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MHD effects on heat transfer over stretching sheet embedded in porous medium with variable viscosity, viscous dissipation and heat source/sink

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Heat source/sink

Abstract In this analysis, MHD boundary layer flow and heat transfer of a fluid with variable viscosity through a porous medium towards a stretching sheet by taking in to the effects of viscous dissipation in presence of heat source/sink is considered. The symmetry groups admitted by the corresponding boundary value problem are obtained by using Lie's scaling group of transformations. These transformations are used to convert the partial differential equations of the governing equations into self-similar non-linear ordinary differential equations. Numerical solutions of these equations are obtained by Runge-Kutta fourth order with shooting method. Numerical results obtained for different parameters such as viscosity variation parameter A, permeability parameter k_1, heat source/sink parameter λ, magnetic field parameter M, Prandtl number Pr, and Eckert number Ec are drawn graphically and effects of different flow parameters on velocity and temperature profiles are discussed. The skin-friction coefficient $-f''(0)$ and heat transfer coefficient $-\theta'(0)$ are presented in tables.

1. Introduction

Flow of an incompressible viscous fluid and heat transfer phenomena over a stretching sheet have received great attention during the past decades owing to the abundance of practical applications in chemical and manufacturing process, such as polymer extrusion, drawing of copper wires, and continuous casting of metals, wire drawing and glass blowing. The study of hydrodynamic flow and heat transfer over a stretching sheet may find its application to sheet extrusion in order to make flat plastic sheets. In doing so, it is important to investigate cooling and heat transfer for the improvement of the final products. The conventional fluids such as water and air are among the most widely used fluids as the cooling medium. However, the rate of heat exchange achievable by the above fluids is realized to be unsuitable for certain sheet materials. Thus, in recent years, it has been proposed to alter flow kinematics that it leads to a slower rate of solidification as compared with water. Among the techniques to control flow kinematics, the idea of using magnetic field appears to be the most attractive one both

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Nomenclature

F	non-dimensional stream function
F^*	variable
k	permeability of the porous medium
k_1	permeability parameter
Pr	Prandtl number
M	magnetic parameter
Ec	Eckert number
Q_0	dimensional heat generation/absorption Coefficient
p, q	variables
T	temperature of the fluid
T_∞	free-stream temperature
T_w	temperature of the wall of the surface
u,v	components of velocity in x and y directions
z	variable

Greek symbols

α_1, α_2, α_3, α_4,α_5, α_6, α', α'	transformation parameters
β', β''	transformation parameters
η	similarity variable
Γ	Lie-group transformations
K	the coefficient of thermal diffusivity
λ	heat source/sink parameter
μ^*	reference viscosity
ψ	stream function
ψ^*	transformed stream function
ρ	density of the fluid
θ	non-dimensional temperature
θ^*, $\bar{\theta}$	variables

because of its ease of implementation and also because of its intrusive nature. The fluid mechanics properties desired for an outcome of such a process would mainly depend on two aspects, one is the rate of cooling of liquid used and the other is the rate of stretching. The rate of cooling and the desired properties of the end product can be controlled by the use of electrically conducting fluid and applications of magnetic fields. The use of magnetic field has been used in the process of purification of molten metal's from non-metallic inclusions (Figs. 1 and 2).

The momentum and heat transfer of boundary layer flow over a stretching sheet have been applied in many chemical engineering processes such as metallurgical process, polymer extrusion process involving cooling of a molten liquid being stretched into a cooling system. These applications involve the cooling of continuous strips of filaments by drawing them through a quiescent fluid. Sakiadis [1] initiated the study of the boundary layer flow over a stretched surface moving with constant velocity and formulated boundary layer equations for two-dimensional and axisymmetric flows. Crane [2] investigated the flow caused by the stretching sheet. Sharidan [3] studied similarity solutions for unsteady boundary layer flow and heat transfer due to stretching sheet. Carraagher et al. [4] studied the flow and heat transfer over a stretching surface when the temperature difference between the surface and an ambient fluid is proportional to the power of distance from a fixed point. Many researchers such as Gupta and Gupta [5], Dutta et al. [6] extended the work of Crane [2] by including the effect of heat and mass transfer analysis under different physical situations. Several authors have considered various aspects of this problem and obtained similarity solutions (Ishak et al. [7–9], Mahapatra et al. [10], Pal [11,12], Aziz et al. [13], Abel et al. [14], Mukhopadhyay and Mondal [15], Zhang et al. [16], Krishnendu [17]). Swati Mukhopadhyay et al. [18] studied the unsteady two-dimensional flow of a non-Newtonian fluid over a stretching surface. The casson fluid model is used to characterize the non-Newtonian fluid behavior.

Viscous dissipation changes the temperature distributions by playing a role like an energy source, which leads to affected heat transfer rates. The merit of the effect of viscous dissipation depends on whether the plate is being cooled or heated. Heat transfer analysis over porous surface is of much practical interest due to its abundant applications. The previous studies are based on the constant physical properties of the fluid. However, it is known that the physical properties of the fluid may change significantly with temperature. The increase in

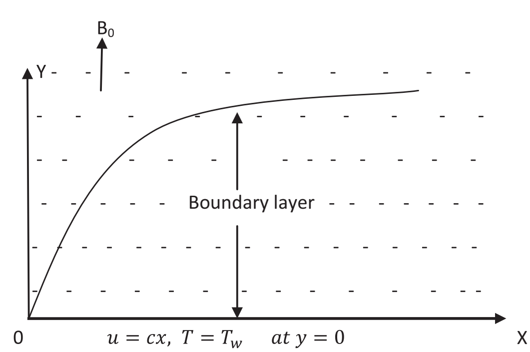

Figure 1 Sketch of the physical problem.

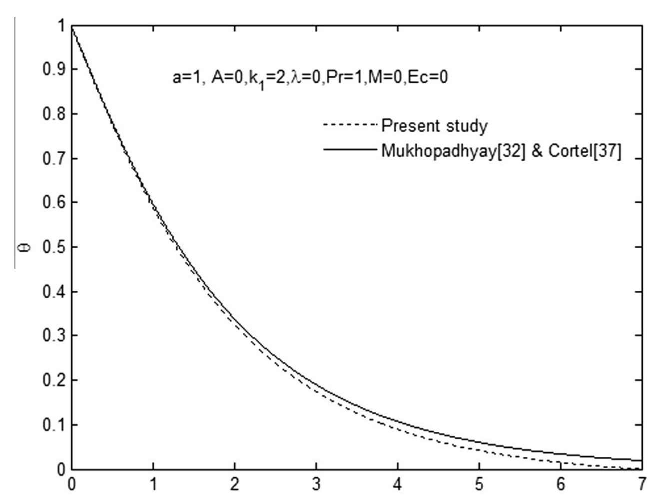

Figure 2 Graphical comparison of the present study with Mukhopadhyay [32] and Cortel [37].

temperature leads to the increase in the transport phenomena by reducing the viscosity across the momentum boundary layer and due to which the heat transfer rate at the wall is also affected. Therefore, to accurately predict the flow and heat transfer rates, it is necessary to take into account the temperature-dependent viscosity of the fluid. The effect of temperature-dependent viscosity on heat and mass transfer laminar boundary layer flow has been discussed by many authors [19–24] in various situations. They showed that when this effect was included, the flow characteristics might change substantially compared with the constant viscosity assumption. Salem [25] investigated variable viscosity and thermal conductivity effects on MHD flow and heat transfer in visco-elastic fluid over a stretching sheet. Anjali Devi and Ganga [26] have considered the viscous dissipation effects on MHD flows past stretching porous surfaces in porous media. From the preceding investigations, it is observed that the variation in viscosity with temperature and viscous dissipation is interesting physical phenomenon in convective fluid flows. A new dimension is added to the above mentioned study of Mukhopadhyay et al. [27] by considering the effects of porous media. Flows through porous media are of principal interest because these are quite prevalent in nature. Such type of flow finds its applications in a broad spectrum of disciplines covering chemical engineering to geophysics. Flow through fluid-saturated porous medium is important in many technological applications, and it has increasing importance with the growth of geothermal energy usage and in astrophysical problems. Several other applications may also benefit from a better understanding of the fundamentals of mass, energy, and momentum transport in porous media, namely cooling of nuclear reactors, underground disposal of nuclear waste, petroleum reservoir operations, building insulation, food processing, and casting and welding in manufacturing processes. In certain porous media applications, working fluid heat generation (source) or absorption (sink) effects are important. Representative studies dealing with these effects have been reported by authors such as Gupta and Sridhar [28], Abel and Veena [29] and Sharma [30]. The effects of variable viscosity and thermal conductivity on an unsteady two-dimensional laminar flow of viscous incompressible conducting fluid past a semi-infinite vertical porous moving plate taking into account the effect of a magnetic field in the presence of variable suction are studied by Seddeek and Salama [31].

Recently Mukhopadhyay et al. [32] studied the effects of variable viscosity on the boundary layer flow and heat transfer of the fluid flow through a porous medium towards a stretching sheet in the presence of heat generation or absorption. In this paper the magneto hydrodynamic flow and heat transfer over a heated stretching sheet immersed in a porous media in the presence of heat source/sink and viscous dissipation have been considered. Fluid viscosity is assumed to vary as a linear function of temperature. In the field of fluid mechanics, most of the researchers try to obtain the similarity solutions in such cases using the similarity variables. In case of scaling group of transformations, the group-invariant solutions are nothing but the well known similarity transformation [33]. A special form of Lie-group of transformations known as scaling is used in this paper to find out the full set of symmetries of the problem and then to study which of them are appropriate to provide group invariant or more specifically similarity solutions. This method reduces the system of non-linear coupled partial differential equations governing the motion of the fluid into a system of coupled ordinary differential equations. In this paper, by applying Lie's scaling group transformations to the problem of boundary layer flow and heat transfer of a fluid with variable viscosity over a stretching sheet embedded in a porous medium by taking the effects of viscous dissipation and heat source /sink in the presence of uniform magnetic field is analyzed. The system remains invariant due to some relations among the parameters of the transformations. With this transformation, a third order and a second order ordinary differential equations corresponding to momentum and energy equations are derived. These equations are solved with the help of Runge-Kutta fourth order method along with shooting technique. The effects of the fluid viscosity parameter, Prandtl number, magnetic parameter, permeability parameter, Eckert parameter and heat source/sink parameter on velocity and temperature fields are investigated and analyzed with the help of graphical representation.

2. Mathematical formulations

Consider a steady two-dimensional forced convection flow of a viscous dissipating incompressible fluid past a heated stretching sheet immersed in a porous medium in the region $y > 0$. Keeping the origin fixed, two equal and opposite forces are applied along the x-axis which results in stretching of the sheet and a uniform magnetic field of strength B_0 is imposed along the y-axis. The temperature of the sheet is different from that of the ambient medium. The fluid viscosity is assumed to vary with temperature while the other fluid properties are assumed constants.

The continuity, momentum and energy equations governing such type of flow are written as

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)-\frac{\sigma B_0^2}{\rho}u-\frac{\mu}{\rho k}u$$
$$=\frac{1}{\rho}\frac{\partial \mu}{\partial T}\frac{\partial T}{\partial y}\frac{\partial u}{\partial y}+\frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}-\frac{\sigma B_0^2}{\rho}u-\frac{\mu}{\rho k}u \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\kappa\frac{\partial^2 T}{\partial y^2}+\frac{\mu}{\rho c_p}\left(\frac{\partial u}{\partial y}\right)^2+\frac{Q_0}{\rho c_p}(T-T\infty) \tag{3}$$

where u and v are components of velocity respectively in x and y directions, T is the temperature, κ is the coefficient of thermal diffusivity, Q_0 (J s^{-1} m^{-3} K^{-1}) is the dimensional heat generation ($Q_0 > 0$) or absorption ($Q_0 < 0$) coefficient, c_p is the specific heat, ρ is the fluid density (assumed constant), μ is the coefficient of fluid viscosity (dependent on temperature), *and* k is the permeability of the porous medium.

2.1. Boundary conditions

The appropriate boundary conditions for the problem are given by

$$u=cx,\quad v=0,\ T=T_w\quad \text{at } y=0,$$

$$u\to 0,\quad T\to T_\infty\quad \text{as } y\to\infty \tag{4}$$

Here c (>0) is constant, T_w is the uniform wall temperature, T_∞ is the temperature far away from the sheet.

2.2. Method of solution

We now introduce the following relations for u, v and θ as

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x} \quad \text{and} \quad \theta = \frac{T - T_\infty}{T_w - T_\infty} \tag{5}$$

where ψ is the stream function.

The temperature dependent fluid viscosity is given by (Batchelor [34]),

$$\mu = \mu^*(a + b(T_w - T)) \tag{6}$$

where μ^* is the constant value of coefficient of viscosity far away from the sheet and a, b are constants and b (>0). We have used viscosity-temperature relation $\mu = a - bT(b > 0)$ which agrees quite well with the relation $\mu = \frac{1}{(b_1 + b_2 T)}$ (Saikrishnan and Roy [35]) when second and higher order terms are neglected.

The viscosity-temperature relation used is $\mu = 1/(b_1 + b_2 T)$ which can be written in expanded form as

$$\mu = \frac{1}{b_1}\left(1 + \frac{b_2}{b_1}T\right)^{-1} = \frac{1}{b_1}\left(1 - \frac{b_2}{b_1}T + \frac{b_2^2}{b_1^2}T^2 - \dots\right)$$

$$\cong \frac{1}{b_1} - \frac{b_2}{b_1^2}T$$

(provided $\left|\frac{b_2}{b_1}T\right|$x < 1) (Neglecting second and higher order terms) $\mu = a - bT$ where $a = \frac{1}{b_1}, b = \frac{b_2}{b_1}$. They took in their study, $b_1 = 53.41$, $b_2 = 2.43$ and so $\left[\frac{b_2}{b_1}T < 1\right]$ gives $0° \leqslant T \leqslant 23°$.

Our viscosity-temperature relation also agrees quite with the relation $\mu = e^{-aT}$ (Bird et al. [36]) when second and higher order terms are neglected in the expansions. Range of temperature i.e ($T_w - T_\infty$) studied here is (0–23 °C). Coefficient of viscosity μ of a large number of liquids agrees very closely with the empirical formula given by $\mu = \frac{C}{(a+bT)^n}$ where a, b, c, n are constants depending on the nature of liquid. This agrees well with $n = 1$ for pure water with our formulation for fluid viscosity.

Using the relations (5) in the boundary layer Eq. (2) and in the energy Eq. (3) we get the following equations

$$\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} = -Av^*\frac{\partial\theta}{\partial y}\frac{\partial^2\psi}{\partial y^2} + v^*[a + A(1-\theta)]\frac{\partial^3\psi}{\partial y^3} - \frac{v^*}{k}[a + A(1-\theta)]\frac{\partial\psi}{\partial y} - \frac{\sigma B_0^2}{\rho}\frac{\partial\psi}{\partial y} \tag{7}$$

$$\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} = k\frac{\partial^2\theta}{\partial y^2} + \frac{Q_0}{\rho c_p}\theta + \frac{v^*}{c_p(T_w - T_\infty)}[a + A(1-\theta)]\left(\frac{\partial^2\psi}{\partial y^2}\right)^2 \tag{8}$$

where $A = b(T_w - T_\infty), v^* = \frac{\mu^*}{\rho}$.

The boundary conditions (4) then becomes

$$\frac{\partial\psi}{\partial y} = cx, \quad \frac{\partial\psi}{\partial x} = 0, \quad \theta = 1 \quad \text{at } y = 0.$$

$$\frac{\partial\psi}{\partial y} \to 0, \quad \theta \to \infty \quad \text{as} \quad y \to \infty. \tag{9}$$

2.3. Scaling group of transformations

Now introduce simplified form of Lie-group transformations namely the scaling group of transformations (Mukhopadhyay et al. [19]),

$$\begin{aligned} \Gamma : x^* &= xe^{\varepsilon\alpha_1} & y^* &= ye^{\varepsilon\alpha_2} \\ \psi^* &= \psi e^{\varepsilon\alpha_3} & u^* &= ue^{\varepsilon\alpha_4} \\ v^* &= ve^{\varepsilon\alpha_5} & \theta^* &= \theta e^{\varepsilon\alpha_6} \end{aligned} \tag{10}$$

Eq. (10) may be considered as a point-transformation which transforms coordinates $(x, y, \psi, u, v, \theta)$ to the coordinates $(x^*, y^*, \psi^*, u^*, v^*, \theta^*)$. Substituting (10) in (7) and (8), we get,

$$\begin{aligned} e^{\varepsilon(\alpha_1+2\alpha_2-2\alpha_3)}&\left(\frac{\partial\psi^*}{\partial y^*}\frac{\partial^2\psi^*}{\partial x^*\partial y^*} - \frac{\partial\psi^*}{\partial x^*}\frac{\partial^2\psi^*}{\partial y^{*2}}\right) \\ &= -Av^*e^{\varepsilon(3\alpha_2-\alpha_3-\alpha_6)}\frac{\partial\theta^*}{\partial y^*}\frac{\partial^2\psi^*}{\partial y^{*2}} + v^*(a + A)v^*e^{\varepsilon(3\alpha_2-\alpha_3)\frac{\partial^3\psi^*}{\partial y^{*3}}} \\ &\quad - Av^*\theta^*e^{\varepsilon(3\alpha_2-\alpha_3-\alpha_6)\frac{\partial^3\psi^*}{\partial y^{*3}}} - \frac{v^*}{k}(a + A)v^*e^{\varepsilon(\alpha_2-\alpha_3)\frac{\partial\psi^*}{\partial y^*}} \\ &\quad + \frac{v^*}{k}Av^*e^{\varepsilon(\alpha_2-\alpha_3-\alpha_6)\frac{\partial\psi^*}{\partial y^*}} - \frac{\sigma B_0^2}{\rho}e^{\varepsilon(\alpha_2-\alpha_3)\frac{\partial\psi^*}{\partial y^*}} \end{aligned} \tag{11}$$

$$\begin{aligned} e^{\varepsilon(\alpha_1+\alpha_2-\alpha_3-\alpha_6)}&\left(\frac{\partial\psi^*}{\partial y^*}\frac{\partial\theta^*}{\partial x^*} - \frac{\partial\psi^*}{\partial x^*}\frac{\partial\theta^*}{\partial y^*}\right) \\ &= \kappa e^{\varepsilon(2\alpha_2-\alpha_6)}\frac{\partial^2\theta^*}{\partial y^{*2}} + \frac{Q_0}{\rho c_p}e^{-\varepsilon\alpha_6} + \frac{v^*}{c_p(T_w - T_\infty)} \\ &\quad \times [a + A]e^{\varepsilon(4\alpha_2-2\alpha_3)}\left(\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2 \\ &\quad - \frac{v^*}{c_p(T_w - T_\infty)}A\theta^*e^{\varepsilon(4\alpha_2-2\alpha_3-\alpha_6)}\left(\frac{\partial^2\psi^*}{\partial y^{*2}}\right)^2 \end{aligned} \tag{12}$$

The system will remain invariant under the group of transformations Γ we would have the following relations among the parameters, namely

$$\alpha_1 + 2\alpha_2 - 2\alpha_3 = 3\alpha_2 - \alpha_3 - \alpha_6 = 3\alpha_2 - \alpha_3 = \alpha_2 - \alpha_3 = \alpha_2 - \alpha_3 - \alpha_6$$

and

$$\alpha_1 + \alpha_2 - \alpha_3 - \alpha_6 = 2\alpha_2 - \alpha_6 = -\alpha_6 = 4\alpha_2 - 2\alpha_3 = 4\alpha_2 - 2\alpha_3 - \alpha_6$$

These relations give $\alpha_1 = \alpha_3$ and $\alpha_2 = 0 = \alpha_6$. The boundary conditions yield $\alpha_1 = \alpha_4$, $\alpha_5 = 0$.

Thus the set reduces to a one parameter group of transformations:

$$x^* = xe^{\varepsilon\alpha_1}, \quad y^* = y, \quad u^* = ue^{\varepsilon\alpha_1},$$

$$v^* = v, \quad \psi^* = \psi e^{\varepsilon\alpha_1}, \quad \theta^* = \theta \tag{13}$$

Expanding by Taylor's series we get

$$x^* - x = x\varepsilon\alpha_1, \quad y^* - y = 0,$$

$$u^* - u = u\varepsilon\alpha_1, \psi^* - \psi = \chi\varepsilon\alpha_1$$

$$v^* - v = 0, \quad \theta^* - \theta = 0 \tag{14}$$

In terms of differentials we get.

$$\frac{dx}{\alpha_1 x} = \frac{dy}{0} = \frac{d\psi}{\alpha_1 \psi} = \frac{du}{\alpha_1 u} = \frac{dv}{0} = \frac{d\theta}{0} \tag{15}$$

From the subsidiary equations $\frac{dx}{\alpha_1 x} = \frac{dy}{0}$ we get $dy = 0$ which on integrations gives

$$y = \eta(\text{constant})/(\text{say}) \tag{15a}$$

From equations $\frac{dx}{\alpha_1 x} = \frac{d\theta}{0}$ we get $d\theta = 0$ which on integration gives us

$$\theta = \theta(\eta)/(\text{say}) \tag{15b}$$

Also integrating the equations $\frac{dx}{\alpha_1 x} = \frac{d\psi}{\alpha_1 \psi}$ we get. $\frac{\psi}{x} =$ constant i.e

$$\psi = xF(\eta)(\text{say}) \tag{15c}$$

where F is an arbitrary function of η.

Thus from Eqs. (15a)–(15c) we obtain,

$$y = \eta, \quad \psi = xF(\eta), \quad \theta = \theta(\eta) \tag{16}$$

Using these transformation Eqs. (11) and (12) becomes

$$F'^2 - FF'' = -Av^*\theta'F'' + v^*[a + A(1-\theta)]F''' - \frac{v^*}{k}[a + A(1-\theta)]F' - \frac{\sigma B_0^2}{\rho}F' \tag{17}$$

$$\kappa\theta'' + F\theta' + \frac{Q_0}{\rho c_p}\theta + \frac{c^2x^2}{c_p(T_w - T_\infty)}[(a + A(1-\theta)]F''^2 = 0 \tag{18}$$

The boundary conditions of Eq. (9) becomes

$$F' = c, \quad F = 0, \quad \theta = 1 \text{ at } \eta = 0.$$

$$F' \to 0, \quad \theta \to 0 \quad \text{as } \eta \to \infty. \tag{19}$$

Introducing $\eta = v^* c^{\beta}\eta^*$, $F = v^*\alpha' c^{\beta'} F^*$, $\theta = v^*\alpha'' c^{\beta''}\bar{\theta}$ in Eqs. (17) and (18) we get

$$\alpha' = \alpha = 1/2, \quad \alpha'' = 0$$

$$\beta' = -\beta = 1/2, \quad \beta'' = 0$$

Eqs. (17) and (18) are transformed to

$$F^{*\prime 2} - F^*F^{*\prime\prime} = -AF^{*\prime\prime}\bar{\theta}' + [a + A(1-\bar{\theta})]F^{*\prime\prime\prime} - k_1[a + A(1-\bar{\theta})]F^{*\prime} - MF^{*\prime} \tag{20}$$

$$\bar{\theta}'' + Pr(F^*\bar{\theta}' + \lambda\bar{\theta} + Ec(a + A(1-\bar{\theta})))F^{*\prime\prime 2} = 0 \tag{21}$$

where $Pr = \frac{v^*}{k}$ is the Prandtl number and $\lambda = \frac{Q_0}{c\rho c_p}$ is the heat source/sink parameter, $M = \frac{\sigma B_0^2}{\rho c}$ is the magnetic parameter, $k_1 = \frac{v^*}{ck}$ is the permeability parameter (Cortell [37]), $Ec = \frac{c^2x^2}{c_p(T_w - T_\infty)}$ is Eckert number.

Taking $F^* = f$ and $\bar{\theta} = \theta$ Eqs. (20) and (21) finally takes the following form

$$[a + A(1-\theta)]f''' + ff'' - Af''\theta' - f'^2 - k_1[a + A(1-\theta)]f' - Mf' = 0 \tag{22}$$

$$\frac{1}{Pr}\theta'' + f\theta' + \lambda\theta + Ec(a + A(1-\theta))f''^2 = 0 \tag{23}$$

The boundary conditions of Eq. (19) take the following form

$$f' = 1, \; f = 0, \quad \theta = 1 \quad \text{at } \eta^* = 0,$$

$$f' \to 0, \; \theta \to 0 \quad \text{as } \eta^* \to \infty. \tag{24}$$

3. Numerical method for solution

The set of coupled non-linear governing boundary layer Eqs. (22) and (23) together with boundary conditions (24) are solved numerically by using Runge-Kutta fourth order technique along with shooting method. First of all, the higher order non-linear differential Eqs. (22) and (23) are converted into simultaneous linear differential equation of first order and they are further transformed into initial value problem by applying the shooting technique. The resultant initial value problem is solved by employing Runge-Kutta fourth order method. The step size $\Delta\eta = 0.001$ is used to obtain the numerical solution with six decimal accuracy as criterion of convergence. The above mentioned third order and second order equations are written in terms of first order equations as follows:

$$\left.\begin{aligned} f' &= z \\ z' &= p \\ p' &= \frac{[z^2 - fp + Apq + k_1[a + A(1-\theta)z + Mz]}{[a + A(1-\theta)]} \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} \theta^* &= q \\ q' &= -Pr(fq + \lambda\theta + Ec(a + A(1-\theta))p^2 \end{aligned}\right\} \tag{26}$$

With boundary conditions

$$f(0) = 0, f'(0) = 1, \theta(0) = 1 \tag{27}$$

In order to integrate (25) and (26) as initial value problem we require a value for $p(0)$ i.e. $f''(0)$ and $q(0)$ i.e $\theta'(0)$ but no such values are given in the boundary. The suitable guess values for $f''(0)$ and $\theta'(0)$ are chosen and then integration is carried out. We compare the calculated values for f' and θ at $\eta = 7$ (say) with the given boundary condition $f'(7) = 0$ and $\theta(7) = 0$ and adjust the estimated values $f''(0)$ and $\theta'(0)$ to give a better approximation for the solution. Different values of η (such as $\eta = 2.5, 3, 6, 7$, etc.) are taken in our numerical computations so that numerical values obtained are independent of η chosen.

We take the series of values for $f''(0)$ and $\theta'(0)$ and apply the fourth order Runge-Kutta method with different step-sizes ($\eta = 0.01, 0.001$, etc.) so that the numerical results obtained are independent of $\Delta\eta$. The above procedure is repeated until we get the results up to the desired degree of accuracy 10^{-6}.

4. Results and discussion

The computations have been carried out for various governing flow parameters such as the viscosity parameter A, permeability parameter k_1, heat source/sink parameter λ, the Prandtl number Pr, magnetic parameter M and Ecket number Ec. For illustrations of the results the numerical values are plotted in figures for dimensionless velocity profile and temperature profiles. In order to access the accuracy of the method the results are compared in case of uniform viscosity and in absence of suction/blowing with Coretell [37] and Mukhopadhyay et al. [32] which are given in Table 1. The results are in good agreement with them. The values of skin friction coefficients $-f''(0)$ and the wall temperature gradient $-\theta'(0)$ are tabulated in Tables 2 and 3. It is evident from the tables that increasing the viscosity parameter A the skin-friction coefficients $-f''(0)$ and temperature gradient $-\theta'(0)$ values increases. The effect of permeability parameter k_1 increases the values of

Table 1 The skin-friction $-f''(0)$ and the wall temperature gradient $-\theta'(0)$ for two values of k_1 with $a = 1$, $A = 0$, $\lambda = 0$, $Pr = 1$, $M = 0$ and $Ec = 0$.

k_1	$-f''(0)$			$-\theta'(0)$		
	Coretel [35]	Mukhopadhyay [31]	Present study	Coretel [35]	Mukhopadhyay [31]	Present study
1	1.414213	1.414213	1.414214	0.500000	0.500001	0.500008
2	1.732051	1.732051	1.732051	0.447552	0.447553	0.447558

Table 2 The skin-friction coefficient $-f''(0)$ and the wall temperature gradient $-\theta'(0)$ values with $a = 1$ and $Pr = 0.71$, $M = 0.5$, $Ec = 0.03$.

A	k_1	λ	$-f''(0)$	$-\theta'$
0.0	0.0	0.0	1.224751	0.413244
1.0	0.0	0.0	1.307859	0.422576
4.0	0.0	0.0	1.546828	0.436481
5.0	0.0	0.0	1.622843	0.438771
10.0	0.0	0.0	1.981813	0.443167
0.0	0.1	0.0	1.264914	0.406634
1.0	0.1	0.0	1.355174	0.412845
0.0	0.0	0.1	1.224751	0.300687
0.0	0.1	0.1	1.264914	0.291309
0.1	0.1	0.1	1.336319	0.300640
0.1	0.1	0.0	1.355174	0.412844
0.1	0.1	−0.1	1.368797	0.499333
0.1	0.1	−0.2	1.379581	0.571603
0.1	0.1	−0.5	1.403208	0.743580
0.1	0.1	−1.0	1.429350	0.959301

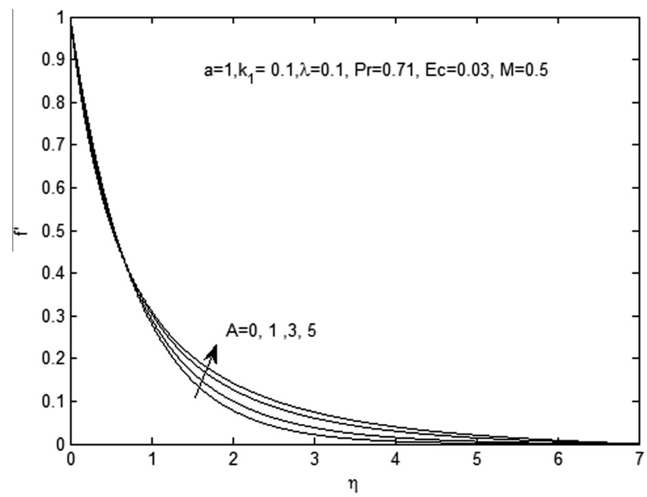

Figure 3a Velocity profiles for different values of viscosity parameter A in case of porous medium and in presence of heat source/sink.

Table 3 The skin-friction coefficient $-f''(0)$and the wall temperature gradient $-\theta'(0)$ values with $a = 1$, $A = 1$, $k_1 = 0.1$, $\lambda = 0.1$.

Pr	M	Ec	$-f''(0)$	$-\theta'(0)$
0.5	0.5	0.03	1.321475	0.234260
0.71	0.5	0.03	1.336319	0.300060
2.0	0.5	0.03	1.411710	0.690783
7.0	0.5	0.03	1.520871	1.519640
10.0	0.5	0.03	1.551378	1.855747
100.0	0.5	0.03	1.705436	6.01850
7.0	1.0	0.03	1.716043	1.449542
7.0	2.0	0.03	2.033797	1.333874
7.0	3.0	0.03	2.297985	1.233453
7.0	5.0	0.03	2.738661	1.094953
7.0	0.5	0.0	1.528324	1.643503
7.0	0.5	0.3	1.453475	0.440350
7.0	0.5	0.7	1.352794	−1.035274
7.0	0.5	0.9	1.302211	−1.711866
7.0	0.5	1.0	1.276931	−2.034022

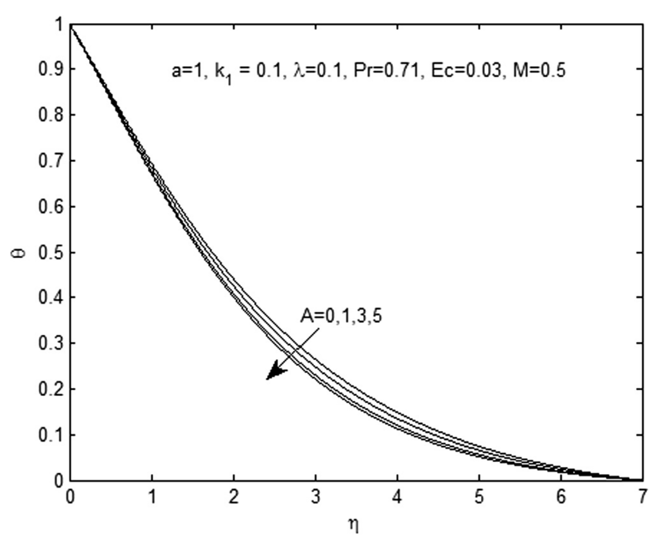

Figure 3b Temperature profiles for different values of viscosity parameter A in case of porous medium and in presence of heat source/sink.

$-f''(0)$ and decreases the values of $-\theta'(0)$. In case of sink ($\lambda < 0$), the values of $-f''(0)$ and $-\theta'(0)$ decreases with the increase of λ, whereas its values increases in case of source ($\lambda > 0$). It can also note that from Table 2 with the increases of Pr the values of and $-\theta'(0)$ increases. The effect of the magnetic parameter M is to increase the values of skin-friction coefficient $-f''(0)$ while the effect of magnetic parameter is to reduce the temperature gradient $-\theta'(0)$. The effect of Eckert number Ec is to reduce the values of skin-friction coefficient $-f''(0)$ and the temperature gradient coefficient $-\theta'(0)$.

Figs. 3a and 3b illustrate the effects of the temperature-dependent fluid viscosity parameter A on velocity and temperature profiles respectively. It is observed that the velocity profiles increases with the increase in viscosity parameter A.

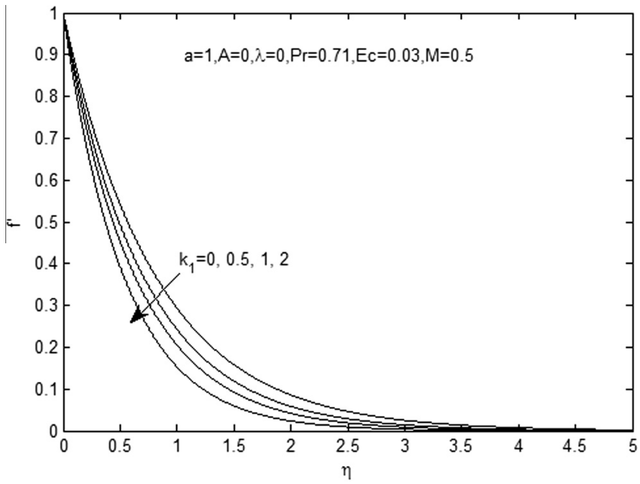

Figure 4a Velocity profiles for different values of permeability k_1 in case of uniform viscosity and in absence of heat source/sink.

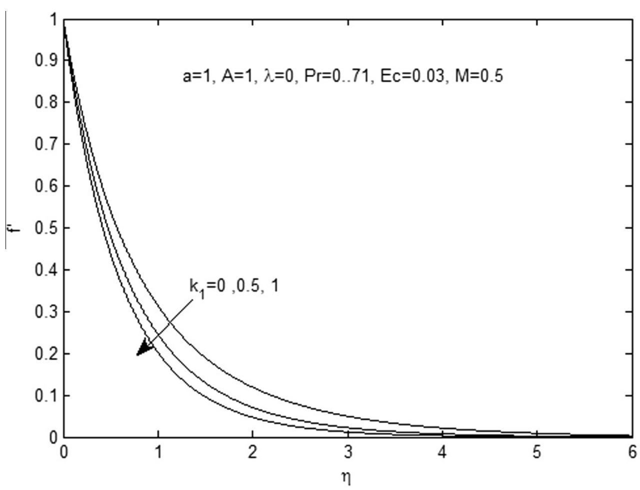

Figure 4b Velocity profiles for different values of permeability k_1 in case of variable viscosity and in absence of heat source/sink.

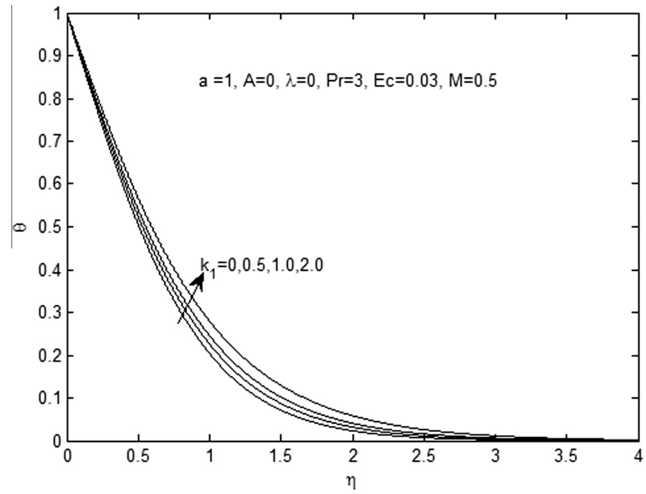

Figure 5a Temperature profiles for different values of permeability parameter k_1 in case of uniform viscosity and in absence of heat source/sink.

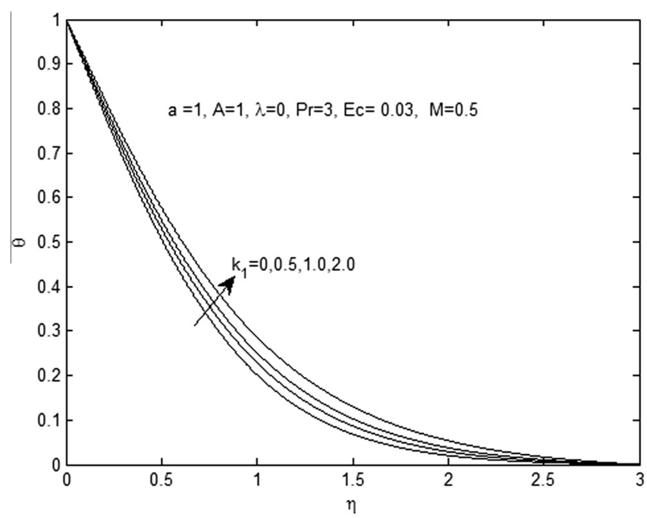

Figure 5b Temperature profiles for different values of permeability parameter k_1 in case of variable viscosity and in absence of heat source/sink.

The effects of the viscosity to reduce the temperature are noticed from Fig. 3b. Figs. 4 and 5 are plotted to the velocity profile f' and temperature profiles θ, respectively, for different values of permeability parameter k_1. The effect of k_1 leads to decrease the velocity profile f' in case of uniform viscosity and variable viscosity. From the Fig. 5 it can be seen that the temperature profiles increases with increase in permeability parameter k_1 in both case of variable and uniform viscosity.

The effects of magnetic field parameter M on velocity and temperature profiles are shown in Fig. 6a–c. The effects of magnetic field are to reduce the velocity profiles, while it increases the temperature profiles. Because of the application of transverse magnetic field in an electrically conducting fluid, a resistive force similar to a drag force is produced, which is Lorentz force. The presence of Lorentz force retards the force on the velocity field and therefore the velocity profiles decreases with the effect of magnetic field parameter. This force has the tendency to slow down the fluid motion and the resistance offered to the flow. Therefore, it is possible for the increase in the temperature. It is also noticed that the thermal boundary layer thickness increases in the presence of a magnetic field. Figs. 7a and 7b depicts the velocity and temperature profiles to the effects of the Prandtl number on momentum and heat transfer. It can be noticed from these figures that the fluid velocity decreases with increasing Prandtl number. An increase in Prandtl number reduces the thermal boundary layer thickness. Prandtl number signifies the ratio of momentum diffusivity to thermal diffusivity. It can be noticed that as Pr decreases, the thickness of the thermal boundary layer becomes greater than the thickness of the velocity boundary layer according to the well-known relation $\delta T/\delta \cong 1/Pr$ where δT the thickness of the velocity thermal boundary layer and δ the thickness of the velocity boundary layer, so the thickness of the thermal boundary layer increases as Prandtl number decreases and hence temperature profile

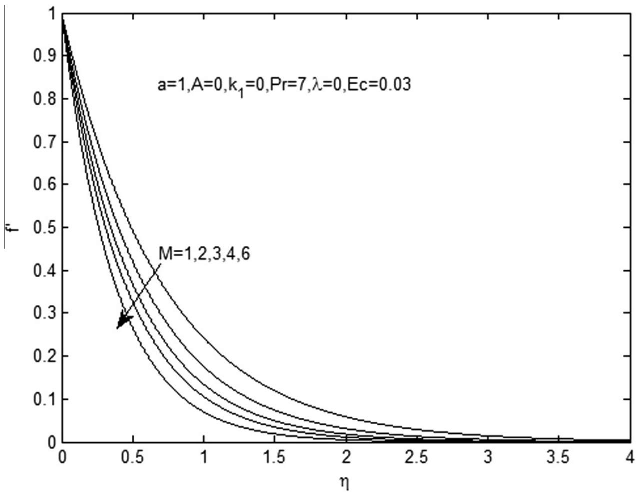

Figure 6a Velocity profiles for different values of magnetic parameter M in case of non-porous, uniform viscosity and in absence of heat source/sink.

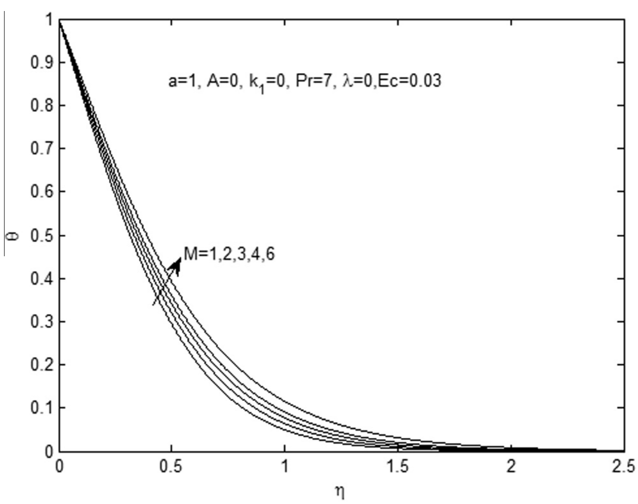

Figure 6b Temperature profiles for different values of magnetic parameter in case of non-porous, uniform viscosity and in absence of heat source/sink.

decreases with increase in Prandtl number (Abel et al. [38]). In heat transfer problems, the Prandtl number controls the relative thickening of momentum and thermal boundary layers. When Prandtl number is small, it means that heat diffuses quickly compared to the velocity (momentum), which means that for liquid metals, the thickness of the thermal boundary layer is much bigger than the momentum boundary layer. Hence Prandtl number can be used to increase the rate of cooling in conducting flows. Fig. 8 is the graphical representation of the dimensionless temperature profiles for different values of heat source/sink parameter λ. From the figure it is noticed that the temperature profiles decreases for increasing of the heat sink ($\lambda < 0$), and due to increase in heat source ($\lambda > 0$) the temperature increases so that the thickness of thermal boundary layer reduces for the increases of heat sink parameter but it decreases with heat source parameter

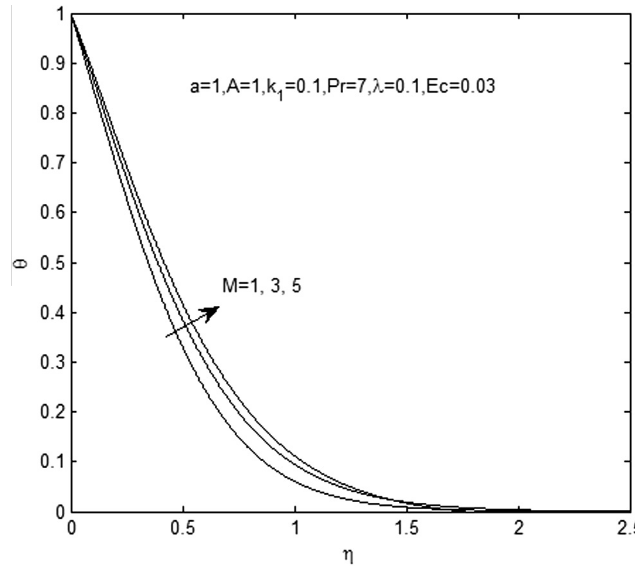

Figure 6c Temperature profiles for different values of magnetic parameter in presence of porous medium variable viscosity and heat source/sink.

($\lambda > 0$). This result is very much significant for the flow where heat transfer is given prime important. In Figs. 9a and 9b the effects of viscous dissipation parameter i.e. the Eckert number Ec on the velocity and temperature profiles exhibited respectively. The Eckert number expresses the relationship between the kinetic energy in the flow and the enthalpy. It embodies the conversion of kinetic energy into internal energy by work done against the viscous fluid stresses. It can be seen from figures the effect of viscous dissipation leads to increase temperature profiles in case of presence /absence of heat source/sink parameter. Interestingly, it also noticed that the thermal

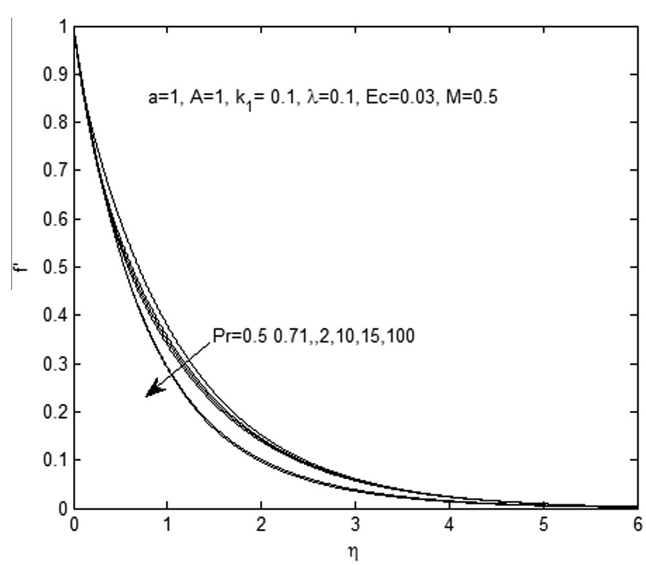

Figure 7a Velocity profiles for different values of Prandtl number Pr in presence of porous medium, variable viscosity and heat source/sink.

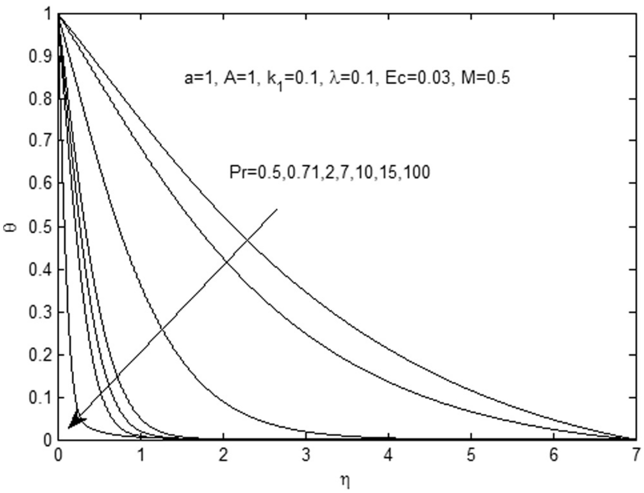

Figure 7b Temperature profiles for different values of Prandl number *Pr* in presence of porous medium, variable viscosity and heat source/sink.

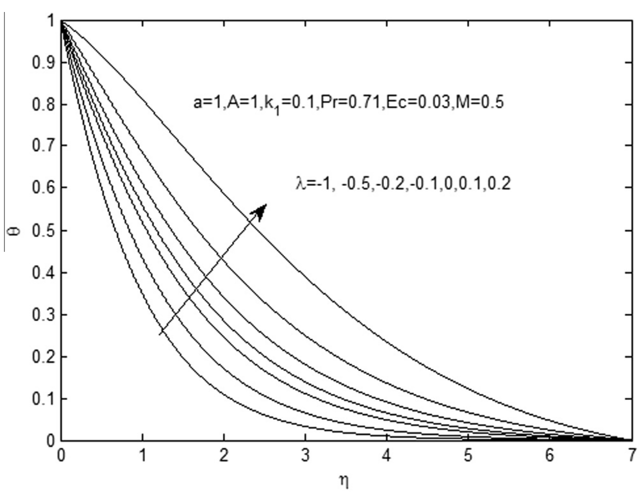

Figure 8 Temperature profiles for different values of heat source/sink λ in case of porous medium and variable viscosity.

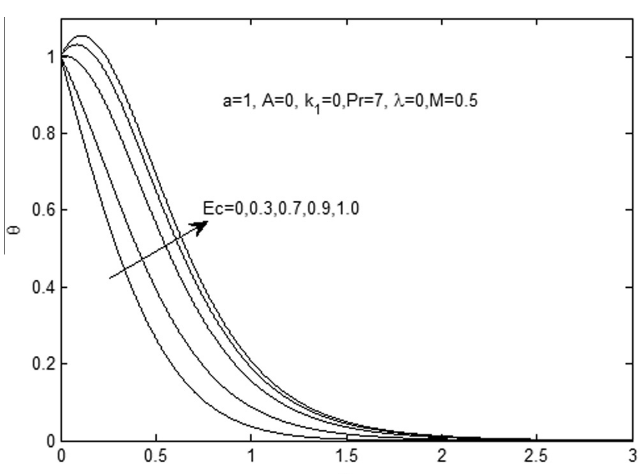

Figure 9a Temperature profiles for different values of Eckert number *Ec* in absence of porous medium and heat source/sink.

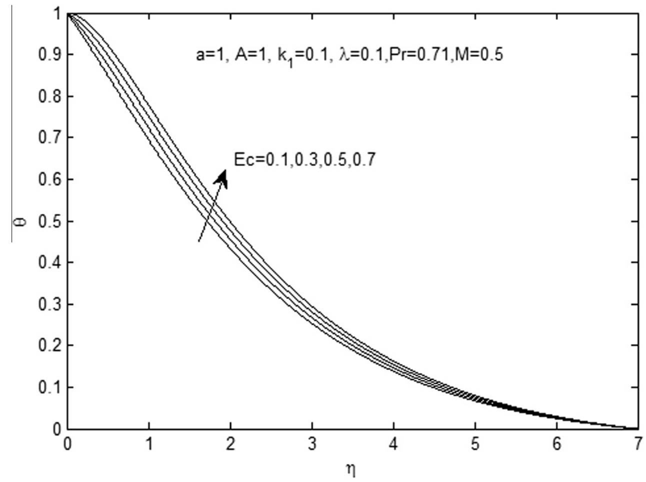

Figure 9b Temperature profiles for different values of Eckert number *Ec* in presence of porous medium variable viscosity and in presence of heat source/sink.

boundary layer thickness is more in presence of viscous dissipation.

5. Conclusions

The present study gives the numerical solution for MHD effects on the boundary layer flow and heat transfer with a variable fluid viscosity on flow past a heated stretching sheet embedded in porous medium in presence of heat source/sink and viscous dissipation. Efficient method of Lie group analysis is used to solve the governing equations of motion. This procedure helps in removing the difficulties faced in solving the equations arising from the non-linear character of the partial differential equations. The scaling symmetry group is very essential procedure to comprehend the mathematical model and to find the similarity solutions for such type of flow which have wider applications in the engineering disciplines related to fluid mechanics. The main findings of this investigation can be summarized as follows.

(i) The effect of transverse magnetic field on a viscous incompressible conducing fluid flow is to suppress the velocity fluid which in turn causes the enhancement of the temperature field. The effect of magnetic field is to decrease both dimensionless velocity profiles and also skin-friction coefficient values.
(ii) Due to the internal heat sink ($\lambda < 0$) the thermal boundary layer increases, whereas it decreases with heat source ($\lambda > 0$). The temperature dependent fluid viscosity plays a significant role in shifting the fluid away from the wall. An increase in Eckert number *Ec* enhances the temperature profiles, where as an increase in Prandtl number *Pr* decrease the temperature profiles.
(iii) The effect of viscosity parameter A is to increase the velocity profiles and the reverse phenomenon is observed in temperature profiles.
(iv) The velocity profiles decreases with the increase in permeability parameter k_1 while temperature profiles increases with the increase in permeability parameter k_1.

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