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INCOMPLETE CONJUNCTIVE INFORMATION

D. Dubois and H. Prade
Laboratoire Langages et Systèmes Informatiques, Université Paul Sabatier, 118, route de Narbonne, 31062 Toulouse Cédex, France

Abstract—Many information systems capable of handling incomplete or fuzzy information manipulate objects with single-valued attributes. Information is then said to be disjunctive. Information is said to be conjunctive when pertaining to many-valued attributes. While a piece of incomplete disjunctive information is easily represented by means of a set of mutually exclusive possible values, modeling incomplete conjunctive information theoretically leads to consider families of sets, since attributes are then set-valued under complete information. Some proposals are made in order to efficiently and rigorously represent incomplete conjunctive information, and deal with query evaluation, especially in the case where only upper and/or lower bounds of the set of values of a many-valued attribute are known. Applications of this approach can be expected for the processing of time intervals, as well as spatial reasoning, among other topics, in knowledge base management.

1. INTRODUCTION

Situations where only incomplete information is available are often encountered. Thus, the representation and the management of such pieces of information are important issues. However there exist various kinds of incomplete (or partial) information. In this paper we make a distinction between a partially-known (or if we prefer ill-known) *value* and a partially-known (or ill-known) *set*. This corresponds to the difference between a single-valued attribute and a multiple-valued attribute in a data base. When the value of a single-valued attribute is incompletely/partially known for a considered object, it is only known that this value belongs to some subset of the attribute domain. In a more general case this subset may be fuzzy and then it restricts the more or less possible values of the attribute. For instance, the attribute "age" is obviously single-valued and in case of incomplete or fuzzy information, we may only know that "John is between 20 and 25 years old" or that "John is young". In that case the (fuzzy) subset under consideration gathers *mutually exclusive* values, since a single-valued attribute has only one value for a given object. This kind of incomplete knowledge is said to be *disjunctive* by some authors [1–5]. In contrast, even when it is perfectly known, the information pertaining to a multiple-valued attribute for a given object can be modelled by a subset which does not necessarily reduce to a singleton. For instance, if we know that "John speaks English and French only", there is no mutual exclusiveness between "English" and "French" in the subset {English, French}. In that case the information is said to be *conjunctive*. More generally, conjunctive information, thus corresponding to a multiple-valued attribute, may be incompletely or even fuzzily known. The representation and the management of such pieces of knowledge in information systems is the topic of this paper. In the next section representation issues are discussed, and a representation by means of pairs of nested sets is more particularly considered for incomplete conjunctive information. Then the evaluation of queries in face of such information is studied.

2. REPRESENTATION ISSUES

2.1. *Upper and lower approximations of an ill-known set*

Let Ω be a referential. Let A be an ordinary set which is ill-known; this means that there exists at least one element $\omega \in \Omega$, for which it is not known if whether ω belongs to A or not. A natural way for representing this state of uncertainty about the elements which do belong to A, is to consider the different possible realization of A, namely the subsets $A_i \subseteq \Omega$, $i \in I$, where I is an index set. Note that the A_is as possible realizations of A are mutually exclusive in 2^Ω. If all the A_is are not regarded as equally probable, a probability assignment m from 2^Ω to [0, 1] can be introduced, such that

$$\sum_{i \in I} m(A_i) = 1 \quad (I \text{ being assumed finite}).$$

$m(A_i)$ is then the probability that $A = A_i$. Then, the ill-known set A is viewed as a random set, and m is very similar to a basic probability assignment in the sense of Shafer [6]. However, we may also use a possibility distribution π from 2^Ω to [0, 1] (see Prade and Testemale [7], Dubois and Prade [8, 9], Yager [10, 11]) in order to model that some realizations are more possible than others. Then $I = \{i \mid \pi(A_i) > 0\}$, and $\pi(A_i)$ is the possibility (in the sense of Zadeh [12]) that $A = A_i$ and we have the normalization condition

$$\max_{i \in I} \pi(A_i) = 1$$

(i.e. at least one realization should be considered as completely possible). Note that $A_i = \varnothing$, for some i is allowed in the model.

A probability assignment or a possibility distribution over 2^Ω may be considered as too sophisticated, hence heavy-to-handle representation. Then it would be desirable to approximate such a representation in a natural way. This is what is proposed now. Let

$$A^- = \bigcap_{i \in I} A_i, \quad A^+ = \bigcup_{i \in I} A_i. \tag{1}$$

Obviously, we are certain that all the elements of A^- belong to A, i.e. $A^- \subseteq A$ and that all the elements outside A^+ do not belong to A, i.e. $A \subseteq A^+$. In that sense A^+ corresponds to a negative information. Moreover, A^+ gathers all the possible elements of A. Note that $A^- \subseteq A^+$ and that A^- may be empty or A^+ may be equal to Ω. The pair (A^-, A^+) provides a lower and an upper approximation of A. Such a pair has been considered by Narin'yani [13, 14] under the name of "sub-definite set". The "flou" sets introduced by Gentilhomme [15] are other examples of pairs of nested sets. Then the ill-known set A is implicitly represented by the subset $\mathscr{A}$ of 2^Ω defined by

$$\mathscr{A} = \{E \in 2^\Omega, A^- \subseteq E \subseteq A^+\}. \tag{2}$$

When $A^- = A^+$, $\mathscr{A} = \{A\}$ and A is perfectly known. Note that, $\forall i \in I$, $A_i \in \mathscr{A}$; however there may be elements of $\mathscr{A}$ which are not A_is, for instance A^+ and A^- themselves. Note that $\mathscr{A}$ contains the empty set as soon as $A^- = \varnothing$.

This representation can be improved by taking into account the possibility distribution π in the definitions of the lower and upper approximations, A^- and A^+. The extent to which it is certain that an element $\omega \in \Omega$ belongs to A (i.e. the degree of membership of this element ω to A^-) corresponds to the extent to which it is impossible to find an A_i such that $\omega \notin A_i$, including $A_i = \varnothing$. This leads to

$$\forall \omega \in \Omega, \quad \mu_{A^-}(\omega) = 1 - \sup_{i:\omega \notin A_i} \pi(A_i) = \inf_{i:\omega \notin A_i} [1 - \pi(A_i)]. \tag{3}$$

Similarly, the degree of membership of an element ω to A^+ corresponds to the extent to which it is possible that this element belongs to A, i.e. to the possibility of finding an A_i such that $\omega \in A_i$. This gives

$$\forall \omega \in \Omega, \quad \mu_{A^+}(\omega) = \sup_{i:\omega \in A_i} \pi(A_i). \tag{4}$$

N.B. $\mu_{A^-}(\omega)$ and $\mu_{A^+}(\omega)$ are just upper necessity and upper possibility measures of the singleton $\{\omega\}$ in the sense of Dubois and Prade [8, 9]. More recently, Yager [11] has considered the quantity $1 - \mu_{A^+}(\omega)$ under the name of "rebuff measure". In this paper, Yager shows how the possibility distribution π can, in some cases, be derived from (fuzzily) quantified statements.

It can be easily checked that if $\forall i \in I$, $\pi(A_i) = 1$, i.e. each A_i is equally possible, then equations (3) and (4) reduce to equation (1). Viewing $\mu_{A^-}(\omega)$ as the degree of certainty (or necessity) that ω belongs to A, i.e. symbolically

$$\forall \omega \in \Omega, \quad \mu_{A^-}(\omega) = \text{Cert}(\omega \in A) \tag{5}$$

we have

$$\forall \omega \in \Omega, \quad \mu_{A^+}(\omega) = 1 - \text{Cert}(\omega \in \bar{A}) \tag{6}$$

since if the ill-known set A set is represented by $\{[A_i, \pi(A_i)], i \in I\}$, its complement $\bar{A}$ is obviously represented by $\{[\bar{A}_i, \bar{\pi}(\bar{A}_i)], i \in I\}$ with

$$\forall i \in I, \quad \bar{\pi}(\bar{A}_i) = \pi(A_i), \tag{7}$$

where $\bar{\pi}$ is the possibility distribution attached to the representation of $\bar{A}$. Equalities (5) and (6) express here the usual relationship between possibility and necessity (certainty) in modal logic or in possibility theory (see Dubois and Prade [16] for instance), i.e. the necessity (certainty) of an event corresponds to the impossibility of the opposite event. It can be easily checked that

$$\forall \omega \in \Omega, \quad \mu_{A^+}(\omega) < 1 \Rightarrow \mu_{A^-}(\omega) = 0, \tag{8}$$

due to the normalization of π. The pair (A^-, A^+) defined by equations (3) and (4) is thus an ill-known set in the sense of Prade [17] or a two-fold fuzzy set in the sense of Dubois and Prade [18, 19]. Section 4 will show that set operations defined for two-fold fuzzy sets, can be applied to pairs of the form (A^-, A^+). Relation (8) expresses that the ωs which are more or less certain elements of A, must also be completely possible elements for A; this is intuitively satisfying. In other words,

$$\text{support}(A^-) \subseteq \text{core}(A^+), \tag{9}$$

where $\text{support}(A^-) = \{\omega \in \Omega, \mu_{A^-}(\omega) > 0\}$ and $\text{core}(A^+) = \{\omega \in \Omega, \mu_{A^+}(\omega) = 1\}$. The equalities (5) and (6) show that the lower and upper approximations of the (ill-known) complement $\bar{A}$ of A are obtained by

$$(\bar{A})^- = \overline{(A^+)}; \quad (\bar{A})^+ = \overline{(A^-)}, \tag{10}$$

from the lower and upper approximations A^- and A^+ of A, where the fuzzy set complementation is defined by

$$\forall \omega \in \Omega, \quad \mu_{\bar{F}}(\omega) = 1 - \mu_F(\omega). \tag{11}$$

This generalizes the fact that, when $\forall i \in I, \pi(A_i) = 1$, we have

$$\overline{(A^-)} = \overline{\bigcap_{i \in I} A_i} = \bigcup_{i \in I} \bar{A}_i = (\bar{A})^+ \quad \text{and} \quad \overline{(A^+)} = \overline{\bigcup_{i \in I} A_i} = \bigcap_{i \in I} \bar{A}_i = (\bar{A})^-.$$

However, note that

$$\{E \in 2^\Omega, \overline{(A^+)} \subseteq E \subseteq \overline{(A^-)}\} \neq \bar{\mathscr{A}}, \tag{12}$$

where $\bar{\mathscr{A}}$ denotes the complement of $\mathscr{A}$ [defined by equation (2)] in 2^Ω.

A similar extension of equation (1) using the probability assignment m instead of the possibility distribution π, is not possible. Indeed, the degree of membership of an element $\omega \in \Omega$ to the fuzzy set A^* defined by

$$\forall \omega \in \Omega, \quad \mu_{A^*}(\omega) = \sum_{i: \omega \in A_i} m(A_i), \tag{13}$$

can be interpreted as the probability that ω belongs to A, namely $\text{Prob}(\omega \in A)$. We can check that $\text{Prob}(\omega \in A) = 1 - \text{Prob}(\omega \in \bar{A})$, due to the normalization of m and representing $\bar{A}$ by $\{(\bar{A}_i, \bar{m}(\bar{A}_i)), i \in I\}$ with

$$\forall i \in I, \bar{m}(\bar{A}_i) = m(A_i). \tag{14}$$

See Dubois and Prade [4] for the introduction of equation (14) in the framework of a set-theoretic view of Shafer's belief functions. Definition (13) is sometimes called the one-point coverage function in the random set literature [20]. Obviously definition (13) does not reduce to equation (1) when m is uniformly distributed.

Although the pair (A^-, A^+) is a natural and easy-to-handle way of modeling partial information about an ill-known set, it does not provide a representation as rich as the one defined by the $\{(A_i, \pi(A_i)), i \in I\}$, as shown on the following example. Let us suppose we have the partial information "John speaks either English and French, or English and German, and no other languages". In that case we have $A_1 = \{\text{English, French}\}$ and $A_2 = \{\text{English, German}\}$, with

$\pi(A_1) = \pi(A_2) = 1$ and $\pi(E) = 0$, $\forall E \neq A_1$, $E \neq A_2$, $E \subseteq \Omega$. Obviously, $A^- = \{\text{English}\}$ and $A^+ = \{\text{English, French, German}\}$. In the representation by means of (A^-, A^+), we have for instance lost the information that John certainly speaks two languages (since $A^- \neq A_1$ and $A^- \neq A_2$), but certainly not both French and German.

Note that, even when A^- and A^+ are fuzzy sets, the ill-known set A under consideration remains an *ordinary* subset. Degrees of membership in A^- or in A^+ only reflect our uncertainty about the belonging of a particular element of Ω to A. Due to condition (9), a distinction is made between the simple possibility of membership and a graded certainty of membership (when possibility is complete, i.e. equal to 1). The approach presented here could be extended to ill-known fuzzy sets (then the possible realizations A_i would become fuzzy too), but situations where sets are both fuzzy and ill-known are not so frequent in practice, and various approaches have been proposed for handling ill-known membership functions. See Dubois and Prade [19] for a discussion of the relation between two-fold fuzzy sets and fuzzy sets with interval-valued membership functions.

2.2. Recovering a possibility distribution on the power set from the upper and lower approximations

Since (A^-, A^+) is only an approximate representation of the information contained in $\{(A_i, \pi(A_i)), i \in I\}$, there are several possibility distributions π in general, which lead via conditions (3) and (4) to the same pair (A^-, A^+). However, among them there exists a possibility distribution π^* which is the largest one in the sense of the inclusion of fuzzy sets defined on 2^Ω [i.e. such that, $\forall B \in 2^\Omega$, $\pi^*(B) \geqslant \pi(B)$ where π satisfies conditions (3) and (4) for a given pair (A^-, A^+)]. Indeed condition (4) can be rewritten as

$$\forall \omega \in \Omega, \quad \mu_{A^+}(\omega) = \sup_{i \in I} \min(\pi(A_i), \mu_R(\omega, A_i)),$$

where

$$\mu_R(\omega, A_i) = \begin{cases} 1, & \text{if } \omega \in A_i, \\ 0, & \text{otherwise,} \end{cases} = \mu_{A_i}(\omega).$$

It is a fuzzy relation equation [21]; this equation has always a solution [e.g. $\pi(\{\omega\}) = \mu_{A^+}(\omega)$ and π is zero for sets which are not singletons]; the largest solution π_1^* is given by

$$\forall B \in 2^\Omega, \quad \pi_1^*(B) = \inf_{\omega \in \Omega} \mu_R(\omega, B) \longrightarrow \mu_{A^+}(\omega),$$

with

$$r \longrightarrow a = \begin{cases} 1, & \text{if } r \leqslant a, \\ a, & \text{if } r > a. \end{cases}$$

Finally, we get, making the membership relation R explicit,

$$\forall B \neq \varnothing, \quad B \in 2^\Omega, \quad \pi_1^*(B) = \inf_{\omega \in B} \mu_{A^+}(\omega), \quad \text{while } \pi_1^*(\varnothing) \text{ is undefined.}$$

Similarly the equation

$$1 - \mu_{A^-}(\omega) = \sup_{i, \omega \notin A_i} \pi(A_i)$$

has always a solution [e.g. $\pi(\Omega - \{\omega\}) = 1 - \mu_{A^-}(\omega)$, and $\pi(B) = 0$, $\forall B$ such that $\nexists \omega$, $B = \Omega - \{\omega\}$]. Its largest solution is

$$\forall B \neq \Omega, \quad B \in 2^\Omega, \quad \pi_2^*(B) = \inf_{\omega \notin B} [1 - \mu_{A^-}(\omega)], \quad \text{and } \pi_2^*(\Omega) \text{ is undefined.}$$

Proposition 1

The possibility distribution π^* defined on 2^Ω by $\pi^*(\varnothing) = \pi_2^*(\varnothing)$, $\pi^*(\Omega) = \pi_1^*(\Omega)$,

$$\forall B \neq \varnothing, \Omega, \quad B \in 2^\Omega, \quad \pi^*(B) = \min\left[\inf_{x \in B} \mu_{A^+}(x), \inf_{y \notin B} (1 - \mu_{A^-}(y))\right], \tag{15}$$

is still a solution of the system of equations (3) and (4) (and by construction the largest one).

Proof. Let us prove that

$$\mu_{A^+}(\omega) = \sup_{\omega \in B} \pi^*(B).$$

If $\omega \notin \text{support}(A^-)$ then

$$\sup_{\omega \in B} \pi^*(B) \geqslant \pi^* [\text{support}(A^-) \cup \{\omega\}] = \mu_{A^+}(\omega),$$

since $\mu_{A^-}(y) = 0, \forall y \notin \text{support}(A^-)$ and $\mu_{A^+}(x) = 1, \forall x \in \text{support}(A^-)$. But $\forall B$, $\omega \in B$ implies $\pi^*(B) \leqslant \mu_{A^+}(\omega)$. Now if $\omega \in \text{support}(A^-)$,

$$\sup_{\omega \in B} \pi^*(B) = \pi^*[\text{support}(A^-)] = 1 = \mu_{A^+}(\omega)$$

due to relation (8). If $\text{support}(A^-) \cup \{\omega\} = \Omega$, then $\Omega = \{\omega\} \cup \text{core}(A^+)$ and

$$\pi^*(\Omega) = \inf_{x \in \Omega} \mu_{A^+}(x) = \mu_{A^+}(\omega),$$

so that the proof still holds.

Similarly, if $\omega \in \text{core}(A^+)$ then

$$\inf_{\omega \notin B} 1 - \pi^*(B) \leqslant 1 - \pi^*[\text{core}(A^+) \cap (\Omega - \{\omega\})] = \mu_{A^-}(\omega),$$

provided that $\text{core}(A^+) \cap (\Omega - \{\omega\}) \neq \varnothing$. But $\forall B, \omega \notin B \Rightarrow \pi^*(B) \leqslant 1 - \mu_{A^-}(\omega)$. The result is still valid of $\omega \notin \text{core}(A^+)$, where $\mu_{A^-}(\omega) = 0$, since $\pi^*[\text{core}(A^+)] = 1$. If $\text{core}(A^+) \cap (\Omega - \{\omega\}) = \varnothing$ then $\text{support}(A^-) \subseteq \{\omega\}$, and

$$\pi^*(\varnothing) = \inf_{x \in \Omega} [1 - \mu_{A^-}(x)] = 1 - \mu_{A^-}(\omega)$$

so that the proof still holds. Q.E.D.

Note that equation (15) gives

$$\forall B, \quad \text{support}(A^-) \subseteq B \subseteq \text{core}(A^+), \quad \pi^*(B) = 1$$

and when A^- and A^+ are ordinary sets, equation (15) reduces to

$$\pi^*(B) = \begin{cases} 1, & \text{if } B \in \mathscr{A}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\mathscr{A}$ is defined by equation (2). In the more general case $\mathscr{A}$ is a fuzzy subset of 2^Ω, defined by equation (15), i.e. $\forall B \in 2^\Omega$, $\mu_{\mathscr{A}}(B) = \pi^*(B)$.

Expression (15) is intuitively satisfying, since it means that B is all the more a possible realization of the ill-known set A, as B includes only elements among the most possible ones (which belong to A^+ with a high degree of membership) and does not exclude any element among the most certain ones (which belong to A^- with a high degree of membership). Expression (15) was proposed by Prade and Testemale [22], in order to estimate to what extent a key-word-based description of a document is possible (admissible) in information retrieval. Indeed such a description must include only "possible" key-words and must not exclude any "necessary" ones with respect to the considered document.

In the next section, we discuss elementary query evaluation in presence of partial conjunctive knowledge represented by pairs of the form (A^-, A^+), where A^- and A^+ may be fuzzy and satisfy condition (9). We also compare this evaluation to the one which would be directly obtained from the representation provided by a possibility distribution on the power set 2^Ω.

3. QUERY EVALUATION IN PRESENCE OF PARTIAL CONJUNCTIVE INFORMATION

The evaluation of vague queries in presence of partial or fuzzy information pertaining to single-valued attribute has been studied at length by Prade and Testemale [23], when the available information is represented by means of possibility distributions. In the case of a single-valued attribute, a possibility distribution is defined on the attribute domain, since the elements in the domain are mutually exclusive as possible values of the attribute. The case of multiple-valued attributes is dealt with by Prade and Testemale [7], where partial information is modeled by means of possibility distributions defined over the power sets of the attribute domains (i.e. using representations of the type $\{(A_i, \pi(A_i)),\ i \in I\}$). In this section we consider the problem of query evaluation when the available information is represented under the form (A^-, A^+) with $\text{support}(A^-) \subseteq \text{core}(A^+) \subseteq \Omega$. The corresponding query will be supposed to be expressed in the same style. Namely, does the attribute under consideration take *at least* all the values in B^- and *at most* all the values in B^+, where $B^- \subseteq B^+ \subseteq \Omega$? More generally vague queries where B^- or B^+ are fuzzy sets satisfying $\text{support}(B^-) \subseteq \text{core}(B^+)$, can be considered. If we are only interested in the "at least" (resp. "at most") part of the query, we take $B^+ = \Omega$(resp. $B^- = \varnothing$). We first consider the case where A^-, A^+, B^- and B^+ are ordinary subsets.

3.1. Non-fuzzy information and query

Knowing that the multiple-valued attribute under consideration takes at least the values in A^- and at most the values in A^+, we are certain that it takes at least the values in B^- if $B^- \subseteq A^-$ and we are certain that it takes at most the values in B^+ if $B^+ \supseteq A^+$. Let us denote by $\text{Cert}(B; A)$ the certainty that the piece of information (A^-, A^+) satisfy the query represented by (B^-, B^+). Thus, we have

$$\text{Cert}(B; A) = \begin{cases} 1, & \text{if } B^- \subseteq A^- \text{ and } A^+ \subseteq B^+, \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

It can be easily checked that condition (16) can be rewritten as

$$\text{Cert}(B; A) = \begin{cases} 1, & \text{if } \mathscr{A} \subseteq \mathscr{B}, \\ 0, & \text{otherwise,} \end{cases} \triangleq N(\mathscr{B}; \mathscr{A}), \tag{17}$$

where $\mathscr{A}$ is defined by equation (2) and similarly $\mathscr{B} = \{F \in 2^\Omega, B^- \subseteq F \subseteq B^+\}$. Besides it is impossible that the datum (A^-, A^+) satisfy the query (B^-, B^+) as soon as $B^- \cap \overline{A^+} \neq \varnothing$ or $A^- \cap \overline{B^+} \neq \varnothing$, since then there are certainly elements of the ill-known set A which are outside B^+, or there are in B^- elements which are certainly outside A. If $\text{Poss}(B; A)$ denotes the possibility that the datum (A^-, A^+) satisfies the query (B^-, B^+), then we have

$$\text{Poss}(B; A) = \begin{cases} 1, & \text{if } B^- \cap \overline{A^+} = \varnothing \text{ and } A^- \cap \overline{B^+} = \varnothing, \\ 0, & \text{otherwise.} \end{cases}$$

$$= \begin{cases} 1, & \text{if } B^- \subseteq A^+ \text{ and } A^- \subseteq B^+, \\ 0, & \text{otherwise.} \end{cases} \tag{18}$$

Note that $B^- \subseteq A^+$ and $A^- \subseteq B^+ \Leftrightarrow \mathscr{A} \cap \mathscr{B} \neq \varnothing$. Indeed if the lefthand part of the equivalence holds then we have $A^- \subseteq A^- \cup B^- \subseteq A^+$ and $B^- \subseteq A^- \cup B^- \subseteq B^+$; thus, $A^- \cup B^- \in \mathscr{A} \cap \mathscr{B}$. If the righthand part of the equivalence holds, then $\exists E$, $A^- \subseteq E \subseteq A^+$ and $B^- \subseteq E \subseteq B^+$ and then the two inclusions of the lefthand part are obtained by transitivity. Thus condition (18) can be rewritten

$$\text{Poss}(B; A) = \begin{cases} 1, & \text{if } \mathscr{A} \cap \mathscr{B} \neq \varnothing, \\ 0, & \text{otherwise,} \end{cases} \triangleq \Pi(\mathscr{B}; \mathscr{A}). \tag{19}$$

It may look strange that from condition (18) $\text{Poss}(B; A) = 1$ when $A^- = B^- = \varnothing$ even if $A^+ \cap B^+ = \varnothing$. However, this case preserves the possibility that $A = B = \varnothing$; this situation occurs in the following example, where the available information is "John was living in Paris at most in 1982 (but maybe he was in Amsterdam that year)", and the query is "Was John living in Paris at most in the seventies?". The query can be answered positively if John had never been living in Paris (a case which is not excluded).

Using conditions (16) and (18), $\text{Cert}(B; A)$ and $\text{Poss}(B; A)$ are easy to evaluate. Moreover conditions (17) and (19) are satisfying from an intuitive point of view. indeed, since the ill-known set A is (approximately) represented by the set $\mathscr{A}$ of the subsets of Ω, then it is certain that the partially-known datum whose content is A satisfy the query represented by (B^-, B^+), as soon as all the subsets in $\mathscr{A}$ are between B^- and B^+ in the sense of inclusion; the same thing is possible as soon as at least one subset in $\mathscr{A}$ is between B^- and B^+. From conditions (17) and (19), we obviously have

$$\mathrm{Cert}(B; A) = 1 - \Pi(\bar{\mathscr{B}}; \mathscr{A}), \tag{20}$$

where $\bar{\mathscr{B}}$ denotes the complement of $\mathscr{B}$ in 2^Ω. $\bar{\mathscr{B}}$ corresponds to the query which is the negation of the one represented by (B^-, B^+), namely "does the attribute take some values outside of B^+ or is there some value in B^- which is not taken by the attribute?" The query corresponding to $(\overline{B^+}, \overline{B^-})$ is obviously different and means "does the attribute take at least all the values outside B^+ and at most all the values outside B^-?" This latter query can be viewed as the antonym of the query associated with (B^-, B^+) rather than its negation.

Let us consider a small example where a query of the form (B^-, B^+) is processed. The available knowledge says that "John was living in Paris at least during the period A^-" (where A^- is a subset of the set Ω of years). Here $A^+ = \Omega$. The answer to a query of the form "Was John living in Paris during the period B?" is "certainly yes" if $B \subseteq A^-$ $(B^- = B)$, and "possibly yes" if $A^- \subseteq B$ (with $B^+ = B$, thus understanding the query as "the period B only"; if $B^+ = \Omega$, the possibility that the answer is "yes" is always equal to 1). For Ω finite, we may think of using the relative cardinality

$$\frac{|B^- \cap \overline{A^-}|}{|B^-|},$$

when $B^- \not\subseteq A^-$ and $B^- \subseteq A^+$, if $B^+ = \Omega$ so that $\mathrm{Cert}(B; A) = 0$ and $\mathrm{Poss}(B; A) = 1$, in order to estimate the proportion of elements in B^- for which we are uncertain about their belonging to the ill-known set A.

3.2. Fuzzy information and vague query

In possibility theory, the extent to which we are certain that a fuzzy set F is included in a fuzzy set G, is estimated by the necessity degree (see Dubois and Prade [24] for instance), defined by

$$N(G; F) = \inf_{\omega \in \Omega} \max(\mu_G(\omega), 1 - \mu_F(\omega)). \tag{21}$$

When F and G are ordinary sets, we have $N(G; F) = 1$ if $F \subseteq G$ and $N(G; F) = 0$ otherwise. More generally, $N(G; F) > 0$ if and only if $\mathrm{core}(F) \subseteq \mathrm{support}(G)$, and $N(G; F) = 1$ if and only if $\mathrm{support}(F) \subseteq \mathrm{core}(G)$.

This enables us to extend conditions (16) and (18) to the case where A^-, A^+, B^- or B^+ are fuzzy sets. We obtain

$$\mathrm{Cert}(B; A) = \min\left[\inf_{\omega \in \Omega} \max(\mu_{A^-}(\omega), 1 - \mu_{B^-}(\omega)), \inf_{\omega \in \Omega} \max(\mu_{B^+}(\omega), 1 - \mu_{A^+}(\omega))\right] \tag{22}$$

$$\mathrm{Poss}(B; A) = \min\left[\inf_{\omega \in \Omega} \max(\mu_{A^+}(\omega), 1 - \mu_{B^-}(\omega)), \inf_{\omega \in \Omega} \max(\mu_{B^+}(\omega), 1 - \mu_{A^-}(\omega))\right]. \tag{23}$$

These expressions are intuitively justified as follows: $\mathrm{Cert}(B; A)$ is the minimum of two terms expressing to what extent B^- is included in A^- and A^+ is included in B^+, respectively. The minimum operation expresses the "and" which appears in condition (16). $\mathrm{Poss}(B; A)$ is similarly interpreted. However, equations (22) and (23) can be rigorously established. Namely $\mathrm{Cert}(B; A)$ is really the degree of inclusion of $\mathscr{A}$ in $\mathscr{B}$ in the sense of equation (21), i.e.

$$\mathrm{Cert}(B; A) = N(\mathscr{B}; \mathscr{A}) = \inf_{C \in 2^\Omega} \max(1 - \mu_{\mathscr{A}}(C), \mu_{\mathscr{B}}(C)). \tag{24a}$$

Similary equation (23) is the degree of intersection of $\mathscr{B}$ and $\mathscr{A}$ in the sense that

$$\mathrm{Poss}(B; A) = \Pi(\mathscr{B}; \mathscr{A}) = \sup_{C \in 2^\Omega} \min(\mu_{\mathscr{A}}(C), \mu_{\mathscr{B}}(C)), \tag{24b}$$

i.e. a degree of possibility of a fuzzy event [12]. These two identitites are proved in the Appendix.

The following properties are easy to establish using inclusion (9) and the properties of the degree of inclusion $N(G; F)$:

$$\mathrm{Cert}(B; A) \leqslant \mathrm{Poss}(B; A);$$

$$\mathrm{Cert}(B; A) > 0 \Rightarrow \mathrm{Poss}(B; A) = 1, \quad \text{when } B^- \text{ and } B^+ \text{ are ordinary subsets}$$

It expresses the certainty that the piece of information (A^-, A^+) satisfies the query can be strictly positive only if the associated possibility is already equal to 1 in case of crisp queries; this is intuitively satisfying, and expected, due to equations (24a) and (24b) since possibility and necessity measures are such that $\Pi(\mathscr{B}; \mathscr{A}) \geqslant N(\mathscr{B}; \mathscr{A})$ when $\mathscr{B}$ is fuzzy and such that $\min(1 - \Pi(\mathscr{B}; \mathscr{A}), N(\mathscr{B}; \mathscr{A})) = 0$, when $\mathscr{B}$ is crisp.

It is interesting to compare the evaluation of a query from the (approximate) knowledge expressed by the pair (A^-, A^+), to the evaluation of the same query from a rich representation, namely the one provided by the fuzzy set $\mathscr{A}_\pi = \{(A_i, \pi(A_i)), i \in I\}$ defined on 2^Ω. When B^- and B^+ are ordinary subsets of Ω, a reasonable estimate of the certainty that the piece of information represented by $\mathscr{A}_\pi$ satisfies the query, is given by

$$\mathrm{Cert}(B; \mathscr{A}_\pi) = \inf_{i,\, B^- \not\subseteq A_i \text{ or } A_i \not\subseteq B^+} 1 - \pi(A_i). \tag{25}$$

Indeed, the extent to which it is certain that B^- (resp. B^+) is included (resp. contains) the ill-known set A, corresponds to the extent to which it is impossible to find an A_i such that $B^- \not\subseteq A_i$ or $A_i \not\subseteq B^+$. This expression extends to the case when B^- and B^+ are fuzzy sets into

$$\mathrm{Cert}(B; \mathscr{A}_\pi) = \inf_i \max(\min(N(A_i; B^-), N(B^+; A_i)), 1 - \pi(A_i)) = N(\mathscr{B}; \mathscr{A}_\pi),$$

noticing that $\mu_{\mathscr{B}}(A_i) = \min(N(A_i; B^-), N(B^+; A_i))$, is just another way of writing expression (15).

We have the following result when B^- and B^+ are ordinary or fuzzy subsets,

$$\mathrm{Cert}(B; \mathscr{A}_\pi) = \mathrm{Cert}(B; A) \tag{26}$$

Proof of equality (26)

$$\mathrm{Cert}(B; \mathscr{A}_\pi) = \inf_i \min(\max(N(A_i; B^-), 1 - \pi(A_i)), \max(N(B^+; A_i), 1 - \pi(A_i)))$$

$$= \min\left(\inf_i \max(N(A_i; B^-), 1 - \pi(A_i)), \inf_i \max(N(B^+; A_i), 1 - \pi(A_i))\right).$$

Letting

$$N(A_i; B^-) = \inf_{x \notin A_i} 1 - \mu_{B^-}(x) \quad \text{and} \quad N(B^+; A_i) = \inf_{y \in A_i} \mu_{B^+}(y)$$

and exchanging

$$\inf_i \text{ and } \inf_{x,\, x \notin A_i} \text{ into } \inf_x \text{ and } \inf_{i,\, x \notin A_i},$$

we get

$$\mathrm{Cert}(B; \mathscr{A}_\pi) = \min\left(\inf_x \max(1 - \mu_{B^-}(x), \inf_{x \notin A_i} 1 - \pi(A_i)), \inf_y \max(\mu_{B^+}(y), \inf_{x \in A_i} 1 - \pi(A_i))\right)$$

$$= \mathrm{Cert}(B; A),$$

since $\mathscr{A}_\pi$ can be represented by (A^+, A^-) via equations (3) and (4) Q.E.D.

Similarly, a reasonable estimate of the possibility that the piece of information represented by $\mathscr{A}_\pi$ satisfies the query, is given by

$$\mathrm{Poss}(B; \mathscr{A}_\pi) = \sup_{i,\, B^+ \supseteq A_i \supseteq B^-} \pi(A_i) \tag{27}$$

Indeed, the extent to which it is possible that B^- (resp. B^+) is included (resp. contains) the ill-known set A, corresponds to the extent to which it exists an A_i such that $A_i \supseteq B^-$ and $A_i \subseteq B^+$.

Equation (27) can be extended to the case when (B^-, B^+) is fuzzy as follows:

$$\text{Poss}(B; \mathscr{A}_\pi) = \sup_i \min(N(B^+; A_i), N(A_i; B^-), \pi(A_i))$$

$$= \Pi(\mathscr{B}; \mathscr{A}_\pi),$$

since once again, $\pi_{\mathscr{B}}(A_i) = \min(N(B^+; A_i), N(A_i; B^-))$ due to equation (15). In contrast with equation (26), we only have the following inequality (a natural one since $\mathscr{A}_\pi \subseteq \mathscr{A}$):

$$\text{Poss}(B; \mathscr{A}_\pi) \leqslant \text{Poss}(B; A). \tag{28}$$

The inequality is simply due to the fact that $\mathscr{A}_\pi \subseteq \mathscr{A}$, where both $\mathscr{A}_\pi$ and $\mathscr{A}$ are approximated by (A^-, A^+), $\mathscr{A}$ being the least specific subset of 2^Ω corresponding to this pair. Inequality (28) only expresses that $\Pi(\mathscr{B}; \mathscr{A}) \geqslant \Pi(\mathscr{B}; \mathscr{A}_\pi)$.

At first glance, it may seem amazing that the equality (26) holds for certainty degrees and that only the inequality (28) holds in case of possibility degrees. In fact, this situation already exists when A^- and A^+ are ordinary subsets (i.e. $\forall i, \pi(A_i) = 1$). Indeed, the requirements $B^- \subseteq A^-$ and $A^+ \subseteq B^+$ in equation (16), along with

$$A^- = \bigcap_i A_i \quad \text{and} \quad A^+ = \bigcup_i A_i,$$

are equivalent to $\forall i, B^- \subseteq A_i$ and $A_i \subseteq B^+$, i.e. to $\text{Cert}(B; \mathscr{A}_\pi) = 1$. Contrastedly, we only have the entailment

$$(\forall i, B^- \subseteq A_i, A_i \subseteq B^+) \Rightarrow B^- \subseteq \bigcup_i A_i, \bigcap_i A_i \subseteq B^+$$

i.e. $\text{Poss}(B; \mathscr{A}_\pi) = 1 \Rightarrow \text{Poss}(B; A) = 1$. Obviously, the converse entailment does not hold.

However, the inequality (28) expresses that using the approximate information conveyed by the pair (A^-, A^+), we obtain an *upper* bound of the possibility degree, which is satisfying. Indeed, overestimating a possibility corresponds to a lack of knowledge, since when the possibility degree of an alternative decreases the certainty degree of the opposite alternative increases [due to relation (20)], the certainty being always total in case of complete information. The equality (26) guarantees that the information which is lost in $\mathscr{A}$ (with respect to the information contained in $\mathscr{A}_\pi$), has no influence on the estimation of the certainty degree in the query evaluation process, which is fortunate.

4. COMBINATION OF CONJUNCTIVE EVIDENCE

A question which naturally arises is to know if the approximation in the sense of equations (3) and (4) of the result of a set operation performed on ill-known sets represented by means of possibility distributions on the power set, is equal to the result of this set operation directly performed on the approximate representations of the ill-known sets under consideration.

Let π and π' be two possibility distributions on 2^Ω. These possibility distributions can be viewed either as fuzzy sets of 2^Ω or as representations of ill-known sets. These two points of view induce two different types of set operations. In the first case we just apply the usual fuzzy set operations to fuzzy sets of 2^Ω. In the second case we estimate the possibility of realization of a particular subset as the result of the set operation on the possible realizations of the ill-known sets under consideration.

Set theoretic operations on twofold fuzzy sets [18, 19] have been suggested. Let (A^-, A^+) and (B^-, B^+) be twofold fuzzy sets, i.e. they satisfy condition (9). The complement of (A^-, A^+), the intersection and the union are defined by

$$\overline{(A^-, A^+)} = (\overline{A^+}, \overline{A^-}), \tag{29}$$

$$(A^-, A^+) \cap (B^-, B^+) = (A^- \cap B^-, A^+ \cap B^+), \tag{30}$$

$$(A^-, A^+) \cup (B^-, B^+) = (A^- \cup B^-, A^+ \cup B^+), \tag{31}$$

where the overbar, $\cap$ and $\cup$ are usual fuzzy set theoretic operations, based on equation (11), min and max, respectively. Let A and B be two ill-known sets characterized by possibility distributions

π_A and π_B on 2^Ω. The complement $\bar{A}$ of A, the union and intersection $A \cup B$ and $A \cap B$ are characterized by possibility distributions $\pi_{\bar{A}}$, $\pi_{A \cup B}$, $\pi_{A \cap B}$ defined by

$$\pi_{\bar{A}}(C) = \pi_A(\bar{C}), \forall C \subseteq 2^\Omega, \tag{32}$$

$$\pi_{A \cup B}(C) = \sup_{i,j \atop C = A_i \cup B_j} \min(\pi_A(A_i), \pi_B(B_j)), \tag{33}$$

$$\pi_{A \cap B}(C) = \sup_{i,j \atop C = A_i \cap B_j} \min(\pi_A(A_i), \pi_B(B_j)). \tag{34}$$

Note that the quantity (34) has been introduced by the authors [9] as a possibilistic counterpart of Dempster rule of combination [6]. The meaning of definitions (32)–(34) can be exemplified as follows.

Let A be the (ill-known) set of years spent by John in Paris, and B the set of years spent by Jack in Paris. $\pi_A(C)$ [resp. $\pi_B(C)$] is the degree of possibility that John (resp. Jack) was in Paris during all years in C and only those. Then $\bar{A}$ is the ill-known set of years when John was *not* in Paris, $A \cap B$ is the ill-known set of years when *both* John and Jack were in Paris, $A \cup B$ is the set of years where at least one of them was in Paris. For instance $\pi_{A \cap B}(C)$ is the degree of possibility that both John and Jack were in Paris exactly during the years in C.

The consistency between conditions (29)–(31) and (32)–(34) is expressed by the following identities:

$$(\bar{A})^+ = \overline{A^-} \qquad (\bar{A})^- = \overline{A^+}, \tag{35}$$

$$(A \cup B)^+ = A^+ \cup B^+ \qquad (A \cup B)^- = A^- \cup B^-, \tag{36}$$

$$(A \cap B)^+ = A^+ \cap B^+ \qquad (A \cap B)^- = A^- \cap B^-. \tag{37}$$

Proof. Construction of (A^-, A^+) out of π_A is done by equation (3) and (4). Identity (35) was proved in Section 2.1.

$$\mu_{(A \cap B)^+}(\omega) = \sup_{\omega \in C} \sup_{i,j \atop A_i \cap B_j = C} \min(\pi_A(A_i), \pi_B(B_j))$$

$$= \sup_{i,j \atop \omega \in A_i, \omega \in B_j} \min(\pi_A(A_i), \pi_B(B_j)) = \min(\mu_{A^+}(\omega), \mu_{B^+}(\omega))$$

$$\mu_{(A \cap B)^-}(\omega) = \inf_{\omega \notin C} \inf_{i,j \atop A_i \cap B_j = C} \max(1 - \pi_A(A_i), 1 - \pi_B(B_j))$$

$$= \inf_{i,j \atop \omega \notin A_i \text{ or } \omega \notin B_j} \max(1 - \pi_A(A_i), 1 - \pi_B(B_j))$$

$$= \min\left(\inf_{i,j \atop \omega \notin A_i} \max(1 - \pi_A(A_i), 1 - \pi_B(B_j)), \inf_{\omega \notin B_j} \max(1 - \pi_A(A_i), 1 - \pi_B(B_j))\right).$$

But

$$\inf_C (1 - \pi_A(C)) = 0 = \inf_C (1 - \pi_B(C))$$

since π_A and π_C are normalized. Hence the arguments of the outer min collapse to $\mu_{A^-}(\omega)$, $\mu_{B^-}(\omega)$, respectively, identity (36) is proved the same way, or noticing that De Morgan Laws are valid on ill-known sets. Q.E.D.

As a consequence twofold fuzzy sets and ill-known sets are homomorphic structures in terms of operations (29)–(31) and (32)–(34), respectively. Namely the function $A \mapsto (A^-, A^+)$ being onto, the relation $\sim$ defined by $A \sim B \Leftrightarrow (A^-, A^+) = (B^-, B^+)$ is an equivalence relation and the set of equivalence classes on $[0, 1]^{2^\Omega}$ is isomorphic to the set of twofold fuzzy sets on Ω which in turn is isomorphic to the set of fuzzy sets on Ω equipped with usual operations [19]. In other words the set of ill-known sets on Ω, is a pseudo-complemented distributive lattice.

These results are counterparts of similar ones for belief functions viewed as random sets [4, 5]. Actually we propose a possibilistic theory of evidence in this paper.

Denoting $\mathscr{A}$ and $\mathscr{B}$ the fuzzy sets on 2^Ω with membership functions $\mu_{\mathscr{A}} = \pi_A$, $\mu_{\mathscr{B}} = \pi_B$, they can be combined via usual fuzzy set operations i.e.

$$\mu_{\bar{\mathscr{A}}}(C) = 1 - \pi_A(C), \tag{38}$$

$$\mu_{\mathscr{A} \cup \mathscr{B}}(C) = \max(\pi_A(C), \pi_B(C)), \tag{39}$$

$$\mu_{\mathscr{A} \cap \mathscr{B}}(C) = \min(\pi_A(C), \pi_B(C)). \tag{40}$$

These operations are clearly different from operations (32)–(34) and do not express the same type of combination. Namely $\bar{\mathscr{A}}$ represents for instance the ill-known set of years which *do not represent* the exact period when John was in Paris; it contains, but is not covered by, the ill-known set of years representing the exact period when John *was not* in Paris (i.e. $\mu_{\bar{\mathscr{A}}}(C) \geq \pi_{\bar{A}}(C)\, \forall C$). $\mathscr{A} \cup \mathscr{B}$ is the ill-known set of years representing either the period when John was in Paris or the period when Jack was in Paris (we no longer know which of the two variables is represented). $\mu_{\mathscr{A} \cap \mathscr{B}}(C)$ is positive if and only if C is possible as the complete period when both John and Jack were in Paris. (This is very drastic because it assumes that Jack and John were always together in Paris!) These operations have less intuitive appeal than operations (32)–(34).

5. POSSIBILITY DISTRIBUTION, LEVEL CUTS AND ILL-KNOWN SETS

A possibility distribution on a power set is an ill-known set. Hence a possibility distribution on a set Ω is an ill-known singleton. Results on ill-known sets can be particularized to standard possibility distributions corresponding to single-valued variables. A possibility distribution on Ω is viewed as an ill-known set A on 2^Ω such that

$$\pi(A) > 0 \Leftrightarrow \exists \{\omega\}, \quad A = \{\omega\} \text{ or } A = \emptyset$$

then equations (3) and (4) reduce to

$$\mu_{A^-}(\omega) = \inf_{\substack{\omega' \\ \omega \neq \omega'}} 1 - \pi(\omega') > 0 \text{ only if } \pi(\omega) = 1$$

and

$$\pi(\omega') \leq 1 - \epsilon, \quad \forall \omega' \neq \omega \text{ with } \epsilon > 0.$$

$$\mu_{A^+}(\omega) = \pi(\omega), \text{ identifying } \pi(\{\omega\}) \text{ with } \pi(\omega).$$

Hence A^- is usually empty, and anyway does not bring new information with regard to A^+, provided that we rule out the possibility that the variable we model has no value as long as μ_{A^+} is normal ($\sup \mu_{A^+} = 1$). Note that $\mathrm{Cert}(B; A) = N(B^+; A^+)$ generally. However $\mathrm{Poss}(B; A) \neq \Pi(B^+; A^+)$. Namely $\mathrm{Poss}(B; A) = 1$ $\forall A, B$, ill-known singletons, when $A^- = B^- = \emptyset$. This behavior has already been mentioned earlier. The proper index of possible matching between B and A is $\Pi(B^+; A^+)$ in the case of ill-known singletons with $\pi(\emptyset) = 0$.

Another interesting particular case is when the A_i s such that $\pi(A_i) > 0$ form a nested sequence, say $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$. Then if $\omega_i \in A_i - A_{i-1}$, $i \geq 2$, equations (3) and (4) become

$$\mu_{A^+}(\omega_i) = \sup_{j \geq i} \pi(A_j),$$

with

$$\mu_{A^+}(\omega_1) = 1, \quad \text{if } \omega_1 \in A_1;$$

$$\mu_{A^-}(\omega_i) = \inf_{j < i} 1 - \pi(A_j).$$

Assuming further that $\pi(A_1) = \alpha_1 = 1 \geq \pi(A_2) = \alpha_2 \geq \cdots \geq \pi(A_n)$, we get

$$\mu_{A^+}(\omega_i) = \alpha_i\, \forall \omega_i \in A_i - A_{i-1}, \quad i \geq 2,$$

$$\begin{aligned} \mu_{A^-}(\omega) &= 0, \quad \forall \omega \notin A_i, \quad i \geq 2 \\ &= 1, \quad \text{if } \omega \in A_1. \end{aligned}$$

The result is similar to the case of ill-known singletons, in that A^+ gathers the whole information, provided that we rule out the possibility that the ill-known set be empty when $A_1 \neq \varnothing$ (i.e. $\sup \mu_{A^+} = 1$). In fact it is clear that the A_i s are the level cuts of A^+, and that equation (4) generalizes Zadeh's [25] representation theorem for fuzzy sets,

$$\mu_{A^+}(\omega) = \sup\{\alpha \mid \omega \in A_\alpha^+\}, \quad \text{where } A_\alpha^+ = \{\omega \mid \mu_{A^+}(\omega) > \alpha\}.$$

However the largest ill-known set equivalent to (A_1, A^+) does not reduce to the set of level-cuts but contains more sets; namely equation (15) reduces to

$$\pi^*(B) = \inf_{\omega \in B} \mu_{A^+}(\omega), \quad \text{if support } (A^+) \supseteq B \supseteq A_1,$$

a richer representation of A^+ than level-cuts. This formula was first used in Dubois and Prade [26] in the definition of an integral over a fuzzy interval.

6. CONCLUDING REMARKS

In Prade and Testemale [7, 23] an approach to the treatment of fuzzy partial information about the values of single-valued or multiple-valued attributes in data bases has been proposed. This information is represented in terms of possibility distributions, which in case of multiple-valuedness, are defined on the power sets of the attribute domains. The evaluation of various kinds of queries is dealt with in this approach and the extent to which an item of the data base satisfies a given query is in any case estimated in terms of a possibility and of a necessity degree.

In the present paper, we have considered the case of multiple-valued attributes, or if we prefer of conjunctive information when fuzzy partial information is available under the form of a lower and of an upper approximation of a set of values (rather than a possibility distribution on a power set). Again the evaluation of a query is made in terms of possibility and necessity degrees. A pair of the form (A^-, A^+) offers a natural and easy-to-handle modeling of incomplete conjunctive information. Although it may not be possible to distinguish with this representation between situations which would be clearly differentiated in a power set-based representation, it is fortunate that the approximative nature of the (A^-, A^+)-representation has no influence on the evaluation of the necessity degrees. Clearly, the approximate representation suggested here allows for the treatment of an interesting class of queries on conjunctive pieces of knowledge. Although having sound foundations, a pair of fuzzy sets has a limited expressive power. Namely there are some kinds of knowledge item which correspond to vacuous approximate representations (i.e. $A^- = \varnothing$, $A^+ = \Omega$). For instance, queries such as "Does John speak at least English or French?" or "Does John speak exactly 2 languages?" cannot be represented by means of (B^-, B^+). What may be useful is some information on the cardinality of the ill-known set, as already suggested by Narin'yani [13]. The treatment of cardinality in fuzzy conjunctive information is a topic for further research. However queries dealing with cardinality can already be handled as suggested in the following example.

An important example of conjunctive information is encountered in the modeling of time. For instance the set of years when John had been more or less certainly (resp. possibly) living in Paris can be conveniently represented by a pair (A^-, A^+). Then, queries such as "Was John living in Paris during the period B^-?", as well as more sophisticated ones such as "Was John living in Paris for at least some years?" can be easily handled in the possibility/necessity setting. This latter query can be dealt with by computing the fuzzy cardinality (which is a possibility distribution on the set of integers) of the ill-known set represented by (A^-, A^+) [27, 28], and then by estimating the matching of this fuzzy cardinality with the fuzzy requirement "at least some years", in terms of possibility and necessity measures using standard methods [29].

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APPENDIX

Proof of the Identity Cert(B; A) = $N(\mathscr{B}; \mathscr{A})$

$$N(\mathscr{B};\mathscr{A}) = \inf_{C\in 2^\Omega} \max(1-\mu_{\mathscr{A}}(C), \mu_{\mathscr{B}}(C))$$

$$= \inf_{C\in 2^\Omega} \max\left[1-\mu_{\mathscr{A}}(C), \min\left(\inf_{y\in C}\mu_{B^+}(y), \inf_{z\notin C} 1-\mu_{B^-}(z)\right)\right]$$

[due to equation (15), expanding $\mu_{\mathscr{B}}(C)$].

Let us apply the following transformations:

- $\max(1-a, \min(b^+, b^-)) = \min(\max(1-a, b^+), \max(1-a, b^-))$
- $\inf_C \min(f(C), g(C)) = \min\left(\inf_C f(C), \inf_{C'} g(C')\right).$

We get

$$N(\mathscr{B};\mathscr{A}) = \min\left[\inf_C \max\left(1-\mu_{\mathscr{A}}(C), \inf_{y\in C}\mu_{B^+}(y)\right), \inf_C \max\left(1-\mu_{\mathscr{A}}(C), \inf_{z\notin C} 1-\mu_{B^-}(z)\right)\right]$$

$$= \min\left[\inf_y \max\left(\inf_{y\in C} 1-\mu_{\mathscr{A}}(C), \mu_{B^+}(y)\right), \inf_z \max\left(\inf_{z\notin C} 1-\mu_{\mathscr{A}}(C), 1-\mu_{B^-}(z)\right)\right]$$

$$= \mathrm{Cert}(B; A).$$

The last equality was obtained noticing that

$$\inf_{(u,v)\in D} f(u,v) = \inf_{u} \inf_{v\in D(u)} f(u,v) = \inf_{v} \inf_{u\in D(v)} f(u,v),$$

where $D(u) = \{u \mid (u,v) \in D\}$; $D(v) = \{v \mid (u,v) \in D\}$.

Proof of the Identity Poss $(B;A) = \Pi(\mathscr{B};\mathscr{A})$

We need the following:

Lemma

Let $\mathscr{A}$ be the largest fuzzy set in 2^{Ω} equivalent to a pair (A^-, A^+) satisfying condition (9); then the level-cut $\mathscr{A}_\alpha$ is equal to $\{E; A^-_{\overline{1-\alpha}} \subseteq E \subseteq A^+_\alpha\}$, where $F_{\bar{\alpha}}$ denotes the strong α-cut ($F_{\bar{\alpha}} = \{x \mid \mu_F(x) > \alpha\}$).

Proof. Let E such that $A^-_{\overline{1-\alpha}} \subseteq E \subseteq A^+_\alpha$. Then

$$\inf_{\omega\in E} \mu_{A^+}(\omega) \geqslant \alpha \quad \text{and} \quad \sup_{\omega\notin A^-_{\overline{1-\alpha}}} \mu_{A^-}(\omega) \leqslant 1-\alpha.$$

Hence

$$\pi_1^*(E) \geqslant \alpha \quad \text{and} \quad \pi_2^*(E) = \inf_{\omega\notin E} 1 - \mu_{A^-}(\omega) \geqslant \alpha,$$

where π_1^* and π_2^* are defined in Section 2.2. Hence $\mu_{\mathscr{A}}(E) \geqslant \alpha$. Now if $E \nsubseteq A^+_\alpha$ then $\exists\, \omega \in E$, $\mu_{A^+}(\omega) < \alpha$ and $\mu_{\mathscr{A}}(E) \leqslant \pi_1^*(E) < \alpha$. Similarly if $A^-_{\overline{1-\alpha}} \nsubseteq E$, $\exists\, \omega \notin E$ and $\omega \in A^-_{\overline{1-\alpha}}$, i.e. $\mu_{A^-}(\omega) > 1-\alpha$. Hence $\mu_{\mathscr{A}}(E) \leqslant \pi_2^*(E) < \alpha$. Q.E.D.

Now assume that $\text{Poss}(B;A) \geqslant \alpha$. Hence the two following inequalities hold:

$$\inf_{\omega\in\Omega} \max(\mu_{A^+}(\omega), 1 - \mu_{B^-}(\omega)) \geqslant \alpha;$$

$$\inf_{\omega\in\Omega} \max(\mu_{B^+}(\omega), 1 - \mu_{A^-}(\omega)) \geqslant \alpha.$$

They are equivalent to $B^-_{\overline{1-\alpha}} \subseteq A^+_\alpha$ and $A^-_{\overline{1-\alpha}} \subseteq B^+_\alpha$. Hence, using condition (9)

$$B^-_{\overline{1-\alpha}} \subseteq B^-_{\overline{1-\alpha}} \cup A^-_{\overline{1-\alpha}} = (B^- \cup A^-)_{\overline{1-\alpha}} \subseteq B^+_\alpha$$

and

$$A^-_{\overline{1-\alpha}} \subseteq A^-_{\overline{1-\alpha}} \cup B^-_{\overline{1-\alpha}} = (A^- \cup B^-)_{\overline{1-\alpha}} \subseteq A^+_\alpha.$$

Using the above lemma, we conclude that $(A^- \cup B^-)_{\overline{1-\alpha}} \in \mathscr{A}_\alpha \cap \mathscr{B}_\alpha$ and so we have proved

$$\text{Poss}(B;A) \geqslant \alpha \Rightarrow \Pi(\mathscr{B};\mathscr{A}) \geqslant \alpha, \quad \forall \alpha \in [0,1].$$

This is equivalent to $\Pi(\mathscr{B};\mathscr{A}) \geqslant \text{Poss}(B;A)$, and the inequality is proved. The reverse inequality is easily obtained using the lemma. Indeed $\Pi(\mathscr{B};\mathscr{A}) \geqslant \alpha \Rightarrow \exists\, E \in \mathscr{A}_\alpha \cap \mathscr{B}_\alpha$. Hence, we have $A^-_{\overline{1-\alpha}} \subseteq E \subseteq A^+_\alpha$ and $B^-_{\overline{1-\alpha}} \subseteq E \subseteq B^+_\alpha$. By transitivity we get $B^-_{\overline{1-\alpha}} \subseteq A^+_\alpha$ and $A^-_{\overline{1-\alpha}} \subseteq B^+_\alpha$, which is equivalent to $\text{Poss}(B;A) \geqslant \alpha$. Q.E.D.