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Submaximal and door compactifications

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ABSTRACT

In this paper, a characterization is given for compact door spaces. We, also, deal with spaces X such that a compactification $K(X)$ of X is submaximal or door.
Let X be a topological space and $K(X)$ be a compactification of X.
We prove, here, that $K(X)$ is submaximal if and only if for each dense subset D of X, the following properties hold:

(i) D is co-finite in $K(X)$;
(ii) for each $x \in K(X) \setminus D$, $\{x\}$ is closed.

If X is a noncompact space, then we show that $K(X)$ is a door space if and only if X is a discrete space and $K(X)$ is the one-point compactification of X.

0. Introduction

A space X is said to be *door* if every subset of X is either open or closed [5]. In 1987, McCartan [4] has classified door spaces as follows:

A topological space $(X, \mathcal{T})$ is door if and only if one of the following properties holds:

- *X is discrete.*
- *X has exactly one accumulation point.*
- *X has a subset A such that the cardinality of $X \setminus A$ is $\geqslant 2$ and there exists an ultrafilter $\mathcal{U}$ on X such that $\mathcal{T} = \mathcal{U} \cup \{G \mid G \subseteq A\}$.*

By a *submaximal* space, we mean a space in which every dense subset is open. Hence, every door space is, clearly, a submaximal space.

This paper deals, essentially, with compact door spaces and compact submaximal spaces. These kind of spaces seem to have connection with compactifications with finite remainder.

The first section is devoted to a short study of compactifications which are submaximal.

The main purpose of the second section is the characterization of compact door spaces.

The third section deals with an intrinsic topological characterization of spaces X such that a compactification $K(X)$ of X is a door space.

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In the fourth section, we are concerned with an example to illustrate the theory: we turn our attention to spaces such that their Wallman compactifications are submaximal (resp., door), such space will be labeled as *w-submaximal* (resp., *w-door*).

1. Submaximal spaces and compactifications

First, recall that a *compactification* of a topological space X is a couple $(K(X), e)$, where $K(X)$ is a compact space and $e: X \longrightarrow K(X)$ is a continuous embedding (e is a continuous one-to-one map and induces a homeomorphism from X onto $e(X)$) such that $e(X)$ is a dense subspace of $K(X)$. When a compactification $(K(X), e)$ of X is given, X will be identified with $e(X)$ and assumed to be dense in $K(X)$.

Let us recall an elementary fact about submaximal spaces.

Theorem 1.1. *([2, Theorem 3.1]) Let X be a topological space. Then the following statements are equivalent*:

(i) *X is submaximal.*
(ii) *$\overline{S} \setminus S$ is closed, for each $S \subseteq X$.*
(iii) *$\overline{S} \setminus S$ is closed and discrete, for each $S \subseteq X$.*

Remark 1.2. One may check easily that, if E is a submaximal space and F is a dense subset of E, then F is submaximal.

Proposition 1.3. *Let X be a topological space and $K(X)$ be a compactification of X. Then the following statements are equivalent*:

(1) *$K(X)$ is submaximal.*
(2) *For each dense subset D of X, the following properties hold*:
  (i) *D is co-finite in $K(X)$;*
  (ii) *for each $x \in K(X) \setminus D$, $\{x\}$ is closed.*

Proof. (1) $\Longrightarrow$ (2) Let D be a dense subset of X. Then D is dense in $K(X)$. Hence, by [2, Theorem 3.1], $K(X) \setminus D$ is a closed discrete subspace of $K(X)$. Thus, $\{x\}$ is closed, for each $x \in K(X) \setminus D$. Besides, as $K(X)$ is compact, $K(X) \setminus D$ is finite; so that D is co-finite in $K(X)$.

(2) $\Longrightarrow$ (1) Let D be a dense set of $K(X)$. Then $D \cap X$ is a dense set of X. By hypothesis, $K(X) \setminus (X \cap D)$ is finite and for each $x \in K(X) \setminus (X \cap D)$, $\{x\}$ is closed. Consequently, $K(X) \setminus D$ is closed, as desired. □

Corollary 1.4. *Let X be a topological space and $K(X)$ be a compactification of X. If, in addition, $K(X)$ is a T_1-space, then the following statements are equivalent*:

(1) *$K(X)$ is submaximal.*
(2) *Each dense subset of X is co-finite in $K(X)$.*

Remark 1.5. Following Proposition 1.3, it is clear that, if a compactification $K(X)$ of X is submaximal, then X is submaximal and the remainder $K(X) \setminus X$ is a finite closed discrete subspace of $K(X)$.

Before giving an example showing that the converse of the previous remark does not hold, let us recall some elementary facts.

Let $(X, \mathcal{T})$ be a T_0-space and $\leqslant_{\mathcal{T}}$ be the ordering defined on X by

$$x \leqslant_{\mathcal{T}} y \quad \text{if and only if} \quad y \in \overline{\{x\}}.$$

The order $\leqslant_{\mathcal{T}}$ will be called *the ordering induced by the topology*.

By a chain of elements of X arriving to x, we mean a chain of the type

$$x_0 <_{\mathcal{T}} x_1 <_{\mathcal{T}} \cdots <_{\mathcal{T}} x_n = x.$$

The integer n is called the *length* of the chain; and the supremum (in $\mathbb{N} \cup \{\infty\}$) of the lengths is called the *Krull dimension* of $(X, \mathcal{T})$ and it is denoted by $\dim_K(X, \mathcal{T})$.

An element x of X is said to be of *height* n – and we write $ht(x) = n$ – if the supremum of the lengths of chains arriving to x is n. We denote $X_n := \{x \in X \mid ht(x) = n\}$, $n \in \mathbb{N}$.

Recall that an *Alexandroff topology* on X is a topology such that any intersection of open sets is open.

Let us recall an elementary construction of Alexandroff spaces (see for instance [3] or [7]). Let X be a set and R be a quasiorder on X. For each $x \in X$, we let $(\downarrow x)_R$ be the set $\{y \in X \mid yRx\}$ and $(x\uparrow)_R$ be the set $\{y \in X \mid xRy\}$. Then the family

$\mathcal{B} := \{(\downarrow x)_R \mid x \in X\}$ is a basis of a topology on X called the *Alexandroff topology associated with* R; we will denote it by $\mathcal{A}(R)$. Conversely, each Alexandroff topology on a given set X is an $\mathcal{A}(R)$, where R is the quasiorder defined on X by

$$xRy \quad \text{if and only if} \quad \overline{\{y\}} \subseteq \overline{\{x\}}.$$

Example 1.6. Let L be an infinite set and $a \notin L$. Set $X := L \cup \{a\}$, and equip X with the topology $\mathcal{T} = \{\emptyset\} \cup \{O \subseteq X\colon\ a \in O\}$. Of course, $(X, \mathcal{T})$ is an Alexandroff T_0-space with Krull dimension 1. Thus, by [1, Proposition 2.2], $(X, \mathcal{T})$ is submaximal. In fact, it is straightforward to see that the dense sets of X are exactly the open sets of X.

However, by [1, Proposition 4.4], the one-point compactification of X is not submaximal (since $\{a\}$ is a dense subset of X which is not co-compact).

2. Door compact spaces

First, we give some basic facts about accumulation points, door spaces and compact door spaces.

Proposition 2.1. *Let $(X, \mathcal{T})$ be a topological space. Then the following properties hold*:

(1) *$m \in X$ is an accumulation point of X if and only if $\{m\}$ is not an open set of X.*
(2) *If X is a door space and $m \in X$ is an accumulation point of X, then $\{m\}$ is a closed set of X (m is a maximal point of X with respect to the order induced by the topology).*
(3) *If X is a door space, then $\dim_K(X, \mathcal{T}) \leqslant 1$.*
(4) *Suppose that X is a T_0-space. Then, any non-minimal point of the ordered set $(X, \leqslant_{\mathcal{T}})$ is an accumulation point of $(X, \mathcal{T})$.*
(5) *If X is a door space and x, y are distinct points of X_1, then $(\downarrow x) \setminus \{x\} = (\downarrow y) \setminus \{y\}$.*
(6) *If X is a compact door space, then $(x\uparrow)$ is finite, for each $x \in X$.*

Proof. (1) and (2) are straightforward.

(3) See [1, Proposition 2.2].

(4) Let m be a non-minimal point of the ordered set $(X, \leqslant_{\mathcal{T}})$. Hence there exists y such that $m \in \overline{\{y\}} \setminus \{y\}$. This implies that m is an accumulation point.

(5) Let $x, y \in X_1$ such that $x \neq y$. Suppose that there exists $x_0 \in (\downarrow x) \cap X_0$ such that $x_0 \notin (\downarrow y)$. The subset $\{x_0, y\}$ is neither open nor closed in X. Contradiction, since X is a door space. Therefore $(\downarrow x) \setminus \{x\} = (\downarrow y) \setminus \{y\}$.

(6) This follows immediately from [1, Proposition 2.7]. □

To give a characterization of compact door spaces, we need the following lemmas.

Lemma 2.2. *Let $(X, \mathcal{T})$ be a compact door space and A_c be the set of accumulation point(s) of X. If the cardinality of A_c is $\geqslant 2$, then the following properties hold and are equivalent.*

(1) *If U is an open set of X such that $U \cap A_c \neq \emptyset$, then U is co-finite.*
(2) *X is finite.*

Proof. As $(X, \mathcal{T})$ has at least two accumulation points, by [4], X has a subset S such that the cardinality of $X \setminus S$ is $\geqslant 2$ and there exists an ultrafilter $\mathcal{U}$ on X such that

$$\mathcal{T} = \mathcal{U} \cup \{G\colon\ G \subseteq S\}. \tag{$\star$}$$

(1) Let U be an open set such that $U \cap A_c \neq \emptyset$. If we suppose that $U \subseteq S$, then by Equality ($\star$), any subset of U is open. Hence for each $x \in U \cap A_c$, $\{x\}$ is open, contradicting the fact that x is an accumulation point. It follows, by Equality ($\star$), that $U \in \mathcal{U}$. Hence $U \cup \{x\} \in \mathcal{U}$, for each $x \in X$. Thus, as X is compact, there exist $x_1, x_2, \ldots, x_n \in X$ such that $X = U \cup \{x_1, x_2, \ldots, x_n\}$. Therefore, U is co-finite.

(2) Suppose that $X \setminus A_c$ is infinite. Let $G \subset X \setminus A_c$ be such that G and $(X \setminus A_c) \setminus G$ are infinite. Let $m \in A_c$; then, according to (1), $G \cup \{m\}$ is neither open nor closed, which is not possible, since X is a door space. Thus, $X \setminus A_c$ is finite.

Now, suppose that A_c is infinite. Let D be an infinite proper subset of A_c such that $A_c \setminus D$ is infinite. Then D is neither open nor closed, by (1).

Therefore, X is finite. □

Lemma 2.3. *Let $(X, \mathcal{T})$ be a compact door space such that $\dim_K(X, \mathcal{T}) = 1$. Then, exactly one of the following statements holds.*

(i) *There exists a unique element m of X such that the cardinality of $(\downarrow m)$ is $\geqslant 2$.*
(ii) *There exists a unique element m of X such that the cardinality of $(m\uparrow)$ is $\geqslant 3$.*

Proof. Suppose that neither (i) nor (ii) is satisfied.

Since $\dim_K(X,\mathcal{T})=1$, there exist $a,b\in X$ such that the cardinalities of $(\downarrow a)$ and $(\downarrow b)$ are $\geqslant 2$.

By Proposition 2.1(5), $(\downarrow a)\setminus\{a\}=(\downarrow b)\setminus\{b\}$. Let a_1 be an element of $(\downarrow a)\setminus\{a\}$. Then the cardinality of $(a_1\uparrow)$ is $\geqslant 3$. As (ii) is not satisfied, there is $b_1\neq a_1$ such that cardinality of $(b_1\uparrow)$ is $\geqslant 3$.

Let $c\in(b_1\uparrow)\setminus\{b_1\}$. Again, according to Proposition 2.1(5),

$$(\downarrow a)\setminus\{a\}=(\downarrow b)\setminus\{b\}=(\downarrow c)\setminus\{c\}.$$

This yields the following inequalities:

$$a_1<a,\qquad a_1<b,\qquad b_1<a,\qquad b_1<b.$$

Consequently, the set $\{a_1,b\}$ is neither open nor closed, contradicting the fact that X is a door space.

We conclude that, at least, one of the statements holds. Suppose that property (ii) is satisfied, then there is three pairwise distinct elements $m,a,b\in X$ such $m<a$ and $m<b$. Then the two elements a, b satisfy the following inequalities $|(\downarrow a)|\geqslant 2$ $|(\downarrow b)|\geqslant 2$, showing that (i) is not satisfied.

Therefore, exactly one of the statements holds. □

Lemma 2.4. *Let $(X,\mathcal{T})$ be a compact door space. If there is a unique element m of X such that $(\downarrow m)\neq\{m\}$, then m is the unique accumulation point of X.*

Proof. Suppose that there exists an accumulation point m' distinct from m. Since $\{m'\}$ is closed, $m'\notin(\downarrow m)$. Let $x\in(\downarrow m)\setminus\{m\}$. As, $m\in\overline{\{m',x\}}$, the set $\{m',x\}$ is not closed; so that it is an open set of X. On the other hand, the fact that m is the unique element of X such that $(\downarrow m)\neq\{m\}$, yields $m'\notin\overline{\{x\}}$. Thus $\{m'\}=\{m',x\}\cap(X\setminus\overline{\{x\}})$. Consequently, $\{m'\}$ is an open set, contradicting the fact that m' is an accumulation point. □

Lemma 2.5. *Let $(X,\mathcal{T})$ be a compact door space. If there exists an element m of X such that $|(m\uparrow)|\geqslant 3$, then X is finite.*

Proof. Let $a\in X$. As $\overline{\{m\}}\nsubseteq\{a,m\}$, the set $\{a,m\}$ is not closed; and consequently, it is an open set of X. Now, since in addition $(X,\mathcal{T})$ is compact, X must be finite. □

In 1976, Lewis and Ohm have introduced the $\mathcal{C}(m)$-topology as follows [6]:

Given an ordered set X and $m\in X$, the $\mathcal{C}(m)$-topology is the topology having the following basis for closed sets:

(i) Finite sets not containing m and closed under specialization.
(ii) Co-finite sets containing m and closed under specialization.

Hence the topology $\mathcal{C}(m)$ is the topology having the following basis for open sets:

(i) Finite sets O closed under generization such that $m\notin O$.
(ii) Co-finite sets O closed under generization such that $m\in O$.

It is clear that the $\mathcal{C}(m)$-topology is always a compact topology.

Now, we are in a position to give a characterization of compact door spaces.

Theorem 2.6. *Let $(X,\mathcal{T})$ be a T_0-space. Then the following statements are equivalent*:

(1) *$(X,\mathcal{T})$ is a compact door space.*
(2) *One of the following properties holds*:
  (i) *X is a finite discrete space (in this case, $\mathcal{T}=\mathcal{C}(m)$, for any $m\in X$).*
  (ii) *X is infinite, $\dim_K(X,\mathcal{T})=0$ and there exists $m\in X$ such that $\mathcal{T}=\mathcal{C}(m)$.*
  (iii) *$\dim_K(X,\mathcal{T})=1$, X has a unique point m such that $|(\downarrow m)|\geqslant 2$ and $\mathcal{T}=\mathcal{C}(m)$.*
  (iv) *X is finite, $\dim_K(X,\mathcal{T})=1$, X has a unique point m such that $|(m\uparrow)|\geqslant 3$ and $\mathcal{T}=\mathcal{C}(m)$.*

Proof. (1) $\Longrightarrow$ (2) By Proposition 2.1(3), $\dim_K(X,\mathcal{T})\leqslant 1$.

Suppose that $\dim_K(X,\mathcal{T})=0$. Then, three cases arise.

1. X has no accumulation points. In this case, X is clearly discrete. As X is compact, X is finite.
2. X has exactly one accumulation point (say m). Then, it is clear that X is infinite. Let us prove that $\mathcal{T}=\mathcal{C}(m)$. Indeed, let $O\in\mathcal{T}$.
   – Suppose that $m\notin O$. Then, clearly, $O\in\mathcal{C}(m)$.

- Suppose that $m \in O$. Since X is compact and every point of X distinct from m is open in $(X, \mathcal{T})$, O is necessarily co-finite. Thus, $O \in \mathcal{C}(m)$.

Conversely, let $O \in \mathcal{C}(m)$. We prove that $O \in \mathcal{T}$. It suffices to do this for the elements of the basis of $\mathcal{C}(m)$ explained above.

- Suppose that O is finite and $m \notin O$. Then, clearly, $O \in \mathcal{T}$, since each point of X distinct from m is open in $(X, \mathcal{T})$.
- Suppose that O is co-finite and $m \in O$. Since X is a T_1-space, $X \setminus O$ is a closed set of $(X, \mathcal{T})$. Therefore, $O \in \mathcal{T}$.

3. X has more than one accumulation point. In this case, by Lemma 2.2, X is finite. Now as $\dim_K(X, \mathcal{T}) = 0$, we see that X is T_1. As X is finite and T_1, X must be discrete. But X has accumulation points, so X cannot be discrete. Therefore, this case cannot happen.

Now, let us suppose that $\dim_K(X, \mathcal{T}) = 1$. Then, by Lemma 2.3, two other cases are to be considered.

4. There exists a unique element m of X such that $|(\downarrow m)| \geqslant 2$.
According to Lemma 2.4, m is the unique accumulation point of $(X, \mathcal{T})$. As in Case 2, one may prove easily that $\mathcal{T} = \mathcal{C}(m)$.
5. There exists a unique element m of X such that $|(m\uparrow)| \geqslant 3$. Hence, by Lemma 2.5, X is finite. Thus, the $\mathcal{C}(m)$-topology on X consists of all subsets of X closed under generization. Consequently, $\mathcal{C}(m) = \mathcal{A}(\leqslant_{\mathcal{T}})$. On the other hand, since X is a finite T_0-space, we have $\mathcal{T} = \mathcal{A}(\leqslant_{\mathcal{T}})$. Therefore, $\mathcal{C}(m) = \mathcal{T}$.

(2) $\Longrightarrow$ (1) To prove that $(X, \mathcal{T})$ is a door compact space, let us discuss the following two cases:

1. $(X, \mathcal{T})$ satisfies one of Properties (i), (ii) and (iii).
Let U be a subset of X. If $m \notin U$, then U is open, since $\{x\}$ is open, for each $x \in X \setminus \{m\}$. If $m \in U$, then $X \setminus U$ is open and hence U is closed. It follows that $(X, \mathcal{C}(m))$ is a door space (we have already mentioned that the $\mathcal{C}(m)$-topology is compact).
2. $(X, \mathcal{T})$ satisfies Property (iv). In this case, each subset U containing m is closed under generization; so it is open. Thus $(X, \mathcal{C}(m))$ is a door space. □

3. Door spaces and compactifications

In [1], Adams et al. called a topological space X *A-door*, if its Alexandroff compactification is door, and they give the following characterization of A-door spaces.

Proposition 3.1. *([1, Proposition 4.3]) Let X be a noncompact space. Then the following statements are equivalent*:

(i) *X is an A-door space.*
(ii) *Every subset of X is either an open set or a compact closed set of X.*

We are interested in topological spaces X such that some compactification $K(X)$ of X is a door space.

Theorem 3.2. *Let X be a noncompact space and $K(X)$ be a compactification of X. Then the following statements are equivalent*:

(1) *$K(X)$ is a door space.*
(2) *X is a discrete space and $K(X)$ is the one-point compactification of X.*

Proof. (1) $\Longrightarrow$ (2) First, let us first show that there exists $m \in K(X)$ such that $\{m\} = K(X) \setminus X$.

By Theorem 2.6, $K(X)$ is equipped with the $\mathcal{C}(m)$-topology with $\dim_K(K(X), \mathcal{C}(m)) = 0$ or $\dim_K(K(X), \mathcal{C}(m)) = 1$ and m is the unique point of $K(X)$ such that $(\downarrow m) \neq \{m\}$.

Anyway, for each $x \in K(X) \setminus \{m\}$, x is a minimal element for the topology induced by the order. As the topology on $K(X)$ is the $\mathcal{C}(m)$-topology, we deduce that $\{x\}$ is an open set of $K(X)$, for each $x \in K(X) \setminus \{m\}$. It follows that X is a discrete space. On the other hand, since X is dense in $K(X)$, we deduce that $\{y\}$ is a non-open set of $K(X)$, for each $y \in K(X) \setminus X$. Therefore, $K(X) \setminus X = \{m\}$.

Now, let us show that the topology on $K(X)$ coincides with the one-point compactification of X.

Indeed, let U be an open set of $K(X)$. We consider two cases:

- If $m \in U$, then $K(X) \setminus U$ is finite. Hence $X \setminus U$ is a closed compact set of X.
- If $m \notin U$, then it is an open set of X.

In both the two previous cases, U is an open set of the one-point compactification of X.

Conversely, let V be an open set of the one-point compactification of X. Again, two cases are to be considered.

– Suppose that $m \in V$. Then $K(X) \setminus V$ is a compact closed set of X. As X is discrete, $K(X) \setminus V$ is finite; and consequently, V is an open set of the $\mathcal{C}(m)$-topology on $K(X)$.
– Now, suppose that $m \notin V$. By definition of the $\mathcal{C}(m)$-topology on $K(X)$, V is an open set of $K(X)$.

We conclude that the topology of $K(X)$ coincides with the one-point compactification of X.
(2) $\Longrightarrow$ (1) follows immediately from Proposition 3.1. □

4. Example

First, let us recall the construction of Wallman compactification of T_1-space (a concept introduced, in 1938, by Wallman [8]).

Let $\mathcal{P}$ be a class of subsets of a topological space X which is closed under finite intersections and finite unions.

A $\mathcal{P}$-filter on X is a collection $\mathcal{F}$ of nonempty elements of $\mathcal{P}$ with the properties:

(i) *$\mathcal{F}$ is closed under finite intersections*;
(ii) $P_1 \subseteq P_2$, and $P_1 \in \mathcal{F}$ *imply* $P_2 \in \mathcal{F}$.

A *$\mathcal{P}$-ultrafilter* is a maximal $\mathcal{P}$-filter. When $\mathcal{P}$ is the class of closed sets of X, then the $\mathcal{P}$-filters are called *closed filters*.

The points of the Wallman compactification wX of a space X are the closed ultrafilters on X. For each closed set $D \subseteq X$, define D^* to be the set $D^* = \{\mathcal{A} \in wX \mid D \in \mathcal{A}\}$. Thus $\mathcal{C} = \{D^* \mid D$ is a closed set of $X\}$ is a base for the closed sets of a topology on wX.

Let U be an open set of X. We define $U^* = \{\mathcal{A} \in wX \mid F \subseteq U$ for some F in $\mathcal{A}\}$. It is easily seen that the class $\{U^* \mid U$ is an open set of $X\}$ is a base for open sets of the topology of wX. The following properties of wX are frequently useful:

Proposition 4.1. *For $x \in X$, let $w_X(x) = \{A \mid A$ is a closed set of X and $x \in A\}$. Then w_X is an embedding of X into wX. Thus, if $x \in X$, then $w_X(x)$ will be identified to x.*

Proposition 4.2. *If $U \subset X$ is open, then $wX \setminus U^* = (X \setminus U)^*$.*

Proposition 4.3. *If $D \subset X$ is closed, then $wX \setminus D^* = (X \setminus D)^*$.*

Proposition 4.4. *If U_1 and U_2 are open in X, then $(U_1 \cap U_2)^* = U_1^* \cap U_2^*$ and $(U_1 \cup U_2)^* = U_1^* \cup U_2^*$.*

Remark 4.5. Let $\mathcal{F} \in wX \setminus X$. Let $F \in \mathcal{F}$; then, clearly, the collection of closed sets $\{G \cap F \mid G \in \mathcal{F}\}$ has the finite intersection property. Nevertheless, $\bigcap[G \cap F\colon\ G \in \mathcal{F}] = \emptyset$; indeed, if it is not the case, there exists an $x \in \bigcap[G\colon\ G \in \mathcal{F}]$. Hence, $\mathcal{F} = w_X(x)$, contradicting the fact that $\mathcal{F} \in wX \setminus X$. It follows that F is a noncompact closed set of X.

We need, also, further new concepts.

Definition 4.6. Let X be a topological space. Let $c\bar{c}(X)$ be the set of all collections $\mathcal{H}$ of closed noncompact sets of X satisfying the following property:

if A, B are distinct elements of $\mathcal{H}$, then $A \cap B = \emptyset$.

The supremum of the cardinalities $|\mathcal{H}|$, when $\mathcal{H}$ lies in $c\bar{c}(X)$ will be denoted by $\dim_{c\bar{c}}(X)$ and called the *closed noncompact dimension* of X.

Remark 4.7. Let X, Y be two disjoint topological spaces. Recall that the free union $X + Y$ is the set $X \cup Y$ equipped with the topology whose open sets are $U \subseteq X \cup Y$ satisfying $U \cap X$ is open in X and $U \cap Y$ is open in Y. Then, it is clear that

$$\dim_{c\bar{c}}(X + Y) = \dim_{c\bar{c}}(X) + \dim_{c\bar{c}}(Y).$$

The following examples show that for each $n \in \mathbb{N} \cup \{\infty\}$, there exists a space X such that $\dim_{c\bar{c}}(X) = n$.

Example 4.8.

(1) A topological space X is compact if and only if $\dim_{c\bar{c}}(X) = 0$.
(2) To construct a space X such that $\dim_{c\bar{c}}(X) = 1$, it suffices to consider a noncompact space X in which two distinct noncompact closed sets meet. To illustrate that space, we let $(X, \leqslant)$ be a totally ordered set with no maximal element. Equip X with the Alexandroff topology associated with the ordering $\leqslant$. Then, one may check easily that $\dim_{c\bar{c}}(X) = 1$.

(3) Now, to get a space X such that $\dim_{c\bar{c}}(X) = n$, it suffices to consider n disjoint spaces $X_1, X_2, \ldots, X_n$ such that $\dim_{c\bar{c}}(X_i) = 1$; and then take $X := X_1 + X_2 + \cdots + X_n$ the free union space. Of course, $\dim_{c\bar{c}}(X) = n$.

Remark 4.9. Let X be a T_1-space. If F is a closed noncompact set of X, then F lies in an ultrafilter $\mathcal{F}$ such that $\bigcap\{G \mid G \in \mathcal{F}\} = \emptyset$ (that is, $\mathcal{F} \in wX \setminus X$).

Proposition 4.10. *Let $n \in \mathbb{N}$ and X be a T_1-space. Then the following statements are equivalent*:

(1) $\dim_{c\bar{c}}(X) = n$.
(2) $|wX \setminus X| = n$.

Proof. Of course, if $n = 0$, then the equivalence is straightforward. Let us suppose that $n \in N \setminus \{0\}$.

(1) $\Longrightarrow$ (2) Suppose that $wX \setminus X$ contains $n+1$ distinct elements, say for instance $\mathcal{F}_1, \ldots, \mathcal{F}_{n+1}$.

Since $\mathcal{F}_1 \neq \mathcal{F}_2$, there exists $F_1 \in \mathcal{F}_1$ such that $F_1 \notin \mathcal{F}_2$. Hence, there is an $F_2 \in \mathcal{F}_2$ such that $F_1 \cap F_2 = \emptyset$.

Suppose that, for $i \leqslant n$, there exist $F_1, F_2, \ldots, F_i$ pairwise disjoint, with $F_j \in \mathcal{F}_j$. Let us show that there are $G_1, G_2, \ldots, G_i, G_{i+1}$ pairwise disjoint, with $G_j \in \mathcal{F}_j$. Indeed, since $\mathcal{F}_{i+1} \neq \mathcal{F}_j$ for $1 \leqslant j \leqslant i$, there exist $H_j \in \mathcal{F}_{i+1}$ and $R_j \in \mathcal{F}_j$ such that $H_j \cap R_j = \emptyset$. For $1 \leqslant j \leqslant i$, set $G_j := R_j \cap F_j$. Consider $G_{i+1} := H_1 \cap H_2 \cap \cdots \cap H_i$. Then $G_1, G_2, \ldots, G_{i+1}$ are pairwise disjoint, and $G_j \in \mathcal{F}_j$, for each $1 \leqslant j \leqslant i+1$.

Thus, there exist pairwise disjoint $G_1, G_2, \ldots, G_{n+1}$ such that $G_i \in \mathcal{F}_i$. Since, in addition each G_i is noncompact (see Remark 4.5), $\dim_{c\bar{c}}(X) \geqslant n+1$, a contradiction. Therefore, $|wX \setminus X| \leqslant n$.

On the other hand, let $F_1, F_2, \ldots, F_n$ be pairwise disjoint noncompact closed sets of X. Then, there exist $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n \in wX \setminus X$, such that $F_i \in \mathcal{F}_i$, by Remark 4.9. Since $F_i \cap F_j = \emptyset$, for each $i \neq j$, we deduce that $\mathcal{F}_i \neq \mathcal{F}_j$. Consequently, $|wX \setminus X| \geqslant n$.

(2) $\Longrightarrow$ (1) Straightforward. □

Now, we are in a position to give a characterization of w-submaximal spaces.

Proposition 4.11. *Let X be a T_1-space. Then the following statements are equivalent*:

(1) *X is a w-submaximal space.*
(2) *X is a submaximal space and $\dim_{c\bar{c}}(X)$ is finite.*

Proof. (1) $\Longrightarrow$ (2) Follows immediately from Remark 1.5.

According to Corollary 1.4, it suffices to prove that each dense subset of X is co-finite in wX. Indeed, let D be a dense subset of X. As X is submaximal, D is co-finite discrete in its closure. On the other hand, by Proposition 4.10 $\dim_{c\bar{c}}(X)$ is finite means that $|wX \setminus X|$ is finite. Thus, D is co-finite in wX. □

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