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A nonlinear programming approach to kinematic shakedown analysis of frictional materials

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Abstract

This paper develops a novel nonlinear numerical method to perform shakedown analysis of structures subjected to variable loads by means of nonlinear programming techniques and the displacement-based finite element method. The analysis is based on a general yield function which can take the form of most soil yield criteria (e.g. the Mohr–Coulomb or Drucker–Prager criterion). Using an associated flow rule, a general yield criterion can be directly introduced into the kinematic theorem of shakedown analysis without linearization. The plastic dissipation power can then be expressed in terms of the kinematically admissible velocity and a nonlinear formulation is obtained. By means of nonlinear mathematical programming techniques and the finite element method, a numerical model for kinematic shakedown analysis is developed as a nonlinear mathematical programming problem subject to only a small number of equality constraints. The objective function corresponds to the plastic dissipation power which is to be minimized and an upper bound to the shakedown load can be calculated. An effective, direct iterative algorithm is then proposed to solve the resulting nonlinear programming problem. The calculation is based on the kinematically admissible velocity with one-step calculation of the elastic stress field. Only a small number of equality constraints are introduced and the computational effort is very modest. The effectiveness and efficiency of the proposed numerical method have been validated by several numerical examples.

Keywords: Shakedown analysis; Frictional materials; Nonlinear programming; Kinematic theorem; Finite element method

1. Introduction

Shakedown analysis is a direct method to calculate the bearing capacity and the stability condition of an elastoplastic structure subjected to variable loads. This can provide a powerful tool for the engineering design and safety estimation of structures. When a variable load is applied to a structure, three conditions may occur:

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- *Purely elastic*: If the applied load magnitude is lower than the elastic limit (first yield load) of the structure, no plastic deformation occurs and the structure undergoes only a purely elastic deformation.
- *Shakedown*: If the load is larger than the elastic limit but is less than a critical limit, plastic deformation takes place in some part of the structure. However, after a number of load cycles, the plastic deformation will cease to develop further and the structure will respond purely elastically to the remaining load cycles. If this happens, then the structure is said to have shakedown. The critical load limit below which shakedown can occur is known as 'shakedown limit'.
- *Non-shakedown*: If the applied load is higher than the shakedown limit, a non-restricted plastic flow will occur and the structure will collapse due to excessive deformation or low cycle fatigue.

If the non-shakedown condition happens, the structure may undergo the failure mode of either incremental collapse (ratchetting) or alternating plasticity (low cycle fatigue). In the first case, plastic strains and displacement increase with the successive loading cycles until the structure fails. In the case of alternating plasticity, materials at the most stressed points begin to break due to low-cycle fatigue damage and high plastic work density, but the plastic deformation still remains small.

By means of shakedown analysis, the shakedown limit and shakedown condition of a structure subjected to variable loads can be found. Shakedown analysis is based on two fundamental shakedown theorems, the static or lower bound theorem (Melan, 1938) and the kinematic or upper bound theorem (Koiter, 1960). The early research works were mainly focused on the theoretical analysis for simple structures. Due to the complexity of engineering problems, numerical techniques are required for shakedown analysis. Over the last two decades, with the rapid development of computational techniques, the numerical methods of shakedown analysis have been developed rapidly (e.g. Maier, 1969; Weichert, 1984; Morelle, 1986; Weichert and Gross, 1986; Genna, 1988; Stein et al., 1992, 1993; Zhang, 1995; Xue et al., 1997; Feng and Liu, 1997; Hachemi and Weichert, 1998; Carvelli et al., 1999; Weichert et al., 1999a,b; Ponter and Engelhardt, 2000; Maier et al., 2000; Chen and Ponter, 2001; Zouain et al., 2002; Khoi et al., 2004). However, these works are mostly for von Mises' yield criterion.

Sharp and Booker (1984) were among the first to apply shakedown theory to the stability analysis of soil structures. By assuming a plane strain deformation normal to the travel direction, Sharp and Booker found that the problem of pavements under repeated surface loads may be analysed as a one-dimensional shakedown problem. As a result, they developed a semi-analytical approach for determining the shakedown loads. Following this work, Raad et al. (1988) and Radovsky and Murashina (1996) applied the lower bound shakedown analysis to pavement engineering for determining the stability condition, where a two-dimensional (2-D) model was assumed.

Instead of using a lower bound method, Collins and Cliffe (1987), Collins and Wang (1992), and Collins and Boulbibane (2000) adopted a kinematic approach to perform shakedown analysis for soil structures. An upper bound to shakedown limit can be obtained.

Yu and Hossain (1998) and Shiau and Yu (2000) developed a linear programming technique to perform shakedown analysis and successfully applied this method to engineering design of layered pavement. Linear finite elements were used, and stress and strain discontinuities were considered. By means of the proposed method, many practical shakedown charts were produced which can be directly used for engineering design of pavements.

Johnson (1992) studied the influence of residual stresses, strain hardening and geometry on the shakedown process and a design criterion in the form of shakedown maps was presented. By means of the static theorem of shakedown analysis, Boulbibane and Weichert (1997), Hamadouche and Weichert (1999) and Yu (2005) theoretically studied the shakedown condition of soil structures subject to variable loads. All of these theoretical works can be used to benchmark numerical shakedown results.

Recently, Li and Yu (2005) developed a novel nonlinear numerical approach to perform limit analysis for a general yield criterion by means of nonlinear mathematical programming technique and the finite element method. Kinematic limit analysis was finally constructed as a nonlinear mathematical programming problem and an upper bound to the limit load of a structure subjected to static loads can be calculated. The proposed method is based entirely on kinematically admissible velocities without calculation of stress fields and only a single equality constraint is introduced into the nonlinear programming problem. Therefore, the computational effort is very modest. Moreover, the proposed method is based on a general yield criterion which covers most of soil yield criteria (e.g. the Mohr–Coulomb or Drucker–Prager yield criterion).

The purpose of this paper is to extend the nonlinear numerical technique of Li and Yu (2005) for limit analysis to shakedown analysis so that it can be used to evaluate the shakedown limit and shakedown condition of frictional materials under repeated/cyclic loading. A general yield criterion defined by a polynomial with both first and second-order terms is used, which makes the developed method suitable for most of frequently-used yield criteria. By means of an associated plastic flow rule, the dissipation power for the general yield criterion is expressed in terms of kinematically admissible velocity only. Nonlinear yield surfaces do not linearized and then can be directly introduced into the kinematic shakedown analysis. This can significantly reduce the number of constraints and the computational error. Using the nonlinear programming theory and the finite element technique, the numerical model of kinematic shakedown analysis is formulated as a nonlinear mathematical programming problem subject to a small number of equality constraints. The objective function corresponding to the plastic dissipation power is to be minimized and then an upper bound to the shakedown limit of a structure subjected to variable loads can be calculated. An effective, direct iterative algorithm is then developed to solve the resulting programming problem. The efficiency and effectiveness of the proposed method are illustrated by numerical examples.

2. Shakedown analysis based on a general yield criterion

According to shakedown theory, a material model of elastic-perfectly plasticity and an associated flow rule is assumed. For soil materials, most yield criteria can be expressed by a polynomial with both first and second-order terms (e.g. the Mohr–Coulomb and Drucker–Prager criteria). Note that a tensile stress is assumed to be positive in this paper. Moreover, in order to use the finite element method, we adopt column vectors to represent strains and stresses. For example, in a two-dimensional model, $\boldsymbol{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, 2\varepsilon_{12}]^{\mathrm{T}}$, and $\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{12}]^{\mathrm{T}}$, and in a three-dimensional model, $\boldsymbol{\varepsilon} = [\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{23}, 2\varepsilon_{31}]^{\mathrm{T}}$, and $\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^{\mathrm{T}}$.

2.1. A general yield criterion

Many widely used yield criteria for frictional materials can be expressed in a general form as follows:

$$F(\boldsymbol{\sigma}) = \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{P}\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{Q} - 1 = 0 \tag{1}$$

where $F(\boldsymbol{\sigma})$ defines a yield function in terms of strength parameters, $\boldsymbol{P}$ and $\boldsymbol{Q}$ are coefficient matrices and related to the strength properties of the material.

Expression (1) can be regarded as a general yield criterion for frictional materials. For example, the Mohr–Coulomb criterion in plane strain can be expressed as

$$F(\sigma_{ij}) = (\sigma_{xx} - \sigma_{yy})^2 + (2\sigma_{xy})^2 - (2c\cos\varphi - (\sigma_{xx} + \sigma_{yy})\sin\varphi)^2 = 0 \tag{2}$$

where c and φ are the cohesion and the internal friction angle of the material, respectively. It can be shown that the Mohr–Coulomb criterion can be expressed in the form of Eq. (1) with the following relations:

$$\boldsymbol{P} = \begin{bmatrix} \dfrac{1}{4c^2} & \dfrac{-1-\sin^2\varphi}{4c^2\cos^2\varphi} & 0 \\ \dfrac{-1-\sin^2\varphi}{4c^2\cos^2\varphi} & \dfrac{1}{4c^2} & 0 \\ 0 & 0 & \dfrac{1}{c^2\cos^2\varphi} \end{bmatrix} \tag{3}$$

$$\boldsymbol{Q} = \begin{bmatrix} \dfrac{\sin\varphi}{c\cos\varphi} & \dfrac{\sin\varphi}{c\cos\varphi} & 0 \end{bmatrix}^{\mathrm{T}} \tag{4}$$

The Drucker–Prager criterion is also frequently used for frictional materials and can be expressed as

$$F(\sigma) = \varphi_0 I_1 + \sqrt{J_2} - c_0 = 0 \tag{5}$$

where I_1 is the first invariant of stress tensor, J_2 is the second invariant of the deviatoric stress tensor, φ_0 and c_0 are strength parameters of the material. In a general stress state, Eq. (1) can also be used to define the Drucker–Prager criterion under the following conditions:

$$\boldsymbol{P} = \begin{bmatrix} \frac{1-3\varphi_0^2}{3c_0^2} & -\frac{1+6\varphi_0^2}{6c_0^2} & -\frac{1+6\varphi_0^2}{6c_0^2} & 0 & 0 & 0 \\ -\frac{1+6\varphi_0^2}{6c_0^2} & \frac{1-3\varphi_0^2}{3c_0^2} & -\frac{1+6\varphi_0^2}{6c_0^2} & 0 & 0 & 0 \\ -\frac{1+6\varphi_0^2}{6c_0^2} & -\frac{1+6\varphi_0^2}{6c_0^2} & \frac{1-3\varphi_0^2}{3c_0^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{c_0^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{c_0^2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{c_0^2} \end{bmatrix} \tag{6}$$

$$\boldsymbol{Q} = \begin{bmatrix} \frac{2\varphi_0}{c_0} & \frac{2\varphi_0}{c_0} & \frac{2\varphi_0}{c_0} & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}} \tag{7}$$

2.2. Kinematic theorem of shakedown analysis

An upper bound to the shakedown limit of a structure can be obtained using the kinematic theorem of shakedown analysis (Koiter, 1960). The kinematic theorem states:

Shakedown cannot occur for a structure subject to repeated or cyclic loads when the rate of plastic dissipation power is less than the work rate done by the applied tractions and body forces for at least one admissible cycle of plastic strain. In other words, shakedown occurs if the rate of plastic dissipation power exceeds the work rate due to external forces for any admissible cycle of plastic strain.

Since the kinematic theorem of shakedown analysis is based on kinematically admissible plastic strain rates, it can be formulated as follows:

$$\lambda_{\mathrm{sd}} \int_0^T \left(\int_{\Gamma_{\mathrm{t}}} t_i \dot{u}_i^* \,\mathrm{d}\Gamma + \int_V f_i \dot{u}_i^* \,\mathrm{d}v \right) \mathrm{d}t \leqslant \int_0^T \int_V D(\dot{\varepsilon}_{ij}^{\mathrm{p}^*}) \,\mathrm{d}v \,\mathrm{d}t \tag{8}$$

where λ_{sd} is the shakedown load multiplier, t_i is the basic load of surface tractions, f_i is the basic load of body force, t_i and f_i are cyclic over a time interval $[0, T]$, $\dot{u}_i$ is the displacement velocity, $\dot{\varepsilon}_{ij}^{\mathrm{p}}$ is the plastic strain rate, "$D(\dot{\varepsilon}_{ij}^{\mathrm{p}^*})$" denotes a function for the rate of plastic dissipation power in terms of the admissible strain rate $\dot{\varepsilon}_{ij}^{\mathrm{p}^*}$, the superscript "*" stands for a parameter corresponding to the kinematically admissible strain field, Γ_{t} denotes the traction boundary, and V represents the space domain of the structure.

By applying the principle of virtual work to the first term of the left-hand side of Eq. (8), one can obtain

$$\int_0^T \int_{\Gamma_{\mathrm{t}}} t_i \dot{u}_i^* \,\mathrm{d}\Gamma \,\mathrm{d}t = \int_0^T \int_V \sigma_{ij}^{\mathrm{e}} (\dot{\varepsilon}_{ij}^{\mathrm{p}^*} + C_{ijkl} \dot{\rho}_{kl}^{\mathrm{s}^*}) \,\mathrm{d}v \,\mathrm{d}t = \int_0^T \int_V \sigma_{ij}^{\mathrm{e}} \dot{\varepsilon}_{ij}^{\mathrm{p}^*} \,\mathrm{d}v \,\mathrm{d}t \tag{9}$$

where σ_{ij}^{e} is the linear elastic stress response to the current external traction t_i, C_{ijkl} is the elastic compliance tensor, and $\dot{\rho}_{kl}^{\mathrm{s}^*}$ is the residual stress rate associated with the admissible plastic strain rate $\dot{\varepsilon}_{ij}^{\mathrm{p}*}$ and $\dot{\rho}_{kl}^{\mathrm{s}^*}$ is self-equilibrated.

Based on the mathematical programming theory and by applying Eq. (9) to Eq. (8), the kinematic shakedown theorem can be re-expressed as the following programming problem, if the body force is omitted:

$$\left\{\begin{array}{lll} \lambda_{sd} = \min\limits_{\dot{\varepsilon}_{ij}^{p^*},\Delta u_i} \int_0^T \int_V D(\dot{\varepsilon}_{ij}^{p^*})\,dv\,dt & & \\ \text{s.t.} \quad \int_0^T \int_V \sigma_{ij}^{e}\dot{\varepsilon}_{ij}^{p^*}\,dv\,dt = 1 & & \\ \qquad \Delta\varepsilon_{ij}^{p} = \int_0^T \dot{\varepsilon}_{ij}^{p^*}\,dt = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}) & \text{in } V \\ \qquad \Delta u_i = \int_0^T \dot{u}_i\,dt & \text{in } V \\ \qquad \Delta u_i = 0 & \text{on } \Gamma_u \end{array}\right. \tag{10}$$

where $\Delta\varepsilon_{ij}^{p}$ and Δu_i are the cumulative plastic strain and displacement fields at the end of one loading cycle over the time interval $[0, T]$, respectively, and Γ_u denotes the displacement boundary.

Finally the kinematic shakedown analysis of a structure is formulated as the calculation of shakedown multiplier λ_{sd}, with $\lambda_{sd} \cdot t_i$ being shakedown limit of the structure.

2.3. Plastic dissipation power for a general yield criterion

Since that the kinematic shakedown analysis is based on displacement modes, the stress terms need to be expressed in terms of the strain terms (i.e. the plastic dissipation power per unit volume in Eq. (10) should be expressed in terms of strain fields which can be obtained by using the yield criterion and a plastic flow rule). The plastic flow rule determines the direction of the plastic strain rate with the following normality relation:

$$\dot{\varepsilon}_{ij}^{p} = \dot{\mu}\frac{\partial\psi(\sigma_{ij})}{\partial\sigma_{ij}} \tag{11}$$

where $\psi(\sigma_{ij})$ denotes a plastic potential function that resembles the yield function and $\dot{\mu}$ is a non-negative plastic proportionality factor. In the theory of shakedown analysis, the flow rule is assumed to be associated, i.e. $\psi(\sigma_{ij}) = F(\sigma_{ij})$. Therefore, the plastic strain rate can be expressed as

$$\dot{\boldsymbol{\varepsilon}}^{p} = 2\dot{\mu}\boldsymbol{P}\boldsymbol{\sigma} + \dot{\mu}\boldsymbol{Q} \tag{12}$$

By introducing Eq. (12) into the yield criterion (1), the plastic proportionality factor $\dot{\mu}$ can be determined by the following formulation:

$$\dot{\mu} = \sqrt{\frac{(\dot{\boldsymbol{\varepsilon}}^{p})^{T}\boldsymbol{P}^{-1}\dot{\boldsymbol{\varepsilon}}^{p}}{4 + \boldsymbol{Q}^{T}\boldsymbol{P}^{-1}\boldsymbol{Q}}} \tag{13}$$

Then, the plastic dissipation power for the general yield criterion (1) can be expressed as

$$\begin{aligned} D(\dot{\varepsilon}_{ij}^{p}) &= \sigma_{ij}\dot{\varepsilon}_{ij}^{p} = \boldsymbol{\sigma}^{T}\dot{\boldsymbol{\varepsilon}}^{p} = \left(\frac{1}{2\dot{\mu}}\mathbf{P}^{-1}\dot{\boldsymbol{\varepsilon}}^{p} - \frac{1}{2}\mathbf{P}^{-1}\mathbf{Q}\right)^{T}\dot{\varepsilon}^{p} = \frac{1}{2\dot{\mu}}(\dot{\boldsymbol{\varepsilon}}^{p})^{T}\mathbf{P}^{-1}\dot{\boldsymbol{\varepsilon}}^{p} - \frac{1}{2}(\dot{\boldsymbol{\varepsilon}}^{p})^{T}\mathbf{P}^{-1}\mathbf{Q} \\ &= \frac{1}{2}\sqrt{((\dot{\boldsymbol{\varepsilon}}^{p})^{T}\mathbf{P}^{-1}\dot{\boldsymbol{\varepsilon}}^{p})\cdot(4 + \mathbf{Q}^{T}\mathbf{P}^{-1}\mathbf{Q})} - \frac{1}{2}(\dot{\boldsymbol{\varepsilon}}^{p})^{T}\mathbf{P}^{-1}\mathbf{Q} \end{aligned} \tag{14}$$

The more detailed description about how to deduce the plastic dissipation power for a general yield criterion can be found in the research of Li and Yu (2005).

As a result, the kinematic shakedown analysis of a structure modelled by the general yield criterion can be formulated as the following nonlinear mathematical programming problem:

$$\left\{\begin{array}{ll} \lambda_{sd} = \min\limits_{\dot{\varepsilon}_{ij}^{p},\Delta u_i} \int_0^T \int_V \left(\frac{1}{2}\sqrt{\left((\dot{\boldsymbol{\varepsilon}}^{p})^{T}\boldsymbol{P}^{-1}\dot{\boldsymbol{\varepsilon}}^{p}\right)\cdot(4 + \mathbf{Q}^{T}\boldsymbol{P}^{-1}\boldsymbol{Q})} - \frac{1}{2}(\dot{\varepsilon}^{p})^{T}\boldsymbol{P}^{-1}\boldsymbol{Q}\right)dv\,dt & \\ \text{s.t.} \quad \int_0^T \int_V \sigma_{ij}^{e}\dot{\varepsilon}_{ij}^{p}\,dv\,dt = 1 & \\ \qquad \Delta\varepsilon_{ij}^{p} = \int_0^T \dot{\varepsilon}_{ij}^{p}\,dt = \frac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}) & \text{in } V \\ \qquad \Delta u_i = \int_0^T \dot{u}_i\,dt & \text{in } V \\ \qquad \Delta u_i = 0 & \text{on } \Gamma_u \end{array}\right. \tag{15}$$

2.4. Removal of the time integration

In order to apply the mathematical programming formulation (15) to a structure, the time integration must be removed because it would be difficult to calculate the integration along a deformation path. To overcome this potential difficulty, König's technique (König, 1979, 1987) is used in this paper.

Due to cyclic loading, the load domain Ω can be thought of as a load space, the shape of which is a hyper polyhedron defined by a convex linear combination of load vertices $\boldsymbol{P}_k$ $(k = 1, 2, \ldots, l)$. It is assumed that if a structure reaches a state of shakedown under any load vertices, then it will shakedown under the whole load domain Ω. The cyclic loading remains constant over a time interval τ_k $\left(\sum_{k=1}^{l} \tau_k = T\right)$ on each vertex, and the admissible plastic strain cycles on these vertices can generate plastic strain increment

$$\boldsymbol{\varepsilon}_k^{\mathrm{p}} = \int_{\tau_k} \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}}\, \mathrm{d}t \quad (k = 1, 2, \ldots, l) \tag{16}$$

Then the cumulative plastic strain at the end of one loading cycle over the time interval $[0, T]$ can be obtained as follows:

$$\Delta\boldsymbol{\varepsilon}^{\mathrm{p}} = \sum_{k=1}^{l} \boldsymbol{\varepsilon}_k^{\mathrm{p}} \tag{17}$$

Finally, the kinematic shakedown analysis of a structure subject to repeated or cyclic loads can be expressed as the following nonlinear mathematical programming problem:

$$\begin{cases} \lambda_{\mathrm{sd}} = \min\limits_{\boldsymbol{\varepsilon}_k^{\mathrm{p}}, \Delta\boldsymbol{u}} \sum\limits_{k=1}^{l} \int_V \left(\frac{1}{2}\sqrt{\left((\boldsymbol{\varepsilon}_k^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{\varepsilon}_k^{\mathrm{p}}\right) \cdot (4 + \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q})} - \frac{1}{2} (\boldsymbol{\varepsilon}_k^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q} \right) \mathrm{d}v \\ \text{s.t.} \quad \sum\limits_{k=1}^{l} \int_V \left(\boldsymbol{\sigma}_k^{\mathrm{e}}\right)^{\mathrm{T}} \varepsilon_k^{\mathrm{p}}\, \mathrm{d}v = 1 \\ \qquad \Delta\boldsymbol{\varepsilon}^{\mathrm{p}} = \sum\limits_{k=1}^{l} \boldsymbol{\varepsilon}_k^{\mathrm{p}} = \Psi(\Delta\boldsymbol{u}) & \text{in } V \\ \qquad \Delta\boldsymbol{u} = 0 & \text{on } \Gamma_u \end{cases} \tag{18}$$

where Ψ is a linear compatibility differential operator which is defined by Eq. $(15)_3$.

3. Finite element modelling

The displacement-based finite element method is used in this paper to perform the numerical calculation for the kinematic limit analysis. The structure is first discretized into finite elements V_e $(V = \bigcup_{e=1}^{N} V_e)$. Then, the displacement velocity and strain rate fields can be interpolated in terms of an unknown nodal displacement velocity vector

$$\Delta\boldsymbol{u}_e(\boldsymbol{x}) = \boldsymbol{N}_e(\boldsymbol{x}) \Delta\boldsymbol{\delta}_e \tag{19}$$

$$\Delta\boldsymbol{\varepsilon}_e(\boldsymbol{x}) = \boldsymbol{B}_e(\boldsymbol{x}) \Delta\boldsymbol{\delta}_e \tag{20}$$

where, with reference to the eth finite element, $\Delta\boldsymbol{\delta}_e$ is the nodal cumulative displacement column vector over a loading cycle, $\boldsymbol{N}_e$ is the interpolation function, and $\boldsymbol{B}_e$ is the strain function

$$\boldsymbol{B}_e = [\boldsymbol{B}_1, \boldsymbol{B}_2, \ldots, \boldsymbol{B}_m] \tag{21}$$

and

$$
\boldsymbol{B}_i = \begin{bmatrix} \frac{\partial N_i}{\partial x} & 0 & 0 \\ 0 & \frac{\partial N_i}{\partial y} & 0 \\ 0 & 0 & \frac{\partial N_i}{\partial z} \\ \frac{\partial N_i}{\partial y} & \frac{\partial N_i}{\partial x} & 0 \\ 0 & \frac{\partial N_i}{\partial z} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial z} & 0 & \frac{\partial N_i}{\partial x} \end{bmatrix} \quad (i = 1, 2, \ldots, m) \tag{22}
$$

where m is the nodal number of elements.

By using the Gaussian integration technique, the objective function in Eq. (18) can be discretized as follows:

$$
\begin{aligned}
&\sum_{k=1}^{l} \int_V \left(\frac{1}{2}\sqrt{\left((\boldsymbol{\varepsilon}_k^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{\varepsilon}_k^{\mathrm{p}}\right) \cdot \left(4 + \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q}\right)} - \frac{1}{2} (\boldsymbol{\varepsilon}_k^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q} \right) \mathrm{d}v \\
&\quad = \sum_{k=1}^{l} \sum_{e=1}^{N} \int_{V_e} \left(\frac{1}{2}\sqrt{\left((\boldsymbol{\varepsilon}_{ek}^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{\varepsilon}_{ek}^{\mathrm{p}}\right) \cdot \left(4 + \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q}\right)} - \frac{1}{2} (\boldsymbol{\varepsilon}_{ek}^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q} \right) \mathrm{d}v \\
&\quad = \sum_{k=1}^{l} \sum_{e=1}^{N} \sum_{i=1}^{IG} (\rho_e)_i |J|_i \left(\frac{1}{2}\sqrt{\left((\boldsymbol{\varepsilon}_{ek}^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{\varepsilon}_{ek}^{\mathrm{p}}\right) \cdot \left(4 + \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q}\right)} - \frac{1}{2} (\boldsymbol{\varepsilon}_{ek}^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q} \right) \\
&\quad = \sum_{k=1}^{l} \sum_{r=1}^{n} \rho_r |J|_r \left(\frac{1}{2}\sqrt{\left((\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{\varepsilon}_{kr}^{\mathrm{p}}\right) \cdot \left(4 + \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q}\right)} - \frac{1}{2} (\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}})^{\mathrm{T}} \boldsymbol{P}^{-1} \boldsymbol{Q} \right)
\end{aligned} \tag{23}
$$

where, with reference to the rth Gaussian integral point, ρ_r is the Gaussian integral weight, $|J|_r$ is the determinant of the Jacobian matrix, IG is the number of Gaussian integral points of the finite element e, and n is the number of Gaussian integral points of the FE-discretized structure.

After the discretization, the normalization condition in Eq. (18) can be rewritten as follows:

$$
\begin{aligned}
\sum_{k=1}^{l} \int_V (\boldsymbol{\sigma}_k^{\mathrm{e}})^{\mathrm{T}} \boldsymbol{\varepsilon}_k^{\mathrm{p}} \, \mathrm{d}v &= \sum_{k=1}^{l} \sum_{e=1}^{N} \int_{V_e} (\boldsymbol{\sigma}_{ek}^{\mathrm{e}})^{\mathrm{T}} \boldsymbol{\varepsilon}_{ek}^{\mathrm{p}} \, \mathrm{d}v = \sum_{k=1}^{l} \sum_{e=1}^{N} \sum_{i=1}^{IG} (\rho_e)_i |J|_i (\boldsymbol{\sigma}_{ek}^{\mathrm{e}})^{\mathrm{T}} \boldsymbol{\varepsilon}_{ek}^{\mathrm{p}} \\
&= \sum_{k=1}^{l} \sum_{r=1}^{n} \rho_r |J|_r (\boldsymbol{\sigma}_{kr}^{\mathrm{e}})^{\mathrm{T}} \boldsymbol{\varepsilon}_{kr}^{\mathrm{p}} = 1
\end{aligned} \tag{24}
$$

Meanwhile, the geometric compatibility of cumulative plastic strains in Eq. (18) can be re-written as

$$
\Delta \boldsymbol{\varepsilon}_r^{\mathrm{p}} = \sum_{k=1}^{l} \boldsymbol{\varepsilon}_{kr}^{\mathrm{p}} = \boldsymbol{B}_r \Delta \boldsymbol{\delta} \quad (r = 1, 2, \ldots, n) \tag{25}
$$

where $\Delta\boldsymbol{\delta}$ is a global nodal cumulative displacement vector of the discretized structure over a loading cycle and $\boldsymbol{B}_r$ is the strain matrix at the rth Gaussian integral point

$$\boldsymbol{B}_r = \boldsymbol{B}_e \cdot \boldsymbol{C}_e \tag{26}$$

where $\boldsymbol{C}_e$ is the transformation matrix which can assemble the finite element matrix into the global matrix.

Finally, the finite element modelling of the kinematic shakedown analysis for a general yield criterion can be expressed as the following nonlinear programming problem:

$$\begin{cases} \lambda_{\mathrm{sd}} = \min\limits_{\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr},\Delta\boldsymbol{\delta}} \sum\limits_{k=1}^{l}\sum\limits_{r=1}^{n}\rho_r|J|_r\left(\frac{1}{2}\sqrt{\left((\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})^{\mathrm{T}}\mathbf{P}^{-1}\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr}\right)\cdot\left(4+\mathbf{Q}^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{Q}\right)} - \frac{1}{2}(\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{Q}\right) & \text{(a)} \\ \text{s.t.} \quad \sum\limits_{k=1}^{l}\sum\limits_{r=1}^{n}\rho_r|J|_r\left(\boldsymbol{\sigma}^{\mathrm{e}}_{kr}\right)^{\mathrm{T}}\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr} = 1 & \text{(b)} \\ \Delta\boldsymbol{\varepsilon}^{\mathrm{p}}_r = \sum\limits_{k=1}^{l}\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr} = \mathbf{B}_r\Delta\boldsymbol{\delta} \quad (r=1,2,\ldots,n) & \text{(c)} \end{cases} \tag{27}$$

After the displacement boundary condition is imposed by means of the conventional finite element technique, a minimum optimized upper bound λ_{sd} to the shakedown limit multiplier of a structure can be obtained by solving the above mathematical programming problem. The shakedown limit of the structure is then given by $\lambda_{\mathrm{sd}}\boldsymbol{F}$.

4. Iterative solution algorithm

The kinematic shakedown analysis of a structure defined by Eq. (27) is a mathematical programming problem subject to equality constraints. The objective function is nonlinear, continuous but may be non-differentiable which results from the calculation of square root. This causes some difficulties in solving the programming problem. For a linear non-differentiable programming problem, if the objective function is finite and continuous in a feasible set, it is not necessary to be differentiable everywhere and an optimal solution can be obtained (Shapiro, 1979). For a nonlinear programming problem similar to Eq. (27), which was constructed to perform limit and shakedown analyses for the von Mises criterion (Zhang et al., 1991; Zhang and Lu, 1995; Liu et al., 1995; Li et al., 2003), was overcome by means of an iterative algorithm (Zhang et al., 1991), where a technique based on distinguishing rigid/plastic areas was developed. Based on this technique, Li and Yu (2005) developed a general iterative algorithm to solve the nonlinear programming problem for limit analysis of frictional materials. This developed algorithm can be extended to solve the nonlinear mathematical programming problem (27).

4.1. Minimum optimization strategy

According to the mathematical programming theory, an equality constraint can be introduced into an optimization problem by means of the Lagrangean method (Himmelblau, 1972) which is used in this paper to remove the constraints from the normalization condition (27b) and the geometric compatibility (27c). As a result, an unconstrained minimum optimization problem can be obtained as follows:

$$\begin{aligned} L(\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr},\Delta\boldsymbol{\delta},\lambda,\boldsymbol{L}_r) = {} & \sum_{k=1}^{l}\sum_{r=1}^{n}\rho_r|J|_r\left(\frac{1}{2}\sqrt{\left((\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr}\right)\cdot\left(4+\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q}\right)} - \frac{1}{2}(\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q}\right) \\ & + \lambda\left(1-\sum_{k=1}^{l}\sum_{r=1}^{n}\rho_r|J|_r\left(\boldsymbol{\sigma}^{\mathrm{e}}_{kr}\right)^{\mathrm{T}}\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr}\right) + \sum_{r=1}^{n}\boldsymbol{L}^{\mathrm{T}}_r\left(\sum_{k=1}^{l}\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr} - \boldsymbol{B}_r\Delta\boldsymbol{\delta}\right) \end{aligned} \tag{28}$$

where λ and $\boldsymbol{L}_r$ are Lagrangean multipliers.

According to the Kuhn–Tucker stationarity conditions (Himmelblau, 1972), the following formulation can be obtained for solving the kinematic shakedown analysis problem (27) by applying $\frac{\partial L}{\partial \boldsymbol{\varepsilon}^{\mathrm{p}}_{kr}} = \mathbf{0}, \frac{\partial L}{\partial \Delta\boldsymbol{\delta}} = \mathbf{0}, \frac{\partial L}{\partial \lambda} = 0, \frac{\partial L}{\partial \boldsymbol{L}_r} = \mathbf{0}$ to Eq. (28)

$$\begin{cases} \dfrac{\mathbf{P}^{-1}\left(4+\mathbf{Q}^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{Q}\right)}{2\sqrt{\left(\left(\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}}\right)^{\mathrm{T}}\mathbf{P}^{-1}\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}}\right)\cdot\left(4+\mathbf{Q}^{\mathrm{T}}\mathbf{P}^{-1}\mathbf{Q}\right)}}\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}} - \lambda\boldsymbol{\sigma}_{kr}^{\mathrm{e}} + \left(\rho_r|J|_r\right)^{-1}\mathbf{L}_r - \frac{1}{2}\mathbf{P}^{-1}\mathbf{Q} = 0 \quad (k=1,2,\ldots,l;\ r=1,2,\ldots,n) & \text{(a)} \\ \sum_{r=1}^{n}(\mathbf{B}_r^{\mathrm{T}}\mathbf{L}_r) = 0 & \text{(b)} \\ \sum_{k=1}^{l}\sum_{r=1}^{n}\rho_r|J|_r\left(\boldsymbol{\sigma}_{kr}^{\mathrm{e}}\right)^{\mathrm{T}}\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}} = 1 & \text{(c)} \\ \sum_{k=1}^{l}\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}} - \mathbf{B}_r\Delta\boldsymbol{\delta} = 0 \quad (r=1,2,\ldots,n) & \text{(d)} \end{cases} \tag{29}$$

It is quite difficult to directly solve the set of equations (29) because it is nonlinear and also non-differentiable. An iteration technique can be used to overcome this difficulty and it will be discussed in detail in Section 4.3. In order to perform this iteration technique, the set of equations (29) need to be re-expressed as follows:

$$\begin{cases} (\mathbf{H}_{kr})_{\mathrm{ICP}}\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}} - \lambda\boldsymbol{\sigma}_{kr}^{\mathrm{e}} + \left(\rho_r|J|_r\right)^{-1}\mathbf{L}_r - \frac{1}{2}\mathbf{P}^{-1}\mathbf{Q} = \mathbf{0} \quad (k=1,2,\ldots,l;\ r=1,2,\ldots,n) & \text{(a)} \\ \sum_{r=1}^{n}(\mathbf{B}_r^{\mathrm{T}}\mathbf{L}_r) = \mathbf{0} & \text{(b)} \\ \sum_{k=1}^{l}\sum_{r=1}^{n}\rho_r|J|_r(\sigma_{kr}^{\mathrm{e}})^{\mathrm{T}}\varepsilon_{kr}^{\mathrm{p}} = 1 & \text{(c)} \\ \sum_{k=1}^{l}\varepsilon_{kr}^{\mathrm{p}} - \mathbf{B}_r\Delta\delta = 0 \quad (r=1,2,\ldots,n) & \text{(d)} \end{cases} \tag{30}$$

where $\boldsymbol{H}_{kr}$ is the coefficient matrix which is defined by

$$(\boldsymbol{H}_{kr})_{\mathrm{ICP}} = \frac{1}{2}\boldsymbol{P}^{-1}(4+\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q})(\mathbf{z}_{kr})_{\mathrm{ICP}}^{-1} \tag{31}$$

and the subscript 'ICP' indicates that an variable is an iteration control parameter. The parameter z_{kr} is defined by

$$z_{kr} = \sqrt{\left((\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}}\right)\cdot\left(4+\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q}\right)} \tag{32}$$

By solving the set of Eqs. (30), the variable $\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}}$ can be calculated and the shakedown load multiplier can be determined as

$$\lambda_{\mathrm{sd}} = \sum_{k=1}^{l}\sum_{r=1}^{n}\rho_r|J|_r\left(\frac{1}{2}\sqrt{\left((\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}}\right)\cdot\left(4+\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q}\right)} - \frac{1}{2}(\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q}\right) \tag{33}$$

4.2. Iterative strategy

Although the unknown fields can be determined by solving the set of Eqs. (30), it is quite difficult to directly solve Eq. (30) because the equations are nonlinear and not smooth. This is because that the objective function in the kinematic shakedown analysis (27) which corresponds to the plastic dissipation power, is nonlinear and not smooth. To overcome the difficulties which may arise from an unsmooth objective function (27a), all of non-differentiable areas need to be identified where the first part of the plastic dissipation power becomes zero (i.e. $z_{kr} = \sqrt{((\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}_{kr}^{\mathrm{p}})\cdot(4+\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q})} = 0$). All non-differentiable areas can be found by an iterative technique and will be regarded as a constraint introduced into the mathematical programming by means of the penalty function method. The iteration starts with the hypothesis that there is no non-differentiable area in the whole structure. By means of a step-by-step technique, all non-differentiable areas will finally be found. The detailed iterative solution algorithm is described as follows:

- *Step* 0: initializing the nonlinear objective functionThe iteration starts from the hypothesis that the plastic strain rate is non-zero everywhere in the structure and the non-differentiable area does not exist at this step, which can guarantee the iterative process will monotonically decrease towards the exact solution (Huh and Yang, 1991; Li and Yu, 2005). The iteration seed can be chosen as

$$(z_{kr})_0 = 1 \quad (k = 1,2,\ldots,l;\, r = 1,2,\ldots,n) \tag{34}$$

where the subscript "0" denotes that the variables is determined at step 0. Then one can obtain

$$(\boldsymbol{H}_{kr})_0 = \frac{1}{2}\boldsymbol{P}^{-1}(4 + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q}) \tag{35}$$

Accordingly, the set of Eqs. (30) becomes linear and the objective variable $(\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})_0$ can be calculated at this step. Then, the initial shakedown load multiplier $(\lambda_{\mathrm{sd}})_0$ can be determined by the following formulation:

$$(\lambda_{\mathrm{sd}})_0 = \sum_{k=1}^{l}\sum_{r=1}^{n}\rho_r|J|_r\left(\frac{1}{2}\sqrt{\left((\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})_0^{\mathrm{T}}\boldsymbol{P}^{-1}(\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})_0\right)\cdot(4 + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q})} - \frac{1}{2}(\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})_0^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q}\right) \tag{36}$$

- *Step* h + 1 $(h = 0,1,2,\ldots)$: distinguishing the non-differentiable areas to revise the objective functionBased on the computational results at the iteration step h, the value of z_{kr} ($z_{kr} = \sqrt{((\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})\cdot(4 + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q})}$) needs to be calculated at every Gaussian integral point to check whether it is in a non-differentiable. Then the Gaussian integral point set I will be subdivided into two subsets: a subset $(I_E)_{h+1}$ where the objective function is not differentiable, and a subset $(I_P)_{h+1}$ where the objective function is differentiable

$$I = (I_E)_{h+1} \cup (I_P)_{h+1} \tag{37a}$$

$$(I_E)_{h+1} = \left\{ r \in I, \sqrt{\left((\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})_h^{\mathrm{T}}\boldsymbol{P}^{-1}(\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})_h\right)\cdot(4 + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q})} = 0 \right\} \tag{37b}$$

$$(I_P)_{h+1} = \left\{ r \in I, \sqrt{\left((\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})_h^{\mathrm{T}}\boldsymbol{P}^{-1}(\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})_h\right)\cdot(4 + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q})} \neq 0 \right\} \tag{37c}$$

It should be mentioned that the subsets $(I_E)_{h+1}$ and $(I_P)_{h+1}$ will automatically change during the iteration. At the beginning of the iteration (*Step* 0), the subset $(I_P)_{h+1}$ is set equal to I while the subset $(I_E)_{h+1}$ is set equal to a null set. Moreover, considering that there is a limitation of storage for a computer and that any attempt to evaluate the gradient of a square root near a zero argument would cause computational overflow, a small real number ζ $(\zeta \to 0)$ is needed in a computer program to distinguish the differentiable and non-differentiable regions. In other words, a region with $z_{kr} = \sqrt{((\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})^{\mathrm{T}}\boldsymbol{P}^{-1}\varepsilon^{\mathrm{p}}_{kr})\cdot(4 + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q})} < \zeta$ can be regarded as non-differentiable.

Once a non-differentiable region is found, the objective function in Eq. (27) will be modified by removing the calculation for this non-differentiable region. The constraint, that the value of z_{kr} ($z_{kr} = \sqrt{\left((\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr}\right)}\cdot(4 + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q})$) is equal to zero, can be introduced into the mathematical programming problem by the penalization function method. Then, the coefficient matrix $\boldsymbol{H}_{kr}$ at this iteration step will be updated as

$$(\boldsymbol{H}_{kr})_{h+1} = \begin{cases} \beta\boldsymbol{P}^{-1} & r \in (I_E)_{h+1} \\ \frac{1}{2}\boldsymbol{P}^{-1}(4 + \boldsymbol{Q}^{\mathrm{T}}\boldsymbol{P}^{-1}\boldsymbol{Q})(z_{kr})^{-1}_{\mathrm{ICP}} & r \in (I_P)_{h+1} \end{cases} \tag{38}$$

where β is the penalization factor which is used to introduce the non-differentiable area as a constraint into the programming problem. In practice, the typical value of β is from 10^6 to 10^{12}.

By solving the linearizied set of Eqs. (30), the objective variables $(\boldsymbol{\varepsilon}^{\mathrm{p}}_{kr})_{h+1}$ can be calculated. Then the shakedown load multiplier $(\lambda_{\mathrm{sd}})_{h+1}$ can be determined by

$$(\lambda_{\rm sd})_{h+1}=\sum_{k=1}^{l}\sum_{r=1}^{n}\rho_r|J|_r\left(\frac{1}{2}\sqrt{((\boldsymbol{\varepsilon}^{\rm p}_{kr})^{\rm T}_{h+1}\boldsymbol{P}^{-1}(\boldsymbol{\varepsilon}^{\rm p}_{kr})_{h+1})\cdot(4+\boldsymbol{Q}^{\rm T}\boldsymbol{P}^{-1}\boldsymbol{Q})}-\frac{1}{2}(\boldsymbol{\varepsilon}^{\rm p}_{kr})^{\rm T}_{h+1}\boldsymbol{P}^{-1}\boldsymbol{Q}\right) \tag{39}$$

The above iterative process is repeated until the following convergence criteria are satisfied:

$$\begin{cases}\dfrac{|(\lambda_{\rm sd})_{h+1}-(\lambda_{\rm sd})_h|}{|(\lambda_{\rm sd})_{h+1}|}\leqslant\eta_1\\[2mm] \dfrac{\|\boldsymbol{\Delta\delta}_{h+1}-\boldsymbol{\Delta\delta}_h\|}{\|\boldsymbol{\Delta\delta}_{h+1}\|}\leqslant\eta_2\end{cases} \tag{40a, b}$$

where η_1 and η_2 are computational error tolerances.

The above iterative process leads to the shakedown load multiplier $\lambda_{\rm sd}$ through a monotonically decreasing convergence sequence and a minimum optimal upper bound to the shakedown multiplier can be obtained.

4.3. Solution of linearized equations

By means of the foregoing iteration technique, the set of Eqs. (30) is linearized at each step of iteration and then solved. However, the linearized set of equations cannot be directly solved to obtain the values of all variables because it is involved in solving a set of implicit equations. Additional manipulations are needed to eliminate the difficulty from the implicit feature. Based on the linearization by means of the proposed iterative algorithm, the set of Eqs. (30) can be solved by the following strategy:

(a) Subtract the equation sets (30a) corresponding to a vertex, say m, from all the other equations to a vertex (k) to obtain

$$\boldsymbol{\varepsilon}^{\rm p}_{kr}=(\mathbf{H}_{kr})^{-1}_{\rm ICP}\{\lambda\boldsymbol{\sigma}^{\rm e}_{kr}-\lambda\boldsymbol{\sigma}^{\rm e}_{mr}+(\mathbf{H}_{mr})_{\rm ICP}\boldsymbol{\varepsilon}^{\rm p}_{mr}\} \tag{41}$$

(b) Substitute Eq. (41) into Eq. (30d), then the latter becomes

$$\boldsymbol{\varepsilon}^{\rm p}_{mr}=(\mathbf{H}_{mr})^{-1}_{\rm ICP}\left(\sum_{k=1}^{l}(\mathbf{H}_{kr})^{-1}_{\rm ICP}\right)^{-1}\left\{\boldsymbol{B}_r\boldsymbol{\Delta\delta}+\lambda\sum_{k=1}^{l}(\mathbf{H}_{kr})^{-1}_{\rm ICP}(\boldsymbol{\sigma}^{\rm e}_{mr}-\boldsymbol{\sigma}^{\rm e}_{kr})\right\} \tag{42}$$

(c) Substitute Eq. (42) into Eq. (30a) first and subsequently, this into Eq. (30b) to obtain

$$\left\{\sum_{r=1}^{n}\rho_r|J|_r\boldsymbol{B}_r^{\rm T}\left(\sum_{k=1}^{l}(\boldsymbol{H}_{kr})^{-1}_{\rm ICP}\right)^{-1}\boldsymbol{B}_r\right\}\boldsymbol{\Delta\delta}-\frac{1}{2}\sum_{r=1}^{n}\rho_r|J|_r\boldsymbol{B}_r^{\rm T}\boldsymbol{P}^{-1}\boldsymbol{Q}$$
$$=\lambda\sum_{r=1}^{n}\rho_r|J|_r\boldsymbol{B}_r^{\rm T}\left(\sum_{k=1}^{l}(\boldsymbol{H}_{kr})^{-1}_{\rm ICP}\right)^{-1}\sum_{k=1}^{l}(\boldsymbol{H}_{kr})^{-1}_{\rm ICP}\boldsymbol{\sigma}^{\rm e}_{kr} \tag{43}$$

(d) Substitute Eq. (42) into Eq. (30c) which thus becomes

$$\sum_{k=1}^{l}\sum_{r=1}^{n}\rho_r|J|_r(\boldsymbol{\sigma}^{\rm e}_{kr})^{\rm T}(\boldsymbol{H}_{kr})^{-1}_{\rm ICP}\left(\sum_{m=1}^{l}(\boldsymbol{H}_{mr})^{-1}_{\rm ICP}\right)^{-1}\cdot\left\{\boldsymbol{B}_r\boldsymbol{\Delta\delta}+\lambda\sum_{m=1}^{l}(\boldsymbol{H}_{mr})^{-1}_{\rm ICP}(\boldsymbol{\sigma}^{\rm e}_{kr}-\boldsymbol{\sigma}^{\rm e}_{mr})\right\}=1 \tag{44}$$

Finally, by solving the linear set of Eqs. (43) and (44), the variables $\boldsymbol{\Delta\delta}$ and λ can be obtained. Then, the remaining unknown variable $\varepsilon^{\rm p}_{kr}$ in Eq. (42) can be calculated and the shakedown load multiplier can be determined by means of Eq. (33).

5. Applications

The proposed numerical method is now applied to the shakedown analysis of some typical soil structures subjected to cyclic loads. Both the Mohr–Coulomb and the Drucker–Prager yield criteria are used to model the plastic behavior of frictional materials. Finite elements with a reduced Gaussian integration strategy are adopted in the numerical implementation.

5.1. A pavement under moving loads

Pavements are civil engineering structures built for the purpose of allowing wheeled vehicles to operate safely and economically. The vehicles include cars and trucks on highway pavements, aircraft on airport runways and taxiways, mobile cranes on port and container terminal pavements together with locomotives and rolling stock on railways. By means of shakedown analysis, the effect of moving loads on the pavements can be revealed and the shakedown condition can also be effectively determined. Sharp and Booker (1984) were among the first to suggest that the shakedown theory may be applied to the design of pavement structures subjected to repeated loading. They developed a semi-analytical approach for determining the shakedown loads based on the static shakedown theorem. Instead, Collins and Cliffe (1987) used the kinematic shakedown theorem to perform shakedown analysis for a pavement subjected to cyclic loads and an upper bound to the shakedown limit can be calculated. Recently, Yu (2005) proposed an analytical solution for shakedown limits of a cohesive-frictional half-space under moving surface loads, where the static shakedown theorem was used and the Mohr–Coulomb yield criterion was assumed. Therefore, a lower bound to the shakedown limit can be obtained.

In this section, a plane strain model is assumed for a pavement under moving loads, as shown in Fig. 1, where p $(0 \leqslant p \leqslant p^{\max})$ is normal load with trapezoidal load distribution applied to the pavement from a repeated loading, p_0 $(0 \leqslant p \leqslant p_0^{\max})$ is the peak value of p, and q is shear force due to the friction between moving wheel and the pavement. Therefore, the relation between p and q can be defined as

$$q = \mu p \tag{45}$$

where μ is the frictional coefficient.

The size of the simulated region is determined as: $L = 10$ m, $H = 4$ m, $B = 1.0$ m, and $a = 0.5$ m, where L and H denote the length and height of the simulated region, respectively, and B and a are the size of trapezoidally loading area. The selected body is discretized with 1750 eight-node quadrilateral finite elements as shown in Fig. 2, and the convergence tolerances adopted in the numerical simulation are $\eta_1 = \eta_2 = 10^{-3}$. The Mohr–Coulomb yield criterion is assumed to model the plastic behavior of the material with the elastic module $E = 100$ MPa and the Poisson's ratio $v = 0.3$. In order to simulate the moving of loads on the pavement, it is assumed that the failure mode varies only along the depth and is same along the horizontal direction.

First, in order to verify the validity of the proposed numerical method, a pavement without surface friction is considered, i.e. the coefficient of surface friction is equal to zero ($\mu = 0$). Therefore, there is only the normal loading p applied to the pavement and no shear force ($q = 0$). By means of the proposed numerical method, the results of the dimensionless shakedown limits λ_{sd} $(\lambda_{sd} = p_0^{\max}/c)$ with the variation of soil internal friction

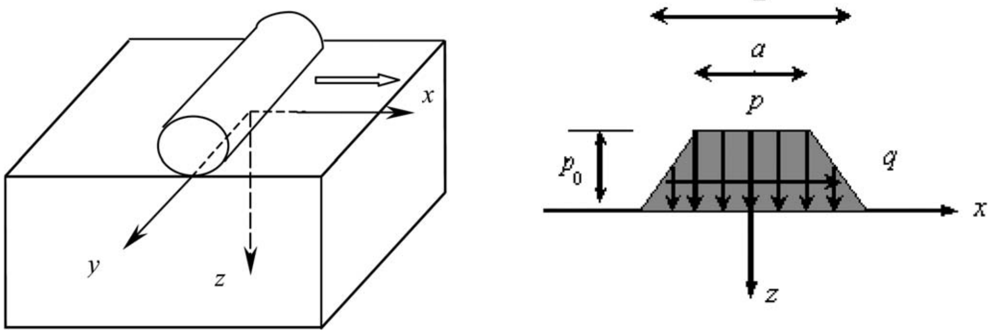

Fig. 1. A 2-D model for a pavement under moving loads.

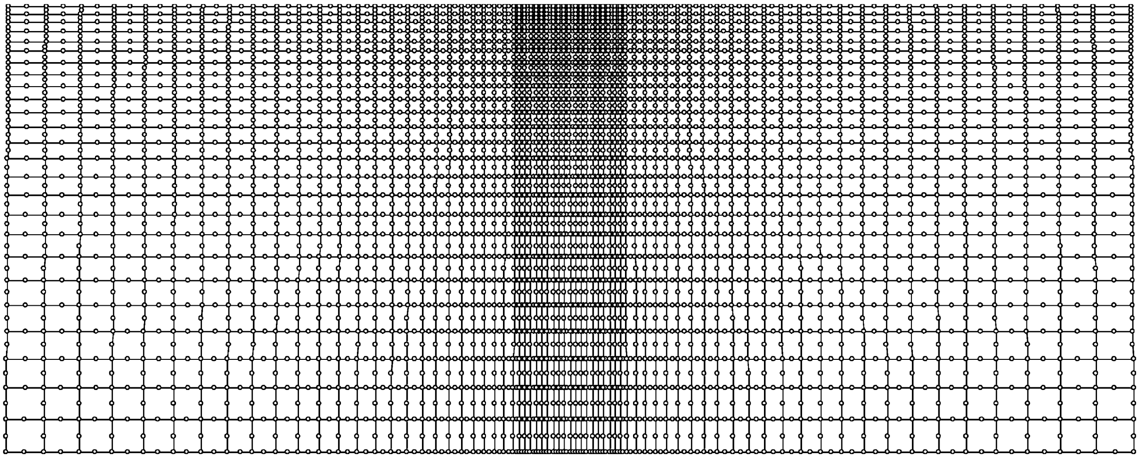

Fig. 2. A typical mesh for a pavement.

angle are presented in Fig. 3, where c and φ are the internal frictional angle and cohesion of material, respectively. It can be drawn that the shakedown limits obtained by the proposed method are closer to those upper bounds provided by Collins and Cliffe (1987), and a little larger than those lower bounds obtained by Sharp and Booker (1984) and Yu (2005).

The interactive effects of the internal frictional angle and the frictional coefficient of pavement surface on the shakedown limits are shown in Figs. 4 and 5. It can be concluded that for a pavement under moving loads, both the frictional angle of materials and the surface frictional coefficient of pavement have a significant effect on the shakedown condition of pavement. For a pavement subjected to compressive forces, the shakedown limits increase significantly with the rising of the frictional angle of materials, while they decrease rapidly with the rising of the surface friction coefficient.

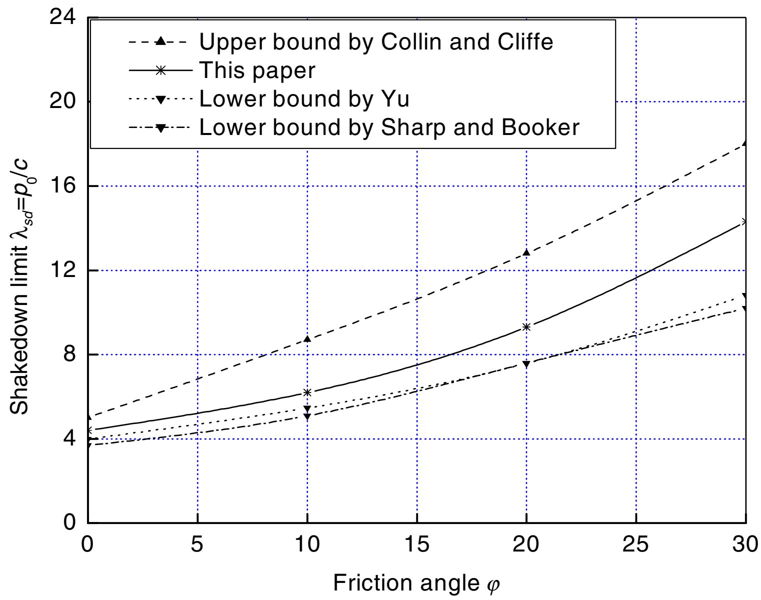

Fig. 3. Effect of friction angle on shakedown limits.

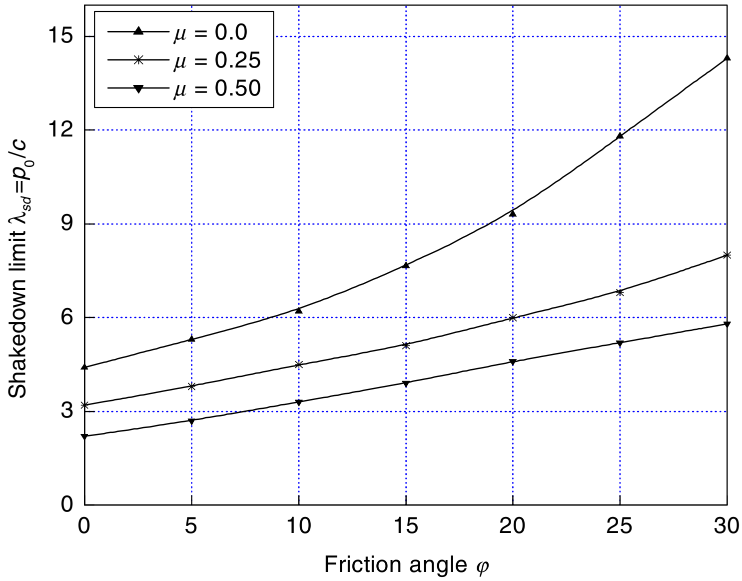

Fig. 4. Shakedown limits with friction angles.

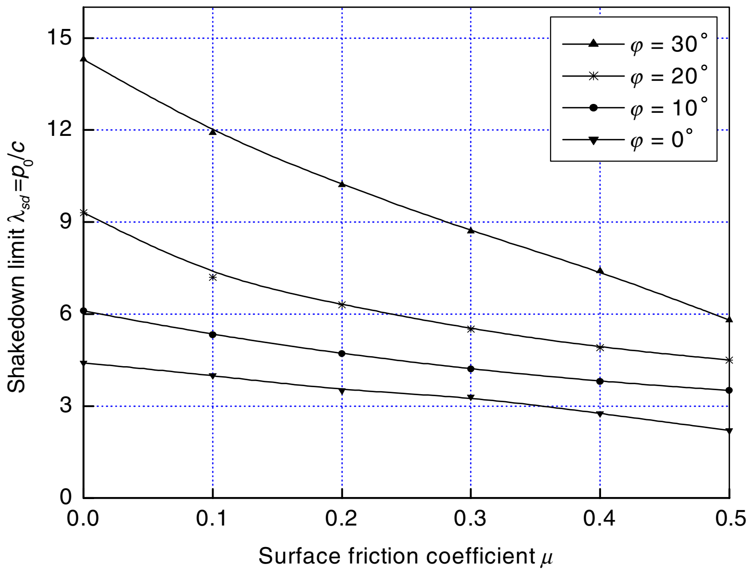

Fig. 5. Shakedown limits with surface friction coefficient.

To further show the failure mechanism of a pavement under moving loads, a typical failure mode is plotted in Fig. 6. From the visualized result, it can be drawn that for a pavement under moving loads, the failure of the body is due to plastic sliding in the surface layer. Therefore, the strength of the surface layer of a pavement is more important to the shakedown condition.

The relation between the iterative convergence sequences λ_{sd} and the iterative step k is shown in Fig. 7 (for the cases $\mu = 0$, $\varphi = 30°$ and $\mu = 0.5$, $\varphi = 30°$). The numerical results show that the efficiency and numerical stability of the proposed algorithm are fairly high and that the amount of computational effort is very small.

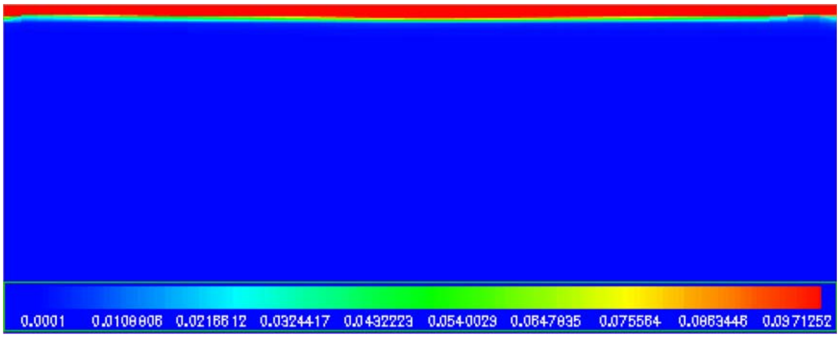

Fig. 6. The failure mode of a pavement ($\mu = 0.0$, $\varphi = 30°$).

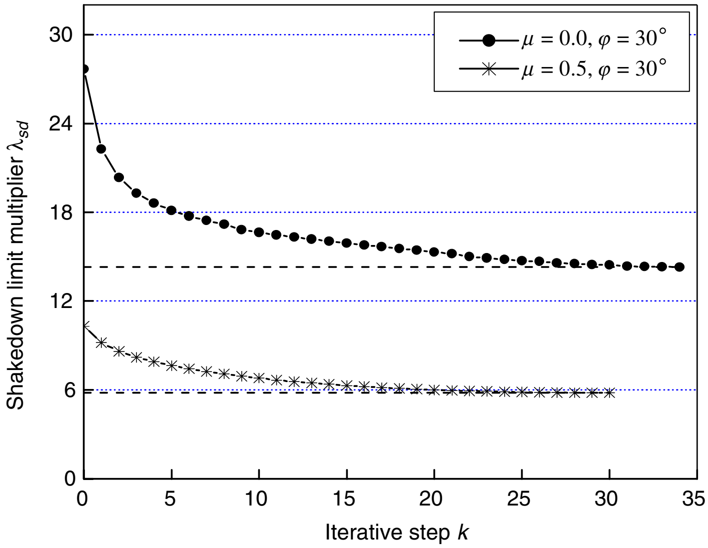

Fig. 7. The convergence sequence λ_{sd} with iterative steps.

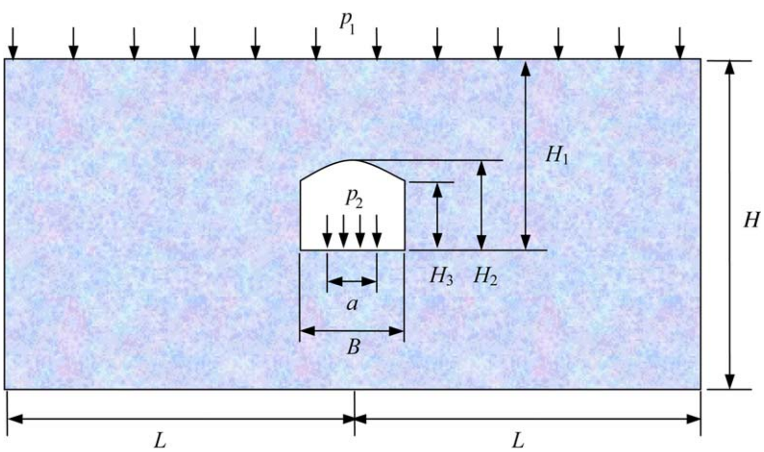

Fig. 8. The geometry of a tunnel.

Table 1
The geometry size of a tunnel (m)

L	B	a	H	H_1	H_2	H_3
25	8	4	30	15	3.5	2.5

5.2. A tunnel under both static and cyclic loads

A tunnel is another important structure of civil engineering which plays a more and more important role in the transport of big cities. How to evaluate the effect of moving trams on the stability and shakedown conditions of tunnels is a complex problem for the engineering design and maintenance. By means of the proposed method, the shakedown limit and condition of a tunnel can be determined. Both static loads from ground buildings and upper soils and cyclic loads from moving trams are considered.

A typical geometry of tunnel is presented in Fig. 8, where p_1 is a static load to simulate the ground building and upper soils while p_2 $(0 \leqslant p_2 \leqslant p_2^{\max})$ is a cyclic load to simulate the moving trams. In this simulation, an unsupported tunnel in frictional soils is considered to be analyzed. The geometry size of tunnel is listed in

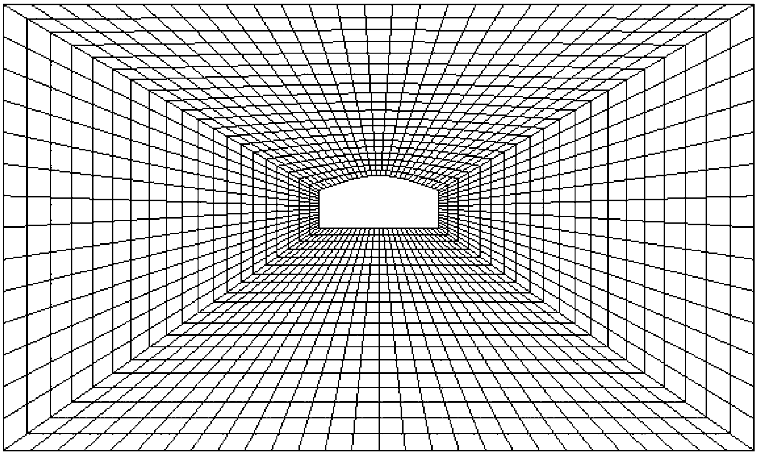

Fig. 9. A FEM mesh of a tunnel.

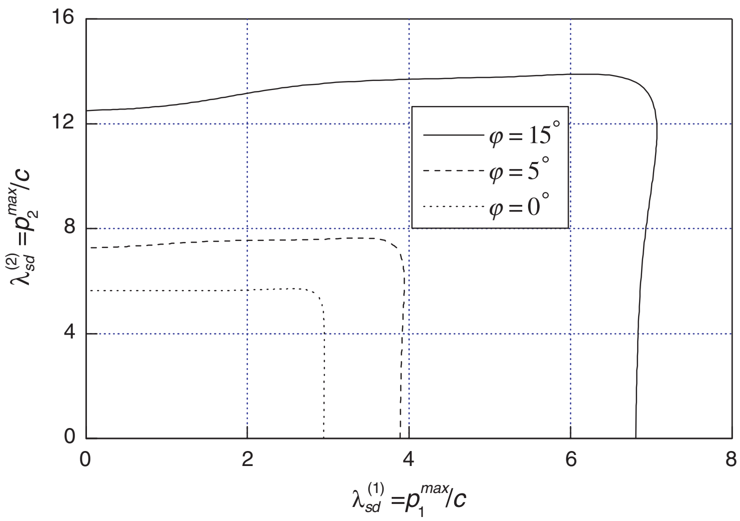

Fig. 10. Shakedown limit domain of a tunnel.

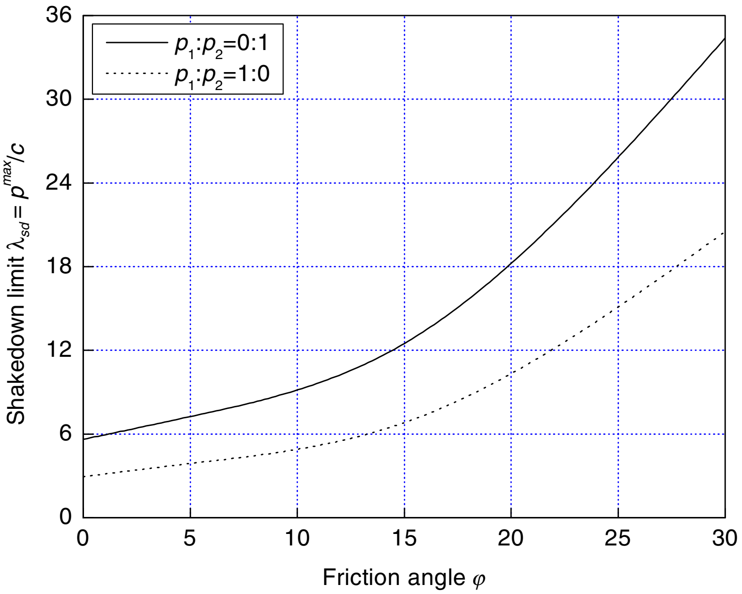

Fig. 11. Shakedown limits with friction angles.

Table 1. The plane strain model is used and the Drucker–Prager yield criterion is assumed to model the plastic behavior of soil materials. In the plane strain model, the strength parameters φ_0 and c_0 in the Drucker–Prager criterion is determined by

$$\varphi_0 = \frac{tg\varphi}{\sqrt{9+12tg^2\varphi}} \tag{46}$$

$$c_0 = \frac{3c}{\sqrt{9+12tg^2\varphi}} \tag{47}$$

A typical mesh of tunnel is plotted in Fig. 9 for the finite element simulation. The selected body is discretized with 1660 eight-node quadrilateral finite elements. The convergence tolerances adopted in the numerical simulation are $\eta_1 = \eta_2 = 10^{-3}$. The Drucker–Prager material is assumed with the elastic module $E = 100$ MPa and the Poisson's ratio $v = 0.3$. The numerical results of the shakedown limits of a tunnel under both static loads and cyclic loads are presented in Figs. 10 and 11, where the dimensionless shakedown limits $\lambda_{sd}^{(i)}$ is

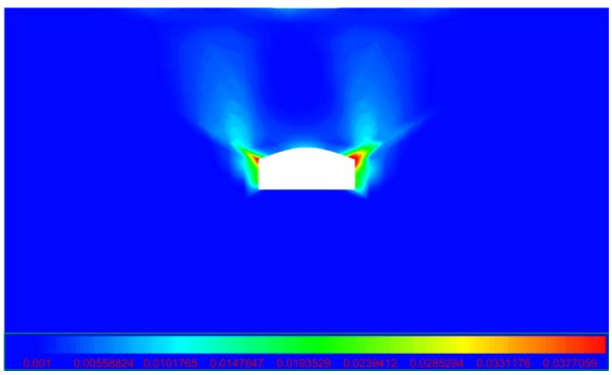

Fig. 12. Failure mode of a tunnel (p_1:p_2= 1:1, $\varphi = 5°$).

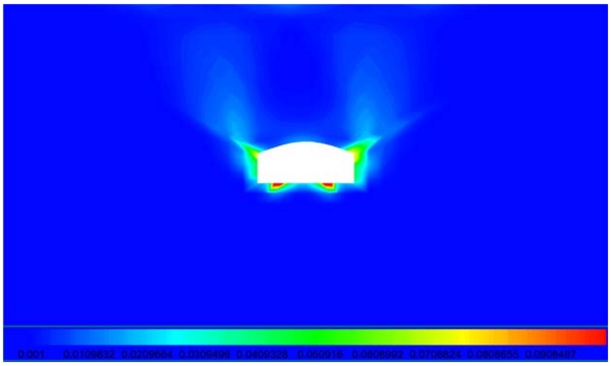

Fig. 13. Failure mode of a tunnel (p_1:$p_2 = 1$:2, $\varphi = 5°$).

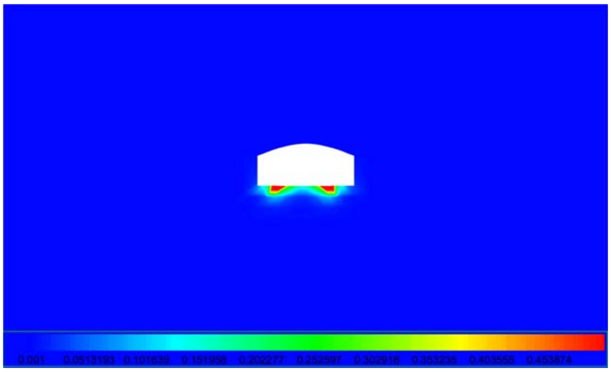

Fig. 14. Failure mode of a tunnel (p_1:$p_2 = 1$:5, $\varphi = 5°$).

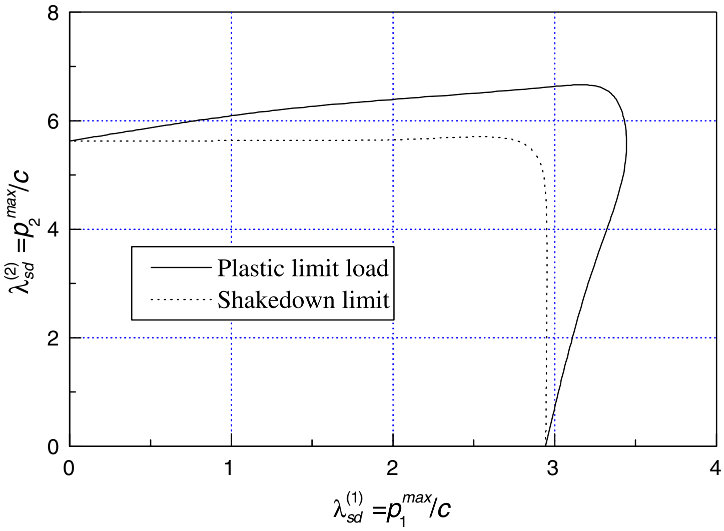

Fig. 15. Limit load and shakedown limit domain of a tunnel ($\varphi = 0°$).

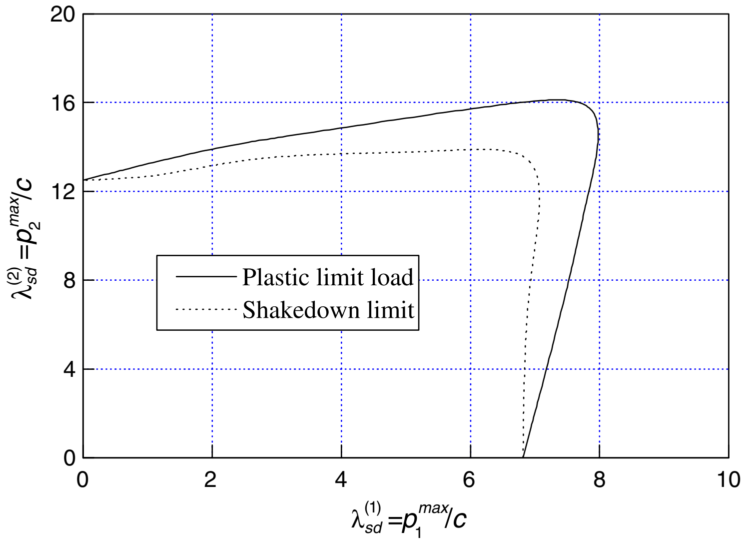

Fig. 16. Limit load and shakedown limit domain of a tunnel ($\varphi = 15°$).

defined by $\lambda_{\mathrm{sd}}^{(i)} = p_i^{\max}/c$, and c and φ are the internal frictional angle and cohesion of material, respectively. From the numerical results, it can be concluded that for an unsupported tunnel, both upper loads and moving trams have a significant effect on the shakedown condition.

To further show the shakedown modes of an unsupported tunnel, some typical failure modes of a tunnel are presented in Figs. 12–14. From the visualized results, it can be seen that as the ratio of the upper loads to the moving trams varies, the failure modes of a tunnel are changed significantly. When the upper loads are larger, the plastic areas develop in the upper layered soil of a tunnel and a plastic sliding occurs there which results in the collapse of structure. If the moving loads from the trams are larger, the plastic areas develop in the bottom of a tunnel to form a plastic sliding and finally the structure fail. The visualized results are very useful for the supporting design of a tunnel.

As a special case of shakedown analysis, limit analysis is to calculate the plastic limit load of a structure subject to static loads (e.g. for this example, both p_1 and p_2 are static, not cyclic). By means of the developed numerical method, the difference between shakedown limit and plastic limit load is presented in Figs. 15 and 16. From the numerical results, it can be drawn that repeated or cyclic loading are more dangerous to the stability of structures than static loading, and the shakedown domain must be smaller than the limit domain.

6. Conclusions

A novel general numerical method has been developed to perform the kinematic shakedown analysis for frictional materials by means of a nonlinear programming technique in conjunction with the displacement-based finite element method. The proposed method is the extension of the nonlinear programming technique applied to the numerical limit analysis (Li and Yu, 2005). By using an associated flow rule, the dissipation work based on a general yield criterion is explicitly expressed in terms of the kinematically admissible velocity. The yield surface does not need to be linearized which can reduce the number of constraints and therefore computational costs. König's technique is used to remove the difficulty from the integration along a deformation path. Then, based on the mathematical programming theory, the finite element model of the kinematic shakedown analysis is proposed as a nonlinear programming problem subject to a small number of equality constraints. The numerical examples show that the proposed iterative algorithm has the advantages of high computational accuracy and good numerical stability. By means of the proposed method, an upper bound to the shakedown limit of a structure under cyclic loads can be calculated and the possible failure mechanisms can also be obtained.

It is well known that for many soil materials which are often modelled by the Mohr–Coulomb or Drucker–Prager yield criterion, the plastic flow rule is non-associated. However, the classical shakedown theory is based on the assumption of an associated flow rule. Therefore, in this paper, the associated flow rule is still used for frictional materials. This may overestimate the bearing-capacity of structures. Under this condition, the kinematic shakedown analysis still provides an overestimated upper bound to shakedown limit while the static shakedown analysis cannot guarantee to necessarily get a lower bound to the actual shakedown limit.

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