Z}^d} \|a_{mn}\|$. If $X_n, Y_n, n \in \mathbb{Z}^d$, are separable Hilbert spaces, we denote by $\mathbb{A}^\star = (a_{mn}^\star)$ the matrix defined by $a_{mn}^\star = (a_{nm})^*$, where $(a_{nm})^* : Y_n \to X_m$ are the (Hilbert) adjoints of the operators a_{nm}. Clearly, $(\mathbb{A}^\star)^\star = \mathbb{A}$.

Remark 2.1. Most of the applications presented in this paper are restricted to operators on $\ell^p = \ell^p(\mathbb{Z}^d, \mathbb{C})$. The reason we introduce the spaces $\mathfrak{L}^p = \ell^p(\mathbb{Z}^d, (X_n))$ is twofold. First, the method we use to prove the abstract results is not affected by this generalization. Secondly, the spaces $\ell^p(\mathbb{Z}^d, X)$ are better suited for applications to differential equations [13] and the spaces $\ell^p(\mathbb{Z}^d, (X_n))$ can be used to study differential equations with unbounded operator coefficients [14,16].

To define certain classes of operator matrices we use the following types of weight functions.

Definition 2.1. A *weight* is a function $\omega : \mathbb{Z}^d \to [1, \infty)$. A weight is *submultiplicative* if

$$\omega(m+n) \leqslant C\omega(m)\omega(n), \quad \text{for some } C > 0.$$

A weight is a *GRS*-weight if it satisfies the Gel'fand–Raĭkov–Šilov condition [26]

$$\lim_{m \to \infty} \omega(mn)^{\frac{1}{m}} = 1, \quad n \in \mathbb{Z}^d.$$

A weight is *balanced* if

$$\sup_{n\in\mathbb{Z}^d} \frac{\omega(kn)}{\omega(n)} < \infty, \quad k \in \mathbb{N}.$$

Finally, an *admissible* weight is an even submultiplicative weight.

Example 2.1. A typical weight on $\mathbb{Z}^d$ is given by

$$\omega(n) = e^{a|n|^b}\left(1+|n|\right)^s, \quad a, b, s \geqslant 0.$$

This weight is admissible when $b \in [0, 1]$, is a GRS-weight when $b \in [0, 1)$ and is balanced when $b = 0$.

Throughout the paper we fix a slant $\alpha \neq 0$. To simplify the notation we use $\beta = \alpha^{-1}$ and $K = \lceil|\beta|\rceil^d$—the dth power of the smallest integer number bigger than or equal to $|\beta|$. By χ_S we denote the characteristic function of a set S.

Definition 2.2. For $\alpha \neq 0$ and $j \in \mathbb{Z}^d$ the matrix $A_j = A_j^\alpha = (a_{mn}^{(j)})$, $m, n \in \mathbb{Z}^d$, defined by

$$a_{mn}^{(j)} = a_{mn} \prod_{k=1}^{d} \chi_{[j_k, j_k+1)}(\alpha m_k - n_k)$$

is called the *jth α-slant* of $\mathbb{A}$.

Observe that for every $m \in \mathbb{Z}^d$ there is at most one $n \in \mathbb{Z}^d$ such that $a_{mn}^{(j)} \neq 0$ and at most K different numbers $\ell \in \mathbb{Z}^d$ such that $a_{\ell m}^{(j)} \neq 0$. Hence, we have $\|A_j\|_p \leqslant K\|A_j\|_{\sup}$ for any $p \in [1, \infty]$. This allows us to define different classes of matrices with decaying α-slants independently of $p \in [1, \infty]$.

Definition 2.3. We consider the following several classes of matrices.

- For some fixed $M \in \mathbb{N}$, $\mathcal{F}_\alpha^M$ will denote the class of matrices $\mathbb{A}$ that satisfy $\mathbb{A} = \sum_{|j|\leqslant M-1} A_j$. Observe that for $\mathbb{A} \in \mathcal{F}_\alpha^M$ we have $a_{mn} = 0$ as soon as $|n - \alpha m| > M - 1$. The class $\mathcal{F}_\alpha = \bigcup_{M\in\mathbb{N}} \mathcal{F}_\alpha^M$ consists of operators with *finitely many α-slants*.
- The class Σ_α^ω of matrices with *ω-summable* α-slants consists of matrices $\mathbb{A}$ such that $\|\mathbb{A}\|_{\Sigma_\alpha^\omega} = K \sum_{j\in\mathbb{Z}^d} \|A_j\|_{\sup}\omega(j) < \infty$, where ω is a weight. We have $\Sigma_\alpha^\omega \subset \Sigma_\alpha^1 = \Sigma_\alpha$—the class of matrices with (unweighted) summable α-slants.
- The class $\mathcal{E}_\alpha$ of matrices with *exponential decay* of α-slants is defined as a subclass of matrices $\mathbb{A}$ from Σ_α such that for some $C \in \mathbb{R}$ and $\tau \in (0, 1)$ we have $\|A_j\|_{\Sigma_\alpha} \leqslant C\tau^{|j|}$.

For $\mathbb{A} \in \Sigma_\alpha^\omega$, we denote by $\mathbb{A}_M \in \mathcal{F}_\alpha^M$, $M \in \mathbb{N}$, the *truncation* of $\mathbb{A}$, i.e., the matrix defined by $a_{mn}^M = a_{mn}$ when $|n - \alpha m| \leqslant M - 1$ and $a_{mn}^M = 0$ otherwise. Equivalently, $\mathbb{A}_M = \sum_{|j|\leqslant M-1} A_j$, where A_j, $j \in \mathbb{Z}^d$, is the jth α-slant of $\mathbb{A}$. By definition of Σ_α^ω, the operators $\mathbb{A}_M$ converge to $\mathbb{A}$ in the norm $\|\cdot\|_{\Sigma_\alpha^\omega}$.

Remark 2.2. Notice that when $\alpha = d = 1$ we get the usual matrix diagonals as a special case of α-slants studied in this paper. We introduce α-slants, in part, to avoid certain reindexing problems that occur in applications. These kind of problems are treated differently in [8,9,40]. It is also not hard to see that in many cases a slanted matrix can be converted into a conventional banded block matrix using an isomorphism between $\ell^p(\mathbb{Z}^d, X)$ and $\ell^p(\mathbb{Z}^d, X^n)$.

Remark 2.3. The observation preceding Definition 2.3 implies that matrices in $\mathcal{F}_\alpha$ define bounded operators in $\mathfrak{c}_0$, and any $\mathfrak{L}^p$, $p \in [1, \infty]$. Since for any $\mathbb{A} \in \Sigma_\alpha^\omega$ we have $\mathbb{A} = \sum_{j \in \mathbb{Z}^d} A_j$, where the series converges in the norm of Σ_α^ω, the matrices in Σ_α^ω and $\mathcal{E}_\alpha$ also define bounded operators in $\mathfrak{c}_0$, and any $\mathfrak{L}^p$, $p \in [1, \infty]$. Moreover, it is not hard to see that Σ_α^ω is a Banach space with respect to the norm given by $\|\cdot\|_{\Sigma_\alpha^\omega}$ and $\|A\|_p \leqslant \|A\|_{\Sigma_\alpha} \leqslant \|A\|_{\Sigma_\alpha^\omega}$ for every $A \in \Sigma_\alpha^\omega$ and $p \in [1, \infty]$. We also point out that all results in Section 2 that are stated for $\mathfrak{L}^\infty$ also hold for $\mathfrak{c}_0$, although we do not mention it later.

Remark 2.4. There is an obvious one to one and onto correspondence between the matrices in Σ_α^ω and a class of operators on $\mathfrak{L}^p$. In particular, given an operator $\mathbb{B} : \ell^p(\mathbb{Z}^d, (X_n)) \to \ell^p(\mathbb{Z}^d, (Y_n))$, we can define its matrix (b_{mn}) so that $b_{mn} = P_m^Y \mathbb{B} P_n^X$, $m, n \in \mathbb{Z}^d$, is an operator from X_n to Y_m, where P_n^X are given by

$$P_n^X(\dots, x_{n-1}, x_n, x_{n+1}, \dots) = (\dots, 0, x_n, 0, \dots), \quad n \in \mathbb{Z}^d,$$

and P_m^Y, $m \in \mathbb{Z}^d$, are defined in a similar way. Below, we do not distinguish between an operator and its matrix when no confusion may arise.

Moreover, one can define a matrix of an operator on any Banach space given a *resolution of the identity*, which is a family of projections with similar properties as P_n^X, $n \in \mathbb{Z}^d$. We refer to [11,12] for details.

2.2. *Slanted matrices: basic properties*

Here we present some basic properties of slanted matrices that are useful for the remainder of the paper.

Lemma 2.1. *For some $p \in [1, \infty]$ we consider two operators $\mathbb{A} : \ell^p(\mathbb{Z}^d, (Y_n)) \to \ell^p(\mathbb{Z}^d, (Z_n))$ and $\mathbb{B} : \ell^p(\mathbb{Z}^d, (X_n)) \to \ell^p(\mathbb{Z}^d, (Y_n))$ and let ω be a submultiplicative balanced weight.*

- *If $\mathbb{A} \in \mathcal{F}_\alpha$ (Σ_α^ω, or $\mathcal{E}_\alpha$) and $\mathbb{B} \in \mathcal{F}_{\tilde\alpha}$ ($\Sigma_{\tilde\alpha}^\omega$, or $\mathcal{E}_{\tilde\alpha}$) then we have $\mathbb{A}\mathbb{B} \in \mathcal{F}_{\alpha\tilde\alpha}$ ($\Sigma_{\alpha\tilde\alpha}^\omega$, or $\mathcal{E}_{\alpha\tilde\alpha}$).*

If, moreover, Y_n, Z_n, $n \in \mathbb{Z}^d$, are Hilbert spaces, then we have $\mathbb{A}^\star : \ell^p(\mathbb{Z}^d, (Z_n)) \to \ell^p(\mathbb{Z}^d, (Y_n))$ and

- *$\mathbb{A}$ is invertible if and only if $\mathbb{A}^\star$ is invertible*;
- *If $\mathbb{A} \in \mathcal{F}_\alpha$ (Σ_α^ω, or $\mathcal{E}_\alpha$) then $\mathbb{A}^\star \in \mathcal{F}_{\alpha^{-1}}$($\Sigma_{\alpha^{-1}}^\omega$, or $\mathcal{E}_{\alpha^{-1}}$).*

Proof. The last two properties are easily verified by direct computation. For the first one, let $\mathbb{D} = (d_{m,n}) = \mathbb{A}\mathbb{B} = (a_{m,n})(b_{m,n})$, and let $\lceil a \rceil = (\lceil a_1 \rceil, \dots, \lceil a_d \rceil) \in \mathbb{Z}^d$, where $a \in \mathbb{R}^d$ and $\lceil a_k \rceil$ is, as before, the smallest integer greater than or equal to a_k, $k = 1, \dots, d$. We have that

$$\begin{aligned}
\|d_{m,\lceil\alpha\tilde{\alpha}m\rceil+j}\| &\leqslant \sum_{k\in\mathbb{Z}^d} \|a_{m,k}\|\|b_{k,\lceil\alpha\tilde{\alpha}m\rceil+j}\| \\
&= \sum_{k\in\mathbb{Z}^d} \|a_{m,\lceil\alpha m\rceil+k-\lceil\alpha m\rceil}\|\|b_{k,\lceil\tilde{\alpha}k\rceil+\lceil\alpha\tilde{\alpha}m\rceil+j-\lceil\tilde{\alpha}k\rceil}\| \\
&\leqslant \sum_{k\in\mathbb{Z}^d} r\big(k-\lceil\alpha m\rceil\big)s\big(\lceil\alpha\tilde{\alpha}m\rceil+j-\lceil\tilde{\alpha}k\rceil\big) \\
&= \sum_{k\in\mathbb{Z}^d} r(k)s\big(\lceil\alpha\tilde{\alpha}m\rceil+j-\big\lceil\tilde{\alpha}k+\tilde{\alpha}\lceil\alpha m\rceil\big\rceil\big),
\end{aligned}$$

where $r(j)=\|A_j\|_{\sup}$ and $s(j)=\|B_j\|_{\sup}$. For $a,b\in\mathbb{R}$ we have

$$\begin{gathered}
\lceil a\rceil+\lceil b\rceil-1\leqslant\lceil a+b\rceil\leqslant\lceil a\rceil+\lceil b\rceil; \\
\big\lceil|a|b\big\rceil\leqslant\big\lceil|a|\lceil b\rceil\big\rceil\leqslant\big\lceil|a|b\big\rceil+\big\lceil|a|\big\rceil.
\end{gathered}$$

Hence,

$$\|d_{m,\lceil\alpha\tilde{\alpha}m\rceil+j}\|\leqslant\sum_{k\in\mathbb{Z}^d} r(k)s\big(j-\lceil\tilde{\alpha}k\rceil+l\big),$$

where $l=l(\alpha,\tilde{\alpha},m,k)\in\mathbb{Z}^d$ is such that $|l|\leqslant\lceil|\tilde{\alpha}|\rceil+1$.

If $\mathbb{A}\in\mathcal{F}_\alpha$ and $\mathbb{B}\in\mathcal{F}_{\tilde{\alpha}}$, the last inequality immediately implies $\mathbb{D}=\mathbb{A}\mathbb{B}\in\mathcal{F}_{\alpha\tilde{\alpha}}$.

If $\mathbb{A}\in\Sigma_\alpha^\omega$ and $\mathbb{B}\in\Sigma_{\tilde{\alpha}}^\omega$, we use the fact that the weight ω is submultiplicative and balanced to obtain

$$\begin{aligned}
&\sum_{j\in\mathbb{Z}^d}\sup_{m\in\mathbb{Z}^d}\|d_{m,\lceil\alpha\tilde{\alpha}m\rceil+j}\|\omega(j) \\
&\quad\leqslant\sum_{j,k\in\mathbb{Z}^d} r(k)s\big(j-\lceil\tilde{\alpha}k\rceil+l\big)\omega(j) \\
&\quad\leqslant Const\cdot\sum_{j,k\in\mathbb{Z}^d} r(k)\omega(k)s\big(j-\lceil\tilde{\alpha}k\rceil\big)\omega\big(j-\lceil\tilde{\alpha}k\rceil\big)\frac{\omega(\lceil\tilde{\alpha}k\rceil)}{\omega(k)} \\
&\quad\leqslant Const\cdot\|\mathbb{A}\|_{\Sigma_\alpha^\omega}\|\mathbb{B}\|_{\Sigma_{\tilde{\alpha}}^\omega}.
\end{aligned}$$

The case $\mathbb{A}\in\mathcal{E}_\alpha$ and $\mathbb{B}\in\mathcal{E}_{\tilde{\alpha}}$ can be treated in a similar way. Since we will not use this result in the paper, we omit the proof. □

2.3. Main result

The property of (left, right) invertibility of operator matrices in certain operator algebras has been studied extensively by many authors (see [11,27,35] and references therein). The main focus in this paper, however, is on a weaker property of boundedness below (or uniform injectivity). As we show in Section 3 matrices with this property play a crucial role in certain applications.

Definition 2.4. We say that the matrix $\mathbb{A}$ is *bounded below in* $\mathfrak{L}^p$ or, shorter, *p-bb*, if

$$\|\mathbb{A}x\|_p \geqslant \wp_p \|x\|_p, \quad \text{for some } \wp_p > 0 \text{ and all } x \in \mathfrak{L}^p. \tag{2.1}$$

Before we state our main result, we note an important spectral property of slanted matrices given by the following lemma due to Pfander [37] (see also [38]). We include the proof for completeness and since the matrices considered here are more general.

Lemma 2.2. *Assume that $X_n = Y_n$, $n \in \mathbb{Z}^d$, and that all these spaces are finite-dimensional. If $\mathbb{A} \in \Sigma_\alpha$, for some $\alpha > 1$, then 0 is an approximate eigenvalue of $\mathbb{A} : \mathfrak{L}^p \to \mathfrak{L}^p$, $p \in [1, \infty]$. Equivalently, for any $\epsilon > 0$ there exists $x \in \mathfrak{L}^p$ such that $\|x\|_p = 1$ and $\|\mathbb{A}x\|_p \leqslant \epsilon$.*

Proof. Let $\mathbb{A} \in \Sigma_\alpha$. For $\epsilon > 0$ choose M so large that $\|\mathbb{A} - \mathbb{A}_M\|_{\Sigma_\alpha} \leqslant \epsilon$. Since $\alpha > 1$, there exists N_0 such that $N = \lceil \alpha N_0 \rceil \geqslant N_0 + 1$. Let $\mathbb{A}_M^N$ be a matrix with an (i, j)-entry coinciding with that of the truncation matrix $\mathbb{A}_M$ if $|i| \leqslant M + N$, $|j| \leqslant M + N$, and equal to 0 otherwise. We have $\mathbb{A}_M x_M^N = \mathbb{A}_M^N x_M^N$ for every $x_M^N \in \mathfrak{L}^p$ such that $x_M^N(i) = 0$ for $|i| > M + N$. By assumption, the subspace $\mathcal{X}_M^N$ of such vectors is finite-dimensional and, by construction, it is invariant with respect to $\mathbb{A}_M^N$. Observe that we chose N so large that the restriction of $\mathbb{A}_M^N$ to $\mathcal{X}_M^N$ cannot be invertible because its matrix has a zero "row." Hence, for $\mathbb{A}_M^N$, we can find a vector $x_M^N \in \mathcal{X}_M^N$ such that $\|x_M^N\| = 1$ and $\mathbb{A}_M x_M^N = \mathbb{A}_M^N x_M^N = 0$. Thus, for any given $\epsilon > 0$, we can find $x_M^N \in \mathcal{X}$ such that $\|x_M^N\| = 1$, and $\|\mathbb{A}x_M^N\|_p = \|\mathbb{A}x_M^N - \mathbb{A}_M x_M^N\|_p \leqslant \epsilon$. $\square$

We note that without the assumption in Lemma 2.2 that $X_n = Y_n$ the lemma may fail.

The following theorem presents our central theoretical result. We observe that to the best of our knowledge it has not been proved before even in the classical case of the slant $\alpha = 1$.

Theorem 2.3. *Let $s > (d + 1)^2$ and $\omega = (1 + |j|)^s$. Then $\mathbb{A} \in \Sigma_\alpha^\omega$ is p-bb for some $p \in [1, \infty]$ if and only if $\mathbb{A}$ is q-bb for all $q \in [1, \infty]$.*

Remark 2.5. Observe that if $X_n = Y_n$, $n \in \mathbb{Z}^d$, Lemma 2.1 allows us to consider only the case $\alpha > 0$. Indeed, if $\mathbb{A} \in \Sigma_\alpha^\omega$ then $\mathbb{A}^2 \in \Sigma_{\alpha^2}^\omega$ and it is immediate that $\mathbb{A}$ is p-bb if and only if $\mathbb{A}^2$ is p-bb. Another way to see that we can disregard the case $\alpha < 0$ (even when $X_n \neq Y_n$) follows from the fact that the lower bound $\wp_p$ does not change when we permute the "rows" of the matrix $\mathbb{A}$. Indeed, if $\mathbb{A} \in \Sigma_\alpha^\omega$, let $\mathbb{B} = (b_{i,j})$ be defined by $b_{i,j} = a_{-i,j}$. Then $\mathbb{B} \in \Sigma_{-\alpha}^\omega$ and $\mathbb{A}$ is p-bb if and only if $\mathbb{B}$ is p-bb.

Observe, also, that Lemma 2.2 implies that often Theorem 2.3 is vacuous for $\alpha > 1$. Remark 2.2, on the other hand, indicates that sometimes the theorem can be reduced to the case $\alpha = 1$. However, we find such a reduction misleading. Firstly, it does not significantly simplify our proof and, secondly, it can make computing explicit estimates in applications more complicated.

The proof of the theorem is preceded by several technical lemmas and observations below. We begin with a lemma that provides some insight into the intuition behind the proof. We should also mention that our approach is somewhat similar to Sjöstrand's proof of a non-commutative Wiener's lemma [39]. We will discuss Wiener-type lemmas in more detail in the next section.

Let $w^N : \mathbb{R}^d \to \mathbb{R}$, $N > 1$, be a family of window functions such that $0 \leqslant w^N \leqslant 1$, $w^N(k) = 0$ for all $|k| \geqslant N$, and $w^N(0) = 1$. By w_n^N we will denote the translates of w^N, i.e., $w_n^N(t) = w^N(t-n)$, and $W_n^N : \ell^p(\mathbb{Z}^d, X) \to \ell^p(\mathbb{Z}^d, X)$ will be the multiplication operator

$$W_n^N x(k) = w_n^N(k)x(k), \quad x \in \mathfrak{L}^p,\ n \in \mathbb{R}^d.$$

Let $x \in \ell^p(\mathbb{Z}^d, X)$, $p \in [1, \infty]$, and define

$$|||x|||_p^p := \sum_{n\in\mathbb{Z}^d} \sum_{j\in\mathbb{Z}^d} \left\| W_n^N x(j) \right\|_p^p = \sum_{n\in\mathbb{Z}^d} \sum_{|j-n|<N} \left\| W_n^N x(j) \right\|_p^p, \quad p \in [1, \infty),$$

$$|||x|||_\infty := \sup_n \left\| W_n^N x \right\|_\infty.$$

Lemma 2.4. *For any $p \in [1, \infty]$, the norms $\|\cdot\|_p$ and $|||\cdot|||_p$ are equivalent norms on $\mathfrak{L}^p$, and we have*

$$\|x\|_p \leqslant |||x|||_p \leqslant (2N)^{d/p} \|x\|_p, \quad p \in [1, \infty),$$

and

$$\|x\|_\infty = |||x|||_\infty.$$

Proof. For $p = \infty$ the equality is obvious. For $p \in [1, \infty)$, the left-hand side inequality follows from the fact $\|x(n)\|^p \leqslant \sum_{|j-n|\leqslant N} \|W_n^N x(j)\|_p^p$, and by summing over n. For the right-hand side inequality we simply note that

$$\sum_{n\in\mathbb{Z}^d} \sum_{|j-n|<N} \left\| w_n^N(j)x(j) \right\|_p^p \leqslant \sum_{n\in\mathbb{Z}^d} \sum_{|j|<N} \left\| x(j+n) \right\|_p^p \leqslant (2N)^d \|x\|_p^p. \qquad \square$$

The above equivalence of norms will supply us with the crucial inequality in the proof of the theorem. The opposite inequality is due to the following observation.

Remark 2.6. We shall make use of the following obvious relation between the norms in finite-dimensional spaces. For every x in a d-dimensional Euclidean space we have

$$\|x\|_p \geqslant \|x\|_\infty \geqslant d^{-\frac{1}{p}} \|x\|_p \quad \text{for any } p \in [1, \infty). \tag{2.2}$$

At this point we choose our window functions to be the family of Cesàro means $\psi^N : \mathbb{R}^d \to \mathbb{R}$, $N > 1$, defined by

$$\psi^N(k) = \begin{cases} (1 - \frac{|k|}{N}), & |k| < N; \\ 0, & \text{otherwise.} \end{cases}$$

Observe that their translates $\psi_n^N(k) = \psi^N(k-n)$, $n \in \mathbb{R}^d$, satisfy

$$\psi_{\alpha n}^{\alpha N}(k) = \psi_n^N\left(\alpha^{-1} k\right) \tag{2.3}$$

for any $\alpha > 0$. Again, by $\Psi_n^N : \mathfrak{L}^p \to \mathfrak{L}^p$, $N > 1$, we will denote the operator of multiplication

$$\Psi_n^N x(k) = \psi_n^N(k)x(k), \quad x \in \mathfrak{L}^p,\ n \in \mathbb{R}^d.$$

The following lemma presents yet another estimate crucial for our proof. To simplify the notation we let $\beta = \alpha^{-1}$.

Lemma 2.5. *The following estimate holds for any $q \in [1,\infty]$, any $\mathbb{A} \in \Sigma_\alpha = \Sigma_\alpha^1$, and all of its truncations $\mathbb{A}_M \in \mathcal{F}_\alpha^M$, $M \in \mathbb{N}$.*

$$\left\| \mathbb{A}_M \Psi_n^N - \Psi_{\beta n}^{\beta N} \mathbb{A}_M \right\|_q \leqslant \frac{(2M)^{d+1}}{2N} \|\mathbb{A}\|_{\sup} =: \aleph/2. \tag{2.4}$$

Proof. Define $J_k = \{i \in \mathbb{Z}^d\colon |i - \alpha k| \leqslant M-1\}$. Using (2.3), we have

$$\left|\psi_n^N(i) - \psi_{\beta n}^{\beta N}(k)\right| \leqslant \frac{M-1}{N}, \quad \text{for } |i-\alpha k| \leqslant M-1.$$

Observe that for any $y \in \mathfrak{L}^q$ we have

$$\left(\mathbb{A}_M \Psi_n^N y\right)(k) = \sum_{i \in J_k} a_{ki} \psi_n^N(i) y(i), \qquad \left(\Psi_{\beta n}^{\beta N} \mathbb{A}_M y\right)(k) = \psi_{\beta n}^{\beta N}(k) \sum_{i \in J_k} a_{ki} y(i).$$

Now the following easy computation shows that (2.4) is true for $q \in [1,\infty)$:

$$\begin{aligned}
\left\| \left(\mathbb{A}_M \Psi_n^N - \Psi_{\beta n}^{\beta N} \mathbb{A}_M\right) y \right\|_q &= \left(\sum_{k \in \mathbb{Z}^d} \left\| \sum_{i \in J_k} a_{ki} \left(\psi_n^N(i) - \psi_{\beta n}^{\beta N}(k)\right) y(i) \right\|^q \right)^{\frac{1}{q}} \\
&\leqslant \frac{M}{N} \|\mathbb{A}\|_{\sup} \left(\sum_{k \in \mathbb{Z}^d} \left(\sum_{i \in J_k} \|y(i)\| \right)^q \right)^{\frac{1}{q}} \\
&\leqslant \frac{(2M)^{d+1}}{2N} \|\mathbb{A}\|_{\sup} \|y\|_q.
\end{aligned}$$

An obvious modification yields it in the case $q = \infty$. $\square$

Observe that for $\mathbb{A}_M \in \mathcal{F}_\alpha^M$ the commutator studied in the above lemma satisfies

$$\left(\Psi_{\beta n}^{\beta N} \mathbb{A}_M - \mathbb{A}_M \Psi_n^N\right) x = \left(\Psi_{\beta n}^{\beta N} \mathbb{A}_M - \mathbb{A}_M \Psi_n^N\right) P_n^{N+M} x, \tag{2.5}$$

where $\beta = \alpha^{-1}$, $P_n^L x(k) = x(k)$ if $|k-n| \leqslant L$, and $P_n^L x(k) = 0$ otherwise, where $L > 1$. Also observe that for any $p \in [1,\infty]$ and any $L > 1$, we have that

$$\left\| P_n^L x \right\|_p \leqslant 2 \left\| \Psi_n^{2L} x \right\|_p. \tag{2.6}$$

Combining the above facts we obtain the following estimate.

Lemma 2.6. *Let $\mathbb{A} \in \Sigma_\alpha$ be p-bb for some $p \in [1,\infty]$. As usual, let $\mathbb{A}_M \in \mathcal{F}_\alpha^M$ be the truncations of $\mathbb{A}$ and $\beta = \alpha^{-1}$. Then for all $n \in \mathbb{Z}^d$, $N > 1$, and all $M \in \mathbb{N}$ with $\gamma_p = \wp_p - \|\mathbb{A} - \mathbb{A}_M\|_p > 0$, we have*

$$\left\|\Psi_n^N x\right\|_p \leqslant \gamma_p^{-1}\left(\left\|\Psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \aleph\left\|\Psi_n^{2(N+M)}x\right\|_p\right). \tag{2.7}$$

Proof. Observe that

$$\left\|\Psi_n^N x\right\|_p \leqslant \wp_p^{-1}\left(\left\|\left(\mathbb{A}_M\Psi_n^N\right)x\right\|_p + \|\mathbb{A} - \mathbb{A}_M\|_p\left\|\Psi_n^N x\right\|_p\right).$$

Hence, using (2.4), (2.5), and (2.6), we get

$$\begin{aligned}\left(1 - \wp_p^{-1}\|\mathbb{A} - \mathbb{A}_M\|_p\right)\left\|\Psi_n^N x\right\|_p &\leqslant \wp_p^{-1}\left\|\mathbb{A}_M\Psi_n^N x\right\|_p \\ &\leqslant \wp_p^{-1}\left(\left\|\Psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \left\|\left(\Psi_{\beta n}^{\beta N}\mathbb{A}_M - \mathbb{A}_M\Psi_n^N\right)x\right\|_p\right) \\ &\leqslant \wp_p^{-1}\left(\left\|\Psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \left\|\left(\Psi_{\beta n}^{\beta N}\mathbb{A}_M - \mathbb{A}_M\Psi_n^N\right)P_n^{N+M}x\right\|_p\right) \\ &\leqslant \wp_p^{-1}\left(\left\|\Psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \frac{\aleph}{2}\left\|P_n^{N+M}x\right\|_p\right) \\ &\leqslant \wp_p^{-1}\left(\left\|\Psi_{\beta n}^{\beta N}\mathbb{A}_M x\right\|_p + \aleph\left\|\Psi_n^{2(N+M)}x\right\|_p\right),\end{aligned}$$

which yields the desired inequality. □

By iterating (2.7) $j-1$ times we get

Lemma 2.7. *Let $\mathbb{A} \in \Sigma_\alpha$ be p-bb for some $p \in [1,\infty]$. Let $\mathbb{A}_M \in \mathcal{F}_\alpha^M$ be the truncations of $\mathbb{A}$ and $\beta = \alpha^{-1}$. Then for all $n \in \mathbb{Z}^d$, $N > 1$, and $M \in \mathbb{N}$ with $\gamma_p = \wp_p - \|\mathbb{A} - \mathbb{A}_M\|_p > 0$, we have*

$$\left\|\Psi_n^N x\right\|_p \leqslant \gamma_p^{-1}\frac{1 - (\aleph\gamma_p^{-1})^j}{1 - (\aleph\gamma_p^{-1})}\left\|\Psi_{\beta n}^{\beta Z_j}\mathbb{A}_M x\right\|_p + \left(\aleph\gamma_p^{-1}\right)^j\left\|\Psi_n^{Z_{j+1}}x\right\|_p, \tag{2.8}$$

where $Z_j = 2^{j-1}N + (2^j - 2)M$, for $j \geqslant 1$.

To simplify the use of (2.8) we let

$$a_{j,p} := \gamma_p^{-1}\frac{1 - (\aleph\gamma_p^{-1})^j}{1 - (\aleph\gamma_p^{-1})} = \frac{1 - \left(\frac{(2M)^{d+1}\|\mathbb{A}\|_{\sup}}{(\wp_p - \|\mathbb{A} - \mathbb{A}_M\|_p)N}\right)^j}{\wp_p - \|\mathbb{A} - \mathbb{A}_M\|_p - \frac{(2M)^{d+1}}{N}\|\mathbb{A}\|_{\sup}} \tag{2.9}$$

and

$$b_{j,p} := \left(\aleph\gamma_p^{-1}\right)^j = \frac{((2M)^{d+1}\|\mathbb{A}\|_{\sup})^j}{(\wp_p - \|\mathbb{A} - \mathbb{A}_M\|_p)^j N^j}. \tag{2.10}$$

Now we are ready to complete the proof of the main result.

Proof of Theorem 2.3. The remainder of the proof will be presented in two major steps. In the first step, we will assume that $\mathbb{A} \in \Sigma_\alpha^\omega$ is ∞-bb and show that this implies that $\mathbb{A}$ is p-bb for all $p \in [1, \infty)$. In the second step we will do the "opposite," that is, assume that $\mathbb{A} \in \Sigma_\alpha^\omega$ is p-bb for some $p \in [1, \infty)$ and show that this implies that $\mathbb{A}$ is ∞-bb. This would obviously be enough to complete the proof.

Step 1. Assume that $\mathbb{A}$ is ∞-bb. Using Hölder's inequality and (2.8), we get for large values of $M \in \mathbb{N}$

$$\left\|\Psi_n^N x\right\|_\infty^p \leqslant 2^{p-1} a_{j,\infty}^p \left\|\Psi_{\beta n}^{\beta Z_j} \mathbb{A}_M x\right\|_\infty^p + 2^{p-1} b_{j,\infty}^p \left\|\Psi_n^{Z_{j+1}} x\right\|_\infty^p.$$

Using (2.2), we get

$$(2N)^{-d} \left\|\Psi_n^N x\right\|_p^p \leqslant 2^{p-1} a_{j,\infty}^p \left\|\Psi_{\beta n}^{\beta Z_j} \mathbb{A}_M x\right\|_p^p + 2^{p-1} b_{j,\infty}^p \left\|\Psi_n^{Z_{j+1}} x\right\|_p^p.$$

Summing over n and using Lemma 2.4, we get

$$\begin{aligned} \|x\|_p^p &\leqslant (2N)^d 2^{p-1} a_{j,\infty}^p (2Z_j)^d \|\mathbb{A}_M x\|_p^p + (2N)^d 2^{p-1} b_{j,\infty}^p (2Z_{j+1})^d \|x\|_p^p \\ &\leqslant N^d 2^{2d+p-1} a_{j,\infty}^p Z_j^d \big(\|\mathbb{A}x\|_p^p + \|\mathbb{A} - \mathbb{A}_M\|_p^p \|x\|_p^p\big) \\ &\quad + N^d 2^{2d+p-1} b_{j,\infty}^p Z_{j+1}^d \|x\|_p^p. \end{aligned} \tag{2.11}$$

At this point we use the assumption $\mathbb{A} \in \Sigma_\alpha^{(1+|j|)^s}$ to get

$$\begin{aligned} \|\mathbb{A} - \mathbb{A}_M\|_p &\leqslant \left\| \sum_{|j| \geqslant M} A_j \right\|_p \leqslant K \sum_{|j| \geqslant M} \|A_j\|_{\sup} \big(1 + |j|\big)^s \big(1 + |j|\big)^{-s} \\ &\leqslant \|\mathbb{A}\|_{\Sigma_\alpha^{(1+|j|)^s}} \sup_{|j| \geqslant M} \big(1 + |j|\big)^{-s} \leqslant \|\mathbb{A}\|_{\Sigma_\alpha^{(1+|j|)^s}} M^{-s}. \end{aligned}$$

Plugging the above estimate into (2.11) we obtain

$$\begin{aligned} \|x\|_p^p &\leqslant 2^{2d+p-1} N^d a_{j,\infty}^p Z_j^d \|\mathbb{A}x\|_p^p \\ &\quad + 2^{2d+p-1} N^d \big(a_{j,\infty}^p Z_j^d \|\mathbb{A}\|_{\Sigma_\alpha^{(1+|j|)^s}}^p M^{-sp} + b_{j,\infty}^p Z_{j+1}^d\big) \|x\|_p^p \\ &= 2^{2d+p-1} a_{j,\infty}^p N^d Z_j^d \|\mathbb{A}x\|_p^p + \widetilde{\aleph} \|x\|_p^p. \end{aligned} \tag{2.12}$$

Hence, to complete Step 1 it suffices to show that one can choose $j, M \in \mathbb{N}$ and $N > 1$ so that $\widetilde{\aleph} < 1$.

We put $N = M^{\delta(d+1)}$ for some $\delta > 1$. From (2.4), (2.9), (2.10), and the definition of Z_j in Lemma 2.7 we get $\aleph = \mathcal{O}(M^{(1-\delta)(d+1)})$, $b_{j,\infty} = \mathcal{O}(M^{(1-\delta)(d+1)j})$, $a_{j,\infty} = \mathcal{O}(1)$, and $Z_j = \mathcal{O}(M^{\delta(d+1)})$ as $M \to \infty$. Hence,

$$\widetilde{\aleph} \leqslant C_1 M^{\delta(d+1)^2 - sp} + C_2 M^{(1-\delta)(d+1)jp + \delta(d+1)^2},$$

where the constants C_1 and C_2 depend on $\mathbb{A}$, s, j, and p but do not depend on M. Since $s > (d+1)^2$, we can choose $\delta \in (1, \frac{sp}{(d+1)^2})$ and $j > \frac{\delta(d+1)}{p(\delta-1)}$. Then, clearly, $\widetilde{\aleph} = \mathcal{O}(1)$ as $M \to \infty$.

Step 2. Now assume that $\mathbb{A}$ is p-bb, for some $p \in [1, \infty)$. Using (2.2) and (2.8), we get

$$\left\|\Psi_n^N x\right\|_\infty \leqslant a_{j,p}(2Z_j)^{d/p}\left\|\Psi_{\beta n}^{\beta Z_j}\mathbb{A}_M x\right\|_\infty + b_{j,p}(2Z_{j+1})^{d/p}\left\|\Psi_n^{Z_{j+1}} x\right\|_\infty.$$

As in Step 1, we have $\|\mathbb{A} - \mathbb{A}_M\|_p \leqslant \|\mathbb{A}\|_{\Sigma_\alpha^{(1+|j|)^s}} M^{-s}$. Using this estimate and Lemma 2.4, we obtain

$$\|x\|_\infty \leqslant a_{j,p}(2Z_j)^{d/p}\|\mathbb{A}x\|_\infty + 2^{d/p}\left(a_{j,p}Z_j^{d/p}\|\mathbb{A}\|_{\Sigma_\alpha^{(1+|j|)^s}} M^{-s} + b_{j,p}Z_{j+1}^{d/p}\right)\|x\|_\infty.$$

Again, as in the previous step, if we choose $\delta \in (1, \frac{sp}{(d+1)^2})$, $N = M^{\delta(d+1)}$, and $j > \frac{\delta(d+1)}{p(\delta-1)}$, we get

$$a_{j,p}Z_j^{d/p}\|\mathbb{A}\|_{\Sigma_\alpha^{(1+|j|)^s}} M^{-s} + b_{j,p}Z_{j+1}^{d/p} = \mathcal{O}(1)$$

as $M \to \infty$ and the proof is complete. $\square$

Careful examination of (2.12) yields the following result.

Corollary 2.8. *Let $s > (d+1)^2$, $\omega = (1+|j|)^s$, and $\mathbb{A} \in \Sigma_\alpha^\omega$ be p-bb for some $p \in [1, \infty]$. Then there exists $\wp > 0$ such that for all $q \in [1, \infty]$*

$$\|\mathbb{A}x\|_q \geqslant \wp\|x\|_q, \quad \textit{for all } x \in \mathfrak{L}^q.$$

As we have seen in the proof above, the group structure of the index set $\mathbb{Z}^d$ has not been used. Thus, it is natural to conjecture that a similar result holds for matrices indexed by much more general (discrete) metric spaces. In this paper, however, we do not pursue this extension. Instead, we prove the result for a class of matrices that define operators of *bounded flow*.

Definition 2.5. A matrix $\mathbb{A}$ is said to have *bounded dispersion* if there exists $M \in \mathbb{N}$ such that for every $m \in \mathbb{Z}^d$ there exists $n_m \in \mathbb{Z}^d$ for which $a_{mn} = 0$ as soon as $|n - n_m| > M$. A matrix $\mathbb{A}$ is said to have *bounded accumulation* if $\mathbb{A}^\star$ has bounded dispersion. Finally, $\mathbb{A}$ is a *bounded flow* matrix if it has both bounded dispersion and bounded accumulation.

Corollary 2.9. *Assume that $\mathbb{A}$ has bounded flow and is p-bb for some $p \in [1, \infty]$. Then $\mathbb{A}$ is q-bb for all $q \in [1, \infty]$.*

Proof. In lieu of the proof it is enough to make the following two observations. First, if a matrix is bounded below then any matrix obtained from the original one by permuting its rows (or columns) is also bounded below with the same bound. Second, if a matrix is bounded below then any matrix obtained from the original one by inserting any number of rows consisting entirely of 0 entries is also bounded below with the same bound. Using these observations we can use row permutations and insertions of zero rows to obtain a slanted matrix in $\mathcal{F}_\alpha^M$ for some $\alpha \in \mathbb{R}$, $|\alpha| > 0$. $\square$

Remark 2.7. The proof of Theorem 2.3 indicates how an explicit bound $\wp_q$ and a universal bound $\wp$ can be obtained in terms of $\wp_p$. We did not compute these bounds because such calculations may be easier and yield better results in specific examples.

2.4. Wiener-type lemma and subspace complementation

The classical Wiener's lemma [42] states that if a periodic function f has an absolutely convergent Fourier series and never vanishes then the function $1/f$ also has an absolutely convergent Fourier series. This result has many extensions (see [7,10–12,27,31,32,34,36,39–41] and references therein), some of which have been used recently in the study of localized frames [8,9,25, 30]. Most of the papers just cited show how Wiener's result can be viewed as a statement about the off-diagonal decay of matrices and their inverses. Using Lemma 2.1 and [12, Theorem 2] we obtain the following result about invertible slanted matrices.

Theorem 2.10. *Let X_n, Y_n, $n \in \mathbb{Z}^d$, be Hilbert spaces and ω be an admissible balanced GRS-weight. If $\mathbb{A} \in \Sigma_\alpha^\omega$ is invertible for some $p \in [1,\infty]$, then $\mathbb{A}$ is invertible for all $q \in [1,\infty]$ and $\mathbb{A}^{-1} \in \Sigma_{\alpha^{-1}}^\omega$. Moreover, if $\mathbb{A} \in \mathcal{E}_\alpha$, then we also have $\mathbb{A}^{-1} \in \mathcal{E}_{\alpha^{-1}}$.*

Proof. First, we observe that $\mathbb{A}^{-1} = (\mathbb{A}^\star\mathbb{A})^{-1}\mathbb{A}^\star$. Second, since Lemma 2.1 implies $\mathbb{A}^\star\mathbb{A} \in \Sigma_1^\omega$ (or $\mathcal{E}_1$), [12, Theorem 2] guarantees that $(\mathbb{A}^\star\mathbb{A})^{-1} \in \Sigma_1^\omega$ (or $\mathcal{E}_1$). Finally, applying Lemma 2.1 once again we get the desired results. □

Remark 2.8. The above result may seem remarkable but Lemma 2.2 shows that in most interesting cases it is vacuous unless $|\alpha| = 1$. The case $\alpha = 1$, however, is standard and to prove the result when $\alpha = -1$ it is enough to recall that if $\mathbb{A} \in \Sigma_{-1}$ then $\mathbb{A}^2 \in \Sigma_1$ by Lemma 2.1 or, if $\mathbb{A}^2$ is not well defined, employ the reindexing trick used in Remark 2.5. The following is a less trivial extension of Wiener's lemma.

Theorem 2.11. *Let $X_n = \mathcal{H}_X$ and $Y_n = \mathcal{H}_Y$ be the same Hilbert (or Euclidean) spaces for all $n \in \mathbb{Z}^d$ and $\mathbb{A} \in \Sigma_\alpha^\omega$ where $\omega(j) = (1+|j|)^s$, $s > (d+1)^2$. Let also $p \in [1,\infty]$.*

(i) *If $\mathbb{A}$ is p-bb, then $\mathbb{A}$ is left invertible for all $q \in [1,\infty]$ and a left inverse is given by $\mathbb{A}^\sharp = (\mathbb{A}^\star\mathbb{A})^{-1}\mathbb{A}^\star \in \Sigma_{\alpha^{-1}}^\omega$.*
(ii) *If $\mathbb{A}^\star$ is p-bb, then $\mathbb{A}$ is right invertible for all $q \in [1,\infty]$ and a right inverse is given by $\mathbb{A}^\flat = \mathbb{A}^\star(\mathbb{A}\mathbb{A}^\star)^{-1} \in \Sigma_{\alpha^{-1}}^\omega$.*

Proof. Since (i) and (ii) are equivalent, we prove only (i). Theorem 2.3 implies that $\|\mathbb{A}x\|_2 \geqslant \wp_2\|x\|_2$ for some $\wp_2 > 0$ and all $x \in \mathfrak{L}^2$. Under the specified conditions the Banach spaces $\ell^2(\mathbb{Z}^d, (X_n))$ and $\ell^2(\mathbb{Z}^d, (Y_n))$ are, however, Hilbert spaces and $\mathbb{A}^\star$ defines the Hilbert adjoint of $\mathbb{A}$. Since $\langle \mathbb{A}^\star\mathbb{A}x, x\rangle = \langle \mathbb{A}x, \mathbb{A}x\rangle \geqslant \wp_2\langle x, x\rangle$, we have that the operator $\mathbb{A}^\star\mathbb{A}$ is invertible in $\mathfrak{L}^2$. It remains to argue as in Theorem 2.10 and apply Lemma 2.1 and [12, Theorem 2 and Corollary 3]. □

Corollary 2.12. *If $\mathbb{A}$ is as in Theorem* 2.11(i) *then, for any $q \in [1,\infty]$,* $\operatorname{Im}\mathbb{A}$ *is a subspace of $\mathfrak{L}^q$ that can be complemented.*

3. Applications

In this section we will address several fundamental questions. Given a sampling set for some $p \in [1,\infty]$ can we deduce that this set is a set of sampling for all p? Under which conditions is a p-frame for some $p \in [1,\infty]$ also a Banach frame for all p? These and a few other questions are discussed in this section and an answer in terms of slanted matrices is presented.

The first part of the section concerns Banach frames and the second one concerns sampling theory.

3.1. Banach frames

The notion of a frame in a separable Hilbert space has already become classical. The pioneering work [22] explicitly introducing it was published in 1952. Its analogues in Banach spaces, however, are non-trivial (see [5,8,9,20,25,30] and references therein). In this subsection we show that in the case of certain localized frames the simplest possible extension of the definition remains meaningful.

Definition 3.1. Let $\mathcal{H}$ be a separable Hilbert space. A sequence $\varphi_n \in \mathcal{H}$, $n \in \mathbb{Z}^d$, is a *frame* for $\mathcal{H}$ if for some $0 < a \leqslant b < \infty$

$$a\|f\|^2 \leqslant \sum_{n\in\mathbb{Z}^d} \left|\langle f,\varphi_n\rangle\right|^2 \leqslant b\|f\|^2 \tag{3.1}$$

for all $f \in \mathcal{H}$.

The operator $T : \mathcal{H} \to \ell^2$, $Tf = \{\langle f,\varphi_n\rangle\}_{n\in\mathbb{Z}^d}$, $f \in \mathcal{H}$, is called an *analysis operator*. It is an easy exercise to show that a sequence $\varphi_n \in \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if its analysis operator has a left inverse. The adjoint of the analysis operator, $T^* : \ell^2 \to \mathcal{H}$, is given by $T^*c = \sum_{n\in\mathbb{Z}^d} c_n\varphi_n$, $c = (c_n) \in \ell^2$. The *frame operator* is $T^*T : \mathcal{H} \to \mathcal{H}$, $T^*Tf = \sum_{n\in\mathbb{Z}^d}\langle f,\varphi_n\rangle\varphi_n$, $f \in \mathcal{H}$. Again, a sequence $\varphi_n \in \mathcal{H}$ is a frame for $\mathcal{H}$ if and only if its frame operator is invertible. The canonical dual frame $\tilde\varphi_n \in \mathcal{H}$ is then $\tilde\varphi_n = (T^*T)^{-1}\varphi_n$ and the (canonical) *synthesis operator* is $T^\sharp : \ell^2 \to H$, $T^\sharp = (T^*T)^{-1}T^*$, so that

$$f = T^\sharp Tf = \sum_{n\in\mathbb{Z}^d}\langle f,\varphi_n\rangle\tilde\varphi_n = \sum_{n\in\mathbb{Z}^d}\langle f,\tilde\varphi_n\rangle\varphi_n$$

for all $f \in \mathcal{H}$.

Generalizing the notion of frames to Banach spaces requires some care. In general Banach spaces one cannot use just the equivalence of norms similar to (3.1). The above construction breaks down because, in this case, the analysis operator ends up being bounded below and not necessarily left invertible. As a result a "frame decomposition" remains possible but "frame reconstruction" no longer makes sense. Theorem 2.11(i) indicates, however, that often this obstruction does not exist. The idea of this section is to make the previous statement precise. To simplify the exposition we remain in the realm of Banach spaces $\ell^p(\mathbb{Z}^d,\mathcal{H})$ and use other chains of spaces such as the one in [30] only implicitly.

Definition 3.2. A sequence $\varphi^n = (\varphi^n_m)_{m\in\mathbb{Z}^d} \in \ell^1(\mathbb{Z}^d, \mathcal{H})$, $n \in \mathbb{Z}^d$, is a *p-frame* (for $\ell^p(\mathbb{Z}^d, \mathcal{H})$) for some $p \in [1, \infty)$ if

$$a\|f\|^p \leqslant \sum_{n\in\mathbb{Z}^d}\left|\sum_{m\in\mathbb{Z}^d}\langle f_m, \varphi^n_m\rangle\right|^p \leqslant b\|f\|^p \tag{3.2}$$

for some $0 < a \leqslant b < \infty$ and all $f = (f_m)_{m\in\mathbb{Z}^d} \in \ell^p(\mathbb{Z}^d, \mathcal{H})$. If

$$a\|f\| \leqslant \sup_{n\in\mathbb{Z}^d}\left|\sum_{m\in\mathbb{Z}^d}\langle f_m, \varphi^n_m\rangle\right| \leqslant b\|f\| \tag{3.3}$$

for some $0 < a \leqslant b < \infty$ and all $f = (f_m)_{m\in\mathbb{Z}^d} \in \ell^\infty(\mathbb{Z}^d, \mathcal{H})$, then the sequence φ^n is called an ∞-frame. It is called a 0-frame if (3.3) holds for all $f \in \mathfrak{c}_0(\mathbb{Z}^d, \mathcal{H})$.

The definition of p-frame above is consistent with the standard one found in the literature [5,21,28]. For example, if $\mathcal{H} = \mathbb{C}$ we obtain the standard definition of p-frames for $\ell^p(\mathbb{Z}^d)$.

The operator $T_\varphi = T : \ell^p(\mathbb{Z}^d, \mathcal{H}) \to \ell^p(\mathbb{Z}^d) = \ell^p(\mathbb{Z}^d, \mathbb{C})$, given by

$$Tf = \langle f, \varphi_n\rangle := \left\{\sum_{m\in\mathbb{Z}^d}\langle f_m, \varphi^n_m\rangle\right\}_{n\in\mathbb{Z}^d}, \quad f \in \ell^p\big(\mathbb{Z}^d, \mathcal{H}\big),$$

is called a *p-analysis operator*, $p \in [1, \infty]$. The 0-analysis operator is defined in the same way for $f \in \mathfrak{c}_0(\mathbb{Z}^d, \mathcal{H})$.

Definition 3.3. A p-frame φ^n with the p-analysis operator T, $p \in \{0\} \cup [1, \infty]$, is (s, α)-*localized* for some $s > 1$ and $\alpha \neq 0$, if there exists an isomorphism $J : \ell^\infty(\mathbb{Z}^d, \mathcal{H}) \to \ell^\infty(\mathbb{Z}^d, \mathcal{H})$ which leaves invariant $\mathfrak{c}_0$ and all $\ell^q(\mathbb{Z}^d, \mathcal{H})$, $q \in [1, \infty)$, and such that

$$TJ_{|\ell^p} \in \Sigma^\omega_\alpha,$$

where $\omega(n) = (1 + |n|)^s$, $n \in \mathbb{Z}^d$, see Remark 2.4.

Remark 3.1. If $\mathcal{H}$ is finite-dimensional, then the above definition is vacuous for $|\alpha| > 1$, due to Lemma 2.2.

As a direct corollary of Theorem 2.11 and the above definition we obtain the following result.

Theorem 3.1. *Let φ^n, $n \in \mathbb{Z}^d$, be an (s, α)-localized p-frame for some $p \in \{0\} \cup [1, \infty]$ with $s > (d + 1)^2$. Then*

(i) *The q-analysis operator T is well defined and left invertible for all $q \in \{0\} \cup [1, \infty]$, and the q-synthesis operator $T^\sharp = (T^*T)^{-1}T^*$ is also well defined for all $q \in \{0\} \cup [1, \infty]$.*
(ii) *The sequence φ^n, $n \in \mathbb{Z}^d$, and its dual sequence $\tilde{\varphi}^n = (T^*T)^{-1}\varphi^n$, $n \in \mathbb{Z}^d$, are both (s, α)-localized q-frames for all $q \in \{0\} \cup [1, \infty]$.*

(iii) *In* $\mathfrak{c}_0$ *and* $\ell^q(\mathbb{Z}^d, \mathcal{H})$, $q \in [1, \infty)$, *we have the reconstruction formula*

$$f = T^\sharp T f = \sum_{n \in \mathbb{Z}^d} \langle f, \varphi_n \rangle \tilde{\varphi}_n = \sum_{n \in \mathbb{Z}^d} \langle f, \tilde{\varphi}_n \rangle \varphi_n.$$

For $f \in \ell^\infty(\mathbb{Z}^d, \mathcal{H})$ *the reconstruction formula remains valid provided the convergence is understood in the weak**-*topology.*

Theorem 3.1(iii) shows that an (s, α)-localized p-frame is a Banach frame for $\mathfrak{c}_0$ and all $\ell^q(\mathbb{Z}^d, \mathcal{H})$, $q \in [1, \infty]$, in the sense of the following definition.

Definition 3.4. (See [29, Definition 13.6.1].) A countable sequence $\{x_n\}_{x_n \in J} \subset X'$ in the dual of a Banach space X is a Banach frame for X if there exist an associated sequence space $X_d(J)$, a constant $C \geqslant 1$, and a bounded operator $R : X_d \to X$ such that for all $f \in X$

$$\frac{1}{C} \|f\|_X \leqslant \big\| \langle f, x_n \rangle \big\|_{X_d} \leqslant C \|f\|_X,$$
$$R\big(\langle f, x_n \rangle_{j \in J}\big) = f.$$

Example 3.1. Following [29], let $g \in \mathcal{S} \subset C^\infty(\mathbb{R}^d)$ be a non-zero window function in the Schwartz class $\mathcal{S}$, and V_g be the short time Fourier transform

$$(V_g f)(x, \omega) = \int_{\mathbb{R}^{2d}} f(t) \overline{g(t - x)} e^{-2\pi i t \cdot \omega} \, dt, \quad x, \omega \in \mathbb{R}^d.$$

Let M^p, $1 \leqslant p \leqslant \infty$, be the modulation spaces of tempered distributions with the norms

$$\|f\|_{M^p} = \bigg(\int_{\mathbb{R}^d} \bigg(\int_{\mathbb{R}^d} \big|V_g f(x, \omega)\big|^p \, dx \bigg) d\omega \bigg)^{1/p}, \quad 1 \leqslant p < \infty,$$
$$\|f\|_{M^\infty} = \|V_g f\|_\infty.$$

It is known that these modulation spaces do not depend on the choice of $g \in \mathcal{S}$ and are isomorphic to $\ell^p(\mathbb{Z}^{2d})$, with isomorphisms provided by the Wilson bases.

Let $g \in M^1$ be a window such that the Gabor system

$$\mathcal{G}(g, a, b) = \big\{ g_{k,n}(x) = e^{-2\pi i (x - ak) \cdot bn} g(x - ak), \ k, n \in \mathbb{Z}^d, \ x \in \mathbb{R}^d \big\}$$

is a *tight Banach frame* for all M^p, $1 \leqslant p \leqslant \infty$. By this we mean that the p-analysis operator $T_{\mathcal{G}} : M^p \to \ell^p$, $T_{\mathcal{G}} f = \{\langle f, g_{k,n} \rangle\}$, is left invertible and the frame operator $T_{\mathcal{G}}^* T_{\mathcal{G}}$ is a scalar multiple of the identity operator for all $p \in [1, \infty]$. Assume that a sequence $\Phi = \{\phi_{i,j}\}_{i,j \in \mathbb{Z}^d}$ of distributions in M^∞ is such that $\{\varphi^{(i,j)} = T_{\mathcal{G}} \phi_{i,j}\}$, $(i, j) \in \mathbb{Z}^{2d}$, is an (s, α)-localized p-frame for some $p \in \{0\} \cup [1, \infty]$, and $s > (d + 1)^2$. Since, by Definition 3.2 $\{\varphi^{(i,j)} = T_{\mathcal{G}} \phi_{i,j}\}$ must be in $\ell^1(\mathbb{Z}^{2d}, \mathbb{C})$, then by [29, Corollary 12.2.8] $\Phi \subset M^1$. Moreover, by Theorem 3.1 we have that

$\{\varphi^{(i,j)}\}$ is an (s,α)-localized q-frame for all $q \in \{0\} \cup [1,\infty]$, and a Banach frame. Finally, since $\mathcal{G}$ is a tight Banach frame for all M^q, $q \in [1,\infty]$, we have that

$$\langle f, \phi_{i,j}\rangle = Const\big\langle T^*_{\mathcal{G}} T_{\mathcal{G}} f, \phi_{i,j}\big\rangle = Const\langle T_{\mathcal{G}} f, T_{\mathcal{G}}\phi_{i,j}\rangle, \quad \text{for all } f \in M^q,$$

and, hence, the frame operator

$$f \mapsto T_{\mathcal{G}} f \mapsto \big\{\langle T_{\mathcal{G}} f, T_{\mathcal{G}}\phi_{i,j}\rangle\big\} \mapsto \big\{\langle f, \phi_{i,j}\rangle\big\} : M^q \to \ell^q\big(\mathbb{Z}^{2d}, \mathbb{C}\big)$$

is left invertible and, therefore, Φ is a Banach frame for all M^q, $q \in [1,\infty]$.

Remark 3.2. A similar example can be produced for more general modulation spaces $M^{p,q}_v$ [29, Chapters 11, 12]. Moreover, the Gabor frame $\mathcal{G}$ can be replaced by any frame in M^1, e.g., a Wilson basis [29, Section 12.3]. We should also mention that a similar result in a special case appears in [23]. The technique employed there, however, uses the frame operator and can be used only if we assume that we start with a Banach frame. If we only assume the p-frame property, the frame operator does not a priori have to be invertible and an interpolation argument similar to that of [23, Corollary 3.7 or Theorem 5.2] cannot be used.

Example 3.2. Here we would like to highlight the role of the slant α in the previous example. Using the same notation as above, let Φ be the frame consisting of two copies of the frame $\mathcal{G}$. Then (renumbering Φ if needed) it is easy to see that the matrix $(\langle \phi_{i,j}, g_{k,n}\rangle)_{(i,j),(k,n)\in\mathbb{Z}^{2d}}$ is $\frac{1}{2}$-slanted. Hence, the slant α serves as a measure of relative redundancy of Φ with respect to $\mathcal{G}$ and a measure of absolute redundancy of Φ if $\mathcal{G}$ is a basis.

Remark 3.3. In the theory of localized frames introduced by K. Gröchenig [30] it is possible to extend a localized (Hilbert) frame to Banach frames for the associated Banach spaces. The technique we developed in Section 2, allows us to start with a localized p-frame and deduce that it is, in fact, a Banach frame for the associated Banach spaces. Slanted matrices provide us with additional information which makes it possible to shift emphasis from the frame operator T^*T to the analysis operator T itself.

3.2. Sampling and reconstruction problems

In this subsection we apply the previous results to handle certain problems in sampling theory. Theorem 3.2 below was the principal motivation for us to prove Theorem 2.3.

The sampling/reconstruction problem includes devising efficient methods for representing a signal (function) in terms of a discrete (finite or countable) set of its samples (values) and reconstructing the original signal from its samples. In this paper we assume that the signal is a function f that belongs to a space

$$V^p(\Phi) = \left\{\sum_{k\in\mathbb{Z}^d} c_k\varphi_k\right\},$$

where $c = (c_k) \in \ell^p(\mathbb{Z}^d)$ when $p \in [1,\infty]$, $c \in \mathfrak{c}_0$ when $p = 0$, and $\Phi = \{\varphi_k\}_{k\in\mathbb{Z}^d} \subset L^p(\mathbb{R}^d)$ is a countable collection of continuous functions. To avoid convergence issues in the definition of $V^p(\Phi)$, we assume that the functions in Φ satisfy the condition

$$m_p\|c\|_{\ell^p} \leqslant \left\| \sum_{k\in\mathbb{Z}^d} c_k\varphi_k \right\|_{L^p} \leqslant M_p\|c\|_{\ell^p}, \quad \text{for all } c \in \ell^p, \tag{3.4}$$

for some $m_p, M_p > 0$ independent of c. This is a p-Riesz basis condition for $p \in [1,\infty) \cup \{0\}$ [5]. Furthermore, we assume that the functions in Φ belong to a Wiener-amalgam space W^1_ω defined as follows.

Definition 3.5. A measurable function φ belongs to W^1_ω for a certain weight ω, if it satisfies

$$\|\varphi\|_{W^1_\omega} = \left(\sum_{k\in\mathbb{Z}^d} \omega(k) \cdot \operatorname{ess\,sup}\bigl\{|\varphi|(x+k)\colon x \in [0,1]^d\bigr\} \right) < \infty. \tag{3.5}$$

When a function φ in W^1_ω is continuous we write $\varphi \in W^1_{0,\omega}$. In many applications $V^p(\Phi)$ is a shift invariant space, that is, $\varphi_k(x) = \varphi(x-k)$, $k \in \mathbb{Z}^d$, for some $\varphi \in W^1_\omega$.

Sampling is assumed to be performed by a countable collection of finite complex Borel measures $\boldsymbol{\mu} = \{\mu_j\}_{j\in\mathbb{Z}^d} \subset \mathcal{M}(\mathbb{R}^d)$. A $\boldsymbol{\mu}$-*sample* is a sequence $f(\boldsymbol{\mu}) = \int f\,d\mu_j$, $j \in \mathbb{Z}^d$. If $f(\boldsymbol{\mu}) \in \ell^p$ and $\|f(\boldsymbol{\mu})\|_{\ell^p} \leqslant C\|f\|_{L^p}$ for all $f \in V^p(\Phi)$, we say that $\boldsymbol{\mu}$ is a (Φ, p)-*sampler*. If a sampler $\boldsymbol{\mu}$ is a collection of Dirac measures then it is called a (Φ, p)-*ideal* sampler. Otherwise, it is a (Φ, p)-*average* sampler.

One of the main goals of sampling theory is to determine when a sampler $\boldsymbol{\mu}$ is *stable*. That is, when f is uniquely determined by its $\boldsymbol{\mu}$-sample and a small perturbation of the sampler results in a small perturbation of $f \in V^p(\Phi)$. The above condition can be formulated as follows [3].

Definition 3.6. A sampler $\boldsymbol{\mu}$ is *stable on* $V^p(\Phi)$ (in other words, $\boldsymbol{\mu}$ is a stable (Φ, p)-sampler) if the bi-infinite matrix $\mathbb{A}^\Phi_\mu$ defined by

$$\bigl(\mathbb{A}^\Phi_\mu c\bigr)(j) = \sum_{k\in\mathbb{Z}^d} \int c_k\varphi_k\,d\mu_j, \quad c \in \ell^p\bigl(\mathbb{Z}^d\bigr),$$

defines a bounded *sampling operator* $\mathbb{A}^\Phi_\mu : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$ which is bounded below in ℓ^p (or p-bb).

We assume that the generator Φ and the sampler $\boldsymbol{\mu}$ are such that the operator $\mathbb{A}^\Phi_\mu$ is bounded on $\mathfrak{c}_0$ and all ℓ^p, $p \in [1,\infty]$; we say that such sampling system $(\Phi, \boldsymbol{\mu})$ is *sparse*. This situation happens, for example, when the generator Φ has sufficient decay at ∞ and the sampler is *separated*. The following theorem is a direct corollary of Theorem 2.11 and the above definitions.

Theorem 3.2. *Assume that* $\omega(n) = (1+|n|)^s$, $n \in \mathbb{Z}^d$, $s > (d+1)^2$, Φ *satisfies* (3.4) *for all* $q \in \{0\} \cup [1,\infty]$, *and* $\boldsymbol{\mu}$ *is a* (Φ, p)-*sampler for every* $p \in [1,\infty]$. *Assume also that the sampling operator* $\mathbb{A}^\Phi_\mu$ *is* p-*bb for some* $p \in \{0\} \cup [1,\infty]$ *and* $\mathbb{A}^\Phi_\mu \in \Sigma^\omega_\alpha$ *for some* $\alpha \neq 0$. *Then* $\boldsymbol{\mu}$ *is a stable sampler on* $V^q(\Phi)$ *for every* $q \in \{0\} \cup [1,\infty]$.

Below we study the case of ideal sampling in shift invariant spaces in greater detail and obtain specific examples of the use of the above theorem. From now on we assume that $\varphi_k(x) = \varphi(x-k)$, $k \in \mathbb{Z}^d$, for some $\varphi \in C \cap W^1_\omega =: W^1_{0,\omega}$.

Definition 3.7. If $\boldsymbol{\mu} = (\mu_j)$ is a stable ideal sampler on $V^p(\Phi)$ and the measures μ_j are supported on $\{x_j\}$, $j \in \mathbb{Z}^d$, then the set $X = \{x_j,\ j \in \mathbb{Z}^d\}$ is called a (stable) *set of sampling* on $V^p(\Phi)$. A set of sampling $X \subset \mathbb{R}^d$ is *separated* if

$$\inf_{j \neq k \in \mathbb{Z}^d} |x_j - x_k| = \delta > 0.$$

A set of sampling $X \subset \mathbb{R}^d$ is *homogeneous* if

$$\#\big\{X \cap [n, n+1)\big\} = M$$

is constant for every $n \in \mathbb{Z}^d$.

We are interested in the homogeneous sets of sampling because of the following result.

Lemma 3.3. *Let $\varphi \in W^1_{0,\omega}$, $\Phi = \{\varphi(\cdot - k)\}$, and $\boldsymbol{\mu} \in \ell^\infty(\mathbb{Z}^d, \mathcal{M}(\mathbb{R}^d))$ be an ideal sampler with a separated homogeneous sampling set X. Then the sampling operator $\mathbb{A}^\Phi_\mu$ belongs to Σ^ω_α for $\alpha = M^{-1}$.*

Proof. Follows by direct computation. □

The following lemma shows that we can restrict our attention to homogeneous sets of sampling without any loss of generality. The intuition behind this result is that we can count each measurement at a point in X not once but finitely many times and still obtain unique and stable reconstructions.

Lemma 3.4. *Let $\mathbb{A}$ be an infinite matrix that defines a bounded operator on ℓ^p, $p \in [1, \infty]$, and $\tilde{\mathbb{A}}$ be a (bounded) operator on ℓ^p obtained from $\mathbb{A}$ by duplicating each row at most M times. Then $\mathbb{A}$ is p-bb if and only if $\tilde{\mathbb{A}}$ is p-bb.*

Proof. The proof for $p < \infty$ follows from the inequalities

$$\|\mathbb{A}x\|_p^p \leqslant \|\tilde{\mathbb{A}}x\|_p^p \leqslant (M+1)\|\mathbb{A}x\|_p^p, \quad x \in \ell^p.$$

For $p = \infty$, we have $\|\mathbb{A}x\|_\infty = \|\tilde{\mathbb{A}}x\|_\infty$, $x \in \ell^\infty$. □

As a direct corollary of Theorems 2.11, 3.2, Lemmas 3.3, 3.4, and Remark 2.3 we obtain the following theorem.

Theorem 3.5. *Let $\omega(n) = (1 + |n|)^s$, $n \in \mathbb{Z}^d$, $s > (d+1)^2$, $\varphi \in W^1_{0,\omega}$, and*

$$a_p \|f\|_{L^p} \leqslant \big\|\big\{f(x_j)\big\}\big\|_{\ell^p} \leqslant b_p \|f\|_{L^p}, \quad \textit{for all } f \in V^p(\Phi),$$

for some $p \in [1, \infty] \cup \{0\}$ and a separated set $X = \{x_j,\ j \in \mathbb{Z}^d\}$. Then X is a stable set of sampling on $V^q(\Phi)$ for all $q \in [1, \infty] \cup \{0\}$.

Now we can prove a Beurling–Landau type theorem [1–4] for shift-invariant spaces generated by piecewise differentiable functions.

Theorem 3.6. *Let Φ be a sequence generated by the translates of a piecewise differentiable function $\varphi \in W^1_{0,\omega}$ such that*

$$a\|c\|_\infty \leqslant \Big\|\sum_{k\in\mathbb{Z}} c_k\varphi_k\Big\|_\infty \leqslant b\|c\|_\infty \quad \text{and} \quad \Big\|\sum_{k\in\mathbb{Z}^d} c_k\varphi_k'\Big\|_\infty \leqslant b'\|c\|_\infty,$$

for all $c \in \mathfrak{c}_0(\mathbb{Z}^d)$. Then every $X = \{x_j\}$ that satisfies $\gamma(X) = \sup(x_{j+1} - x_j) < 2a/b'$ is a set of sampling for $V^p(\Phi)$ for all $p \in \{0\} \cup [1,\infty]$.

Proof. We prove the result for everywhere differentiable functions φ and omit the obvious generalization.

Let $f \in V^0(\Phi)$ be such that $f' = \sum_{k\in\mathbb{Z}^d} c_k\varphi_k'$, where the series has finitely many non-zero terms. The set of such functions is dense in $V^0(\Phi)$ and if we prove that for all such f

$$\big\|\{f(x_j)\}\big\|_\infty = \sup_{j\in\mathbb{Z}^d}\big|f(x_j)\big| \geqslant \wp_\infty\|c\|_\infty,$$

the result would follow immediately from Theorem 3.5.

Let $x^* \in \mathbb{R}$ be such that $\|f\|_\infty = |f(x^*)|$. There exists $j \in J$ such that $|x_j - x^*| \leqslant \frac{1}{2}\gamma(X)$. Using the Fundamental theorem of calculus, we get

$$\begin{aligned}\big|f(x_j)\big| &= \left|\int_{x_j}^{x^*} f'(t)\,dt - f\big(x^*\big)\right| \geqslant \|f\|_\infty - \left|\int_{x_j}^{x^*}\sum_{k\in\mathbb{Z}^d} c_k\varphi_k'(t)\,dt\right| \\ &\geqslant \|f\|_\infty - \left|\int_{x_j}^{x^*}\Big\|\sum_{k\in\mathbb{Z}^d} c_k\varphi_k'\Big\|_\infty dt\right| \geqslant \left(a - \frac{1}{2}b'\gamma(X)\right)\|c\|_\infty.\end{aligned}$$

Since $\gamma(X) < \frac{2a}{b'}$, we have $\wp_\infty > a - \frac{1}{2}b' \cdot \frac{2a}{b'} = 0$. $\square$

Corollary 3.7. *Let Φ be a sequence generated by the translates of a piecewise twice differentiable function $\varphi \in W^1_\omega$ such that*

$$a\|c\|_\infty \leqslant \Big\|\sum_{k\in\mathbb{Z}^d} c_k\varphi_k\Big\|_\infty \leqslant b\|c\|_\infty \quad \text{and} \quad \Big\|\sum_{k\in\mathbb{Z}^d} c_k\varphi_k''\Big\|_\infty \leqslant b''\|c\|_\infty,$$

for all $c \in \mathfrak{c}_0(\mathbb{Z})$. Then every $X = \{x_j\}$ that satisfies $\gamma(X) = \sup(x_{j+1} - x_j) < \sqrt{\frac{8a}{b''}}$ is a set of sampling for $V^p(\Phi)$ for all $p \in \{0\} \cup [1,\infty]$.

Proof. Using the same notation as in the proof of the theorem, we see that $f'(x^*) = 0$ and, therefore,

$$\left|f(x_j)\right| = \left|\int_{x_j}^{x^*} f'(t)\,dt - f\left(x^*\right)\right| \geqslant \|f\|_\infty - \left|\int_{x_j}^{x^*}\int_{t}^{x^*} f''(u)\,du\,dt\right|$$
$$\geqslant \|f\|_\infty - \frac{b''}{2}|x^* - x_j|^2\|c\|_\infty \geqslant \left(a - \frac{1}{8}b''\gamma^2(X)\right)\|c\|_\infty.$$

At this point the statement easily follows. □

In the next two examples we apply the above theorem and its corollary to spaces generated by B-splines $\beta_1 = \chi_{[0,1]} * \chi_{[0,1]}$ and $\beta_2 = \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]}$.

Example 3.3. Let $\varphi = \beta_1$. This function satisfies the conditions of Theorem 3.6 with $a = 1$ and $b' = 2$. Hence, if $\gamma(X) < 1$, we have that X is a set of sampling for $V^0(\varphi)$ with the lower bound $1 - \gamma(X)$. Using the estimates in the proof of Theorem 2.3 one can obtain explicit lower bounds for any $V^p(\varphi)$, $p \in [1,\infty]$, and a universal bound for all $p \in [1,\infty]$ (see Remark 2.7).

Example 3.4. Let $\varphi = \beta_2$. This function satisfies the conditions of Corollary 3.6 with $a = \frac{1}{2}$ and $b'' = 4$. Hence, if $\gamma(X) < 1$, we have that X is a set of sampling for $V^0(\varphi)$ with the lower bound $\frac{1}{2}(1 - \gamma^2(X))$. Again, using the estimates in the proof of Theorem 2.3, one can obtain explicit lower bounds for any $V^p(\varphi)$, $p \in [1,\infty]$, and a universal bound for all $p \in [1,\infty]$.

3.3. Other applications

Slanted matrices have also been studied in wavelet theory and signal processing (see e.g. the book of Bratteli and Jorgensen [18], the papers [17,19,24,33], and references therein). They also occur in K-theory of operator algebras and its applications to topology of manifolds [43]. Our technique may be applied to these situations as well. Finally, our results may be useful in the study of differential equations with unbounded operator coefficients similar to the ones described in [13,14,16].

Acknowledgments

First and foremost we would like to thank K. Gröchenig for his comments on [4] which inspired us to embark on this project. Secondly, we would like to thank all those people that attended our talks on the above results and shared their valuable opinions, to name just a few, R. Balan, C. Heil, P. Jorgensen, G. Pfander, R. Tessera. Finally, we thank the cat Rosie for gracefully allowing us to divert our attention from him and type this paper.

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