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Monotone generalized contractions in partially ordered probabilistic metric spaces

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ABSTRACT

In this paper, a concept of monotone generalized contraction in partially ordered probabilistic metric spaces is introduced and some fixed and common fixed point theorems are proved. Presented theorems extend the results in partially ordered metric spaces of Nieto and Rodriguez-Lopez [Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order 22 (2005) 223–239; Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations, Acta Math. Sin. (Engl. Ser.) 23 (2007) 2205–2212], Ran and Reurings [A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc. 132 (2004) 1435–1443] to a more general class of contractive type mappings in partially ordered probabilistic metric spaces and include several recent developments.

1. Introduction

The Banach fixed point theorem for contraction mappings has been generalized and extended in many directions, see e.g., [1–7,9–19,22,25,26] and references therein. Recently Nieto and Rodriguez-Lopez [21], Ran and Reurings [24], Petruşel and Rus [23] presented some new results for contractions in partially ordered metric spaces. The main idea in [20,21,24] involve combining the ideas of iterative technique in the contraction mapping principle with those in the monotone technique.

Recall that if $(X, \leqslant)$ is a partially ordered set and $F : X \to X$ is such that for $x, y \in X$, $x \leqslant y$ implies $F(x) \leqslant F(y)$, then a mapping F is said to be non-decreasing. The main result of Nieto and Rodriguez-Lopez [20,21] and Ran and Reurings [24] is the following fixed point theorem.

Theorem 1.1. *Let $(X, \leqslant)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Suppose F is a non-decreasing mapping with*

$$d\big(F(x), F(y)\big) \leqslant kd(x, y) \tag{1.1}$$

for all $x, y \in X$, $x \leqslant y$, where $0 < k < 1$. Also suppose either

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(a) *F is continuous or*
(b) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to x$ in X, then $x_n \leqslant x$ for all n hold.*

If there exists an $x_0 \in X$ with $x_0 \leqslant F(x_0)$, then F has a fixed point.

The works of Nieto and Rodriguez-Lopez [20,21] and Ran and Reurings [24] have motivated Agarwal et al. [1], Bhaskar and Lakshmikantham [2] and Lakshmikantham and Ćirić [11] to undertake further investigation of fixed points in the area of ordered metric spaces. Hence, the following question is bound to arise:

Question 1.2. Is it possible to obtain a probabilistic metric space version of Theorem 1.1 and prove fixed point theorems for mappings satisfying a more general contraction condition than (1.1).

The purpose of this paper is to give an affirmative answer of Question 1.2. We prove the existence and approximation results for a wide class of contractive mappings in probabilistic metric space. Our results are an extension and improvement of the results of Nieto and Rodriguez-Lopez [20,21] and Ran and Reurings [24] to a more general class of contractive type mappings and include several recent developments.

2. Preliminaries

K. Menger introduced the notion of a probabilistic metric space in 1942 and since then the theory of probabilistic metric spaces has developed in many directions [27]. The idea of K. Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis.

Throughout this paper, the space of all probability distribution functions (briefly, d.f.'s) is denoted by $\Delta^+ = \{F : \mathbb{R} \cup \{-\infty, +\infty\} \longrightarrow [0,1]$: F is left-continuous and non-decreasing on $\mathbb{R}$, $F(0) = 0$ and $F(+\infty) = 1\}$ and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+\colon\ l^- F(+\infty) = 1\}$. Here $l^- f(x)$ denotes the left limit of the function f at the point x, $l^- f(x) = \lim_{t \to x^-} f(t)$. The space Δ^+ is partially ordered by the usual point-wise ordering of functions, i.e., $F \leqslant G$ if and only if $F(t) \leqslant G(t)$ for all t in $\mathbb{R}$. The maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leqslant 0, \\ 1, & \text{if } t > 0. \end{cases}$$

Definition 2.1. ([27]) A mapping $T : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous *t-norm* if T satisfies the following conditions:

(a) T is commutative and associative;
(b) T is continuous;
(c) $T(a,1) = a$ for all $a \in [0,1]$;
(d) $T(a,b) \leqslant T(c,d)$ whenever $a \leqslant c$ and $c \leqslant d$, and $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $T_P(a,b) = ab$ and $T_M(a,b) = Min(a,b)$.

Now t-norms are recursively defined by $T^1 = T$ and

$$T^n(x_1, \ldots, x_{n+1}) = T\big(T^{n-1}(x_1, \ldots, x_n), x_{n+1}\big)$$

for $n \geqslant 2$ and $x_i \in [0,1]$, for all $i \in \{1, 2, \ldots, n+1\}$.

A t-norm T is said to be *of Hadžić type* if the family $\{T^n\}_{n \in \mathbb{N}}$ is equicontinuous at $x = 1$, that is,

$$\forall \varepsilon \in (0,1)\ \exists \delta \in (0,1)\colon\ a > 1 - \delta \Rightarrow T^n(a) > 1 - \varepsilon \quad (n \geqslant 1).$$

T_M is a trivial example of a t-norm of Hadžić type, but there exist t-norms of Hadžić type weaker than T_M [8].

Definition 2.2. A *Menger probabilistic metric space* (briefly, Menger PM-space) is a triple (X, F, T), where X is a nonempty set, T is a continuous t-norm, and F is a mapping from $X \times X$ into D^+ such that, if $F_{x,y}$ denotes the value of F at the pair (x, y), the following conditions hold:

(PM1) $F_{x,y}(t) = \varepsilon_0(t)$ for all $t > 0$ if and only if $x = y$;
(PM2) $F_{x,y}(t) = F_{y,x}(t)$ for all $x, y \in X, t > 0$;
(PM3) $F_{x,z}(t+s) \geqslant T(F_{x,y}(t), F_{y,z}(s))$ for all $x, y, z \in X$ and $t, s \geqslant 0$.

Definition 2.3. Let (X, F, T) be a Menger PM-space.

(1) A sequence $\{x_n\}_n$ in X is said to be *convergent* to x in X if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $F_{x_n,x}(\epsilon) > 1 - \lambda$ whenever $n \geqslant N$.
(2) A sequence $\{x_n\}_n$ in X is called *Cauchy sequence* if, for every $\epsilon > 0$ and $\lambda > 0$, there exists positive integer N such that $F_{x_n,x_m}(\epsilon) > 1 - \lambda$ whenever $n, m \geqslant N$.
(3) A Menger PM-space (X, F, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X.

Definition 2.4. Let (X, F, T) be a Menger PM space. For each p in X and $\lambda > 0$, the strong λ-*neighborhood* of p is the set

$$N_p(\lambda) = \{q \in X\colon\ F_{p,q}(\lambda) > 1 - \lambda\},$$

and the strong neighborhood system for X is the union $\bigcup_{p \in V} \mathcal{N}_p$ where $\mathcal{N}_p = \{N_p(\lambda)\colon\ \lambda > 0\}$.

The strong neighborhood system for X determines a Hausdorff topology for X.

Theorem 2.5. *([27]) If (X, F, T) is a Menger PM-space and $\{p_n\}$, and $\{q_n\}$ are sequences such that $p_n \to p$ and $q_n \to q$, then $\lim_{n\to\infty} F_{p_n,q_n}(t) = F_{p,q}(t)$ for every continuity point t of $F_{p,q}$.*

3. Main results

Definition 3.1. Suppose $(X, \leqslant)$ is a partially ordered set and $A, h : X \to X$ are mappings of X into itself. We say A is a h-non-decreasing if for $x, y \in X$,

$$h(x) \leqslant h(y) \quad \text{implies} \quad A(x) \leqslant A(y). \tag{3.1}$$

In the proof of our first theorem we use the following two lemmas:

Lemma 3.2. *([8]) Let (X, F, T) be a Menger PM space with T of Hadžić-type and $\{x_n\}$ be a sequence in X such that, for some $k \in (0, 1)$,*

$$F_{x_n,x_{n+1}}(kt) \geqslant F_{x_{n-1},x_n}(t) \quad (n \geqslant 1,\ t > 0).$$

Then $\{x_n\}$ is a Cauchy sequence.

Lemma 3.3. *If $F, G \in D^+$ and, for some $k \in (0, 1)$,*

$$F(kt) \geqslant \min\{G(t), F(t)\}, \quad \forall t > 0,$$

then $F(kt) \geqslant G(t)$, $\forall t > 0$.

Proof. Suppose, with the view to obtain a contradiction, that there exists $t_0 > 0$ such that $G(t_0) > F(kt_0)$. Since by assumption $F(kt_0) \geqslant \min\{G(t_0), F(t_0)\}$, it follows that $F(kt_0) \geqslant F(t_0)$. As F is non-decreasing and $k < 1$, one then has that $F(t) = F(t_0)$ for all $kt_0 \leqslant t \leqslant t_0$. So in fact $G(t_0) > F(t_0)$. Let $m = \sup\{t > 0\colon\ F(t) = F(t_0)\}$. Since $F \in D^+$, it follows that $m < \infty$ and choose $t_1 \in (km, m)$ and $t_2 \in (m, t_1/k)$. Then $t_2 > m$ and $kt_2 < t_1$ and so we have, as F is non-decreasing and $t_1 < m$,

$$F(kt_2) \leqslant F(t_1) \leqslant F(t_0) < F(t_2).$$

This implies $F(kt_2) \geqslant G(t_2)$ (as $F(kt_2) \geqslant \min\{G(t_2), F(t_2)\}$). Since $G(t_0) > F(t_0)$, we have

$$G(t_0) > F(t_0) \geqslant F(kt_2) \geqslant G(t_2) \geqslant G(t_0),$$

a contradiction. Thus our assumption $G(t_0) > F(t_0)$ is wrong. The proof is complete. □

Theorem 3.4. *Let $(X, \leqslant)$ be a partially ordered set and (X, F, T) be a complete Menger PM-space under a t-norm T of Hadžić-type. Let $A, h : X \to X$ be two self-mappings of X such that $A(X) \subseteq h(X)$, A is a h-non-decreasing mapping and, for some $k \in (0, 1)$,*

$$F_{A(x),A(y)}(kt) \geqslant Min\{F_{h(x),h(y)}(t), F_{h(x),A(x)}(t), F_{h(y),A(y)}(t)\}, \tag{3.2}$$

for all $x, y \in X$ for which $h(x) \leqslant h(y)$ and all $t > 0$.

Also suppose that $h(X)$ is closed and

$$\begin{aligned} &\text{if } \{h(x_n)\} \subset X \text{ is a non-decreasing sequence with } h(x_n) \to h(z) \text{ in } h(X), \\ &\text{then } h(z) \leqslant h(h(z)) \text{ and } h(x_n) \leqslant h(z) \text{ for all } n \text{ hold.} \end{aligned} \tag{3.3}$$

If there exists an $x_0 \in X$ with $h(x_0) \leqslant A(x_0)$, then A and h have a coincidence. Further, if A and h commute at their coincidence points, then A and h have a common fixed point.

Proof. Let $x_0 \in X$ be such that $h(x_0) \leqslant A(x_0)$. Since $A(X) \subseteq h(X)$, we can choose $x_1 \in X$ such that $h(x_1) = A(x_0)$. Again from $A(X) \subseteq h(X)$ we can choose $x_2 \in X$ such that $h(x_2) = A(x_1)$. Continuing this process we can choose a sequence $\{x_n\}$ in X such that

$$h(x_{n+1}) = A(x_n) \quad \text{for all } n \geqslant 0. \tag{3.4}$$

Since $h(x_0) \leqslant A(x_0)$ and $h(x_1) = A(x_0)$, we have $h(x_0) \leqslant h(x_1)$. Then from (3.1),

$$A(x_0) \leqslant A(x_1),$$

that is, by (3.4), $h(x_1) \leqslant h(x_2)$. Again from (3.1),

$$A(x_1) \leqslant A(x_2),$$

that is, $h(x_2) \leqslant h(x_3)$. Continuing we obtain

$$A(x_0) \leqslant A(x_1) \leqslant A(x_2) \leqslant A(x_3) \leqslant \cdots \leqslant A(x_n) \leqslant A(x_{n+1}) \leqslant \cdots. \tag{3.5}$$

Since from (3.4) and (3.5) we have $h(x_{n-1}) \leqslant h(x_n)$, from (3.2) with $x = x_n$ and $y = x_{n+1}$,

$$F_{A(x_n),A(x_{n+1})}(kt) \geqslant Min\bigl\{F_{h(x_n),h(x_{n+1})}(t), F_{h(x_n),A(x_n)}(t), F_{h(x_{n+1}),A(x_{n+1})}(t)\bigr\}.$$

So by (3.4),

$$F_{A(x_n),A(x_{n+1})}(kt) \geqslant Min\bigl\{F_{A(x_{n-1}),A(x_n)}(t), F_{A(x_{n-1}),A(x_n)}(t), F_{A(x_n),A(x_{n+1})}(t)\bigr\},$$

hence

$$F_{A(x_n),A(x_{n+1})}(kt) \geqslant Min\bigl\{F_{A(x_{n-1}),A(x_n)}(t), F_{A(x_n),A(x_{n+1})}(t)\bigr\} \quad (n \in \mathbb{N},\ t > 0).$$

By Lemma 3.3, it follows that

$$F_{A(x_n),A(x_{n+1})}(kt) \geqslant F_{A(x_{n-1}),A(x_n)}(t) \quad (n \in \mathbb{N},\ t > 0).$$

Now, by Lemma 3.2, $\{A(x_n)\}$ is a Cauchy sequence.

Since $h(X)$ is closed and as $A(x_n) = h(x_{n+1})$, there is some $z \in X$ such that

$$\lim_{n\to\infty} h(x_n) = h(z). \tag{3.6}$$

Now we show that z is a coincidence of A and h. Since from (3.3) and (3.6) we have $h(x_n) \leqslant h(z)$ for all n, then from (3.2) we have

$$F_{A(x_n),A(z)}(kt) \geqslant Min\bigl\{F_{h(x_n),h(z)}(t), F_{h(x_n),A(x_n)}(t), F_{h(z),A(z)}(t)\bigr\} \quad (t > 0).$$

Letting $n \to \infty$ we get

$$F_{h(z),A(z)}(kt) \geqslant Min\bigl\{F_{h(z),h(z)}(t), F_{h(z),h(z)}(t), F_{h(z),A(z)}(t)\bigr\} \tag{3.7}$$

for all $t > 0$. Therefore,

$$F_{h(z),A(z)}(t) \geqslant F_{h(z),A(z)}\left(\frac{t}{k}\right) \quad (t > 0).$$

From here we get

$$F_{h(z),A(z)}(t) \geqslant F_{h(z),A(z)}\left(\frac{t}{k^n}\right) \to 1 \quad \text{as } n \to \infty, \quad \text{for all } t > 0,$$

concluding that $F_{h(z),A(z)}(t) = 1$ for all $t > 0$. Then, by (PM1), $A(z) = h(z)$. Thus we proved that A and h have a coincidence.

Suppose now that A and h commute at z. Set $w = h(z) = A(z)$. Then

$$A(w) = A\bigl(h(z)\bigr) = h\bigl(A(z)\bigr) = h(w).$$

Since from (3.3) we have $h(z) \leqslant h(h(z)) = h(w)$ and as $h(z) = A(z)$ and $h(w) = A(w)$, from (3.2) we get

$$F_{w,A(w)}(kt) = F_{A(z),A(w)}(kt) \geqslant Min\bigl\{F_{h(z),h(w)}(t), F_{h(z),A(z)}(t), F_{h(w),A(w)}(t)\bigr\}, \tag{3.8}$$

that is,

$$F_{A(z),A(w)}(kt) \geqslant F_{A(z),A(w)}(t),$$

hence, similarly as above, $A(w) = A(z)$. Since $A(z) = h(z) = w$, we have

$$A(w) = h(w) = w.$$

Thus we proved that A and h have a common fixed point. $\square$

Remark 3.5. Note A is h-non-decreasing can be replaced by A is h-non-increasing in Theorem 3.4 provided $h(x_0) \leqslant A(x_0)$ is replaced by $A(x_0) \geqslant h(x_0)$ in Theorem 3.4.

Corollary 3.6. *Let $(X, \leqslant)$ be a partially ordered set and (X, F, T) be a complete Menger PM-space under a t-norm T of Hadžić-type. Let $A : X \to X$ be a non-decreasing self-mapping of X for which there exists $k \in (0, 1)$ such that*

$$F_{A(x),A(y)}(kt) \geqslant Min\left\{F_{x,y}(t), F_{x,A(x)}(t), F_{y,A(y)}(t)\right\}$$

for all $x, y \in X$ with $x \leqslant y$ and all $t > 0$. Also suppose either

(i) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$ in X, then $x_n \leqslant z$ for all n hold or*
(ii) *A is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leqslant A(x_0)$, then A has a fixed point.

Proof. Taking $h = I$ ($I =$ the identity mapping) in Theorem 3.4, then (3.3) reduces to the hypothesis (i).
Suppose now that A is continuous. Since $x_{n+1} = A(x_n)$, for all $n \geqslant 0$, and $x_n \to z$, then

$$A(z) = A(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} A(x_n) = z. \qquad \square$$

Theorem 3.7. *Let $(X, \leqslant)$ be a partially ordered set and (X, F, T_M) be a complete Menger PM-space. Let $A, h : X \to X$ be two self-mappings of X such that $A(X) \subseteq h(X)$, A is a h-non-decreasing mapping and, for some $k \in (0, 1)$,*

$$F_{A(x),A(y)}(kt) \geqslant Min\left\{F_{h(x),h(y)}(t), F_{h(x),A(x)}(t), F_{h(y),A(y)}(t), F_{h(x),A(y)}((1+q)t), F_{h(y),A(x)}(1-q)t)\right\} \tag{3.9}$$

for all $x, y \in X$ for which $h(x) \leqslant h(y)$ and all $t > 0$, $q \in (0, 1)$.
Also suppose that

$$\begin{aligned}&\text{if } \left\{h(x_n)\right\} \subset X \text{ is a non-decreasing sequence with } h(x_n) \to h(z) \text{ in } h(X),\\ &\text{then } h(z) \leqslant h\big(h(z)\big) \text{ and } h(x_n) \leqslant h(z) \text{ for all } n \text{ hold.}\end{aligned} \tag{3.10}$$

and that $h(X)$ is closed. If there exists an $x_0 \in X$ with $h(x_0) \leqslant A(x_0)$, then A and h have a coincidence. Further, if A and h commute at their coincidence points, then A and h have a common fixed point.

Proof. As in the proof of the preceding theorem, starting with $x_0 \in X$ be such that $h(x_0) \leqslant A(x_0)$, we can choose a sequence

$$A(x_0) \leqslant A(x_1) \leqslant A(x_2) \leqslant A(x_3) \leqslant \cdots \leqslant A(x_n) \leqslant A(x_{n+1}) \leqslant \cdots. \tag{3.11}$$

Since $h(x_{n-1}) \leqslant h(x_n)$, it follows that

$$\begin{aligned}F_{A(x_n),A(x_{n+1})}(kt) \geqslant Min\big\{&F_{h(x_n),h(x_{n+1})}(t), F_{h(x_n),A(x_n)}(t), F_{h(x_{n+1}),A(x_{n+1})}(t),\\ &F_{h(x_n),A(x_{n+1})}\big((1+q)t\big), F_{h(x_{n+1}),A(x_n)}\big((1-q)t\big)\big\},\end{aligned}$$

and thus

$$F_{A(x_n),A(x_{n+1})}(kt) \geqslant Min\left\{F_{A(x_{n-1}),A(x_n)}(t), F_{A(x_{n-1}),A(x_n)}(t), F_{A(x_n),A(x_{n+1})}(t), F_{A(x_{n-1}),A(x_{n+1})}\big((1+q)t\big), 1\right\}.$$

Since by (PM3),

$$F_{A(x_{n-1}),A(x_{n+1})}\big((1+q)t\big) \geqslant Min\left\{F_{A(x_{n-1}),A(x_n)}(t), F_{A(x_n),A(x_{n+1})}(qt)\right\},$$

we have

$$F_{A(x_n),A(x_{n+1})}(kt) \geqslant Min\left\{F_{A(x_{n-1}),A(x_n)}(t), F_{A(x_n),A(x_{n+1}}(t), F_{A(x_n),A(x_{n+1})}(qt)\right\}.$$

Letting $q \to 1$ we get

$$F_{A(x_n),A(x_{n+1})}(kt) \geqslant Min\left\{F_{A(x_{n-1}),A(x_n)}(t), F_{A(x_n),A(x_{n+1})}(t)\right\}.$$

Now, as in the proof of the preceding theorem, it follows that $\{A(x_n)\}$ is a Cauchy sequence.
Since $h(X)$ is closed and as $A(x_n) = h(x_{n+1})$, there is some $z \in X$ such that

$$\lim_{n\to\infty} h(x_n) = h(z). \tag{3.12}$$

Now we show that z is a coincidence of A and h. As $h(x_n) \leqslant h(z)$ for all n,

$$F_{A(x_n),A(z)}(kt) \geqslant Min\{F_{h(x_n),h(z)}(t), F_{h(x_n),A(x_n)}(t), F_{h(z),A(z)}(t), F_{h(x_n),A(z)}((1+q)t), F_{h(z),A(x_n)}((1-q)t)\}. \tag{3.13}$$

Letting $n \to \infty$ we get

$$F_{h(z),A(z)}(kt) \geqslant Min\{F_{h(z),h(z)}(t), F_{h(z),h(z)}(t), F_{h(z),A(z)}(t), F_{h(z),A(z)}((1+q)t), F_{h(z),h(z)}((1-q)t)\} \tag{3.14}$$

for all $t > 0$. Therefore,

$$F_{h(z),A(z)}(t) \geqslant F_{h(z),A(z)}\left(\frac{t}{k}\right) \quad (t > 0).$$

Hence we get $A(z) = h(z)$, proving that A and h have a coincidence.

Suppose now that A and h commute at z. Set $w = h(z) = A(z)$. Then

$$A(w) = A(h(z)) = h(A(z)) = h(w).$$

Since $h(z) \leqslant h(h(z)) = h(w)$ and as $h(z) = A(z)$ and $h(w) = A(w)$, we get

$$\begin{aligned} F_{w,A(w)}(kt) &= F_{A(z),A(w)}(kt) \\ &\geqslant Min\{F_{h(z),h(w)}(t), F_{h(z),F(z)}(t), F_{h(w),F(w)}(t), F_{h(w),A(z)}((1+q)t), F_{h(z),A(w)}((1-q)t)\} \\ &= F_{A(z),A(w)}((1-q)t). \end{aligned} \tag{3.15}$$

Letting $q \to 0$ we get

$$F_{A(z),A(w)}(kt) \geqslant F_{A(z),A(w)}(t),$$

hence $A(w) = A(z)$. Since $A(z) = h(z) = w$, we conclude that

$$A(w) = h(w) = w,$$

that is, A and h have a common fixed point. □

Corollary 3.8. *Let $(X, \leqslant)$ be a partially ordered set and (X, F, T_M) be a complete Menger PM-space. Let $A : X \to X$ be a non-decreasing self-mapping of X such that, for some $k \in (0, 1)$,*

$$F_{A(x),A(y)}(kt) \geqslant Min\{F_{x,y}(t), F_{x,A(x)}(t), F_{y,A(y)}(t), F_{x,A(y)}((1+q)t), F_{y,A(x)}((1-q)t)\} \tag{3.16}$$

for all $x, y \in X$ for which $x \leqslant y$ and all $t > 0$, $q \in (0, 1)$. Also suppose either

(i) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$ in X then $x_n \leqslant z$ for all n hold or*
(ii) *A is continuous.*

If there exists an $x_0 \in X$ with $x_0 \leqslant A(x_0)$, then A has a fixed point.

Proof. Taking $h = I$ ($I =$ the identity mapping) in Theorem 3.7, then (3.10) reduces to the hypothesis (i).

Suppose now that A is continuous. Since $x_{n+1} = A(x_n)$, for all $n \geqslant 0$, and $x_n \to z$, then

$$A(z) = A(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} A(x_n) = z. \qquad \square$$

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