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G-frames and g-Riesz bases ☆

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Abstract

G-frames are generalized frames which include ordinary frames, bounded invertible linear operators, as well as many recent generalizations of frames, e.g., bounded quasi-projectors and frames of subspaces. G-frames are natural generalizations of frames and provide more choices on analyzing functions from frame expansion coefficients. We give characterizations of g-frames and prove that g-frames share many useful properties with frames. We also give a generalized version of Riesz bases and orthonormal bases. As an application, we get atomic resolutions for bounded linear operators.

Keywords: Frames; g-Frames; g-Riesz bases; g-Orthonormal bases; Atomic resolution

1. Introduction

Frames were first introduced in 1952 by Duffin and Schaeffer [9], reintroduced in 1986 by Daubechies, Grossmann, and Meyer [6], and popularized from then on. Frames have many nice properties which make them very useful in the characterization of function spaces, signal processing and many other fields. We refer to [4,7,11,14–16,20] for an introduction to frame theory and its applications. One of the main virtues of frames is that, given a frame, we can get properties of a function and reconstruct it only from the frame coefficients, a sequence of complex numbers. For example, let $\{a^{j/2}\psi_\ell(a^j\cdot -bk)\colon\ 1\leqslant \ell\leqslant r,\ \ j,k\in\mathbb{Z}\}$ be a

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multi-wavelet frame for $L^2(\mathbb{R})$. Then every $f \in L^2(\mathbb{R})$ can be reconstructed by the sequence $\{\langle f, a^{j/2}\psi_\ell(a^j \cdot -bk)\rangle\colon 1 \leqslant \ell \leqslant r,\ j,k \in \mathbb{Z}\}$ which satisfies

$$A\|f\|_2^2 \leqslant \sum_{1\leqslant \ell\leqslant r}\sum_{j,k\in\mathbb{Z}} \left|\left\langle f, a^{j/2}\psi_\ell\left(a^j \cdot -bk\right)\right\rangle\right|^2 \leqslant B\|f\|_2^2$$

for some positive constants A and B. Put

$$c_{j,k}(f) = \left(\left\langle f, a^{j/2}\psi_1\left(a^j \cdot -bk\right)\right\rangle, \ldots, \left\langle f, a^{j/2}\psi_r\left(a^j \cdot -bk\right)\right\rangle\right)^T \in \mathbb{C}^r.$$

Then the above inequalities turn out to be

$$A\|f\|_2^2 \leqslant \sum_{j,k\in\mathbb{Z}} \left\|c_{j,k}(f)\right\|^2 \leqslant B\|f\|_2^2.$$

This prompts us to give the following generalization of frames.

Throughout this paper, $\mathcal{U}$ and $\mathcal{V}$ are two Hilbert spaces and $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$ is a sequence of subspaces of $\mathcal{V}$, where $\mathbb{J}$ is a subset of $\mathbb{Z}$. $\mathcal{L}(\mathcal{U}, \mathcal{V}_j)$ is the collection of all bounded linear operators from $\mathcal{U}$ into $\mathcal{V}_j$.

Note that for any sequence $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$ of Hilbert spaces, we can always find a larger Hilbert space $\mathcal{V}$ to contain all the $\mathcal{V}_j$ by setting $\mathcal{V} = \bigoplus_{j\in\mathbb{J}} \mathcal{V}_j$.

Definition 1.1. We call a sequence $\{\Lambda_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j)\colon j \in \mathbb{J}\}$ a generalized frame, or simply a g-frame, for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$ if there are two positive constants A and B such that

$$A\|f\|^2 \leqslant \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathcal{U}. \tag{1.1}$$

We call A and B the lower and upper frame bounds, respectively.

We call $\{\Lambda_j\colon j \in \mathbb{J}\}$ a tight g-frame if $A = B$.

We call $\{\Lambda_j\colon j \in \mathbb{J}\}$ an exact g-frame if it ceases to be a g-frame whenever any one of its elements is removed.

We call this family a g-frame for $\mathcal{U}$ whenever the space sequence $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$ is clear.

We call this family a g-frame for $\mathcal{U}$ with respect to $\mathcal{V}$ whenever $\mathcal{V}_j = \mathcal{V}$, $\forall j \in \mathbb{J}$.

We observe that various generalizations of frames have been proposed recently. For example, bounded quasi-projectors [12,13], frames of subspaces [2,3], pseudo-frames [17], oblique frames [5,10], and outer frames [1]. All of these generalizations have proved to be useful in many applications. Here we point out that they can be regarded as special cases of g-frames (see examples below) and many basic properties can be derived within this more general context.

While we were preparing this paper we learned that another generalization of frames in the context of numerical analysis, called stable space splittings, have been studied in [18,19]. We prove at the end of Section 3 that they are equivalent to g-frames. We point out that the approaches are quite different from each other. In particular, the adjoint operators of Λ_j are used in the definition of stable space splittings. Moreover, we give a characterization of g-frames and study g-Riesz bases and g-orthonormal bases.

Example 1.1. Let $\mathcal{H}$ be a separable Hilbert space and $\{f_j\colon j \in \mathbb{J}\}$ be a frame for $\mathcal{H}$. Let Λ_{f_j} be the functional induced by f_j, i.e.,

$$\Lambda_{f_j} f = \langle f, f_j\rangle, \quad \forall f \in \mathcal{H}.$$

It is easy to check that $\{\Lambda_{f_j}\colon j \in \mathbb{J}\}$ is a g-frame for $\mathcal{H}$ with respect to $\mathbb{C}$.

By the Riesz Representation Theorem, to every functional $\Lambda \in \mathcal{L}(\mathcal{U}, \mathbb{C})$, one can find some $\varphi \in \mathcal{U}$ such that $\Lambda f = \langle f, \varphi \rangle$, $\forall f \in \mathcal{U}$. Hence we have the following.

Lemma 1.1. *A frame is equivalent to a g-frame whenever $\mathcal{V}_j = \mathbb{C}$, $j \in \mathbb{J}$.*

Example 1.2. Pseudo-frames (Li and Ogawa[17]), or similar, oblique frames (Christensen and Eldar [5,10]) or outer frames (Aldroubi et al. [1]) have been studied recently in literature. Here we point out that they are all classes of g-frames.

Let $\mathcal{H}_0$ be a closed subspace of $\mathcal{H}$. Let $\{f_j\colon j \in \mathbb{J}\} \subset \mathcal{H}$ be a Bessel sequence in $\mathcal{H}_0$ and $\{\tilde{f}_j\colon j \in \mathbb{J}\} \subset \mathcal{H}$ be a Bessel sequence in $\mathcal{H}$. Recall that $\{f_j\colon j \in \mathbb{J}\}$ is said to be a pseudo-frame for $\mathcal{H}_0$ with respect to $\{\tilde{f}_j\colon j \in \mathbb{J}\}$ [17, Definition 1] if

$$f = \sum_{j \in \mathbb{J}} \langle f, f_j \rangle \tilde{f}_j, \quad \forall f \in \mathcal{H}_0.$$

Since both $\{f_j\colon j \in \mathbb{J}\}$ and $\{\tilde{f}_j\colon j \in \mathbb{J}\}$ are Bessel sequences in $\mathcal{H}_0$, it is easy to check from the above equation that we can find some constants $A, B > 0$ such that $A\|f\|^2 \leqslant \sum_{j \in \mathbb{J}} |\langle f, f_j \rangle|^2 \leqslant B\|f\|^2$, $\forall f \in \mathcal{H}_0$. Let Λ_{f_j} be the functional induced by f_j, $j \in \mathbb{J}$. Then we have

$$A\|f\|^2 \leqslant \sum_{j \in \mathbb{J}} |\Lambda_{f_j} f|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathcal{H}_0.$$

In other words, $\{\Lambda_{f_j}\colon j \in \mathbb{J}\}$ is a g-frame for $\mathcal{H}_0$ with respect to $\mathbb{C}$.

Example 1.3. Bounded quasi-projectors (Fornasier [12,13]).

It was shown in [12, Lemma 1] that if a system of bounded quasi-projectors $\{P_j\colon j \in \mathbb{J}\}$ is self-adjoint and compatible with the canonical projections (see [12,13] for details), then for any $f \in \mathcal{H}$,

$$A\|f\|^2 \leqslant \sum_{j \in \mathbb{J}} \|P_j f\|^2 \leqslant B\|f\|^2.$$

In this case, $\{P_j\colon j \in \mathbb{J}\}$ is a g-frame for $\mathcal{H}$ with respect to $\mathcal{H}$.

Example 1.4. Frames of subspaces (Casazza and Kutyniok [3] and Asgari and Khosravi [2]).

Let $\{W_j\colon j \in \mathbb{J}\}$ be a sequence of subspaces of $\mathcal{H}$ and P_{W_j} be the orthogonal projection on W_j. $\{W_j\colon j \in \mathbb{J}\}$ is called a frame of subspaces if there exist positive constants A and B such that

$$A\|f\|^2 \leqslant \sum_{j \in \mathbb{J}} \|P_{W_j} f\|^2 \leqslant B\|f\|^2, \quad \forall f \in \mathcal{H}.$$

Obviously, a frame of subspaces is a g-frame for $\mathcal{H}$ with respect to $\{W_j\colon j \in \mathbb{J}\}$.

Example 1.5. Time-frequency localization operators (Dörfler et al. [8]).

For $f, g \in L^2(\mathbb{R}^d)$, define the windowed Fourier transform of f with respect to g by

$$(V_g f)(t,\omega) = \int_{\mathbb{R}^d} f(x)\overline{g(x-t)}e^{-i2\pi x\omega}\,dx.$$

Let $S_0(\mathbb{R}^d) := \{g \in L^2(\mathbb{R}^d)\colon\ V_g g \in L^1(\mathbb{R}^{2d})\}$ be the Feichtinger algebra. Take some $\varphi \in S_0(\mathbb{R}^d)$ with $\|\varphi\|_2 = 1$. Let σ be a bounded function on $\mathbb{R}^{2d}$ with compact support and $\sigma(x) \geqslant 0$. Define the time-frequency localization operator H_σ corresponding to σ and φ by $H_\sigma f = V_\varphi^* \sigma V_\varphi f$. If $\sigma \in S_0(\mathbb{R}^{2d})$ and

$$C_1 \leqslant \sum_{k\in\mathbb{Z}^{2d}} \sigma(x-k) \leqslant C_2,$$

for some constants $C_1, C_2 > 0$, then it is shown in [8] that one can find some constants $A, B > 0$ such that

$$A\|f\|_2^2 \leqslant \sum_{k\in\mathbb{Z}^{2d}} \|H_{\sigma(\cdot-k)} f\|_2^2 \leqslant B\|f\|_2^2, \quad \forall f \in L^2\big(\mathbb{R}^d\big).$$

Hence $\{H_{\sigma(\cdot-k)}\colon\ k \in \mathbb{Z}^{2d}\}$ is a g-frame for $L^2(\mathbb{R}^d)$ with respect to $L^2(\mathbb{R}^d)$. We refer to [8] for details.

Example 1.6. Every bounded invertible linear operator itself forms a g-frame.

We see from the above examples that g-frames are natural generalizations of frames and provide more choices on analyzing functions from frame expansion coefficients. In the following sections we first study g-frame operators and get the dual g-frames, then give definitions of g-Riesz bases and g-orthonormal bases and present characterizations of generalized frames and bases. As an application of g-frames, we get atomic resolutions of bounded linear operators.

2. G-frame operators and dual g-frames

Let $\{\Lambda_j\colon\ j \in \mathbb{J}\}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon\ j \in \mathbb{J}\}$. Define the g-frame operator S as follows:

$$Sf = \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j f, \quad \forall f \in \mathcal{U}, \tag{2.1}$$

where Λ_j^* is the adjoint operator of Λ_j. First of all, S is well defined on $\mathcal{U}$. To see this, let $n_1 < n_2$ be integers. Then we have

$$\begin{aligned}\left\|\sum_{j=n_1}^{n_2} \Lambda_j^*\Lambda_j f\right\| &= \sup_{h\in\mathcal{U},\,\|h\|=1}\left|\left\langle \sum_{j=n_1}^{n_2} \Lambda_j^*\Lambda_j f, h\right\rangle\right| = \sup_{\|h\|=1}\left|\sum_{j=n_1}^{n_2}\langle \Lambda_j f, \Lambda_j h\rangle\right| \\ &\leqslant \sup_{\|h\|=1}\left(\sum_{j=n_1}^{n_2}\|\Lambda_j f\|^2\right)^{1/2}\cdot\left(\sum_{j=n_1}^{n_2}\|\Lambda_j h\|^2\right)^{1/2} \\ &\leqslant B^{1/2}\left(\sum_{j=n_1}^{n_2}\|\Lambda_j f\|^2\right)^{1/2}.\end{aligned}$$

Now we see from (1.1) that the series in (2.1) are convergent. Therefore, Sf is well defined for any $f \in \mathcal{U}$.

On the other hand, it is easy to check that for any $f_1, f_2 \in \mathcal{U}$,

$$\langle Sf_1, f_2\rangle = \sum_{j\in\mathbb{J}} \langle \Lambda_j^*\Lambda_j f_1, f_2\rangle = \sum_{j\in\mathbb{J}} \langle f_1, \Lambda_j^*\Lambda_j f_2\rangle = \langle f_1, Sf_2\rangle$$

and therefore,

$$\|S\| = \sup_{\|f\|=1} \langle Sf, f\rangle = \sup_{\|f\|=1} \sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 \leqslant B.$$

Hence S is a bounded self-adjoint operator.

Since $A\|f\|^2 \leqslant \langle Sf, f\rangle \leqslant \|Sf\| \cdot \|f\|$, we have

$$\|Sf\| \geqslant A\|f\|,$$

which implies that S is injective and $S\mathcal{U}$ is closed in $\mathcal{U}$. Let $f_2 \in \mathcal{U}$ be such that $\langle Sf_1, f_2\rangle = 0$ for every $f_1 \in \mathcal{U}$. Then we have $\langle f_1, Sf_2\rangle = 0$, $\forall f_1 \in \mathcal{U}$. This implies that $Sf_2 = 0$ and therefore $f_2 = 0$. Hence $S\mathcal{U} = \mathcal{U}$. Consequently, S is invertible and

$$\|S^{-1}\| \leqslant \frac{1}{A}.$$

For any $f \in \mathcal{U}$, we have

$$f = SS^{-1}f = S^{-1}Sf = \sum_{j\in\mathbb{J}} \Lambda_j^*\Lambda_j S^{-1} f = \sum_{j\in\mathbb{J}} S^{-1}\Lambda_j^*\Lambda_j f.$$

Let $\tilde{\Lambda}_j = \Lambda_j S^{-1}$. Then the above equalities become

$$f = \sum_{j\in\mathbb{J}} \Lambda_j^*\tilde{\Lambda}_j f = \sum_{j\in\mathbb{J}} \tilde{\Lambda}_j^*\Lambda_j f. \tag{2.2}$$

We now prove that $\{\tilde{\Lambda}_j\colon j \in \mathbb{J}\}$ is also a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$.

In fact, for any $f \in \mathcal{U}$, we have

$$\begin{aligned}\sum_{j\in\mathbb{J}} \|\tilde{\Lambda}_j f\|^2 &= \sum_{j\in\mathbb{J}} \|\Lambda_j S^{-1} f\|^2 = \sum_{j\in\mathbb{J}} \langle \Lambda_j S^{-1} f, \Lambda_j S^{-1} f\rangle = \sum_{j\in\mathbb{J}} \langle \Lambda_j^*\Lambda_j S^{-1} f, S^{-1} f\rangle \\ &= \langle SS^{-1} f, S^{-1} f\rangle = \langle f, S^{-1} f\rangle \leqslant \frac{1}{A}\|f\|^2.\end{aligned}$$

On the other hand, since

$$\begin{aligned}\|f\|^2 &= \sum_{j\in\mathbb{J}} \langle \tilde{\Lambda}_j^*\Lambda_j f, f\rangle = \sum_{j\in\mathbb{J}} \langle \Lambda_j f, \tilde{\Lambda}_j f\rangle \\ &\leqslant \Big(\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2\Big)^{1/2} \cdot \Big(\sum_{j\in\mathbb{J}} \|\tilde{\Lambda}_j f\|^2\Big)^{1/2} \\ &\leqslant B^{1/2}\|f\| \Big(\sum_{j\in\mathbb{J}} \|\tilde{\Lambda}_j f\|^2\Big)^{1/2},\end{aligned}$$

we have

$$\sum_{j\in\mathbb{J}}\|\tilde{\Lambda}_j f\|^2 \geqslant \frac{1}{B}\|f\|^2.$$

Hence, $\{\tilde{\Lambda}_j\colon j\in\mathbb{J}\}$ is a g-frame for $\mathcal{U}$ with frame bounds $1/B$ and $1/A$. We call it the (canonical) dual g-frame of $\{\Lambda_j\colon j\in\mathbb{J}\}$.

Let $\tilde{S}$ be the g-frame operator associated with $\{\tilde{\Lambda}_j\colon j\in\mathbb{J}\}$. Then we have

$$S\tilde{S}f=\sum_{j\in\mathbb{J}}S\tilde{\Lambda}_j^*\tilde{\Lambda}_j f=\sum_{j\in\mathbb{J}}SS^{-1}\Lambda_j^*\Lambda_j S^{-1}f=\sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_j S^{-1}f$$
$$=SS^{-1}f=f,\quad \forall f\in\mathcal{U}.$$

Hence $\tilde{S}=S^{-1}$ and $\tilde{\Lambda}_j\tilde{S}^{-1}=\Lambda_j S^{-1}S=\Lambda_j$. In other words, $\{\Lambda_j\colon j\in\mathbb{J}\}$ and $\{\tilde{\Lambda}_j\colon j\in\mathbb{J}\}$ are dual g-frames with respect to each other.

Remark. We see from the above arguments that g-frames behave very similarly to frames. For example, we can always get a tight g-frame from any g-frame $\{\Lambda_j\colon j\in\mathbb{J}\}$. In fact, put

$$Q_j=\Lambda_j S^{-1/2}.$$

It is easy to check that $\{Q_j\colon j\in\mathbb{J}\}$ is a tight g-frame with the frame bound 1.

Moreover, the canonical dual g-frames give rise to expansion coefficients with the minimal norm.

Lemma 2.1. *Let $\{\Lambda_j\colon j\in\mathbb{J}\}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon j\in\mathbb{J}\}$ and $\tilde{\Lambda}_j=\Lambda_j S^{-1}$. Then for any $g_j\in\mathcal{V}_j$ satisfying $f=\sum_{j\in\mathbb{J}}\Lambda_j^*g_j$, we have*

$$\sum_{j\in\mathbb{J}}\|g_j\|^2=\sum_{j\in\mathbb{J}}\|\tilde{\Lambda}_j f\|^2+\sum_{j\in\mathbb{J}}\|g_j-\tilde{\Lambda}_j f\|^2.$$

Proof. It is easy to check that

$$\sum_{j\in\mathbb{J}}\|\tilde{\Lambda}_j f\|^2=\sum_{j\in\mathbb{J}}\langle\tilde{\Lambda}_j f,\Lambda_j S^{-1}f\rangle=\sum_{j\in\mathbb{J}}\langle\Lambda_j^*\tilde{\Lambda}_j f,S^{-1}f\rangle=\sum_{j\in\mathbb{J}}\langle\Lambda_j^*g_j,S^{-1}f\rangle$$
$$=\sum_{j\in\mathbb{J}}\langle g_j,\Lambda_j S^{-1}f\rangle=\sum_{j\in\mathbb{J}}\langle g_j,\tilde{\Lambda}_j f\rangle,\quad \forall f\in\mathcal{U}.$$

Now the conclusion follows. □

In Example 1.1, we show that every frame $\{f_j\colon j\in\mathbb{J}\}$ for $\mathcal{H}$ induces a g-frame $\{\Lambda_{f_j}\colon j\in\mathbb{J}\}$ for $\mathcal{H}$ with respect to $\mathbb{C}$ via the induced functionals Λ_{f_j}.

Let $\{\tilde{f}_j\colon j\in\mathbb{J}\}$ be the canonical dual frame of $\{f_j\colon j\in\mathbb{J}\}$. We conclude that $\{\Lambda_{\tilde{f}_j}\colon j\in\mathbb{J}\}$ is the canonical dual g-frame of $\{\Lambda_{f_j}\colon j\in\mathbb{J}\}$.

In fact, it is easy to see that $\Lambda_{f_j}^*c=cf_j$ for any $c\in\mathbb{C}$, which implies that the corresponding g-frame operator and frame operator are the same. Consequently,

$$\Lambda_{f_j}S^{-1}f=\langle S^{-1}f,f_j\rangle=\langle f,S^{-1}f_j\rangle=\langle f,\tilde{f}_j\rangle=\Lambda_{\tilde{f}_j}f,\quad \forall f\in\mathcal{H}.$$

Hence $\Lambda_{\tilde{f}_j}=\Lambda_{f_j}S^{-1}$. In other words, $\{\Lambda_{\tilde{f}_j}\colon j\in\mathbb{J}\}$ is the dual g-frame of $\{\Lambda_{f_j}\colon j\in\mathbb{J}\}$.

3. Generalized Bessel sequences, Riesz bases and orthonormal bases

Similarly to generalized frames, we can define generalized Bessel sequences, Riesz bases, and orthonormal bases.

Definition 3.1. Let $\Lambda_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j)$, $j \in \mathbb{J}$.

(i) If the right-hand inequality of (1.1) holds, then we say that $\{\Lambda_j\colon\ j \in \mathbb{J}\}$ is a g-Bessel sequence for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon\ j \in \mathbb{J}\}$.
(ii) If $\{f\colon\ \Lambda_j f = 0,\ j \in \mathbb{J}\} = \{0\}$, then we say that $\{\Lambda_j\colon\ j \in \mathbb{J}\}$ is g-complete.
(iii) If $\{\Lambda_j\colon\ j \in \mathbb{J}\}$ is g-complete and there are positive constants A and B such that for any finite subset $\mathbb{J}_1 \subset \mathbb{J}$ and $g_j \in \mathcal{V}_j,\ j \in \mathbb{J}_1$,

$$A \sum_{j \in \mathbb{J}_1} \|g_j\|^2 \leqslant \left\| \sum_{j \in \mathbb{J}_1} \Lambda_j^* g_j \right\|^2 \leqslant B \sum_{j \in \mathbb{J}_1} \|g_j\|^2, \tag{3.1}$$

then we say that $\{\Lambda_j\colon\ j \in \mathbb{J}\}$ is a g-Riesz basis for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon\ j \in \mathbb{J}\}$.
(iv) We say $\{\Lambda_j\colon\ j \in \mathbb{J}\}$ is a g-orthonormal basis for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon\ j \in \mathbb{J}\}$ if it satisfies the following:

$$\langle \Lambda_{j_1}^* g_{j_1}, \Lambda_{j_2}^* g_{j_2} \rangle = \delta_{j_1, j_2} \langle g_{j_1}, g_{j_2} \rangle, \quad \forall j_1, j_2 \in \mathbb{J},\ g_{j_1} \in \mathcal{V}_{j_1},\ g_{j_2} \in \mathcal{V}_{j_2}, \tag{3.2}$$

$$\sum_{j \in \mathbb{J}} \|\Lambda_j f\|^2 = \|f\|^2, \quad \forall f \in \mathcal{U}. \tag{3.3}$$

Example 3.1. As in Example 1.1, the induced functionals of any Bessel sequence (respectively Riesz basis, orthonormal basis) form a g-Bessel sequence (respectively g-Riesz basis, g-orthonormal basis).

Example 3.2. The sequence containing only the identity mapping $\{I_{\mathcal{U}}\}$ is a g-Bessel sequence, g-Riesz basis, and a g-orthonormal basis for $\mathcal{U}$ with respect to $\mathcal{U}$.

Example 3.3. Let $(X, \mathcal{B}, m)$ be a measure space and $\{X_j\colon\ j \in \mathbb{J}\}$ be a sequence of measurable sets. Let Λ_j be the orthonormal projection from $L^2(X)$ onto $L^2(X_j)$, i.e., $\Lambda_j f = f \cdot \chi_{X_j}$. Then we have

(i) $\{\Lambda_j\colon\ j \in \mathbb{J}\}$ is a g-frame for $L^2(X)$ with respect to $\{L^2(X_j)\colon\ j \in \mathbb{J}\}$ if and only if $\bigcup_{j \in \mathbb{J}} X_j = X$ and $\sup_{j \in \mathbb{J}} \#\{j'\colon\ m(X_j \cap X_{j'}) > 0\} < +\infty$.
(ii) $\{\Lambda_j\colon\ j \in \mathbb{J}\}$ is a g-Riesz basis for $L^2(X)$ with respect to $\{L^2(X_j)\colon\ j \in \mathbb{J}\}$ if and only if $\bigcup_{j \in \mathbb{J}} X_j = X$ and $m(X_j \cap X_{j'}) = 0,\ j \neq j'$. If it is the case, it is also a g-orthonormal basis.

3.1. Characterizations of g-frames, g-Riesz bases and g-orthonormal bases

Let $\Lambda_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j)$. We do not have other assumptions on Λ_j at the moment. Suppose that $\{e_{j,k}\colon\ k \in \mathbb{K}_j\}$ is an orthonormal basis for $\mathcal{V}_j$, where $\mathbb{K}_j$ is a subset of $\mathbb{Z}$, $j \in \mathbb{J}$. Then

$$f \mapsto \langle \Lambda_j f, e_{j,k} \rangle$$

defines a bounded linear functional on $\mathcal{U}$. Consequently, we can find some $u_{j,k} \in \mathcal{U}$ such that

$$\langle f, u_{j,k}\rangle = \langle \Lambda_j f, e_{j,k}\rangle, \quad \forall f \in \mathcal{U}. \tag{3.4}$$

Hence

$$\Lambda_j f = \sum_{k\in\mathbb{K}_j} \langle f, u_{j,k}\rangle e_{j,k}, \quad \forall f \in \mathcal{U}. \tag{3.5}$$

Since $\sum_{k\in\mathbb{K}_j} |\langle f, u_{j,k}\rangle|^2 = \|\Lambda_j f\|^2 \leqslant \|\Lambda_j\|^2 \cdot \|f\|^2$, $\{u_{j,k}\colon k \in \mathbb{K}_j\}$ is a Bessel sequence for $\mathcal{U}$. It follows that for any $f \in \mathcal{U}$ and $g \in \mathcal{V}_j$,

$$\langle f, \Lambda_j^* g\rangle = \langle \Lambda_j f, g\rangle = \sum_{k\in\mathbb{K}_j} \langle f, u_{j,k}\rangle \cdot \langle e_{j,k}, g\rangle = \left\langle f, \sum_{k\in\mathbb{K}_j} \langle g, e_{j,k}\rangle u_{j,k}\right\rangle.$$

Hence

$$\Lambda_j^* g = \sum_{k\in\mathbb{K}_j} \langle g, e_{j,k}\rangle u_{j,k}, \quad \forall g \in \mathcal{V}_j. \tag{3.6}$$

In particular,

$$u_{j,k} = \Lambda_j^* e_{j,k}, \quad j \in \mathbb{J},\ k \in \mathbb{K}_j. \tag{3.7}$$

We call $\{u_{j,k}\colon j \in \mathbb{J}, k \in \mathbb{K}_j\}$ the sequence induced by $\{\Lambda_j\colon j \in \mathbb{J}\}$ with respect to $\{e_{j,k}\colon j \in \mathbb{J}, k \in \mathbb{K}_j\}$.

With above representations of Λ_j and Λ_j^*, we get characterizations of generalized frames, Riesz bases, and orthonormal bases.

Theorem 3.1. *Let $\Lambda_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j)$ and $u_{j,k}$ be defined as in* (3.7). *Then we have the following*:

(i) $\{\Lambda_j\colon j \in \mathbb{J}\}$ *is a g-frame* (*respectively g-Bessel sequence, tight g-frame, g-Riesz basis, g-orthonormal basis*) *for $\mathcal{U}$ if and only if $\{u_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ is a frame* (*respectively Bessel sequence, tight frame, Riesz basis, orthonormal basis*) *for $\mathcal{U}$.*

(ii) *If $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-frame, then*

$$\sum_{j\in\mathbb{J}} \dim \mathcal{V}_j \geqslant \dim \mathcal{U}$$

and the equality holds whenever $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-Riesz basis.

(iii) *Moreover, the g-frame operator for $\{\Lambda_j\colon j \in \mathbb{J}\}$ coincides with the frame operator for $\{u_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$.*

(iv) *Furthermore, $\{\Lambda_j\colon j \in \mathbb{J}\}$ and $\{\tilde{\Lambda}_j\colon j \in \mathbb{J}\}$ are a pair of* (*canonical*) *dual g-frames if and only if the induced sequences are a pair of* (*canonical*) *dual frames.*

Proof. (i) We see from (3.5) that

$$\sum_{j\in\mathbb{J}} \|\Lambda_j f\|^2 = \sum_{j\in\mathbb{J}} \sum_{k\in\mathbb{K}_j} \left|\langle f, u_{j,k}\rangle\right|^2, \quad \forall f \in \mathcal{U}.$$

Hence $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-frame (respectively g-Bessel sequence, tight g-frame) for $\mathcal{U}$ if and only if $\{u_{j,k}\colon j \in \mathbb{J}, k \in \mathbb{K}_j\}$ is a frame (respectively Bessel sequence, tight frame) for $\mathcal{U}$.

Next we assume that $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-Riesz basis for $\mathcal{U}$. Since $\{e_{j,k}\colon k \in \mathbb{K}_j\}$ is an orthonormal basis for $\mathcal{V}_j$, every $g_j \in \mathcal{V}_j$ has an expansion of the form $g_j = \sum_{k\in\mathbb{K}_j} c_{j,k}e_{j,k}$, where $\{c_{j,k}\colon k \in \mathbb{K}_j\} \in \ell^2(\mathbb{K}_j)$. It follows that

$$A\sum_{j\in\mathbb{J}_1}\|g_j\|^2 \leqslant \left\|\sum_{j\in\mathbb{J}_1}\Lambda_j^* g_j\right\|^2 \leqslant B\sum_{j\in\mathbb{J}_1}\|g_j\|^2$$

is equivalent to

$$A\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{K}_j}|c_{j,k}|^2 \leqslant \left\|\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{K}_j}c_{j,k}u_{j,k}\right\|^2 \leqslant B\sum_{j\in\mathbb{J}_1}\sum_{k\in\mathbb{K}_j}|c_{j,k}|^2.$$

On the other hand, we see from $\Lambda_j f = \sum_{k\in\mathbb{K}_j}\langle f, u_{j,k}\rangle e_{j,k}$ that $\{f\colon \Lambda_j f = 0,\ j \in \mathbb{J}\} = \{f\colon \langle f, u_{j,k}\rangle = 0,\ j \in \mathbb{J},\ k \in \mathbb{K}_j\}$. Hence $\{\Lambda_j\colon j \in \mathbb{J}\}$ is g-complete if and only if $\{u_{j,k}\colon j \in \mathbb{J}, k \in \mathbb{K}_j\}$ is complete. Therefore, $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-Riesz basis if and only if $\{u_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ is a Riesz basis.

Now we assume that $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-orthonormal basis. It follows from (3.2) and (3.4) that

$$\begin{aligned}\langle u_{j_1,k_1}, u_{j_2,k_2}\rangle &= \langle \Lambda_{j_2}u_{j_1,k_1}, e_{j_2,k_2}\rangle = \overline{\langle \Lambda_{j_2}^* e_{j_2,k_2}, u_{j_1,k_1}\rangle} = \overline{\langle \Lambda_{j_1}\Lambda_{j_2}^* e_{j_2,k_2}, e_{j_1,k_1}\rangle}\\ &= \langle \Lambda_{j_1}^* e_{j_1,k_1}, \Lambda_{j_2}^* e_{j_2,k_2}\rangle = \delta_{j_1,j_2}\delta_{k_1,k_2}, \quad \forall j_1, j_2 \in \mathbb{J},\ k_1 \in \mathbb{K}_{j_1},\ k_2 \in \mathbb{K}_{j_2}.\end{aligned}$$

Hence $\{u_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ is an orthonormal sequence. Moreover, observe that

$$\|f\|^2 = \sum_{j\in\mathbb{J}}\|\Lambda_j f\|^2 = \sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{K}_j}\left|\langle f, u_{j,k}\rangle\right|^2, \quad \forall f \in \mathcal{U}.$$

We have $\{u_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ is an orthonormal basis.

For the converse, we need only to show that (3.2) holds. In fact, we see from (3.6) that for any $j_1 \neq j_2 \in \mathbb{J}$, $g_{j_1} \in \mathcal{V}_{j_1}$ and $g_{j_2} \in \mathcal{V}_{j_2}$,

$$\langle \Lambda_{j_1}^* g_{j_1}, \Lambda_{j_2}^* g_{j_2}\rangle = \left\langle \sum_{k_1\in\mathbb{K}_{j_1}}\langle g_{j_1}, e_{j_1,k_1}\rangle u_{j_1,k_1}, \sum_{k_2\in\mathbb{K}_{j_2}}\langle g_{j_2}, e_{j_2,k_2}\rangle u_{j_2,k_2}\right\rangle = 0$$

and for $g_1, g_2 \in \mathcal{V}_j$,

$$\langle \Lambda_j^* g_1, \Lambda_j^* g_2\rangle = \left\langle \sum_{k_1\in\mathbb{K}_j}\langle g_1, e_{j,k_1}\rangle u_{j,k_1}, \sum_{k_2\in\mathbb{K}_j}\langle g_2, e_{j,k_2}\rangle u_{j,k_2}\right\rangle = \langle g_1, g_2\rangle.$$

Now the conclusion follows.

(ii) Since the cardinity of a frame is no less than that of a basis, we have $\#\{u_{j,k}\colon j \in \mathbb{J}, k \in \mathbb{K}_j\} \geqslant \dim\mathcal{U}$. Hence $\sum_{j\in\mathbb{J}}\dim\mathcal{V}_j \geqslant \dim\mathcal{U}$. Moreover, we see from (i) that the equality holds whenever $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-Riesz basis.

(iii) We see from (3.5) and (3.6) that

$$\begin{aligned}\sum_{j\in\mathbb{J}}\Lambda_j^*\Lambda_j f &= \sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{K}_j}\langle \Lambda_j f, e_{j,k}\rangle u_{j,k}\\ &= \sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{K}_j}\left\langle \sum_{k'\in\mathbb{K}_j}\langle f, u_{j,k'}\rangle e_{j,k'}, e_{j,k}\right\rangle u_{j,k}\\ &= \sum_{j\in\mathbb{J}}\sum_{k\in\mathbb{K}_j}\langle f, u_{j,k}\rangle u_{j,k}, \quad \forall f \in \mathcal{U}.\end{aligned}$$

Hence the g-frame operator for $\{\Lambda_j\colon j \in \mathbb{J}\}$ coincides with the frame operator for $\{u_{j,k}\colon j \in \mathbb{J}, k \in \mathbb{K}\}$.

(iv) This is a consequence of (i) and (iii). This completes the proof. □

The following are immediate consequences. We leave the proofs to interested readers.

Corollary 3.2. $\{\Lambda_j\colon j \in \mathbb{J}\}$ *is a g-Bessel sequence with an upper bound* B *if and only if for any finite subset* $\mathbb{J}_1 \subset \mathbb{J}$,

$$\Bigl\| \sum_{j\in\mathbb{J}_1} \Lambda_j^* g_j \Bigr\|^2 \leqslant B \sum_{j\in\mathbb{J}_1} \|g_j\|^2, \quad g_j \in \mathcal{V}_j.$$

Corollary 3.3. *A g-Riesz basis* $\{\Lambda_j\colon j \in \mathbb{J}\}$ *is an exact g-frame. Moreover, it is g-biorthonormal with respect to its dual* $\{\tilde{\Lambda}_j\colon j \in \mathbb{J}\}$ *in the following sense*:

$$\bigl\langle \Lambda_{j_1}^* g_{j_1}, \tilde{\Lambda}_{j_2}^* g_{j_2} \bigr\rangle = \delta_{j_1,j_2} \langle g_{j_1}, g_{j_2}\rangle, \quad \forall j_1, j_2 \in \mathbb{J},\ g_{j_1} \in \mathcal{V}_{j_1},\ g_{j_2} \in \mathcal{V}_{j_2}.$$

Corollary 3.4. *A sequence* $\{\Lambda_j\colon j \in \mathbb{J}\}$ *is a g-Riesz basis for* $\mathcal{U}$ *with respect to* $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$ *if and only if there is a g-orthonormal basis* $\{Q_j\colon j \in \mathbb{J}\}$ *for* $\mathcal{U}$ *and a bounded invertible linear operator* T *on* $\mathcal{U}$ *such that* $\Lambda_j = Q_j T$, $j \in \mathbb{J}$.

Proof. Let $\{e_{j,k}\colon k \in \mathbb{K}_j\}$ be an orthonormal basis for $\mathcal{V}_j$, $j \in \mathbb{J}$. First, we assume that $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-Riesz basis for $\mathcal{U}$. By Theorem 3.1, we can find some Riesz basis $\{u_{j,k}\colon j \in \mathbb{J}, k \in \mathbb{K}_j\}$ for $\mathcal{U}$ such that

$$\Lambda_j f = \sum_{k\in\mathbb{K}_j} \langle f, u_{j,k}\rangle e_{j,k}.$$

Take an orthonormal basis $\{u^{\circ}_{j,k}\colon j \in \mathbb{J}, k \in \mathbb{K}_j\}$ for $\mathcal{U}$ and define the operator T on $\mathcal{U}$ by

$$T^* u^{\circ}_{j,k} = u_{j,k}.$$

Obviously, T is a bounded invertible operator. Let $Q_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j)$ be such that $Q_j f = \sum_{k\in\mathbb{K}_j} \langle f, u^{\circ}_{j,k}\rangle e_{j,k}$. By Theorem 3.1, $\{Q_j\colon j \in \mathbb{J}\}$ is a g-orthonormal basis. Moreover, for any $f \in \mathcal{U}$,

$$Q_j T f = \sum_{k\in\mathbb{K}_j} \bigl\langle Tf, u^{\circ}_{j,k}\bigr\rangle e_{j,k} = \sum_{k\in\mathbb{K}_j} \bigl\langle f, T^* u^{\circ}_{j,k}\bigr\rangle e_{j,k} = \sum_{k\in\mathbb{K}_j} \langle f, u_{j,k}\rangle e_{j,k} = \Lambda_j f.$$

Hence $\Lambda_j = Q_j T$, $\forall j \in \mathbb{J}$.

Next we assume that $\{Q_j\colon j \in \mathbb{J}\}$ is a g-orthonormal basis and $\Lambda_j = Q_j T$ for some bounded invertible operator T. Then $\{\Lambda_j\colon j \in \mathbb{J}\}$ is g-complete in $\mathcal{U}$ and we can find some orthonormal basis $\{u^{\circ}_{j,k}\colon j \in \mathbb{J}, k \in \mathbb{K}_j\}$ for $\mathcal{U}$ such that $Q_j f = \sum_{k\in\mathbb{K}_j} \langle f, u^{\circ}_{j,k}\rangle e_{j,k}$. Hence $\Lambda_j f = \sum_{k\in\mathbb{K}_j} \langle Tf, u^{\circ}_{j,k}\rangle e_{j,k} = \sum_{k\in\mathbb{K}_j} \langle f, T^* u^{\circ}_{j,k}\rangle e_{j,k}$. Now we see from Theorem 3.1 that $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-Riesz basis. □

3.2. Excess of g-frames

By Theorem 3.1, g-frames, g-Riesz bases and g-orthonormal bases have properties similar to those of frames, Riesz bases and orthonormal bases, respectively. However, not all the properties are similar. For example, Riesz bases are equivalent to exact frames. But it is not the case for g-Riesz bases and exact g-frames. In fact, we see from Theorem 3.1 that a g-Riesz basis is also an exact g-frame while the converse is not true, which is not surprising since one element of a g-frame might correspond to several elements of the induced frame.

Example 3.4. Let $\{\varphi_j\colon j \in \mathbb{J}\}$ be a Riesz basis for some Hilbert space $\mathcal{H}$. Define $\Lambda_j : \mathcal{H} \mapsto \mathbb{C}^2$ as follows:

$$\Lambda_j f = \big(\langle f, \varphi_j\rangle, 0\big)^T.$$

Then $\{\Lambda_j\colon j \in \mathbb{J}\}$ is an exact g-frame. By Theorem 3.1, it is not a g-Riesz basis for $\mathcal{H}$ with respect to $\mathbb{C}^2$. However, it is a g-Riesz basis for $\mathcal{H}$ with respect to $\mathbb{C} \times \{0\}$.

The above example shows that an exact g-frame may be a g-Riesz basis when we change the reference. Does this hold in general? The answer is negative.

Example 3.5. Let $\{\varphi_j\colon j \in \mathbb{Z}\}$ a Riesz basis for some Hilbert space $\mathcal{H}$. Define $\Lambda_j\colon \mathcal{H} \mapsto \mathbb{C}^3$ as follows:

$$\Lambda_j f = \big(\langle f, \varphi_{2j-1}\rangle, \langle f, \varphi_{2j}\rangle, \langle f, \varphi_{2j+1}\rangle\big)^T.$$

Then $\{\Lambda_j\colon j \in \mathbb{Z}\}$ is an exact g-frame. However, $\{\Lambda_j\colon j \in \mathbb{Z}\}$ is not a g-Riesz basis for $\mathcal{H}$ with respect to any $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$, thanks to Theorem 3.1.

On the other hand, it is well known (e.g., see [20]) that a frame either remains a frame or is incomplete whenever any one of its elements is removed. This fails for g-frames. The following is a counterexample.

Example 3.6. Let $g(x) = e^{-x^2/2}$ be the Gaussian and $\{\alpha_{m,n}\colon m, n \in \mathbb{Z}\}$ be an orthonormal basis for $\ell^2(\mathbb{Z}^2)$. Define

$$\Lambda_j f = \sum_{m,n\in\mathbb{Z}} \big\langle f(x), e^{i2\pi mx} g(x - 2n - j)\big\rangle \alpha_{m,n}, \quad j = 1, 2,$$

$$\Lambda_3 f = \sum_{m,n\in\mathbb{Z}} \big\langle f(x), e^{i2\pi mx} g(x - n + 1/2)\big\rangle \alpha_{m,n}, \quad f \in L^2(\mathbb{R}).$$

We see from Theorem 3.1 and the frame theory (e.g., see [7, pp. 84–86]) that $\{\Lambda_1, \Lambda_2, \Lambda_3\}$ is a g-frame for $L^2(\mathbb{R})$ with respect to $\ell^2(\mathbb{Z}^2)$. However, $\{\Lambda_1, \Lambda_2\}$ is not a g-frame but g-complete.

A natural problem arises: given a subsequence of a g-frame for which only one element is removed, when is it a g-frame? To this problem, we have the following.

Theorem 3.5. *Let $\{\Lambda_j\colon j \in \mathbb{J}\}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$ and $\{\tilde{\Lambda}_j\colon j \in \mathbb{J}\}$ be the canonical dual g-frame. Suppose that $j_0 \in \mathbb{J}$.*

- (i) *If there is some $g_0 \in \mathcal{V}_{j_0} \setminus \{0\}$ such that $\tilde{\Lambda}_{j_0}\Lambda_{j_0}^* g_0 = g_0$, then $\{\Lambda_j\colon j \in \mathbb{J},\ j \neq j_0\}$ is not g-complete in $\mathcal{U}$.*
- (ii) *If there is some $f_0 \in \mathcal{U} \setminus \{0\}$ such that $\Lambda_{j_0}^* \tilde{\Lambda}_{j_0} f_0 = f_0$, then $\{\Lambda_j\colon j \in \mathbb{J},\ j \neq j_0\}$ is not g-complete in $\mathcal{U}$.*

(iii) *If $I-\Lambda_{j_0}\tilde{\Lambda}^*_{j_0}$ or $I-\tilde{\Lambda}_{j_0}\Lambda^*_{j_0}$ is bounded invertible on $\mathcal{V}_{j_0}$, then $\{\Lambda_j\colon\ j\in\mathbb{J},\ j\neq j_0\}$ is a g-frame for $\mathcal{U}$.*

Proof. (i) Since $\Lambda^*_{j_0}g_0\in\mathcal{U}$, we have

$$\Lambda^*_{j_0}g_0=\sum_{j\in\mathbb{J}}\Lambda^*_j\tilde{\Lambda}_j\Lambda^*_{j_0}g_0.$$

Hence, $0=\sum_{j\in\mathbb{J},j\neq j_0}\Lambda^*_j\tilde{\Lambda}_j\Lambda^*_{j_0}g_0$. Put $v_{j_0,j}=\delta_{j_0,j}g_0$. We have

$$\Lambda^*_{j_0}g_0=\sum_{j\in\mathbb{J}}\Lambda^*_j v_{j_0,j}.$$

It follows from Lemma 2.1 that

$$\sum_{j\in\mathbb{J}}\|v_{j_0,j}\|^2=\sum_{j\in\mathbb{J}}\big\|\tilde{\Lambda}_j\Lambda^*_{j_0}g_0\big\|^2+\sum_{j\in\mathbb{J}}\big\|\tilde{\Lambda}_j\Lambda^*_{j_0}g_0-v_{j_0,j}\big\|^2.$$

Consequently,

$$\|g_0\|^2=\|g_0\|^2+2\sum_{j\neq j_0}\big\|\tilde{\Lambda}_j\Lambda^*_{j_0}g_0\big\|^2.$$

Hence, $\tilde{\Lambda}_j\Lambda^*_{j_0}g_0=0$. Therefore, $\Lambda_j\tilde{\Lambda}^*_{j_0}g_0=\Lambda_jS^{-1}\Lambda^*_{j_0}g_0=\tilde{\Lambda}_j\Lambda^*_{j_0}g_0=0$, $j\neq j_0$. But

$$\big\langle\Lambda^*_{j_0}g_0,\tilde{\Lambda}^*_{j_0}g_0\big\rangle=\big\langle\tilde{\Lambda}_{j_0}\Lambda^*_{j_0}g_0,g_0\big\rangle=\|g_0\|^2>0,$$

which implies that $\tilde{\Lambda}^*_{j_0}g_0\neq 0$. Hence $\{\Lambda_j\colon\ j\in\mathbb{J},\ j\neq j_0\}$ is not g-complete in $\mathcal{U}$.

(ii) Since $\Lambda^*_{j_0}\tilde{\Lambda}_{j_0}f_0=f_0\neq 0$, we have $\tilde{\Lambda}_{j_0}f_0\neq 0$ and $\tilde{\Lambda}_{j_0}\Lambda^*_{j_0}\tilde{\Lambda}_{j_0}f_0=\tilde{\Lambda}_{j_0}f_0$. Now the conclusion follows from (i).

(iii) Since $\tilde{\Lambda}_j=\Lambda_jS^{-1}$, where S is the g-frame operator for $\{\Lambda_j\colon\ j\in\mathbb{J}\}$, we have

$$I-\Lambda_{j_0}\tilde{\Lambda}^*_{j_0}=I-\Lambda_{j_0}S^{-1}\Lambda^*_{j_0}=I-\tilde{\Lambda}_{j_0}\Lambda^*_{j_0}.$$

Let A and B be the lower and upper frame bounds for $\{\Lambda_j\colon\ j\in\mathbb{J}\}$, respectively. For any $f\in\mathcal{U}$, we have

$$f=\sum_{j\in\mathbb{J}}\tilde{\Lambda}^*_j\Lambda_jf.$$

Hence

$$\Lambda_{j_0}f=\sum_{j\in\mathbb{J}}\Lambda_{j_0}\tilde{\Lambda}^*_j\Lambda_jf.$$

Therefore,

$$\big(I-\Lambda_{j_0}\tilde{\Lambda}^*_{j_0}\big)\Lambda_{j_0}f=\sum_{j\neq j_0}\Lambda_{j_0}\tilde{\Lambda}^*_j\Lambda_jf. \tag{3.8}$$

Note that

$$\Big\|\sum_{j\neq j_0}\Lambda_{j_0}\tilde{\Lambda}^*_j\Lambda_jf\Big\|^2=\sup_{g\in\mathcal{V}_{j_0},\|g\|=1}\Big|\Big\langle\sum_{j\neq j_0}\Lambda_{j_0}\tilde{\Lambda}^*_j\Lambda_jf,g\Big\rangle\Big|^2$$

$$= \sup_{\|g\|=1} \left| \sum_{j \neq j_0} \langle \Lambda_j f, \tilde{\Lambda}_j \Lambda_{j_0}^* g \rangle \right|^2$$
$$\leqslant \sum_{j \neq j_0} \|\Lambda_j f\|^2 \cdot \sup_{\|g\|=1} \sum_{j \in \mathbb{J}} \left\| \tilde{\Lambda}_j \Lambda_{j_0}^* g \right\|^2$$
$$\leqslant \frac{1}{A} \|\Lambda_{j_0}\|^2 \sum_{j \neq j_0} \|\Lambda_j f\|^2.$$

We see from (3.8) that

$$\|\Lambda_{j_0} f\|^2 \leqslant \left\| \left(I - \Lambda_{j_0} \tilde{\Lambda}_{j_0}^*\right)^{-1} \right\|^2 \frac{1}{A} \|\Lambda_{j_0}\|^2 \sum_{j \neq j_0} \|\Lambda_j f\|^2.$$

Hence

$$\sum_{j \in \mathbb{J}} \|\Lambda_j f\|^2 \leqslant C \sum_{j \neq j_0} \|\Lambda_j f\|^2.$$

Therefore,

$$\frac{A}{C} \|f\|^2 \leqslant \sum_{j \neq j_0} \|\Lambda_j f\|^2 \leqslant B \|f\|^2, \quad \forall f \in \mathcal{U}.$$

This completes the proof. □

Corollary 3.6. *Let $\{\Lambda_j\colon j \in \mathbb{J}\}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$. If $\dim \mathcal{V}_j < +\infty$, $j \in \mathbb{J}$, then $\{\Lambda_j\colon j \in \mathbb{J},\ j \neq j_0\}$ is either g-incomplete in $\mathcal{U}$ or a g-frame for $\mathcal{U}$ for any $j_0 \in \mathbb{J}$.*

Proof. If there is some $g_0 \in \mathcal{V}_{j_0} \setminus \{0\}$ such that $\tilde{\Lambda}_{j_0} \Lambda_{j_0}^* g_0 = g_0$, then Theorem 3.5(i) shows that $\{\Lambda_j\colon j \in \mathbb{J},\ j \neq j_0\}$ is not g-complete in $\mathcal{U}$. Otherwise, $I - \tilde{\Lambda}_{j_0} \Lambda_{j_0}^*$ is injective. Consequently, $(I - \Lambda_{j_0} \tilde{\Lambda}_{j_0}^*)\mathcal{V}_{j_0}$ is dense in $\mathcal{V}_{j_0}$. Since $\dim \mathcal{V}_{j_0} < +\infty$, we have $(I - \Lambda_{j_0} \tilde{\Lambda}_{j_0}^*)\mathcal{V}_{j_0} = \mathcal{V}_{j_0}$. Therefore, $I - \Lambda_{j_0} \tilde{\Lambda}_{j_0}^*$ is bounded invertible. Now the conclusion follows from Theorem 3.5(iii). □

3.3. Equivalence between stable space splittings and g-frames

Stable space splittings are generalizations of frames which lead to a better understanding of iterative solvers (multigrid/multilevel respectively domain decomposition methods) for large-scale discretization of elliptic operator equations (see [19] and references therein). Here we prove that stable space splittings are equivalent to g-frames.

Let $\mathcal{V}$ and $\mathcal{V}_j$, $j \in \mathbb{J}$, be Hilbert spaces. Let b_j be a bilinear form on $\mathcal{V}_j \times \mathcal{V}_j$ satisfying

$$b_j(u,u) \geqslant C_j \|u\|^2 \quad \text{and} \quad b_j(u,v) = b_j(v,u) \leqslant C_j' \|u\| \cdot \|v\|, \quad \forall u, v \in \mathcal{V}_j. \tag{3.9}$$

Suppose that $R_j \in \mathcal{L}(\mathcal{V}_j, \mathcal{V})$. Recall that a system $\{(\{\mathcal{V}_j, b_j\}, R_j)\colon j \in \mathbb{J}\}$ is called a stable space splitting of $\mathcal{V}$ if there are some positive constants C, C' such that

$$C\|u\|^2 \leqslant \inf_{u=\sum_{j \in \mathbb{J}} R_j u_j} \sum_{j \in \mathbb{J}} b_j(u_j, u_j) \leqslant C' \|u\|^2, \quad \forall u \in \mathcal{V},$$

where $u_j \in \mathcal{V}_j$. It was shown in [19, Theorem 4] (see also [18, pp. 73–75]) that a stable space splitting $\{(\{\mathcal{V}_j, b_j\}, R_j)\colon j \in \mathbb{J}\}$ satisfies $A \leqslant \sum_{j\in\mathbb{J}} R_j R_j^* \leqslant B$ for some constants $A, B > 0$, which is equivalent to

$$A\|u\|^2 \leqslant \sum_{j\in\mathbb{J}} \left\| R_j^* u \right\|^2 \leqslant B\|u\|^2, \quad \forall u \in \mathcal{V}.$$

Hence $\{R_j^*\colon j \in \mathbb{J}\}$ is a g-frame for $\mathcal{V}$ with respect to $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$.

For the converse, we need $b_j(u,u)$ to be uniformly bounded, i.e., we assume that

$$C_1\|u\|^2 \leqslant b_j(u,u) \leqslant C_2\|u\|^2, \quad \forall u \in \mathcal{V}. \tag{3.10}$$

Suppose that $\{\Lambda_j\colon j \in \mathbb{J}\}$ is a g-frame for $\mathcal{V}$ with respect to $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$. Let A and B be the frame bounds and $\{\tilde{\Lambda}_j\colon j \in \mathbb{J}\}$ be the canonical dual g-frame. Put $R_j = \Lambda_j^*$. We see from (3.10) and Lemma 2.1 that

$$\inf_{u=\sum_{j\in\mathbb{J}} R_j u_j} \sum_{j\in\mathbb{J}} b_j(u_j,u_j) \geqslant \inf_{u=\sum_{j\in\mathbb{J}} R_j u_j} \sum_{j\in\mathbb{J}} C_1\|u_j\|^2 = \sum_{j\in\mathbb{J}} C_1\|\tilde{\Lambda}_j u\|^2 \geqslant \frac{C_1}{B}\|u\|^2.$$

Similarly we can prove that

$$\inf_{u=\sum_{j\in\mathbb{J}} R_j u_j} \sum_{j\in\mathbb{J}} b_j(u_j,u_j) \leqslant \frac{C_2}{A}\|u\|^2.$$

Hence $\{(\{\mathcal{V}_j, b_j\}, R_j)\colon j \in \mathbb{J}\}$ is a stable space splitting.

4. Applications of g-frames

4.1. Atomic resolution of bounded linear operators

Here we give an application of g-frames.

Let $\{\Lambda_j\colon j \in \mathbb{J}\}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$. Suppose that $\{\tilde{\Lambda}_j\colon j \in \mathbb{J}\}$ is the canonical dual g-frame. Then for any $f \in \mathcal{U}$, we have

$$f = \sum_{j\in\mathbb{J}} \Lambda_j^* \tilde{\Lambda}_j f = \sum_{j\in\mathbb{J}} \tilde{\Lambda}_j^* \Lambda_j f, \quad f \in \mathcal{U}.$$

It follows that

$$I_{\mathcal{U}} = \sum_{j\in\mathbb{J}} \Lambda_j^* \tilde{\Lambda}_j = \sum_{j\in\mathbb{J}} \tilde{\Lambda}_j^* \Lambda_j, \tag{4.1}$$

where the convergence is in weak* sense. Let T be a bounded linear operator on $\mathcal{U}$. We see from (4.1) that

$$T = \sum_{j\in\mathbb{J}} T\Lambda_j^* \tilde{\Lambda}_j = \sum_{j\in\mathbb{J}} T\tilde{\Lambda}_j^* \Lambda_j = \sum_{j\in\mathbb{J}} \Lambda_j^* \tilde{\Lambda}_j T = \sum_{j\in\mathbb{J}} \tilde{\Lambda}_j^* \Lambda_j T. \tag{4.2}$$

We call (4.2) atomic resolutions of an operator T.

4.2. Construction of frames via g-frames

Let $\{\Lambda_j\colon j \in \mathbb{J}\}$ be a g-frame for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$. We see from Theorem 3.1 that $\{u_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\} = \{\Lambda_j^* e_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ is a frame for $\mathcal{U}$, where $\{e_{j,k}\colon k \in \mathbb{K}_j\}$ is an orthonormal basis for $\mathcal{V}_j$. However, it might be difficult to find an orthonormal basis for $\mathcal{V}_j$ in practice. Fortunately, the orthonormality is not necessary to get a frame. In fact, we have the following.

Theorem 4.1. *Let $\{\Lambda_j\colon j \in \mathbb{J}\}$ and $\{\tilde{\Lambda}_j\colon j \in \mathbb{J}\}$ be a pair of dual g-frames for $\mathcal{U}$ with respect to $\{\mathcal{V}_j\colon j \in \mathbb{J}\}$ and $\{g_{j,k}\colon k \in \mathbb{K}_j\}$ and $\{\tilde{g}_{j,k}\colon k \in \mathbb{K}_j\}$ be a pair of dual frames for $\mathcal{V}_j$, respectively. Then $\{\Lambda_j^* g_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ and $\{\tilde{\Lambda}_j^* \tilde{g}_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ are a pair of dual frames for $\mathcal{U}$ provided the frame bounds for $\{g_{j,k}\colon k \in \mathbb{K}_j\}$ satisfying $C_1 \leqslant A_j \leqslant B_j \leqslant C_2$ for some constants $C_1, C_2 > 0$.*

Moreover, suppose that $\{\Lambda_j\colon j \in \mathbb{J}\}$ and $\{\tilde{\Lambda}_j\colon j \in \mathbb{J}\}$ are canonical dual g-frames, $\{g_{j,k}\colon k \in \mathbb{K}_j\}$ and $\{\tilde{g}_{j,k}\colon k \in \mathbb{K}_j\}$ are canonical dual frames, and that $\{g_{j,k}\colon k \in \mathbb{K}_j\}$ is a tight g-frame with frame bounds $A_j = B_j = A$, $\forall j \in \mathbb{J}$. Then $\{\Lambda_j^* g_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ and $\{\tilde{\Lambda}_j^* \tilde{g}_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ are canonical dual frames.

Proof. Note that

$$\langle f, \Lambda_j^* g_{j,k}\rangle = \langle \Lambda_j f, g_{j,k}\rangle.$$

It is easy to see that both $\{\Lambda_j^* g_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ and $\{\tilde{\Lambda}_j^* \tilde{g}_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ are frames for $\mathcal{U}$. On the other hand, For any $f \in \mathcal{U}$, we have

$$\sum_{j\in\mathbb{J}} \sum_{k\in\mathbb{K}_j} \langle f, \Lambda_j^* g_{j,k}\rangle \tilde{\Lambda}_j^* \tilde{g}_{j,k} = \sum_{j\in\mathbb{J}} \tilde{\Lambda}_j^* \sum_{k\in\mathbb{K}_j} \langle \Lambda_j f, g_{j,k}\rangle \tilde{g}_{j,k} = \sum_{j\in\mathbb{J}} \tilde{\Lambda}_j^* \Lambda_j f = f.$$

Similarly we can get that

$$\sum_{j\in\mathbb{J}} \sum_{k\in\mathbb{K}_j} \langle f, \tilde{\Lambda}_j^* \tilde{g}_{j,k}\rangle \Lambda_j^* g_{j,k} = f.$$

Hence $\{\Lambda_j^* g_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ and $\{\tilde{\Lambda}_j^* \tilde{g}_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$ are dual frames for $\mathcal{U}$.

Next we assume that $\{\Lambda_j\colon j \in \mathbb{J}\}$ and $\{\tilde{\Lambda}_j\colon j \in \mathbb{J}\}$ are canonical dual g-frames and $\{g_{j,k}\colon k \in \mathbb{K}_j\}$ is a tight frame with frame bounds $A_j = B_j = A$, $j \in \mathbb{J}$. Then $\tilde{g}_{j,k} = \frac{1}{A} g_{j,k}$. Let S_Λ and $S_{\Lambda,g}$ be the frame operators associated with $\{\Lambda_j\colon j \in \mathbb{J}\}$ and $\{\Lambda_j^* g_{j,k}\colon j \in \mathbb{J},\ k \in \mathbb{K}_j\}$, respectively. Then we have

$$\begin{aligned} S_{\Lambda,g} f &= \sum_{j\in\mathbb{J}} \sum_{k\in\mathbb{K}_j} \langle f, \Lambda_j^* g_{j,k}\rangle \Lambda_j^* g_{j,k} = \sum_{j\in\mathbb{J}} \Lambda_j^* \sum_{k\in\mathbb{K}_j} \langle \Lambda_j f, g_{j,k}\rangle g_{j,k} \\ &= A \sum_{j\in\mathbb{J}} \Lambda_j^* \Lambda_j f = A S_\Lambda f, \quad \forall f \in \mathcal{U}. \end{aligned}$$

Hence

$$S_{\Lambda,g}^{-1} \Lambda_j^* g_{j,k} = \frac{1}{A} S_\Lambda^{-1} \Lambda_j^* g_{j,k} = \tilde{\Lambda}_j^* \tilde{g}_{j,k}, \quad j \in \mathbb{J},\ k \in \mathbb{K}_j.$$

This completes the proof. □

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