Physics Letters B 724 (2013) 11–16

Contents lists available at SciVerse ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Thermodynamics of quasi-topological cosmology

M.H. Dehghani [a,b,*], A. Sheykhi [a,b], R. Dehghani [a]

[a] Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran
[b] Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), P.O. Box 55134-441, Maragha, Iran

ARTICLE INFO

Article history:
Received 30 January 2013
Received in revised form 25 May 2013
Accepted 28 May 2013
Available online 31 May 2013
Editor: M. Trodden

ABSTRACT

In this Letter, we study thermodynamical properties of the apparent horizon in a universe governed by quasi-topological gravity. Our aim is twofold. First, by using the variational method we derive the general form of Friedmann equation in quasi-topological gravity. Then, by applying the first law of thermodynamics on the apparent horizon, after using the entropy expression associated with the black hole horizon in quasi-topological gravity, and replacing the horizon radius, r_+, with the apparent horizon radius, $\tilde{r}_A$, we derive the corresponding Friedmann equation in quasi-topological gravity. We find that these two different approaches yield the same result which shows the profound connection between the first law of thermodynamics and the gravitational field equations of quasi-topological gravity. We also study the validity of the generalized second law of thermodynamics in quasi-topological cosmology. We find that, with the assumption of the local equilibrium hypothesis, the generalized second law of thermodynamics is fulfilled for the universe enveloped by the apparent horizon for the late time cosmology.

1. Introduction

The most general Lagrangian which keeps the field equations of motion for the metric of second-order, as the pure Einstein–Hilbert action, is Lovelock Lagrangian [1]. This Lagrangian is constructed from the dimensionally extended Euler densities and can be written as

$$\mathcal{L} = \sum_{p=0}^{m} \alpha_p \mathcal{L}_p, \tag{1}$$

where α_p and $\mathcal{L}_p$ are arbitrary constant and Euler density, respectively. In an $(n+1)$-dimensional spacetime $m = [n/2]$. $\mathcal{L}_0$ set to be one, and therefore α_0 plays the role of the cosmological constant. Because of the topological origin of the Lovelock terms, the second-order (Gauss–Bonnet) term does not have any dynamical effect in four dimensions. Similarly, the cubic interaction only contributes to the equations of motion when the bulk dimension is seven or greater. In other words, although the equations of motion of pth-order Lovelock gravity are second-order differential equations, the pth-order Lovelock term has no contribution to the field equations in $2p$ and lower dimensions. Is it possible to construct a gravitational action with cubic curvature interactions or higher which has contribution in five dimension? The answer is positive and the corresponding theory is called "quasi-topological" gravity which was recently proposed in Refs. [2–4] with cubic and quartic terms of Riemann tensor, respectively. This new gravitational theory provides a useful toy model to study a broader class of four (and higher) dimensional CFT's, involving three or more independent parameters [5]. Various aspects of pth-order quasi-topological terms which have at most second-order derivatives of the metric in the field equations for spherically symmetric spacetimes in five and higher dimensions except $2p$ dimensions have been investigated [6–9].

Nowadays, it is a general belief that there is a profound connection between the gravitational field equations and the laws of thermodynamics. It was shown that the gravitational field equation of a static spherically symmetric spacetime in Einstein, Gauss–Bonnet and more general Lovelock gravity can be recast as the first law of thermodynamics [10]. The studies were also extended to other gravity theories such as $f(R)$ gravity [11] and scalar-tensor gravity [12]. In the cosmological setup, it was shown that the differential form of the Friedmann equation of Friedmann–Robertson–Walker (FRW) universe can be transformed to the first law of thermodynamics on the apparent horizon [13,14]. In the context of brane cosmology, it was shown that the Friedmann equations on the brane can be expressed as $dE = TdS + WdV$ on the apparent horizon [15–17]. This procedure also leads to extract an expression for the entropy at the apparent horizon on the brane, which is useful

* Corresponding author at: Physics Department and Biruni Observatory, College of Sciences, Shiraz University, Shiraz 71454, Iran.
E-mail addresses: mhd@shirazu.ac.ir (M.H. Dehghani), asheykhi@shirazu.ac.ir (A. Sheykhi).

0370-2693
http://dx.doi.org/10.1016/j.physletb.2013.05.065

in studying the thermodynamical properties of the black hole horizon on the brane [15–17].

Is the inverse procedure also possible? That is starting from the first law of thermodynamics to extract the general field equations of gravitational theory. Jacobson [18] was the first who disclosed that the Einstein field equation can be derived from the relation between the horizon area and entropy, together with the Clausius relation $\delta Q = T\delta S$. Also, in the cosmological setup, it was shown that the corresponding Friedmann equations of Einstein, Gauss–Bonnet and Lovelock gravity can be derived by applying the energy balance relation $-dE = TdS$ to the apparent horizon of a Friedmann–Robertson–Walker universe (FRW) with any spatial curvature [19]. Here, $-dE$ is actually just the heat flux δQ in [18] crossing the apparent horizon within an infinitesimal internal of time dt. In the framework of Horava–Lifshitz gravity, it was shown that the corresponding Friedmann equation cannot be derived by applying the first law of thermodynamics on the apparent horizon and using the entropy expression for static spherically symmetric black holes in this gravity theory [20]. The reason of failure seems to be due to the fact that Horava–Lifshitz gravity is not diffeomorphism invariant [21]. Indeed, the action of Horava–Lifshitz gravity is invariant only under a restricted class of diffeomorphism [22]. This implies that the connection between first law of thermodynamics and gravitational field equations is not a generic feature of any theory of gravity.

In this Letter we will address the question on the connection between thermodynamics and gravity by investigating whether and how the relation can be found in quasi-topological cosmology. This is the first study on the quasi-topological cosmology and in particular investigating thermodynamical aspects of this gravity theory. For this purpose, we first derive the Friedmann equations in quartic and higher-order quasi-topological gravity by varying the action of the quasi-topological gravity. Then, to show the consistency of this theory with thermodynamics, we extract the corresponding Friedmann equations by applying the first law of thermodynamics, $dE = T_h dS_h + WdV$, on the apparent horizon of a FRW universe governed by quasi-topological gravity. Our strategy is to pick up the entropy expression associated with the black hole horizon in quasi-topological gravity and replacing the black hole horizon radius r_+ by the apparent horizon radius $\tilde{r}_A$. We will also examine the time evolution of the total entropy, including the entropy associated with the apparent horizon in quasi-topological gravity together with the matter field entropy inside the apparent horizon. We find that, in the late time, the generalized second law (GSL) of thermodynamics is fulfilled for the universe governed by quasi-topological gravity.

This Letter is outlined as follows. In the next section, we introduce the action of the quasi-topological gravity and derive the general form of the Friedmann equation by using the variational method in this gravity theory. In Section 3, we extract the Friedmann equation of quartic quasi-topological cosmology by applying the first law of thermodynamics, $dE = T_h dS_h + WdV$, on the apparent horizon. We also generalize our study to higher-order quasi-topological theory in this section. We investigate the validity of GSL of thermodynamics for a universe enveloped by the apparent horizon in quasi-topological gravity in Section 4. We finish our Letter with conclusions in Section 5.

2. Quasi-topological cosmology

In this section we derive the field equations governing the evolution of the universe in quasi-topological gravity. The most general gravitational theory which produces second-order equation of motion is the i-order Lovelock gravity with action [1]

$$I_G = \frac{1}{16\pi G_{n+1}} \int d^{n+1}x \sqrt{-g} \left(-2\Lambda + \sum_{i=0}^{m} \alpha_i \mathcal{L}_i + \mathcal{L}_M \right), \tag{2}$$

where Λ is the cosmological constant, and the α_i's are Lovelock coefficients with dimensions $(\text{length})^{2i-2}$, and $\mathcal{L}_i$ is the ith-order Lovelock Lagrangian

$$\mathcal{L}_i = \frac{1}{2^i} \delta^{\mu_1\mu_2\cdots\mu_{2i}}_{\nu_1\nu_2\cdots\nu_{2i}} R_{\mu_1\mu_2}{}^{\nu_1\nu_2} \cdots R_{\mu_{2i-1}\mu_{2i}}{}^{\nu_{2i-1}\nu_{2i}}. \tag{3}$$

In the action (2), the term proportional to α_i contributes to the equations of motion in dimensions with $n \geqslant 2i$. For example, the terms associated to $i = 3$ or higher do not contribute to the field equations in five dimensions. Recently, a new gravity theory called quasi-topological gravity has been introduced, which has contribution to the field equations in five dimensions from the i-order ($i \geqslant 3$) term in Riemann tensor.

The gravity part of the action of the quartic quasi-topological theory in $(n+1)$-dimensions in the absence of cosmological constant is given by [4]

$$I = \int d^{n+1}x (\mathcal{L}_G + \mathcal{L}_M), \tag{4}$$

where $\mathcal{L}_M$ is the Lagrangian of the matter and

$$\mathcal{L}_G = \frac{\sqrt{-g}}{16\pi G_{n+1}} (\mu_1 \mathcal{L}_1 + \mu_2 \mathcal{L}_2 + \mu_3 \mathcal{X}_3 + \mu_4 \mathcal{X}_4). \tag{5}$$

In Eq. (5) $\mathcal{L}_1 = R$ is the Einstein–Hilbert Lagrangian, $\mathcal{L}_2 = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2$ is the second-order Lovelock (Gauss–Bonnet) Lagrangian, $\mathcal{X}_3$ is the curvature-cubed Lagrangian [3]

$$\begin{aligned}\mathcal{X}_3 = {} & R^{cd}_{ab} R_c{}^e{}_d{}^f R_e{}^a{}_f{}^b + \frac{1}{(2n-1)(n-3)} \left(\frac{3(3n-5)}{8} R_{abcd}R^{abcd}R \right. \\ & - 3(n-1)R_{abcd}R^{abc}{}_e R^{de} + 3(n+1)R_{abcd}R^{ac}R^{bd} \\ & + 6(n-1)R_a{}^b R_b{}^c R_c{}^a - \frac{3(3n-1)}{2} R_a{}^b R_b{}^a R \\ & \left. + \frac{3(n+1)}{8} R^3 \right)\end{aligned} \tag{6}$$

and $\mathcal{X}_4$ is the fourth-order term of quasi-topological gravity [4]

$$\begin{aligned}\mathcal{X}_4 = {} & c_1 R_{abcd}R^{cdef}R^{hg}{}_{ef}R_{hg}{}^{ab} + c_2 R_{abcd}R^{abcd}R_{ef}R^{ef} \\ & + c_3 RR_{ab}R^{ac}R_c{}^b + c_4 \left(R_{abcd}R^{abcd}\right)^2 + c_5 R_{ab}R^{ac}R_{cd}R^{db} \\ & + c_6 RR_{abcd}R^{ac}R^{db} + c_7 R_{abcd}R^{ac}R^{be}R^d{}_e \\ & + c_8 R_{abcd}R^{acef}R^b{}_e R^d{}_f + c_9 R_{abcd}R^{ac}R_{ef}R^{bedf} \\ & + c_{10}R^4 + c_{11}R^2 R_{abcd}R^{abcd} + c_{12}R^2 R_{ab}R^{ab} \\ & + c_{13}R_{abcd}R^{abef}R_{ef}{}^c{}_g R^{dg} + c_{14}R_{abcd}R^{aecf}R_{gehf}R^{gbhd},\end{aligned} \tag{7}$$

where the coefficients c_i are given by

$$\begin{aligned}c_1 &= -(n-1)\left(n^7 - 3n^6 - 29n^5 + 170n^4 - 349n^3 + 348n^2 \right. \\ &\quad \left. - 180n + 36\right), \\ c_2 &= -4(n-3)\left(2n^6 - 20n^5 + 65n^4 - 81n^3 + 13n^2 + 45n - 18\right), \\ c_3 &= -64(n-1)\left(3n^2 - 8n + 3\right)\left(n^2 - 3n + 3\right), \\ c_4 &= -\left(n^8 - 6n^7 + 12n^6 - 22n^5 + 114n^4 - 345n^3 + 468n^2 \right. \\ &\quad \left. - 270n + 54\right), \\ c_5 &= 16(n-1)\left(10n^4 - 51n^3 + 93n^2 - 72n + 18\right), \\ c_6 &= -32(n-1)^2(n-3)^2\left(3n^2 - 8n + 3\right),\end{aligned}$$

$$c_7 = 64(n-2)(n-1)^2\left(4n^3-18n^2+27n-9\right),$$
$$c_8 = -96(n-1)(n-2)\left(2n^4-7n^3+4n^2+6n-3\right),$$
$$c_9 = 16(n-1)^3\left(2n^4-26n^3+93n^2-117n+36\right),$$
$$c_{10} = n^5-31n^4+168n^3-360n^2+330n-90,$$
$$c_{11} = 2\left(6n^6-67n^5+311n^4-742n^3+936n^2-576n+126\right),$$
$$c_{12} = 8\left(7n^5-47n^4+121n^3-141n^2+63n-9\right),$$
$$c_{13} = 16n(n-1)(n-2)(n-3)\left(3n^2-8n+3\right),$$
$$c_{14} = 8(n-1)\left(n^7-4n^6-15n^5+122n^4-287n^3+297n^2 -126n+18\right).$$

The action (4) not only works in five dimensions, but also yields second-order equations of motion for spherically symmetric spacetimes [4].

Our aim here is to derive the corresponding Friedmann equations of quartic and higher-order quasi-topological gravity. We consider a homogeneous and isotropic FRW universe in $(n+1)$-dimensions which is described by the line element

$$ds^2 = -N(t)\,dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\,d\Omega_{n-1}^2\right], \tag{8}$$

where k is the spatial curvature constant with values $1, 0$ and -1 correspond to closed, flat and open universe, respectively, and $d\Omega_{n-1}^2$ represents the line elements of an $(n-1)$-dimensional unit sphere.

We use the variational method for deriving the Friedmann equation from the action principle. Using metric (8), the Lagrangian of the fourth-order quasi-topological gravity can be written as

$$\mathcal{L}_G = \frac{n(n-1)}{16\pi G_{n+1}}\frac{\sqrt{-\gamma}\,a^n}{N^{(n+3)/2}a^8}\left(-N^4b_1 + N^3b_2\dot{a}^2 + \frac{1}{3}N^2b_3\dot{a}^4 + \frac{1}{5}Nb_4\dot{a}^6 + \frac{1}{7}b_5\dot{a}^8\right), \tag{9}$$

where γ is the determinant of the metric of t-constant hypersurface, b_i's are

$$b_1 = a^6k + a^4k^2\hat{\mu}_2l^2 + a^2k^3\hat{\mu}_3l^4 + k^4\hat{\mu}_4l^6,$$
$$b_2 = a^6 + 2a^4k\hat{\mu}_2l^2 + 3a^2k^2\hat{\mu}_3l^4 + 4k^3\hat{\mu}_4l^6,$$
$$b_3 = a^4\hat{\mu}_2l^2 + 3a^2k\hat{\mu}_3l^4 + 6k^2\hat{\mu}_4l^6,$$
$$b_4 = a^2\hat{\mu}_3l^4 + 4k\hat{\mu}_4l^6,$$
$$b_5 = \hat{\mu}_4l^6,$$

and the dimensionless parameters $\hat{\mu}_j$'s are

$$\hat{\mu}_1 = 1, \qquad \hat{\mu}_2 = \frac{(n-2)(n-3)}{l^2}\mu_2,$$
$$\hat{\mu}_3 = \frac{(n-2)(n-5)(3n^2-9n+4)}{8(2n-1)l^4}\mu_3,$$
$$\hat{\mu}_4 = \frac{n(n-1)(n-2)^2(n-3)(n-7)(n^5-15n^4+72n^3-156n^2+150n-42)}{l^6}\mu_4.$$

Varying the Lagrangian (9) with respect to $N(t)$, we arrive at

$$\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}_G}{\delta N} = -\frac{n(n-1)}{32\pi G_{n+1}}\frac{1}{N^{(n+6)/2}a^8} \times\left[N^4b_1 + N^3b_2\dot{a}^2 + N^2b_3\dot{a}^4 + Nb_4\dot{a}^6 + b_5\dot{a}^8\right]. \tag{10}$$

On the other hand the variation of the Lagrangian of matter with respect to $g_{00} = -N$ leads to

$$\frac{1}{\sqrt{-g}}\frac{\delta\mathcal{L}_M}{\delta N} \equiv \frac{T_0^0}{2N} = \frac{\rho}{2N}. \tag{11}$$

Now, one can absorb $N(t)$ in t coordinate. That is, one can set $N(t) = 1$. Using (10) and (11), one obtains

$$\sum_{i=1}^{4}\hat{\mu}_il^{2i-2}\left(H^2 + \frac{k}{a^2}\right)^i = \frac{16\pi G_{n+1}}{n(n-1)}\rho. \tag{12}$$

As in the case of black hole solutions presented in [4], the form of the field equation (12) allows us to generalize this equation to the case of mth-order quasi-topological gravity:

$$\sum_{i=1}^{m}\hat{\mu}_il^{2i-2}\left(H^2 + \frac{k}{a^2}\right)^i = \frac{16\pi G_{n+1}}{n(n-1)}\rho. \tag{13}$$

Here, we pause to study the field equations under a small perturbation around the Friedmann–Robertson–Walker metric. The authors of [3], examined the linearized equations of motion for a graviton perturbation around the AdS metric in cubic quasi-topological gravity, and showed that the linearized graviton equation in an AdS background is only a second-order equation. Here, we examine the same fact under a small perturbation around the FRW metric and find that the linearized field equation is a second-order equation. This fact is different from the small perturbation around FRW metric in the new massive gravity [23]. In the latter case, the linearized field equation contains more than two-derivative and one may have ghosty vacuum at initial times while it becomes free of ghosts at later times in the cosmological scenario [23].

3. Friedman equation from the first law

In this section we would like to derive the Friedmann equation of quasi-topological cosmology by applying the first law of thermodynamics on the apparent horizon of FRW universe. The entropy associated with the event horizon of higher dimensional static spherically symmetric black holes in cubic quasi-topological gravity has the following form [3,6]

$$S_h = \frac{A}{4G_{n+1}}\left[1 + \frac{2(n-1)}{n-3}\frac{\hat{\mu}_2l^2}{r_+^2} + \frac{3(n-1)}{n-5}\frac{\hat{\mu}_3l^4}{r_+^4}\right], \tag{14}$$

where $n \neq 5$ and G_{n+1} is the $(n+1)$-dimensional gravitational constant. Here $A = n\Omega_nr_+^{n-1}$ is the surface area of the black hole horizon, and

$$\Omega_n = \frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})}, \qquad \Gamma\left(\frac{n+2}{2}\right) = \left(\frac{n}{2}\right)\left(\frac{n-2}{2}\right)!. \tag{15}$$

We further assume the entropy expression (14) is also valid for the apparent horizon of the FRW universe in quasi-topological gravity. Replacing the horizon radius r_+ with the apparent horizon radius $\tilde{r}_A$, the entropy expression (29) can be written

$$S_h = \frac{n\Omega_n}{4G_{n+1}}\tilde{r}_A^{n-1}\left[1 + \frac{\beta}{\tilde{r}_A^2} + \frac{\gamma}{\tilde{r}_A^4}\right], \tag{16}$$

where we have defined

$$\beta = \frac{2(n-1)}{n-3}\hat{\mu}_2l^2, \qquad \gamma = \frac{3(n-1)}{n-5}\hat{\mu}_3l^4. \tag{17}$$

Taking differential form of relation (16), we have

$$dS = \frac{\partial S}{\partial \tilde{r}_A} d\tilde{r}_A = \frac{n\Omega_n}{4G_{n+1}}\left[(n-1)\tilde{r}_A^{n-2} + \beta(n-3)\tilde{r}_A^{n-4} + \gamma(n-5)\tilde{r}_A^{n-6}\right] d\tilde{r}_A.$$

We rewrite the line element of the FRW metric as

$$ds^2 = h_{ab}dx^a dx^b + \tilde{r}^2 d\Omega_{n-1}^2, \tag{18}$$

where $x^0 = t$, $x^1 = r$, $\tilde{r} = a(t)r$, and $h_{ab} = \mathrm{diag}(-1, a^2/(1-kr^2))$ represents the two dimensional metric. The dynamical apparent horizon, a marginally trapped surface with vanishing expansion, is determined by the relation $h^{ab}\partial_a \tilde{r}\partial_b \tilde{r} = 0$. It is a matter of calculation to show that the radius of the apparent horizon for the FRW universe becomes [24]

$$\tilde{r}_A = \frac{1}{\sqrt{H^2 + k/a^2}}. \tag{19}$$

The temperature associated with the apparent horizon is defined as $T_h = \kappa/2\pi$, where $\kappa = \frac{1}{2\sqrt{-h}}\partial_a(\sqrt{-h}h^{ab}\partial_b \tilde{r})$ is the surface gravity. It is easy to show that the surface gravity at the apparent horizon of FRW universe can be written as

$$\kappa = -\frac{1}{\tilde{r}_A}\left(1 - \frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right). \tag{20}$$

Since for $\dot{\tilde{r}}_A < 2H\tilde{r}_A$, we have $\kappa < 0$, which leads to the negative temperature, thus one may, in general, define the temperature on the apparent horizon as $T_h = |\kappa|/2\pi$. In addition, since we associate with the apparent horizon a temperature, thus one may expect that the apparent horizon have a kind of Hawking radiation just like a black hole event horizon. This issue was previously addressed [25], by showing the connection between temperature on the apparent horizon and the Hawking radiation. This study gives more solid physical implication of the temperature associated with the apparent horizon.

The energy conservation law $\nabla_\mu T^{\mu\nu} = 0$ leads to the continuity equation in the form

$$\dot{\rho} + nH(\rho + p) = 0. \tag{21}$$

The next quantity we need to have is the work density. In our case it can be calculated as in [24]

$$W = -\frac{1}{2}T^{\mu\nu}h_{\mu\nu} = \frac{1}{2}(\rho - p). \tag{22}$$

The work density is regarded as the work done when the apparent horizon radius changes from $\tilde{r}_A$ to $\tilde{r}_A + d\tilde{r}_A$. Then, we suppose the first law of thermodynamics on the apparent horizon of the universe in quasi-topological gravity holds and has the form

$$dE = T_h dS_h + W dV, \tag{23}$$

where S_h is the entropy associated with the apparent horizon in quasi-topological cosmology given in Eq. (16). The term WdV in the first law comes from the fact that we have a volume change for the total system enveloped by the apparent horizon. For a pure de Sitter space, $\rho = -p$, and the work term reduces to the standard $-pdV$, thus we obtain exactly the standard first law of thermodynamics, $dE = TdS - pdV$.

Assuming the total energy content of the universe inside an n-sphere of radius $\tilde{r}_A$ is $E = \rho V$, where $V = \Omega_n \tilde{r}_A^n$ is the volume enveloped by an n-dimensional sphere. Taking differential form of the total energy, after using the continuity equation (21), we obtain

$$dE = \rho n\Omega_n \tilde{r}_A^{n-1} d\tilde{r}_A + \Omega_n \tilde{r}_A^n \dot{\rho}\, dt = \rho n\Omega_n \tilde{r}_A^{n-1} d\tilde{r}_A - nH\Omega_n \tilde{r}_A^n(\rho + p)\, dt. \tag{24}$$

Substituting Eqs. (17), (22) and (24) in the first law (23) and using the definition of the temperature associated with the apparent horizon, we get the differential form of the Friedmann equation in cubic quasi-topological gravity as

$$\frac{1}{8\pi G_{n+1}\tilde{r}_A}\left[\frac{n-1}{\tilde{r}_A^2} + \frac{\beta(n-3)}{\tilde{r}_A^4} + \frac{\gamma(n-5)}{\tilde{r}_A^6}\right] d\tilde{r}_A = H(\rho + p)\, dt. \tag{25}$$

Using the continuity equation (21), we obtain

$$\left[\frac{n-1}{\tilde{r}_A^3} + \frac{\beta(n-3)}{\tilde{r}_A^5} + \frac{\gamma(n-5)}{\tilde{r}_A^7}\right] d\tilde{r}_A = -\frac{8\pi G_{n+1}}{n} d\rho. \tag{26}$$

Integrating (26) yields

$$\frac{1}{\tilde{r}_A^2} + \frac{\hat{\mu}_2 l^2}{\tilde{r}_A^4} + \frac{\hat{\mu}_3 l^4}{\tilde{r}_A^6} = \frac{16\pi G_{n+1}}{n(n-1)}\rho, \tag{27}$$

where an integration constant has been absorbed into the energy density ρ. Substituting $\tilde{r}_A$ from Eq. (19) we obtain

$$H^2 + \frac{k}{a^2} + \hat{\mu}_2 l^2\left(H^2 + \frac{k}{a^2}\right)^2 + \hat{\mu}_3 l^4\left(H^2 + \frac{k}{a^2}\right)^3 = \frac{16\pi G_{n+1}}{n(n-1)}\rho. \tag{28}$$

In this way we derived the $(n+1)$-dimensional Friedmann equation governing the evolution of the universe in cubic-order quasi-topological gravity by applying the first law of thermodynamics on the apparent horizon.

The above analysis can be extended to higher-order quasi-topological gravity. The entropy associated with spherically symmetric black hole solutions in quartic-order of quasi-topological gravity is given by [4]

$$S_h = \frac{A}{4G_{n+1}}\left[\hat{\mu}_1 + \frac{2(n-1)}{(n-3)}\frac{\hat{\mu}_2 l^2}{r_+^2} + \frac{3(n-1)}{(n-5)}\frac{\hat{\mu}_3 l^4}{r_+^4} + \frac{4(n-1)}{(n-7)}\frac{\hat{\mu}_4 l^6}{r_+^6}\right]. \tag{29}$$

The extension to higher-order quasi-topological black holes is quite straightforward and can be written in the compact form

$$S_h = \frac{A}{4G_{n+1}}\sum_{i=1}^{m} i\frac{(n-1)}{(n+1-2i)}\frac{\hat{\mu}_i l^{2i-2}}{r_+^{2i-2}}. \tag{30}$$

Using the same formalism as we have done in this section, we obtain the general form of the Friedmann equation as

$$\hat{\mu}_1\left(H^2 + \frac{k}{a^2}\right) + \hat{\mu}_2 l^2\left(H^2 + \frac{k}{a^2}\right)^2 + \hat{\mu}_3 l^4\left(H^2 + \frac{k}{a^2}\right)^3 + \hat{\mu}_4 l^6\left(H^2 + \frac{k}{a^2}\right)^4 + \cdots = \frac{16\pi G_{n+1}}{n(n-1)}\rho \tag{31}$$

or in a compact form as

$$\sum_{i=1}^{\infty}\hat{\mu}_i l^{2i-2}\left(H^2 + \frac{k}{a^2}\right)^i = \frac{16\pi G_{n+1}}{n(n-1)}\rho. \tag{32}$$

Which is compatible with the field equation derived through the use of the variation of the action.

4. GSL in quasi-topological gravity

Next, we examine the time evolution of the total entropy. For this purpose, we take the time derivative of Eq. (31) and get

$$\left[\frac{1}{\tilde{r}_A^3}+\frac{2\hat{\mu}_2 l^2}{\tilde{r}_A^5}+\frac{3\hat{\mu}_3 l^4}{\tilde{r}_A^7}+\frac{4\hat{\mu}_4 l^6}{\tilde{r}_A^9}+\cdots\right]\dot{\tilde{r}}_A=-\frac{8\pi G_{n+1}}{n(n-1)}\dot{\rho}. \quad (33)$$

Using the continuity equation (21) and solving for $\dot{\tilde{r}}_A$ we obtain

$$\dot{\tilde{r}}_A=\frac{8\pi G_{n+1}}{(n-1)}H(\rho+p)\times\left[\frac{1}{\tilde{r}_A^3}+\frac{2\hat{\mu}_2 l^2}{\tilde{r}_A^5}+\frac{3\hat{\mu}_3 l^4}{\tilde{r}_A^7}+\frac{4\hat{\mu}_4 l^6}{\tilde{r}_A^9}+\cdots\right]^{-1}. \quad (34)$$

Calculating $T_h\dot{S}_h$,

$$T_h\dot{S}_h=\frac{1}{2\pi\tilde{r}_A}\left(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right)\frac{n(n-1)\Omega_n}{4G_{n+1}}\tilde{r}_A^{n+1}\times\left[\frac{1}{\tilde{r}_A^3}+\frac{2\hat{\mu}_2 l^2}{\tilde{r}_A^5}+\frac{3\hat{\mu}_3 l^4}{\tilde{r}_A^7}+\frac{4\hat{\mu}_4 l^6}{\tilde{r}_A^9}+\cdots\right]\dot{\tilde{r}}_A, \quad (35)$$

and substituting $\dot{\tilde{r}}_A$ from Eq. (34) in it, we obtain

$$T_h\dot{S}_h=n\Omega_n\tilde{r}_A^n H\rho(1+w)\left(1-\frac{\dot{\tilde{r}}_A}{2H\tilde{r}_A}\right), \quad (36)$$

where we have defined the equation of state $w=p/\rho$, as usual. Since $T\geqslant 0$, we have $\dot{\tilde{r}}_A\leqslant 2H\tilde{r}_A$, thus the sign of Eq. (36) depends on w. If $w\geqslant -1$, then $\dot{S}_h\geqslant 0$ and the second law of thermodynamics is fulfilled. However, some astrophysical evidences show that our universe is currently accelerating and in particular the equation of state parameter can cross the phantom line, i.e. $w<-1$, indicating that the second law of thermodynamics, $\dot{S}_h\geqslant 0$, does not hold. Nevertheless, as we shall see below, the GSL of thermodynamics, $\dot{S}_h+\dot{S}_m\geqslant 0$, is still preserved throughout the history of the universe. In order to verify the GSL of thermodynamics, we have to study the time evolution of the total entropy including the entropy S_h associated with the apparent horizon together with the matter field entropy S_m inside the apparent horizon. The entropy of the universe inside the horizon can be related to its energy and pressure by Gibbs equation [26]

$$T_m dS_m=d(\rho V)+p dV=V d\rho+(\rho+p)dV=\Omega_n\tilde{r}_A^n\,d\rho+(\rho+p)n\Omega_n\tilde{r}_A^{n-1}\,d\tilde{r}_A, \quad (37)$$

where T_m is the temperature of the matter field inside the apparent horizon. We assume the temperature of the perfect fluid inside the apparent horizon scales as the temperature of the apparent horizon T_h. Thus, we suppose the temperature $T_m=T_h$. We limit ourselves to the assumption of the local equilibrium hypothesis, that the energy would not spontaneously flow between the horizon and the fluid, the latter would be at variance with the FRW geometry. Therefore, from the Gibbs equation, we get

$$T_m\dot{S}_m=n\Omega_n\tilde{r}_A^{n-1}(\rho+p)\dot{\tilde{r}}_A-n\Omega_n\tilde{r}_A^n H(\rho+p), \quad (38)$$

where we have used the continuity equation (21). Adding Eqs. (36) and (38), after substituting $\dot{\tilde{r}}_A$ from Eq. (34), we obtain

$$T_h(\dot{S}_h+\dot{S}_m)=\frac{4\pi G_{n+1}}{n-1}AH(\rho+p)^2\tilde{r}_A^3\times\left[1+\frac{2\hat{\mu}_2 l^2}{\tilde{r}_A^2}+\frac{3\hat{\mu}_3 l^4}{\tilde{r}_A^4}+\frac{4\hat{\mu}_4 l^6}{\tilde{r}_A^6}+\cdots\right]^{-1}. \quad (39)$$

Expanding the r.h.s. of the above equation for the late time where $\tilde{r}_A\gg l$, we arrive at

$$T_h(\dot{S}_h+\dot{S}_m)=\frac{4\pi G_{n+1}}{n-1}AH(\rho+p)^2\tilde{r}_A^3\times\left[1-\frac{2\hat{\mu}_2 l^2}{\tilde{r}_A^2}-\frac{3\hat{\mu}_3 l^4}{\tilde{r}_A^4}-\frac{4\hat{\mu}_4 l^6}{\tilde{r}_A^6}+\cdots\right]. \quad (40)$$

The expression in the bracket, in the late time cosmology is positive which indicate that $\dot{S}_h+\dot{S}_m\geqslant 0$. This implies that for the late time cosmology, the GSL of thermodynamics is fulfilled in the universe governed by quasi-topological gravity, regardless of the nature of the energy content of the universe.

5. Conclusions

In this Letter we investigated the thermodynamical properties of the apparent horizon in quasi-topological gravity. We first derived the Friedmann equation governing the evolution of the universe in quartic and higher-order quasi-topological gravity by varying the corresponding action of quasi-topological gravity. Then, by applying the first law of thermodynamics, $dE=T_h dS_h+W dV$ on the apparent horizon of FRW universe, we extracted the Friedmann equation of quartic and higher-order quasi-topological gravity. Here $E=\rho V$ is the total energy inside the apparent horizon and T_h and S_h are the temperature and entropy associated with the apparent horizon, respectively. The entropy expression depends on the gravity theory and it is generally accepted that the apparent horizon entropy in each gravity theory has the same expression as the entropy of black hole horizon, but replacing the black hole horizon radius r_+ with the apparent horizon radius $\tilde{r}_A$. We found that the result obtained by the first law is exactly coincides with the ones derived from the action principle. We also studied the time evolution of the total entropy, including the entropy associated with the apparent horizon together with the matter field entropy enveloped by the apparent horizon. Our study implies that, with the assumption of the local equilibrium hypothesis, the GSL of thermodynamics is preserved for the late time cosmology in the universe governed by quasi-topological gravity.

Acknowledgements

We are grateful to the referee for constructive comments which helped us to improve the paper significantly. We also thank Shiraz University Research Council. This work has been supported financially by Research Institute for Astronomy and Astrophysics of Maragha (RIAAM), Iran.

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