On the Length of the Conjugacy Classes of Finite Groups*

DAVID CHILLAG AND MARCEL HERZOG

Department of Mathematics, Technion, Israel Institute of Technology, Haifa 32000, Israel, and
School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel-Aviv University, Tel-Aviv, Israel

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Let $G$ be a finite group. The question of how certain arithmetical conditions on the degrees of the irreducible characters of $G$ influence the group structure was studied by several authors. Our purpose here is to impose analogous conditions on the lengths of the conjugacy classes of $G$ and to describe the group structure under these conditions.

In this paper $G$ denotes a finite group. Let $\text{Irr}(G)$ be the set of all the irreducible characters of $G$ over the field of complex numbers and let $\text{Con}(G)$ be the set of all the conjugacy classes of $G$. The fact that $|\text{Irr}(G)| = |\text{Con}(G)|$ yields many analogous studies of $\text{Irr}(G)$ and $\text{Con}(G)$. See [1] for a survey of such studies.

One of the questions that was studied extensively is what can be said about the structure of the group $G$, if some information is known about the arithmetical structure of $\text{Irr}(G)$. Answers in many cases were given by Isaacs, Passman, Ito, Thompson, Manz, Huppert, Willems, and others. See Manz [20] for a survey of these type of questions and results. Our main purpose in this article is to obtain some analogous results on $\text{Con}(G)$ and to compare them with the corresponding ones on $\text{Irr}(G)$. Our Theorem 1

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and Corollaries 2.3 and 2.4 are the analogues of [10, 14], while Theorem 2 and Corollaries 2.1 and 2.2 are the analogues of [21, 22]. Analogues of [23, 18] are also proved.

To state our results we need some notation. A solvable group is called an $A$-group if all its Sylow subgroups are abelian. The Fitting subgroup of $G$ is denoted by $F(G)$ and its center by $Z(G)$. The derived length of a solvable group is denoted by $dl(G)$ and the nilpotent class of a nilpotent group by $cl(G)$. Let $\rho c(G)$ be the cardinality of the set of the "relevant primes" of $G$, i.e.,

$$\rho c(G) = \left| \left\{ p ; p \text{ a prime, } p \mid |C| \text{ for some } C \in \text{Con}(G) \right\} \right|.$$ 

The set of character degrees of $G$ is denoted by c.d.(G). The rest of the notation is standard. Some additional notation will be introduced as we go along.

**Theorem 1.** Suppose that $|C|$ is a squarefree number for each $C \in \text{Con}(G)$. Then $G$ is supersolvable and both $|G/F(G)|$ and $|F(G)'|$ are squarefree numbers. In particular, $G' \leq F(G)$, $G/F(G)$ is cyclic, $dl(G) \leq 3$ and $cl(F(G)) \leq 2$.

**Theorem 2.** Each $C \in \text{Con}(G)$ has length a power of a prime if and only if either $G$ is nilpotent with at most one non-abelian Sylow subgroup, or $G$ is an $A$-group and there exist two distinct primes $p$ and $q$ such that $G = A \times PO_q(G)$, where $A$ is an abelian $\{p, q\}$-subgroup of $G$, $P$ is a non-normal Sylow $p$-subgroup of $G$ and $P \cap P^g = O_p(G)$ for each $g \in G - N_G(P)$.

**Corollary 2.1.** If $|C|$ is a prime power for each $C \in \text{Con}(G)$, then either $G$ is nilpotent with $\rho c(G) \leq 1$ and of unbounded $dl(G)$, or $G$ is solvable, but non-nilpotent, with $\rho c(G) = dl(G) = 2$.

**Corollary 2.2.** If $|C|$ is a prime power for each $C \in \text{Con}(G)$ and if $G$ is non-nilpotent, then $G/Z(G)$ is a Frobenius group. In the notation of Theorem 2, $O_q(G)Z(G)/Z(G)$ is the Frobenius kernel and $PZ(G)/Z(G)$ is a Frobenius complement.

**Corollary 2.3.** If, for each $C \in \text{Con}(G)$, $|C|$ is either one or a prime number, then:

1. Each of $|G/F(G)|$ and $|F(G)'|$ is either one or a prime number, and
2. Either $G$ is nilpotent with $cl(G) \leq 2$ and $\rho c(G) \leq 1$, or $G/Z(G)$ is a Frobenius group of order $pq$, where $p$ and $q$ are primes and $\rho c(G) = 2$. 
Corollary 2.4. If, for each $C \in \text{Con}(G)$, $|C|$ is either one or a prime number or a product of two distinct primes, then each of $|G/F(G)|$ and $|F(G)|$ is either one or a prime number or a product of two distinct primes.

In order to prove Theorem 1, we need the following proposition, which is of independent interest.

Proposition 3. If the fixed prime $p$ divides $|C_G(x)|$ for all $x \in G$, then $G$ is not a nonabelian simple group.

This proposition follows from the Michler-Willems theorem [24, 29] about the existence of $p$-blocks of defect zero in finite simple groups. Their theorem uses the classification of the finite simple groups. Neither the proposition nor the classification is needed for the proof of Theorem 2 and Corollaries 2.1, 2.2, and 2.3 (the proof of Corollary 2.3 relies only on the solvable case of Theorem 1).

In [3, Theorem, p. 27], Reinhold Baer characterized all finite groups satisfying the following property: the class length of every element of $G$ of prime power order, is also a power of a prime. Since Baer's proof is quite complicated, our proof of Theorem 2 is independent of this result.

In the proof of Theorem 2 we use the following result of Wielandt [3, Lemma 6]:

WIELANDT'S LEMMA. Let $p$ be a prime and let $g \in G$. If both the order of $g$ and the class-length of $g$ are powers of $p$, then $g \in O_p(G)$.

Before stating the analogues of [23, 18], we will compare our results with the corresponding ones on character degrees.

Remarks. (1) If all nonlinear $\chi \in \text{Irr}(G)$ have prime degrees, then the Isaacs-Passman theorem [14] states that $d(G) \leq 3$ and $\rho(G) \leq 2$ (where $\rho(G)$ is the character-degrees equivalent of our $\rho C(G)$). Similar conclusions were reached in the analogous Corollary 2.3.

(2) By a theorem of Ito [15, 16], a prime $p | \chi(1)$ for all $\chi \in \text{Irr}(G)$ if and only if $G$ has a normal abelian Sylow $p$-group. The well-known analogue, whose short proof we give in Section I, is:

Proposition 4. If $p$ is a prime, then $p | |C|$ for each $C \in \text{Con}(G)$ if and only if $G$ has the Sylow $p$-subgroup in its center.

(3) By [12, pp. 81, 84], $\chi(1)$ is a power of a fixed prime $p$ for all $\chi \in \text{Irr}(G)$ if and only if $G$ has an abelian normal $p$-complement. Proposition 4 implies the following analogue: $|C|$ is a power of a fixed prime $p$ for each $C \in \text{Con}(G)$ if and only if $G = P \times H$, where $P$ is a Sylow $p$-subgroup of $G$ and $H$ is abelian. In fact, a more general result follows
from Proposition 4: If $\pi$ is a set of primes, then $|C|$ is a $\pi$-number for each $C \in \text{Con}(G)$ if and only if $G = G_1 \times G_2$, where $G_1$ is a $\pi$-subgroup of $G$ and $G_2$ is an abelian $\pi'$-subgroup. This result serves also as an analogue to Gallagher's theorem [5] stating that if $G$ is a solvable and each character-degree is a $\pi$-number, then $G$ has a normal Hall $\pi$-subgroup. We also mention here that if $|C| = p^aq^b$ for all $C \in \text{Con}(G)$, where $p$ and $q$ are fixed primes, then $G$ is solvable (see [2, p. 461] for a proof; a simpler one can be obtained as well).

(4) If $\chi(1)$ is a prime power for each $\chi \in \text{Irr}(G)$ and at least two distinct primes occur, then $G$ is not necessarily solvable (see [22, 28]). However, if solvability is assumed, then $\rho(G) = 2$ and $dl(G) \leq 5$ [21]. In the classes analogue, as stated in Corollary 2.1, $G$ is always solvable and $\rho_c(G) = dl(G) = 2$.

(5) If all $\chi \in \text{Irr}(G)$ have squarefree degrees, then again $G$ can be nonsolvable, but if $G$ is assumed to be solvable, then $dl(G) \leq 4$ [10]. In the class-length analogue, as stated in Theorem 1, $G$ is necessarily supersolvable with $dl(G) \leq 3$.

(6) By a result of Willems [28, p. 483], if $4|\chi(1)$ for each $\chi \in \text{Irr}(G)$, then either $G$ is solvable or it has a normal solvable subgroup with the factor group $A_7$. Using the Michler-Willems theorem, the following class-length analogue will be proved:

**Proposition 5.** If $4|C|$ for each $C \in \text{Con}(G)$, then $G$ is solvable.

(7) We mention that Ito [17] proved that if there are only two class-lengths, 1 and $m$, then $m = p^a$ for some prime $p$ and $G$ is a direct product of a $p$-group with an abelian group. Ito also showed that if there are three class-lengths, then $G$ is solvable. Compare with the weaker character-degree analogues [12, Chap. 12].

Naturally, the proofs of our results do not resemble the proofs of their character-degree analogues. However, the following analogue of Manz-Leisering result [18] can be obtained using the Manz-Leisering methods with only a few adjustments. In fact, even the proofs of Manz's inequalities [23] for solvable groups can be transferred in terms of class-lengths with minor changes (the class-lengths proof is even easier). These inequalities are stated and proved in Section II.

To state the class-length analogue of [18] we need to introduce some more notation. If $n = \prod_{i=1}^{s} p_i^{n_i}$ is the prime number decomposition of the natural number $n$, then the largest $n_i$ is denoted by $\tau(n)$. We set $\tau_c(G) = \max \{\tau(|C|); C \in \text{Con}(G)\}$. Finally, denote the nilpotent length of the solvable group $G$ by $n(G)$.
Theorem 8. Let $G$ be a finite solvable group. Then:

1. $n(G) \leq 2\tau c(G) + 4$ and $n(G) \leq \tau c(G) + 4$ for $|G|$ odd.
2. $dl(G) \leq 6\tau c(G) + 2$ and $dl(G) \leq 5\tau c(G) + 2$ for $|G|$ odd.

The first section of this paper is devoted to the proofs of Theorems 1 and 2, of Corollaries 2.1–2.4 and of Propositions 3, 4, and 5. In the second section we prove Theorem 8 and Propositions 6 and 7, which are the analogues of [18, 23]. Also in Section II we remark on what happens when the assumptions on the character-degrees or class-lengths which are considered in this paper, are applied to the set \{ $|C_G(x)|$; $x \in G$ \}.

I. PROOFS OF THEOREMS 1, 2, COROLLARIES, AND PROPOSITIONS 3, 4, 5

We first prove the propositions.

Proof of Proposition 3. Suppose, to the contrary, that $G$ is a nonabelian simple group and $p$ is a prime dividing $|C_G(x)|$ for all $x \in G$. If $G$ has a $p$-block of defect zero, say $B$, and if $x$ is an element of a defect class of $B$, then $|C_G(x)|/p = 1$, a contradiction. Thus $G$ has no $p$-blocks of defect zero and by the Michler–Willems theorem [27, Corollary] we conclude that $G$ is either an alternating group or one of the following sporadic simple groups: $M_{12}$, $M_{22}$, $M_{24}$, $J_2$, $HS$, $Ru$, $Co_1$, $Co_3$, and $B$. By [4], each of these sporadic groups, as well as each of the alternating groups of degree less than 11, has two elements $a$ and $b$ with $(|C_G(a)|, |C_G(b)|) = 1$, so that $p$ cannot divide both centralizers. Thus by the classification of simple groups $G \cong A_n$, an alternating group of degree $n \geq 11$. If $n$ is odd, let $x$ be an $n$-cycle and $y$ an $(n-2)$-cycle. Then $x, y \in A_n$ and $|C_{s_n}(x)|/n$, $|C_{s_n}(y)|/2(n-2)$ (see [19, p. 108]). So $|C_{s_n}(x)|$ divides $n$ and $|C_{s_n}(y)|$ divides $2(n-2)$. As $n$ is odd, $p | (n, n-2) = 1$, a contradiction. Thus $n$ is even. Let $a, b, c \in A_n$ be permutations with the cycle-structures $(n-1, 1)$, $(n-3, 3)$, and $(n-5, 5)$, respectively. Since $n > 10$, it follows that $n-3 > 3$ and $n-5 > 5$, and hence $|C_{s_n}(a)| = n-1$, $|C_{s_n}(b)| = 3(n-3)$, and $|C_{s_n}(c)| = 5(n-5)$. As above $p | (n-1, 3(n-3), 5(n-5))$ and since $n-1$ is odd, $(n-1, 3(n-3)) = 1$ or 3. Thus $p = 3$ and $3 | ((n-1)-(n-5)) = 4$, a final contradiction.

We need some more notation. If $A \leq G$ and $x \in A$, we denote the conjugacy class of $x$ in $A$ by $C_{s_n}(x)$. We write $Cl(x)$ for $C_{s_n}(x)$. The Frattini subgroup of $G$ will be denoted by $\Phi(G)$. If $G$ is a $p$-group and $x \in G$, we define $\beta(x)$ by $p^{\beta(x)} = |Cl(x)|$ and $\beta(G) = \max \{ \beta(x); x \in G \}$.

Proof of Proposition 4. If a Sylow $p$-subgroup $P$ of $G$ is in the center of $G$, then $P \leq C_G(x)$ for all $x \in G$ and $p | |Cl(x)|$ for all $x \in G$. Conversely, suppose that $p$ is a prime such that $p | |Cl(x)|$ for all $x \in G$ and let $P \in Syl_p(G)$.  


Take \( x \in G \), then some conjugate of \( P \) centralizes \( x \), so that \( P \leq C_G(x^g) \) for some \( g \in G \). If \( C_1, C_2, \ldots, C_k \) are the conjugacy classes of \( G \), it follows that for each \( i = 1, 2, \ldots, k \) there exists \( x_i \in C_i \) with \( P \leq C_G(x_i) \). Thus \( P \) centralizes \( H = \langle x_1, x_2, \ldots, x_k \rangle \). Since \( H \) contains an element from each conjugacy class of \( G \), we conclude that \( G \) is the union of all conjugates of \( H \). Hence \( G = H \) and \( P \leq Z(G) \).

The proofs of Proposition 5 and of Theorems 1 and 2 are via a series of lemmas. The first lemma implies that the assumptions of Proposition 5 and of Theorems 1 and 2 are inherited by normal subgroups and quotient groups.

**Lemma 1.1.** Let \( N \leq G \), \( x \in N \), and \( y \in G \). Then

\[
|Cl_N(x)| \mid |Cl_G(x)| \quad \text{and} \quad |Cl_{G/N}(yN)| \mid |Cl_G(y)|.
\]

**Proof.** Clearly,

\[
|Cl_N(x)| = |N : C_N(x)| = |N : N \cap C_G(x)| = |N : C_G(x)| = |Cl_G(x)|.
\]

Also, \( C_G(y)N/N \leq C_{G/N}(yN) \) and consequently

\[
|Cl_{G/N}(yN)| = |G/N : C_{G/N}(yN)| \mid |G/N : C_G(y)N/N| = |G : C_G(y)N| \mid |G : C_G(y)| = |Cl_G(y)|. \]

**Proof of Proposition 5.** Suppose, first, that \( G \) is non-abelian simple. As in the proof of Proposition 3, if \( G \) has a 2-block of defect zero, then there is an \( x \in G \) with \( |C_G(x)| \) odd. But then \( |Cl(x)| \) is divisible by 4, a contradiction. Hence there is no 2-block of defect zero and, by [28], \( G \) is one of the groups which appear in the proof of Proposition 3. It can be easily checked that each such group has a class-length divisible by 4. So \( G \) is not simple. The solvability of \( G \) now follows by Lemma 1.1 and induction on \( |G| \).

In the proofs of the following lemmas we freely use properties of solvable and supersolvable groups, which can be found in [7, Chap. VI, Sections 8, 9; or 26, Sections 5.4, 9.4].

**Lemma 1.2.** Suppose that \( |C| \) is a squarefree number for all \( C \in \text{Con}(G) \). Then \( G \) is supersolvable.

**Proof.** Let \( G \) be a minimal counterexample. If \( G \) is simple, then Proposition 3 implies that \( |C_G(x)| \) is odd for some \( x \in G \). As \( G \) is simple, \( 4 \mid |G| \) and so \( |Cl(x)| = |G : C_G(x)| \) is divisible by four, a contradiction.
Hence $G$ is not simple and the solvability of $G$ now follows by induction and Lemma 1.1.

By [7, VI, 9.5] $G$ has a maximal subgroup $M$, such that $|G : M|$ is not a prime number. As $G$ is solvable, it follows that $|G : M| = p^n$ with $p$ a prime and $n \geq 2$. Suppose that $M$ contains a subgroup $T$ such that $1 < T \leq M$ and $T \lhd G$. Again, by Lemma 1.1 and induction, $M/T$ is a maximal subgroup of the supersolvable group $G/T$ whose index is not a prime number—a contradiction. Thus no normal subgroup of $G$ is contained in $M$. Since $O_p(G) \leq M$, we get that $O_p(G) = 1$ and hence $F(G) = O_p(G)$. By the above observation $F(G) \leq M$ and so $G = MF(G)$.

If $\Phi(G) > 1$, then induction implies that $G/\Phi(G)$ is supersolvable and hence $G$ is supersolvable, a contradiction. Therefore $\Phi(G) = 1$ and, by [7, III, 4.5], $F(G)$ is an elementary abelian $p$-subgroup of $G$. Clearly $M \cap F(G) \lhd M$, and as $F(G)$ is abelian, $M \cap F(G) \lhd MF(G) = G$. By the previous paragraph we get that $M \cap F(G) = 1$, so that $G = MF(G)$ is a semidirect product.

Now $M \cong G/F(G)$ and hence $M$ is supersolvable by induction. Let $Q$ be a minimal normal subgroup of $M$. Then $|Q| = q$, a prime. Set $Q = \langle x \rangle$, let $R \in \text{Syl}_p(C_G(x))$ an choose $P \in \text{Syl}_p(G)$ such that $R \leq P$. Recall that $F(G) = O_p(G)$. If $R \cap O_p(G) = 1$, then $RO_p(G) \leq P$ and $|R| | O_p(G) | | P|$. Hence $p^n = |O_p(G)| | |P : R|$. But $|\text{Cl}(x)|_p = |P : R|$ is squarefree and therefore $n \leq 1$, a contradiction. Therefore, $1 < R \cap O_p(G) \leq C_G(Q) \leq N_G(Q)$. It follows that $\langle R \cap O_p(G), M \rangle \leq N_G(Q)$ and as $M$ is maximal in $G$, we get that $Q \lhd G$. Since $1 < Q \leq M$, this contradicts the second paragraph of the proof. 

**Lemma 1.3.** Suppose that $|C|$ is a squarefree number for all $C \in \text{Con}(G)$. Then $G/F(G)$ is a cyclic group of squarefree order.

**Proof.** Let $G$ be a minimal counterexample and denote $F(G)$ by $F$. Since by Lemma 1.2 $G$ is supersolvable, it follows that $G/F$ is abelian and it suffices to show that $|G/F|$ is a squarefree number.

Let $A$ be either $\Phi(G)$ or $Z(G)$. In both cases $F(G/A) = F/A$. If $A > 1$, Lemma 1.1 and induction imply that $|G/A : F(G/A)| = |G/A : F/A| = |G : F|$ is a squarefree number, a contradiction. Thus $\Phi(G) = Z(G) = 1$ and, by [7, III, 4.5], $F$ is abelian and it is a direct product of minimal normal subgroups of $G$, each of which is of prime order.

As $G$ is a counterexample, $|G/F|$ is not a prime number and as $G/F$ is abelian, there exists $H \lhd G$ such that $F < H < G$ and $|G : H| = |G/F : H/F| = p$ for some prime $p$. Now $F(H)$ is a characteristic subgroup of $H$ and so $F(H) \lhd G$, implying that $F(H) \leq F$. But $F \leq F(H)$ so that $F(H) = F$. As $H \lhd G$, induction implies that $|H/F(H)| = |H/F|$ is squarefree. Suppose that a prime $q \neq p$ divides $|G/F|$. Then, as above, we can find
Let $L \triangleleft G$ such that $F < L < G$, $F(L) = F$, $|G/L| = q$, and $|L/F(L)| = |L/F|$ is squarefree. Since $|G/F| = p$, $|H/F| = q$, $|L/F|$, we get that $|G/F|$ is squarefree, a contradiction. So no such $q$ exists.

We conclude that $G/F$ is an abelian $p$-group. But $|G/F| = p$, $|H/F|$ and $|H/F|$ is squarefree, hence $|G/F| = p^2$. As $F$ is abelian, it has a complement $A$ in $G$ (see [7, III, 4.4]). Thus $G = FA$, where $A$ is a subgroup of order $p^2$ and $A \cap F = 1$. We claim that $F$ is a $p'$-group. If not, then $F$ contains a subgroup $P$ of order $p$ which is normal in $G$ (see the second paragraph). As each element of $A$ normalizes $P$ and $|\text{Aut}(P)| = p - 1$, we get $A \leq C_A(P)$. But $F$ is abelian, so $F \leq C_A(P)$ and hence $1 < P \leq Z(G) = 1$, a contradiction. Thus $F$ is a $p'$-group.

Now $A$ acts on $F$ coprimely and since $C_G(F) \leq F$, also faithfully. Since both $A$ and $F$ are abelian, we can use [13, p. 210] to conclude that there exists a $y \in F$ such that $C_A(y) = 1$. But then $C_G(y) = FC_A(y) = F$ and so $|C(y)| = |G : C_G(y)| = |A| = p^2$, a final contradiction.

**Proof of Theorem 1.** Denote $F(G)$ by $F$. By Lemmas 1.2 and 1.3 we know that $G$ is supersolvable and that $G/F$ is a cyclic group of squarefree order. Let $P \in \text{Syl}_p(F)$; then by Lemma 1.1 each class-length in $P$ is squarefree, and hence either 1 or $p$. Thus either $P$ is abelian or $P = 1$, in which case $|P| = p$ by [7, III, p. 309] and $\text{cl}(P) = 2$. In particular, $F'$ is an abelian group of squarefree order and $\text{cl}(F) \leq 2$. As $G$ is supersolvable, it follows that $G' \leq F$ and hence $G'' \leq F'' = 1$. The proof of Theorem 1 is now complete.

**Lemma 1.4.** If $|C|$ is a prime power for each $C \in \text{Con}(G)$, then $G/F(G)$ is a $p$-group for some prime $p$. In particular, $G$ is solvable.

**Proof.** Let $G$ be a minimal counterexample. First we observe that by Burnside's theorem [7, V, 7.2], $G$ is not a nonabelian simple group. Hence, by induction and Lemma 1.1, $G$ is solvable.

Arguing as in the second paragraph of the proof of Lemma 1.3 we conclude that $Z(G) = \Phi(G) = 1$ and $F(G) = F$ is abelian. By [7, III, 4.4], there exists $A \leq G$ with $G = FA$ and $A \cap F = 1$.

By refining the series $1 < F < G$, we can find $H \leq G$ such that $F < H < G$ and $|G:H| = p$, a prime. As in Lemma 1.3 (the third paragraph of the proof) we get that $H/F = F$ and induction implies that $H/F(H) = H/F$ is a $q$-group for some prime $q$. Hence $|A| = |G:F| = p$, $|H:F| = pq^n$ for some $n \geq 1$ and $p \neq q$. Since $H/F$ is a normal Sylow $q$-subgroup of $G/F \cong A$, we conclude that $A$ has a normal Sylow $q$-subgroup $Q$ and $A = QF$ with $|P| = p$. Clearly $FQ \triangleleft G$ of index $p$.

We want to show that also $FP \triangleleft G$. Let $R = F_q$ be the $q'$-part of $F$ and denote $C_G(R)$ by $C$. As $FQ \triangleleft G$, it follows that $R \neq 1$. If $P \leq C$, then since $F$ is abelian and $Z(G) = 1$, we may conclude that $F \leq FP \leq C < G$. Since
$C \triangleleft G$, it follows that $F(C) = F$ and by the inductive assumption $|C : F|$ is a prime power. Thus, in this case, $C = FP \triangleleft G$. If $P \not\triangleleft C$, then, letting $P = \langle x \rangle$, we conclude that $|\text{Cl}_{G}(x)| = r^k$ for some prime number $r \neq q$. But then, by Lemma 1.1, $x^F \in Z(G/F)$ and again $FP \triangleleft G$, as claimed. Thus $P = FP \cap A \triangleleft A$ and, since $Q \triangleleft A$, we get $A = P \times Q$. Let $U$ be a normal subgroup of $G$ such that $FP \leq U < G$ and $|G : U| = q$. As before we conclude that $F(U) = F$ and $|U : F|$ is a prime power, forcing $U = FP$. Thus $|Q| = q$ and $A = P \times Q$ is an abelian group of order $pq$.

Let $P = \langle x \rangle$ and $Q = \langle y \rangle$. As $C_{G}(F) \leq F$, we know that $C_{1} = C_{F}(x) < F$ and $C_{2} = C_{F}(y) < F$. This implies that $C_{1} \cup C_{2} \neq F$. Pick $t \in F - (C_{1} \cup C_{2})$. It follows that $C_{G}(t) = 1$ and as $F$ is abelian, $C_{G}(t) = FC_{A}(t) = F$. Thus $|\text{Cl}_{G}(t)| = |G : F| = pq$, $p \neq q$, a final contradiction. □

Proof of Theorem 2. First we want to show that our conditions are sufficient. If $G$ is nilpotent with at most one non-abelian Sylow subgroup, then clearly the class-length of each element of $G$ is a prime power. So suppose that $G$ is non-nilpotent. Since $A \triangleleft Z(G)$, it suffices to show that the class-length of each $\{p, q\}$-element of $G$ is a prime power. Since the Sylow subgroups of $G$ are abelian, if $g$ is a $p$-element of $G$ (a $q$-element of $G$), then $C_{A}(g)$ contains a Sylow $p$-subgroup (a $q$-subgroup) of $G$ and hence the class-length of $g$ is a prime power. Let, finally, $g$ be an element of $G$ of order $p^{n}q^{m}$ with $n, m > 0$. Then $g = xy$, where $x \in G$ of order $p^{n}$ and $y \in G$ of order $q^{m}$, and $xy = yx$. Without loss of generality, we may assume that $x \in P$. If $x \in O_{p}(G)$, then $x \in Z(G)$ and $C_{G}(g) = C_{G}(y)$ which, by the previous argument, forces the class-length of $g$ to be a prime power. So suppose that $x \notin O_{p}(G)$. As $x^{y} = x$ and since $P^{y} \cap P = O_{p}(G)$ if $y \notin N_{G}(P)$, it follows that $y \in N_{G}(P)$. Since $G$ is an $A$-group and since $N_{G}(P)$ is a system-normalizer of $G$, it follows by [26, Theorem 9.2.4] that $N_{G}(P)$ is a nilpotent $A$-group, hence abelian. Thus $y \in N_{G}(P)$ implies that $y \in C_{G}(P)$, and hence $y \in Z(G)$. We conclude that $C_{G}(g) = C_{G}(x)$ and by the previous argument the class-length of $g$ is a prime power. The proof of the sufficiency of our conditions is complete.

It remains to prove that our conditions are necessary. If $G = H \times K$ with $(|H|, |K|) = 1$ and if $H$ is non-abelian, then $K$ must be abelian. Indeed, if $K$ is not abelian, and if $h \in H - Z(H)$ and $k \in K - Z(K)$, then $C_{G}(h) = C_{H}(h) \times K < G$, $C_{G}(k) = H \times C_{K}(k) < G$ and $C_{G}(hk) = C_{H}(h) \times C_{K}(k)$, forcing $pq | |\text{Cl}_{G}(hk)|$, where $p$ and $q$ are distinct primes, in contradiction to our assumptions. It follows, in particular, that if $G$ is nilpotent, then at most one Sylow subgroup of $G$ is non-abelian. So assume, from now on, that $G$ is a non-nilpotent group. By Lemma 1.4, $G = PF$ with $P$ a Sylow $p$-subgroup of $G$ and $F = F(G)$.

Claim 1. $F_{p'}$, the $p'$-part of $F(G)$, is abelian.
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Proof. Suppose that \( F_q \) is non-abelian for some \( q \neq p \) and let \( Z = C_G(F_q) \). If \( x \in G - Z \), then the class-length of \( x \) is a power of \( q \). Thus, by Lemma 1.1, \( G = G/Z \) satisfies the following property: the class-length of each \( \bar{x} \in \bar{G} \) is a power of \( q \). But then, by Proposition 4, \( \bar{G} = \bar{Q} \times \bar{A} \), where \( \bar{Q} \) is a \( q \)-group and \( \bar{A} \) is an abelian \( q' \)-group. Since, by Lemma 1.4, \( G = ZF_p \) it follows that \( \bar{A} = PZ/Z \). Let \( x \in F_q - Z \); such an \( x \) exists since \( F_q \) is non-abelian. Clearly \( Z \leq C_G(x) \) and \( |G/Z : C_G(x)/Z| = |G : C_G(x)| \) is a power of \( q \). Thus \( PZ/Z \leq C_G(x)/Z \) and \( P \leq C_G(x) \). If \( z \) is any element of \( F_q \cap Z \), then \( x, xz \in F_q - Z \) and by the above argument \( P \leq C_G(x) \cap C_G(xz) \leq C_G(z) \). Consequently \( P \leq C_G(F_q) = Z \) and since \( Z \) also contains \( F_q \), it follows that \( G = F_q \times PF_q \). As shown above, \( F_q \) being non-abelian forces \( PF_q \) to be abelian. But then \( G \) is nilpotent, a contradiction. The proof of Claim 1 is complete.

Claim 2. \( G \) is an \( A \)-group.

Proof. By Lemma 1.4 and Claim 1, it suffices to prove that \( P \) is abelian. Since \( G \) is non-nilpotent, there exists \( x \in P - F_p \). By Wielandt’s lemma, \( |Cl_G(x)| \) is a \( q \)-power with \( q \neq p \). Hence \( C_G(x) \) contains a Sylow \( p \)-subgroup of \( G \) and it follows that \( F_p \leq C_G(x) \) and \( x \in C_G(F_p) \). Now \( xF_p \leq P - F_p \), so by the above argument \( xF_p \leq C_G(F_p) \). But then \( F_p \leq C_G(F_p) \), forcing \( F_p \) to be abelian. Moreover, since \( F_p \leq C_G(x) \) for each \( x \in P - F_p \), it follows that \( F_p \leq Z(P) \). Thus, if \( x \) is any \( p \)-element of \( G \), then either \( x \in F_p \) and \( |Cl_G(x)| \) is a \( q \)-power, with \( q \neq p \), or \( x \) is conjugate to some \( y \) in \( P - F_p \) and again, by Wielandt’s lemma, \( |Cl_G(x)| \) is a power of a prime different from \( p \). We may conclude that if \( \bar{x} \in \bar{G} = G/F_p \), then, in view of Lemma 1.1, \( p \) does not divide \( |Cl_G(\bar{x})| \). But \( \bar{G} \cong P \), a \( p \)-group, hence \( |Cl_G(x)| = 1 \) for each \( \bar{x} \in \bar{G} \). It follows that \( P \) is abelian, as required.

Claim 3. \( G = A \times PO_q(G) \), where \( q \) is a prime different from \( p \), \( A \) is an abelian \( \{p, q\}' \)-subgroup of \( G \), \( P \trianglelefteq G \) and \( P \cap P^g = O_p(G) \) for each \( g \in G - N_G(P) \).

Proof. Since \( G \) is non-nilpotent and \( G = PF \), \( P \trianglelefteq G \). Hence \( P \not\trianglelefteq C_G(F) = F \). Thus there exists a prime \( q \neq p \) such that \( P \not\trianglelefteq C_G(F_q) \) and consequently there exists \( x \in P \) such that \( x \not\in C_G(F_q) \) and \( |Cl_G(x)| = q^i \) with \( i > 0 \). If there exists another such prime \( r \neq q \) and an element \( y \in P \) such that \( y \not\in C_G(F_r) \) and \( |Cl_G(y)| = r^j \), \( j > 0 \), consider the element \( xy \in P \). Clearly \( F_r \leq C_G(x) \) and \( F_q \leq C_G(y) \), and there exist elements \( a \in F_q - C_G(x) \) and \( b \in F_r - C_G(y) \). But then

\[
(xy)^a = x^ay \neq xy \quad \text{and} \quad (xy)^b = xy^b \neq xy
\]

and so \( C_G(xy) \) contains neither \( F_q \) nor \( F_r \). Hence \( qr \) divides \( |Cl_G(xy)| \), a contradiction. So there exists exactly one prime \( q \) such that \([P, F_q] \neq 1\) and
we conclude that \( G = A \times P \cdot O_q(G) \), with \( A \) an abelian \( \{ p, q \} \)-subgroup of \( G \). Suppose that \( x \in P \cap P^9 \), \( P \neq P^9 \) and \( x \notin O_p(G) \). Since \( AP \leq C_G(x) \), we have \( C_G(x) = A \times P \cdot Q \), where \( Q = O_q(G) \cap C_G(x) \). Clearly \( Q \geq Z_q \), the \( q \)-component of \( Z(G) \). If equality holds, then \( C_G(x) = A \times P \times Q \) and \( P^9 \leq C_G(x) \) forces \( P = P^9 \), a contradiction. So there exists \( y \in O_q(G) \) such that \( xy = yx \). Since \( x \) and \( y \) are of coprime orders, \( C_G(xy) = C_G(x) \cap C_G(y) \). As \( x \notin O_p(G) \), hence \( O_q(G) \leq C_G(x) \) and \( O_q(G) \not\leq C_G(xy) \). Moreover, by Claim 2, \( C_G(y) = FT \) with \( T \not= P \) as \( y \notin Z_q \). Thus \( C_G(xy) \), and hence also \( C_G(xy) \), contains no conjugate of \( P \). We conclude that \( pq \) divides \( |C_G(xy)| \), a final contradiction.

It follows from Claims 2 and 3 that our conditions are necessary as well and the proof of Theorem 2 is complete.

**Proof of Corollary 2.1.** In view of Theorem 2, we need only to prove that \( \rho c(G) = 2 \) in the non-nilpotent case. If \( O_q(G) \leq C_G(x) \) for each \( x \in P \), then \( P \cdot O_q(G) = P \times O_q(G) \) and \( G \) is nilpotent, a contradiction. Thus \( q \) is a “relevant prime” of \( G \). If, on the other hand, each \( C_G(y) \), \( y \in O_q(G) \), contains a conjugate of \( P \), then \( O_q(G) \) being abelian forces \( O_q(G) \not\leq Z(G) \), in contradiction to the non-nilpotency of \( G \). Hence both \( p \) and \( q \) are “relevant primes” of \( G \) and \( \rho c(G) = 2 \).

**Proof of Corollary 2.2.** Denote \( Z = Z(G) \) and \( \bar{G} = G/Z \) and suppose that \( G \) is non-nilpotent. Then, by Theorem 2 and using the bar convention, we get \( \bar{G} = \bar{P} \cdot \bar{O}_q(G) \) with \( \bar{P}, \bar{O}_q(G) \neq 1 \) and \( \bar{P} \) is a trivial intersection proper subgroup of \( \bar{G} \). If \( P^q Z/Z = PZ/Z \), then \( P^q \leq PZ \) and since \( PZ \) is an abelian group, \( P^q = P \). Thus \( g \in N_G(P) \). Since \( N_G(P) \) is a system-normalizer in the \( A \)-group \( G \), hence \( N_G(P) \) is abelian and \( g \in C_G(P) \). Clearly \( C_G(P) \geq PZ \). As \( O_q(G) \) is abelian, \( C_G(P) \cap O_q(G) \leq Z \) and it follows that \( g \in C_G(P) = PZ \). We conclude that \( \bar{P} \) is a proper, trivial-intersection, and self-normalizing subgroup of \( \bar{G} \). Hence \( \bar{G} \) is a Frobenius group with \( \bar{P} \) as a complement and with \( \bar{O}_q(G) \) as the kernel. The proof of the corollary is complete.

**Proof of Corollary 2.3.** Denote \( F(G) \) by \( F \) and \( Z(G) \) by \( Z \). By Theorems 1 and 2, each of \( |G/F| \) and \( |F'| \) is a squarefree prime power, so that each of them is a prime or one, proving (1). If \( G \) is nilpotent, then by Theorem 1, \( cl(G) \leq 2 \) and, by Corollary 2.1, \( \rho c(G) \leq 1 \). So it suffices to deal with the non-nilpotent case. The opening remarks, together with Theorem 2, imply that in the notation of Theorem 2 we have: \( G = FP \), \( |G : F| = p \), and \( |O_q(G)Z/Z| = q' > 1 \). Thus \( p = |G : F| = |FP : F| = |P/F \cap P| \) and, since \( F \cap P \leq Z \cap P \) and \( P \not\leq Z \), we conclude that \( |PZ/Z| = |P/Z \cap P| = p \). By Corollary 2.2, \( G/Z \) is a Frobenius group of order \( pq' \) with \( O_q(G)Z/Z \) as the Frobenius kernel and with \( PZ/Z \) as a Frobenius complement. Let \( x \in P - F \); then \( G = F\langle x \rangle \). If \( y \in O_q(G) - Z \), then \( y \in Z(F) \),
since $G$ is an $A$-group, and hence $y \notin C_G(x)$. Thus $O_q(G)Z \cap C_G(x) = Z$ and hence $|O_q(G)ZC_G(x) : C_G(x)| = |O_q(G)Z : Z| = q^i$. We conclude that $q^i | G : C_G(x)| = |Cl_G(x)|$. It follows from our assumptions that $i = 1$ and consequently $G/Z$ is a Frobenius group of order $pq$ and $\rho c(G) = 2$, as required.

**Proof of Corollary 2.4.** By Theorem 1, $G$ is supersolvable and $G/F(G)$ is cyclic of squarefree order. Denote $F(G)$ by $F$.

We first show, by induction on $|G|$, that $|G/F|$ is either one, or a prime, or a product of two distinct primes. As in the proof of Lemma 1.3, we have $Z(G) = \Phi(G) = 1$, $F$ is abelian and $G = FA$, where $A \cap F = 1$. Thus $A \cong G/F$ is a cyclic group of squarefree order and it suffices to show that it is not divisible by three primes. So assume that $p$, $q$, $r$ are distinct prime divisors of $|A|$ and let $x, y, z \in A$ be of orders $p, q, r$, respectively. Then $P = \langle x \rangle$, $Q = \langle y \rangle$ and $R = \langle z \rangle$ are the unique Sylow subgroups of $A$ for their respective primes. Consider the following subgroups of $F$: $C_1 = C_F(x)$, $C_2 = C_F(y)$, and $C_3 = C_F(z)$. Each of them is a proper subgroup of $F$ as $C_G(F) \not\subseteq F$. If $F = C_1 \cup C_2 \cup C_3$, then $|F|$ is even by [11]. But then, since $G$ is supersolvable, it has a minimal normal subgroup $N$ of order 2 and $N \leq Z(G) = 1$, a contradiction. Hence there exists $u \in F - C_1 \cup C_2 \cup C_3$ and, since $F \leq C_G(u)$, we get $C_G(u) = FC_A(u)$. But $C_A(u) \cap P = C_A(u) \cap Q = C_A(u) \cap R = 1$, so that $(|C_A(u)|, pqr) = 1$. It follows that $pq r | |Cl_G(u)|$, a contradiction.

Finally, we show that $|F'|$ is either one, or a prime, or a product of two distinct primes. Since, by Theorem 1, $|F'|$ is squarefree, it suffices to show that at most two Sylow subgroups of $F$ are nonabelian. So suppose that $F$ has nonabelian Sylow $p_i$-subgroups $A_i$, $i = 1, 2, 3$, and $p_1, p_2, p_3$ are distinct primes. Set $B = A_1 \times A_2 \times A_3$ and pick $x_i \in A_i - Z(A_i)$ for $i = 1, 2, 3$. Clearly $p_1 p_2 p_3 | |Cl_B(x_1 x_2 x_3)|$. As $B \triangleleft G$, Lemma 1.1 implies that $p_1 p_2 p_3 | |Cl_G(x_1 x_2 x_3)|$, in contradiction to our assumptions.

II. **Upper Bounds**

**Notation.** 1. If $n = \prod_{i=1}^k p_i^{n_i}$ is the prime number decomposition of the natural number $n$, we denote

$$\omega_{p_i}(n) = n_i \quad \text{and} \quad \tau(n) = \max_{1 \leq i \leq k} n_i.$$  

2. Let

$$\omega p(G) = \max_{C \in Con(G)} \omega_p(|C|) \quad \text{and} \quad \tau c(G) = \max_{C \in Con(G)} \tau(|C|).$$
3. Let $G$ be a solvable group and let $p$ be a prime number. The $p$-rank of $G$, which will be denoted by $r_p(G)$, is the maximum dimension of all the chief $p$-factors of $G$. Denote by $r(G)$ the maximum dimension of all the chief factors of $G$.

The statements and proofs of this section are the class lengths versions of those of [23, 18].

**Proposition 6.** Let $G$ be a finite solvable group and let $p$ be a prime. Assume that $O_p(G) = 1$. Then

1. $r_p(G) \leq 2\omega c_p(G)$.
   
2. If $|G|$ is odd, then $r_p(G) \leq \omega c_p(G)$.

**Proof.** Let $G$ be a minimal counterexample. Let $V$ be a minimal normal subgroup of $G$. As $O_p(G) = 1$, hence $p \nmid |V|$. Let $N/V = O_p(G/V)$ and $P \in \text{Syl}_p(N)$. If $O_p(G/V) = 1$, then induction and Lemma 1.1 imply that

$$r_p(G) = r_p(G/V) \leq 2\omega c_p(G/V) \leq 2\omega c_p(G)$$

and if $|G|$ is odd then $r_p(G) \leq \omega c_p(G)$, a contradiction. Thus $O_p(G/V) \neq 1$. Note that $N = VP$ and $|P| = |O_p(G/V)|$. Also $C_p(V) \triangleleft PV = N$, so $C_p(V) \leq O_p(N) \leq O_p(G) = 1$. Hence $C_p(V) = 1$.

Set $|P| = p^n$, where $n$ is a natural number. By a theorem of Passman [25, Theorem 1.1], there exists a $v \in V$ such that $|C_p(v)| = p^m$, where $m \leq n/2$. If $|G|$ is odd, then there even exists a $v \in V$ with $|C_p(v)| = 1$. As $C_N(v) = VC_p(v)$, we conclude that $|C_N(v)| = p^u$, where $u \geq n/2$ and $|C_N(v)| = p^u$ if $|G|$ is odd. It follows that $\omega c_p(N) \geq n/2$ and if $|G|$ is odd then $\omega c_p(N) \geq n$. Hence, in view of Lemma 1.1, we get $n \leq 2\omega c_p(N) = 2\omega c_p(G)$ and $n \leq \omega c_p(G)$ for $|G|$ odd. Thus all the chief $p$-factors of $G$ between 1 and $N$ have rank at most $2\omega c_p(G)$ (at most $\omega c_p(G)$ for $|G|$ odd).

As $O_p(G/N) = 1$, we can apply induction to $G/N$ and get $r_p(G/N) \leq 2\omega c_p(G/N) \leq 2\omega c_p(G)$ and $r_p(G/N) \leq \omega c_p(G)$ for $|G|$ odd. It follows that $r_p(G) \leq 2\omega c_p(G)$ and $r_p(G) \leq \omega c_p(G)$ for $|G|$ odd. 

**Proposition 7.** Let $G$ be a finite solvable group. Then $r(G/F(G)) \leq 2\tau c(G)$. If $|G|$ is odd, then $r(G/F(G)) \leq \tau c(G)$.

**Proof.** Let $p$ be a prime such that $r(G/F) = r_p(G/F)$, where $F = F(G)$. Clearly, $\omega c_p(G) \leq \tau c(G)$ and Proposition 6 implies that

$$r_p(G/O_p(G)) \leq 2\omega c_p(G/O_p(G)) \leq 2\omega c_p(G) \leq 2\tau c(G).$$

Write $G/O_p(G) = G$ and $F/O_p(G) = F$. Then

$$r(G/F) = r_p(G/F) = r_p(G/F) \leq r_p(G/O_p(G)) \leq 2\tau c(G)$$

as desired. If $|G|$ is odd, the same proof yields $r(G/F) \leq \tau c(G)$. 

Proof of Theorem 8. Set $\bar{G} = G/F$ and $F_2/F = F(\bar{G})$, where $F = F(G)$. Then $G/F_2 \cong \bar{G}/F(\bar{G})$ is isomorphic to a completely reducible subgroup of a direct product of some $GL(r(\bar{G}), p)$, where $p$ runs through the prime divisors of $|F(\bar{G})|$ (see Lemma 2.3 in [18]). By Proposition 7 we know that $r(\bar{G}) \leq 2\tau_c(G)$ and so $G/F_2 \cong \bar{G}/F(\bar{G})$ is isomorphic to a completely reducible subgroup of a direct product of some $GL(2\tau_c(G), p)$ for the above primes $p$. By Theorem 2.5 of [18], $\text{dl}(G/F_2) \leq 2\tau_c(G) + 2$ and consequently,

$$n(G) = 2 + n(G/F_2) \leq 2 + \text{dl}(G/F_2)$$

$$\leq 2 + 2\tau_c(G) + 2 = 4 + 2\tau_c(G)$$

which is the first part of (1). The inequality for $|G|$ odd is obtained similarly.

Let $P$ be a (normal) Sylow $p$-subgroup of either $F$ or $F_2/F$. Applying Lemma 1.1 several times, we get that $\tau_c(P) \leq \tau_c(G)$. Recall that $p^{\beta(P)} = \max_{C \in \text{Con}(P)} |C|$. As $\text{dl}(P) \leq \text{cl}(P)$, [8, p. 346] implies that

$$\text{dl}(P) \leq \text{cl}(P) \leq 2\beta(P) = 2\tau_c(P) \leq 2\tau_c(G).$$

Choose $Q \in \text{Syl}_q(F)$ and $R \in \text{Syl}_r(F_2/F)$, where $q$ and $r$ are primes such that $\text{dl}(Q) = \text{dl}(F)$ and $\text{dl}(R) = \text{dl}(F_2/F)$. This choice is possible since $F$ and $F_2/F$ are nilpotent. Then

$$\text{dl}(G) \leq \text{dl}(F) + \text{dl}(F_2/F) + \text{dl}(G/F_2)$$

$$= \text{dl}(Q) + \text{dl}(R) + \text{dl}(G/F_2).$$

It follows that $\text{dl}(G) \leq 2\tau_c(G) + 2\tau_c(G) + (2\tau_c(G) + 2)$, as claimed in (2).

The inequality for $|G|$ odd is obtained similarly.

Better bounds can be obtained if the better bounds of Theorem 2.5 of [18] are used. We note that by substituting the bounds obtained in Proposition 6 into Huppert's estimates [9; 23, Theorem 2.4], a bound for the $p$-length, $l_p(G)$, can be obtained in terms of $\omega_c_p(G)$. Namely:

**Proposition 9.** Let $G$ be a finite solvable group and $p$ an odd prime. Define $b = 1$ if $G$ is of odd order and $b = 2$ if $G$ is an arbitrary solvable group. Then

$$l_p(G) \leq \begin{cases} 3 + \log_p(b\omega_c_p(G)) & \text{for } p \neq 2, \text{ } p \text{ not a Fermat prime} \\ 2 + \log_{p-2}(b\omega_c_p(G)(p - 3) + 1) & \text{for } p \neq 3, \text{ } p \text{ a Fermat prime} \\ 3 + \log_2(b\omega_c_p(G)) & \text{for } p = 3. \end{cases}$$
Remarks on the Order of Centralizers. We conclude our paper with a list of analogous results on $\{ |C_G(x)|; x \in G \}$. The first four are easy to see and number 5 is our Proposition 3. Number 6 (the only one in which non-solvable groups are possible) can be proved using elementary methods and [27].

1. If $|C_G(x)|$ is a power of a fixed prime $p$ for all $x \in G - Z(G)$, then $G$ is either abelian or a $p$-group.

2. If $|C_G(x)|$ is prime for all $x \in G - Z(G)$, then $G$ is either abelian or a Frobenius group of order $pq$, where $p$ and $q$ are primes.

3. If $|C_G(x)|$ is squarefree for all $x \in G - Z(G)$, then either $G$ is abelian or $|G|$ is squarefree.

4. If $p \nmid |C_G|$ for all $x \in G - Z(G)$, then either $G$ is abelian or $p \nmid |G|$.

5. If $|C_G(x)|$ is divisible by the prime $p$ for all $x \in G$, then $G$ is not a nonabelian simple group.

6. Suppose that $|C_G(x)|$ is a prime power for all $x \in G - Z(G)$. Then one of the following holds:
   a. $G$ is either abelian or a $p$-group for some $p$.
   b. $G$ is solvable, $Z(G) = 1$, $|G| = p^aq^b$ for some primes $p$ and $q$ and $l_p(G) \leq 2$.
   c. $G/O_2(G)$ is a group from the following list: $PSL_2(q)$, $Sz(q)$, $PSL_3(4)$, $M_9$ (the stabilizer of a point in $M_{11}$), for some values of $q$.

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