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# The Codazzi equation for surfaces ☆

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**Abstract**

In this paper we study the classical Codazzi equation in space forms from an abstract point of view, and use it as an analytic tool to derive global results for surfaces in different ambient spaces. In particular, we study the existence of holomorphic quadratic differentials, the uniqueness of immersed spheres in geometric problems, height estimates, and the geometry and uniqueness of complete or properly embedded Weingarten surfaces.


*Keywords:* Codazzi equation; Holomorphic quadratic differentials; Height estimates; Uniqueness and completeness of surfaces

## 1. Introduction

The *Codazzi equation* for an immersed surface $\Sigma$ in Euclidean 3-space $\mathbb{R}^3$ yields

$$\nabla_X SY - \nabla_Y SX - S[X,Y] = 0, \qquad X, Y \in \mathfrak{X}(\Sigma). \tag{1}$$

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Here $\nabla$ is the Levi-Civita connection of the first fundamental form $I$ of $\Sigma$ and $S$ is the shape operator, defined by $II(X,Y) = I(S(X),Y)$, where $II$ is the second fundamental form of the surface. This Codazzi equation is, together with the Gauss equation, one of the two classical integrability conditions for surfaces in $\mathbb{R}^3$, and it yields true if we substitute the ambient space $\mathbb{R}^3$ by other space forms $\mathbb{S}^3$ or $\mathbb{H}^3$.

It is remarkable that some crucial results of surface theory in $\mathbb{R}^3$ only depend, in essence, on the Codazzi equation. This is the case, for instance, of Hopf's theorem (resp. Liebmann's theorem) on the uniqueness of round spheres among immersed constant mean curvature spheres (resp. among complete surfaces of constant positive curvature). This suggests the possibility of adapting these results to an abstract setting of *Codazzi pairs* (i.e. pairs of real quadratic forms $(I, II)$ on a surface verifying (1), where $I$ is Riemannian), and to explore their possible consequences in surface theory (see, for instance, [5,14,18] and references therein).

The basic idea in this sense is to use the Codazzi pair $(I, II)$ as a geometric object in a *non-standard* way, i.e. so that $(I, II)$ are no longer the first and second fundamental forms of a surface in a space form. This abstract approach has some very definite applications to the study of complete, or properly embedded surfaces in homogeneous spaces. For instance:

1. It reveals the existence of holomorphic quadratic differentials for some classes of surfaces in space forms and, more generally, in homogeneous 3-spaces.
2. It unifies the proofs of apparently non-related theorems.
3. It gives an analytic tool to prove uniqueness results for complete or compact Weingarten surfaces in space forms.

These applications show the flexibility of the use of Codazzi pairs in surface theory, and suggest the possibility of obtaining further global results with the techniques that we employ here.

We have organized this paper as follows. In Section 2 we recall the notions of fundamental pairs and Codazzi pairs, and some of their associated invariants such as the mean curvature $H$, the extrinsic curvature $K$ and the Hopf differential. We shall also recall two basic lemmas by T. Klotz Milnor [18] which relate the concepts of Codazzi pair, constant mean curvature (or positive constant extrinsic curvature) and holomorphic Hopf differential.

In Section 3 we study when a real quadratic form $II$ on a Riemannian surface is conformal to the metric, even if the Codazzi equation is not satisfied. For that, we shall define the *Codazzi function* on a surface associated with its induced metric $I$ and a real quadratic form $II$. This function will measure how far is the pair $(I, II)$ from satisfying the Codazzi equation.

We devote Section 4 to analyze the fundamental relation between the Codazzi equation and the existence of holomorphic quadratic differentials. Thus, given a Codazzi pair on a surface $\Sigma$, we shall find under certain conditions the existence of a new Codazzi pair on $\Sigma$ whose Hopf differential is holomorphic. This new pair will provide geometric information about the initial one.

We shall apply this result to the study of Codazzi pairs of special Weingarten type, that is, pairs satisfying $H = f(H^2 - K)$ for a certain function $f$. The corresponding problem for surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ was studied by Bryant in [7]. We shall also prove that every Codazzi pair on a surface $\Sigma$ satisfying $H = f(H^2 - K)$ can be recovered in terms of a Riemannian metric on $\Sigma$ and a holomorphic quadratic form.

In Section 5 we review some facts on surfaces in the homogeneous product spaces and explain how some Codazzi pairs and fundamental pairs have been used in the classification of spheres with constant mean curvature, complete surfaces with constant Gauss curvature or complete surfaces with positive constant extrinsic curvature in these ambient spaces.

Finally, in Section 6, we give some applications of our abstract approach to surfaces in space forms. We begin by obtaining height estimates for a wide family of surfaces of elliptic type. Although these estimates are not optimal, the existence of such height estimates with respect to planes constitutes a fundamental tool for studying the behavior of complete embedded surfaces.

Following the ideas developed by Rosenberg and Sa Earp in [23], we show that the theory developed by Korevaar, Kusner, Meeks and Solomon [19–21] for constant mean curvature surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ remains valid for some families of surfaces satisfying the maximum principle (Theorem 4).

In particular, when Theorem 4 is applied to a properly embedded Weingarten surface $\Sigma$ of elliptic type satisfying $H = f(H^2 - K)$ in $\mathbb{R}^3$ or $\mathbb{H}^3$, we obtain the following conclusion: If $\Sigma$ has finite topology and $k$ ends, then $k \geqslant 2$. Moreover, $\Sigma$ is rotational if $k = 2$, and it is contained in a slab if $k = 3$.

To finish the paper, we shall study the problem of classifying Weingarten surfaces of elliptic type satisfying $H = f(H^2 - K)$ in $\mathbb{R}^3$, and for which $K$ does not change sign [26]. We shall show that, in the above conditions, we have: (a) if $\Sigma$ is a complete surface with $K \geqslant 0$ then it must be a totally umbilical sphere, a plane or a right circular cylinder, and (b) if $\Sigma$ is properly embedded with $K \leqslant 0$, then it is a right circular cylinder or a surface of minimal type (i.e. $f(0) = 0$).

## 2. Preliminaries

In this section we recall some classical results about fundamental pairs and Codazzi pairs, and establish some notation. For our approach, a classical reference about this topic is [18] (see also [14]). Besides we point out that, although we shall always assume that the differentiability used is always $\mathcal{C}^\infty$, the differentiability requirements are actually much lower.

We shall denote by $\Sigma$ an orientable (and oriented) smooth surface. Otherwise we would work with its oriented two-sheeted covering.

**Definition 1.** A fundamental pair on $\Sigma$ is a pair of real quadratic forms $(I, II)$ on $\Sigma$, where $I$ is a Riemannian metric.

Associated with a fundamental pair $(I, II)$ we define the shape operator $S$ of the pair as

$$II(X, Y) = I\big(S(X), Y\big) \tag{2}$$

for any vector fields $X$, $Y$ on $\Sigma$.

Conversely, it is clear from (2) that the quadratic form $II$ is totally determined by $I$ and $S$. In other words, to give a fundamental pair on $\Sigma$ is equivalent to give a Riemannian metric on $\Sigma$ together with a self-adjoint endomorphism $S$.

We define the *mean curvature*, the *extrinsic curvature* and the *principal curvatures* of $(I, II)$ as one half of the trace, the determinant and the eigenvalues of the endomorphism $S$, respectively.

In particular, given local parameters $(x, y)$ on $\Sigma$ such that

$$I = E\,dx^2 + 2F\,dx\,dy + G\,dy^2, \qquad II = e\,dx^2 + 2f\,dx\,dy + g\,dy^2,$$

the mean curvature and the extrinsic curvature of the pair are given, respectively, by

$$H = H(I, II) = \frac{Eg + Ge - 2Ff}{2(EG - F^2)}, \qquad K = K(I, II) = \frac{eg - f^2}{EG - F^2}.$$

Moreover, the principal curvatures of the pair are $H \pm \sqrt{H^2 - K}$.

We shall say that the pair $(I, II)$ is *umbilical* at $p \in \Sigma$ if $II$ is proportional to $I$ at $p$, or equivalently:

- if both principal curvatures coincide at $p$, or
- if $S$ is proportional to the identity map on the tangent plane at $p$, or
- if $H^2 - K = 0$ at $p$.

We define the *Hopf differential* of the fundamental pair $(I, II)$ as the $(2,0)$-part of $II$ for the Riemannian metric $I$. In other words, if we consider $\Sigma$ as a Riemann surface with respect to the metric $I$ and take a local conformal parameter $z$, then we can write

$$\begin{aligned} I &= 2\lambda\,|dz|^2, \\ II &= Q\,dz^2 + 2\lambda H\,|dz|^2 + \overline{Q}\,d\bar{z}^2. \end{aligned} \tag{3}$$

The quadratic form $Q\,dz^2$, which does not depend on the chosen parameter $z$, is known as the *Hopf differential* of the pair $(I, II)$. We note that $(I, II)$ is umbilical at $p \in \Sigma$ if, and only if, $Q(p) = 0$.

All the above definitions can be understood as natural extensions of the corresponding ones for isometric immersions of a Riemannian surface in a 3-dimensional ambient space, where $I$ plays the role of the induced metric and $II$ the role of its second fundamental form.

A specially interesting case happens when the fundamental pair satisfies, in an abstract way, the Codazzi equation for surfaces in $\mathbb{R}^3$.

**Definition 2.** We say that a fundamental pair $(I, II)$, with shape operator $S$, is a Codazzi pair if

$$\nabla_X SY - \nabla_Y SX - S[X, Y] = 0, \quad X, Y \in \mathfrak{X}(\Sigma), \tag{4}$$

where $\nabla$ stands for the Levi-Civita connection associated with the Riemannian metric $I$ and $\mathfrak{X}(\Sigma)$ is the set of smooth vector fields on $\Sigma$.

Many Codazzi pairs appear in a natural way in the study of surfaces. For instance, the first and second fundamental forms of a surface isometrically immersed in a 3-dimensional space form constitute a Codazzi pair. The same happens for spacelike surfaces in a 3-dimensional Lorentzian space form. More generally, if the surface is immersed in an $n$-dimensional (semi-Riemannian) space form and has a parallel unit normal vector field $N$, then its induced metric and its second fundamental form associated with $N$ constitute a Codazzi pair.

Classically, Codazzi pairs also arise in the study of harmonic maps. Many other examples of Codazzi pairs also appear in [3,5,14,18,22] and references therein. All of this shows that the results on Codazzi pairs can be used in different contexts.

Let us also observe that, by (3) and (4), a fundamental pair $(I, II)$ is a Codazzi pair if and only if

$$Q_{\bar{z}} = \lambda H_z.$$

Thus, one has (see [18]):

**Lemma 1.** *Let $(I, II)$ be a fundamental pair. Then, any two of the conditions* (i)–(iii) *imply the third:*

(i) *$(I, II)$ is a Codazzi pair.*
(ii) *$H$ is constant.*
(iii) *The Hopf differential of the pair is holomorphic.*

We observe that Hopf's theorem [13, p. 138], on the uniqueness of round spheres among immersed constant mean curvature spheres in Euclidean 3-space, can be easily obtained from this result.

For our purposes, we shall need the following lemma [18], which also shows the existence of a geometric holomorphic quadratic form associated with any Codazzi pair with positive constant extrinsic curvature.

**Lemma 2.** *Let $(I, II)$ be a fundamental pair with positive extrinsic curvature $K$. Then, any two of the conditions* (i)–(iii) *imply the third:*

(i) *$(I, II)$ is a Codazzi pair.*
(ii) *$K$ is constant.*
(iii) *The Hopf differential of the new fundamental pair $(II, I)$ is holomorphic. That is, the* $(2,0)$-*part of $I$ with respect to the Riemannian metric $II$ is holomorphic.*

Liebmann theorem, about the characterization of the round spheres among complete surfaces of constant positive curvature in the Euclidean 3-space, can be easily obtained from Lemma 2 (see [16]).

## 3. The Codazzi function

Under certain natural conditions, it is possible to obtain important consequences about a surface endowed with a fundamental pair although the Codazzi equation is not satisfied. In order to study these conditions, next we define a tensor associated with the pair and the *Codazzi function*, which will play an essential role in our study.

**Definition 3.** Given a fundamental pair $(I, II)$ on a surface $\Sigma$ with associated shape operator $S$, we define the tensor $T_S : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ by

$$T_S(X, Y) = \nabla_X SY - \nabla_Y SX - S[X, Y], \quad X, Y \in \mathfrak{X}(\Sigma).$$

Although the definition above has been made in an abstract context, this tensor appears naturally in the study of isometric immersions of surfaces. Specifically, the Codazzi equation of a surface isometrically immersed in a 3-dimensional manifold $M^3$ is

$$\nabla_X SY - \nabla_Y SX - S[X,Y] = -\overline{R}(X,Y)N, \quad X,Y \in \mathfrak{X}(\Sigma).$$

Here $N$ is the unit normal vector field of the immersion, $S$ the associated shape operator and $\overline{R}$ the curvature tensor of $M^3$

$$\overline{R}(X,Y)Z = \overline{\nabla}_X\overline{\nabla}_Y Z - \overline{\nabla}_Y\overline{\nabla}_X Z - \overline{\nabla}_{[X,Y]}Z,$$

where $\overline{\nabla}$ is the Levi-Civita connection of $M^3$.

A straightforward computation shows that the tensor $T_S$ satisfies the following properties:

**Lemma 3.** *Let $(I, II)$ be a fundamental pair on a surface $\Sigma$ with associated shape operator $S$. Then:*

1. *$T_S$ is skew-symmetric, i.e. $T_S(X,Y) = -T_S(Y,X)$ for all $X,Y \in \mathfrak{X}(\Sigma)$.*
2. *$T_S$ is $\mathcal{C}^\infty(\Sigma)$-bilinear, that is,*

$$T_S(f_1X_1 + f_2X_2, Y) = f_1T_S(X_1,Y) + f_2T_S(X_2,Y)$$

*for all vector fields $X_1, X_2, Y \in \mathfrak{X}(\Sigma)$ and all smooth real functions $f_1$, $f_2$.*

3. *Moreover, given vector fields $X,Y \in \mathfrak{X}(\Sigma)$ and a smooth real function $f$ on $\Sigma$, it holds*

$$T_{fS}(X,Y) = fT_S(X,Y) + X(f)SY - Y(f)SX.$$

Associated with the tensor $T_S$ we define the Codazzi function, from which we can measure how distant the pair is from satisfying the Codazzi equation.

**Definition 4.** Let $(I, II)$ be a fundamental pair on a surface $\Sigma$ with associated shape operator $S$. We call the map $\mathcal{T}_S : \Sigma \to \mathbb{R}$ the *Codazzi function* of $(I, II)$, given by

$$I\big(T_S(v_1,v_2), T_S(v_1,v_2)\big) = \mathcal{T}_S(p)\big(I(v_1,v_1)I(v_2,v_2) - I(v_1,v_2)^2\big),$$

where $v_1, v_2 \in T_p\Sigma$, $p \in \Sigma$.

Observe that $\mathcal{T}_S$ is a well-defined smooth function since $T_S$ is skew-symmetric. Besides, $\mathcal{T}_S$ vanishes identically if, and only if, $(I, II)$ is a Codazzi pair.

**Lemma 4.** *Let $(I, II)$ be a fundamental pair on a surface $\Sigma$ with associated shape operator $S$, mean curvature $H$ and extrinsic curvature $K$. Let $z$ be a local conformal parameter for $I$ such that*

$$I = 2\lambda\,|dz|^2, \qquad II = Q\,dz^2 + 2\lambda H\,|dz|^2 + \overline{Q}\,d\bar{z}^2. \tag{5}$$

*Then*

$$|Q_{\bar{z}}|^2 = \frac{\lambda \mathcal{T}_{\widetilde{S}}}{2(H^2 - K)} |Q|^2,$$

*where $\widetilde{S}$ is the traceless operator $S - H\mathrm{id}$. Here, $\mathrm{id}_p$ is the identity map on the tangent plane at $p \in \Sigma$.*

**Proof.** Let $z$ be a local conformal parameter for the Riemannian metric $I$. From (5) the Levi-Civita connection is given by

$$\nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial z} = \frac{\lambda_z}{\lambda} \frac{\partial}{\partial z}, \qquad \nabla_{\frac{\partial}{\partial z}} \frac{\partial}{\partial \bar{z}} = 0$$

and the shape operator $S$ becomes

$$S \frac{\partial}{\partial z} = H \frac{\partial}{\partial z} + \frac{Q}{\lambda} \frac{\partial}{\partial \bar{z}}.$$

Hence,

$$K = H^2 - \frac{|Q|^2}{\lambda^2}. \tag{6}$$

Thus, using the definition of $T_S$ we have

$$\begin{aligned} T_{\widetilde{S}}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) &= \nabla_{\frac{\partial}{\partial z}} \widetilde{S} \frac{\partial}{\partial \bar{z}} - \nabla_{\frac{\partial}{\partial \bar{z}}} \widetilde{S} \frac{\partial}{\partial z} = \nabla_{\frac{\partial}{\partial z}} \frac{\overline{Q}}{\lambda} \frac{\partial}{\partial z} - \nabla_{\frac{\partial}{\partial \bar{z}}} \frac{Q}{\lambda} \frac{\partial}{\partial \bar{z}} \\ &= \frac{1}{\lambda}\left(\overline{Q}_z \frac{\partial}{\partial z} - Q_{\bar{z}} \frac{\partial}{\partial \bar{z}}\right). \end{aligned} \tag{7}$$

Therefore, the Codazzi function satisfies $\frac{2}{\lambda}|Q_{\bar{z}}|^2 = \mathcal{T}_{\widetilde{S}}\lambda^2$, and the proof finishes using (6). □

Given a fundamental pair $(I, II)$ on a surface $\Sigma$, we shall denote by $\Sigma_U \subseteq \Sigma$ the set of umbilical points of the pair. Then we have

**Theorem 1.** *Let $(I, II)$ be a fundamental pair on a surface $\Sigma$ with associated shape operator $S$, mean curvature $H$ and extrinsic curvature $K$. Let us suppose that every point $p \in \partial\Sigma_U$ has a neighborhood $V_p$ such that*

$$\frac{\mathcal{T}_{\widetilde{S}}}{H^2 - K} \quad \text{is bounded in } V_p \cap (\Sigma - \Sigma_U),$$

*where $\widetilde{S} = S - H\mathrm{id}$. Then either the Hopf differential of $(I, II)$ vanishes identically or its zeroes are isolated and of negative index.*

*In particular, if $\Sigma$ is a topological sphere then the pair is totally umbilical.*

**Proof.** If $p \in \partial \Sigma_U$, then there exists an open neighborhood $V_p$ and a constant $m_0$ such that $\frac{\mathcal{T}_{\widetilde{S}}}{H^2-K} \leqslant m_0$ in $V_p \cap (\Sigma - \Sigma_U)$. Thus, using Lemma 4,

$$|Q_{\bar{z}}|^2 \leqslant m_0 \frac{\lambda}{2} |Q|^2 \tag{8}$$

in $V_p \cap (\Sigma - \Sigma_U)$. Besides, since this inequality is also satisfied in the interior of $\Sigma_U$, we conclude that (8) holds in $V_p$.

Therefore, using [15, Lemma 2.7.1], we have that $p$ is an isolated zero of negative index of the Hopf differential, as we wanted to prove.

In particular, if $\Sigma$ is a topological sphere, we get by the Poincaré index theorem that the Hopf differential $Q\,dz^2$ must vanish identically on $\Sigma$. Therefore, the pair is totally umbilical. □

**Remark 1.** The above result can be globally used not only for topological spheres. Indeed, if $\Sigma$ is a topological torus that satisfies the assumptions of Theorem 1, then we deduce that the pair $(I, II)$ is either totally umbilical or umbilically free. Analogously, if $\Sigma$ is a closed topological disk and its boundary $\partial\Sigma$ is a line of curvature for $(I, II)$, then the pair is totally umbilical. (See [9] for the case of surfaces with non-regular boundary and applications to homogeneous spaces.)

If $(I, II)$ is a Codazzi pair we have

$$\mathcal{T}_{\widetilde{S}}\left(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}}\right) = -\nabla_{\frac{\partial}{\partial z}} H \frac{\partial}{\partial \bar{z}} + \nabla_{\frac{\partial}{\partial \bar{z}}} H \frac{\partial}{\partial z} = H_{\bar{z}} \frac{\partial}{\partial z} - H_z \frac{\partial}{\partial \bar{z}}.$$

Therefore, its Codazzi function is

$$\mathcal{T}_{\widetilde{S}} = \frac{2}{\lambda} |H_z|^2 = \|\nabla H\|^2,$$

where $\|\nabla H\|$ stands for the modulus of the gradient of $H$ for the Riemannian metric $I$. Thus, Theorem 1 can be applied for Codazzi pairs whenever the quotient $\|\nabla H\|^2/(H^2 - K)$ is locally bounded.

However, as we will see in Section 5, this result can be also applied to fundamental pairs which are not Codazzi pairs.

## 4. Holomorphic quadratic differentials

In this section we shall see that, under certain assumptions on a Codazzi pair, it is possible to obtain a new Codazzi pair with vanishing constant mean curvature which is geometrically related to the first one. By using this second Codazzi pair, we shall show the existence of a holomorphic quadratic differential which will provide important information on the geometric behavior of the initial pair.

If $(u, v)$ are doubly orthogonal parameters for a fundamental pair $(I, II)$, then we can write

$$I = E\,du^2 + G\,dv^2, \qquad II = k_1 E\,du^2 + k_2 G\,dv^2,$$

where $k_1, k_2$ are the principal curvatures of the pair.

Hence, if $S$ is the shape operator of the pair, the tensor $T_S$ can be expressed as

$$\begin{aligned} T_S\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) &= \nabla_{\frac{\partial}{\partial u}} S\frac{\partial}{\partial v} - \nabla_{\frac{\partial}{\partial v}} S\frac{\partial}{\partial u} = \nabla_{\frac{\partial}{\partial u}} k_2\frac{\partial}{\partial v} - \nabla_{\frac{\partial}{\partial v}} k_1\frac{\partial}{\partial u} \\ &= (k_2)_u\frac{\partial}{\partial v} - (k_1)_v\frac{\partial}{\partial u} + (k_2 - k_1)\left(\frac{E_v}{2E}\frac{\partial}{\partial u} + \frac{G_u}{2G}\frac{\partial}{\partial v}\right) \\ &= -\frac{1}{E}\big((k_1E)_v - HE_v\big)\frac{\partial}{\partial u} + \frac{1}{G}\big((k_2G)_u - HG_u\big)\frac{\partial}{\partial v}, \end{aligned} \tag{9}$$

where $H$ is the mean curvature of $(I, II)$.

We observe that doubly orthogonal parameters exist in a neighborhood of every non-umbilical point, as well as in a neighborhood of every point in the interior of $\Sigma_U = \{\text{umbilical points of } (I, II)\}$. Thus, the set of points around which doubly orthogonal parameters exist is dense in $\Sigma$. Consequently, all the properties that we prove by using this kind of parameters will be extended to the whole surface by continuity.

In what follows, let $II'$ denote the quadratic form given by

$$II' = II - HI,$$

where $(I, II)$ is a fundamental pair.

**Lemma 5.** *Let $(I, II)$ be a Codazzi pair on a surface $\Sigma$ with mean and extrinsic curvatures $H$ and $K$, respectively. Let $\varphi$ be a smooth function on $\Sigma$ such that the function $\sinh\varphi/\sqrt{H^2 - K}$ can be smoothly extended to $\Sigma$. Then*

$$A = \cosh\varphi I + \frac{\sinh\varphi}{\sqrt{H^2 - K}} II',$$
$$B = \sqrt{H^2 - K}\sinh\varphi I + \cosh\varphi II',$$

*is a fundamental pair with mean curvature $H(A, B) = 0$ and extrinsic curvature $K(A, B) = -(H^2 - K)$. Moreover, if we denote by $\widetilde{S}$ the shape operator associated with $(A, B)$, then*

$$T_{\widetilde{S}}(X, Y) = \omega(Y)X - \omega(X)Y, \qquad \omega := \frac{1}{2}\big(dH - \sqrt{H^2 - K}\, d\varphi\big), \tag{10}$$

*for all $X, Y \in \mathfrak{X}(\Sigma)$.*

**Proof.** Let $(u, v)$ be doubly orthogonal parameters for the Codazzi pair $(I, II)$ such that

$$I = E\,du^2 + G\,dv^2, \qquad II = k_1E\,du^2 + k_2G\,dv^2,$$

being $k_1 \geqslant k_2$.

Then we can write $(A, B)$ as

$$A = e^{\varphi}E\,du^2 + e^{-\varphi}G\,dv^2, \qquad B = \frac{k_1 - k_2}{2}\big(e^{\varphi}E\,du^2 - e^{-\varphi}G\,dv^2\big).$$

Hence, $A$ is a Riemannian metric on $\Sigma$ and the mean and extrinsic curvatures of the pair are given by $H(A,B)=0$ and $K(A,B)=-(H^2-K)$.

In addition, from (9) and taking into account that $(I,II)$ is a Codazzi pair, we get

$$\begin{aligned}
T_{\widetilde{S}}\left(\frac{\partial}{\partial u},\frac{\partial}{\partial v}\right) &= -\frac{1}{e^{\varphi}E}\left(\frac{k_1-k_2}{2}\left(e^{\varphi}E\right)\right)_v\frac{\partial}{\partial u}-\frac{1}{e^{-\varphi}G}\left(\frac{k_1-k_2}{2}\left(e^{-\varphi}G\right)\right)_u\frac{\partial}{\partial v}\\
&= -\frac{1}{2}\big((k_1)_v-(k_2)_v+(k_1-k_2)\varphi_v-2(k_1)_v\big)\frac{\partial}{\partial u}\\
&\quad -\frac{1}{2}\big((k_1)_u-(k_2)_u-(k_1-k_2)\varphi_u+2(k_2)_u\big)\frac{\partial}{\partial v}\\
&= \omega\left(\frac{\partial}{\partial v}\right)\frac{\partial}{\partial u}-\omega\left(\frac{\partial}{\partial u}\right)\frac{\partial}{\partial v}.
\end{aligned}$$

Finally, we obtain (10) from the linearity of $T_{\widetilde{S}}$. □

Under the assumptions of Lemma 5, if $z$ is a conformal parameter for $A$ and we consider

$$A=2\lambda\,|dz|^2,\qquad B=Q\,dz^2+\overline{Q}\,d\bar{z}^2, \tag{11}$$

then from (7) and (10) we get

$$Q_{\bar{z}}=\lambda\omega\left(\frac{\partial}{\partial z}\right)=\frac{\lambda}{2}\big(H_z-\sqrt{H^2-K}\varphi_z\big).$$

Moreover, we obtain from (10) that the pair $(A,B)$ given by (11) is a Codazzi pair if and only if $dH-\sqrt{H^2-K}\,d\varphi=0$, or equivalently $Q_{\bar{z}}=0$.

With all of this we have

**Corollary 1.** *Let $(I,II)$ be a Codazzi pair on a surface $\Sigma$ with mean and extrinsic curvatures $H$ and $K$, respectively. Let $\varphi$ be a smooth function on $\Sigma$ such that the function $\sinh\varphi/\sqrt{H^2-K}$ can be smoothly extended to $\Sigma$. Then the fundamental pair*

$$\begin{aligned}
A&=\cosh\varphi I+\frac{\sinh\varphi}{\sqrt{H^2-K}}II',\\
B&=\sqrt{H^2-K}\,\sinh\varphi I+\cosh\varphi II',
\end{aligned}\tag{12}$$

*has mean curvature $H(A,B)=0$ and extrinsic curvature $K(A,B)=-(H^2-K)$. In addition, the following conditions are equivalent:*

- *$(A,B)$ is a Codazzi pair,*
- *the Hopf differential of $(A,B)$ is holomorphic for the conformal structure induced by $A$,*
- *$dH-\sqrt{H^2-K}\,d\varphi=0$.*

Next, we see some situations where the corollary above can be used. In order to do that, and following the classical notation, we give the following definition.

**Definition 5.** We say that a Codazzi pair $(I, II)$ is a special Weingarten pair if there exists a smooth function $f$ defined on an interval $\mathcal{J} \subseteq [0, \infty)$ such that its mean curvature $H$ and extrinsic curvature $K$ satisfy

$$H = f\big(H^2 - K\big).$$

Now let us assume that the mean and extrinsic curvatures of a Codazzi pair $(I, II)$ satisfy a general Weingarten relation $W(H, K) = 0$, where $W$ is a smooth function defined on an open set of $\mathbb{R}^2$ containing the set of points $\{(H(p), K(p))\colon\ p \in \Sigma\}$. Let us parametrize $H = H(t)$, $K = K(t)$, for $t$ varying in a certain interval. Then, if we look for a solution of the type $\varphi = \varphi(t)$ for the previous equation $dH - \sqrt{H^2 - K}\, d\varphi = 0$, we have

$$\sqrt{H(t)^2 - K(t)}\varphi'(t) = H'(t).$$

Therefore, if there exists a primitive $\varphi(t)$ of the function

$$\frac{H'(t)}{\sqrt{H(t)^2 - K(t)}}$$

for which $\sinh\varphi(t)/\sqrt{H(t)^2 - K(t)}$ is well defined even at the umbilical points, then the Codazzi pair $(A, B)$ given as (12) will exist on the whole surface $\Sigma$.

In addition, if $(I, II)$ is a special Weingarten pair satisfying $H = f(H^2 - K)$, then we can consider $H^2 - K = t^2$ and $H = f(t^2)$ as above. This allows us to take

$$\varphi(t) = \int_0^t 2f'\big(s^2\big)\, ds \tag{13}$$

whenever there exist umbilical points (i.e. $t = 0$ has sense), or any primitive of $2f'(t^2)$ otherwise.

Thus, for special Weingarten pairs, the metric $A$ defined as in Corollary 1 is always well defined, because so is the function $\sinh\varphi(t)/t$.

The metric $A$ for special Weingarten surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ was first defined by R.L. Bryant in [7]. In that work, he also found a holomorphic quadratic form for the metric $A$ which agrees with the Hopf differential of the pair $(A, B)$. Using this, Bryant provided an easy proof of the fact that every topological sphere in $\mathbb{R}^3$ or $\mathbb{H}^3$ satisfying a special Weingarten relation must be a totally umbilical round sphere.

The abstract formulation which we have adopted in this work allows us to extend this result to general special Weingarten surfaces.

Another remarkable consequence of our abstract approach is that we can describe all special Weingarten pairs as follows:

**Corollary 2.** *Let $\Sigma$ be a surface and $f$ a smooth function defined on an interval $\mathcal{J} \subseteq [0, \infty)$. Let us take a primitive $\varphi(t)$ of $2f'(t^2)$ on that interval such that the function* $\sinh\varphi(t)/t$ *is well defined. Then every special Weingarten pair $(I, II)$ on $\Sigma$ satisfying $H = f(H^2 - K)$ is given by*

$$I = -\frac{\sinh\varphi(t)}{t}Q + \cosh\varphi(t)A - \frac{\sinh\varphi(t)}{t}\overline{Q},$$
$$II - f\left(t^2\right)I = -\cosh\varphi(t)Q + t\sinh\varphi(t)A - \cosh\varphi(t)\overline{Q}, \tag{14}$$

*where $A$ is a Riemannian metric on $\Sigma$ and $Q$ is a holomorphic 2-form for $A$ such that the image of the function $t:\Sigma \to [0,\infty)$ defined as $2|Q| = tA$ is contained in $\mathcal{J}$. In particular, $t^2 = H^2 - K$.*

**Proof.** It suffices to observe that given a special Weingarten pair $(I, II)$, if we take $H^2 - K = t^2$ (and therefore $H = f(t^2)$), we have already proved that there exists a primitive $\varphi = \varphi(t)$ of $2f'(t^2)$ in the conditions of Corollary 1. Thus, there exists a fundamental pair $(A, B)$ for which $B$ can be written as $B = Q + \overline{Q}$, since $H(A, B) = 0$. Besides, the Hopf differential $Q$ of $(A, B)$ is a holomorphic 2-form for the metric $A$ and

$$t^2 = H^2 - K = -K(A, B) = 4\frac{|Q|^2}{|A|^2}.$$

So, using (12), we can see that the pair $(I, II)$ can be written as (14) in terms of the pair $(A, B)$. Finally, it is a straightforward computation to check that any pair $(A, B)$ as above gives a Codazzi pair $(I, II)$ which is special Weingarten. □

## 5. Surfaces in homogeneous product spaces

A remarkable theorem by U. Abresch and H. Rosenberg showed that every surface with constant mean curvature in the homogeneous product spaces $\mathbb{S}^2 \times \mathbb{R}$ or $\mathbb{H}^2 \times \mathbb{R}$ has a holomorphic quadratic differential (see [1] and also [2]). This fact was used to classify the spheres with constant mean curvature in these spaces, and has inspired many research works on surfaces in these ambient spaces. In this section we explain how some natural Codazzi pairs appear in the context of homogeneous product spaces.

Let $\mathbb{M}^2(\varepsilon)$ denote $\mathbb{H}^2$ or $\mathbb{S}^2$ depending on whether $\varepsilon = -1$ or $\varepsilon = 1$, respectively. Consider a surface $\Sigma$ and an immersion $\psi : \Sigma \to \mathbb{M}^2(\varepsilon) \times \mathbb{R}$, with unit normal vector field $N$. Let $\pi : \mathbb{M}^2(\varepsilon) \times \mathbb{R} \to \mathbb{R}$ be the usual projection and consider the *height function* $h := \pi \circ \psi$ and the *angle function* $\nu := d\pi(N)$ on the surface $\Sigma$.

If $\psi$ is an immersion with constant mean curvature $H$, then it was proved by Abresch and Rosenberg [1] that the quadratic differential

$$\left(2HQ - \varepsilon h_z^2\right)dz^2$$

is holomorphic. Here, $z$ is a conformal parameter for the induced metric $I$ and $Q\,dz^2$ is the Hopf differential of the pair $(I, II)$.

The real quadratic form

$$B = 2HII - \varepsilon\,dh^2 + \frac{\varepsilon}{4}\left(1 - \nu^2\right)I$$

is then locally given by

$$B = \left(2HQ - \varepsilon h_z^2\right)dz^2 + 4H^2\lambda\,|dz|^2 + \left(2H\overline{Q} - \varepsilon h_{\bar z}^2\right)d\bar z^2,$$

where we have used [4, Eq. 2.2].

Thus, since the mean curvature of the pair $(I, B)$ is given by $H(I, B) = 2H^2$ then $(I, B)$ is a Codazzi pair from Lemma 1. It is expected that this pair could play an important role in the research of surfaces with constant mean curvature in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$.

For an immersion $\psi : \Sigma \to \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ with constant Gauss curvature $K(I)$, a Codazzi pair is found when the quadratic form

$$A = I + \frac{1}{\varepsilon K(I) - 1}\, dh^2$$

is Riemannian [3]. In particular, if $K(I) \notin [0, 1]$ in $\mathbb{S}^2 \times \mathbb{R}$ or $K(I) \notin [-1, 0]$ in $\mathbb{H}^2 \times \mathbb{R}$, then $A$ is a Riemannian metric.

The pair $(A, II)$ is a Codazzi pair with constant extrinsic curvature $K(A, II) = K(I) - \varepsilon$. Thus, from Lemma 2, if $K(I) > 1$ in $\mathbb{S}^2 \times \mathbb{R}$ or if $K(I) > 0$ in $\mathbb{H}^2 \times \mathbb{R}$, the $(2, 0)$-part of $A$ with respect to $II$ is holomorphic. This fact was used in [3] in order to classify the complete surfaces of constant Gauss curvature. In addition, the existence of a Codazzi pair for constant $K(I) < -1$ was used to obtain a Hilbert type theorem in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ (see [3]).

The case of immersions in $\mathbb{M}^2(\varepsilon) \times \mathbb{R}$ with positive constant extrinsic curvature is of special interest since a Codazzi pair has not yet been found, but there exists a fundamental pair for which Theorem 1 applies.

In fact, if $\psi : \Sigma \to \mathbb{M}^2(\varepsilon) \times \mathbb{R}$ is an immersion with positive constant extrinsic curvature $K = K(I, II)$ then $(II, C)$ is a fundamental pair where

$$C = I + g(\nu)\, dh^2.$$

Here, $g : \mathbb{R} \to \mathbb{R}$ is the analytic function given by

$$g(t) = \frac{t^2 - 1 + \varepsilon K (e^{\frac{\varepsilon(1-t^2)}{K}} - 1)}{(1 - t^2)^2}.$$

From [10, Lemma 7.1] this pair fulfills the conditions of Theorem 1. This fact was used in order to classify the complete immersions in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ with positive constant extrinsic curvature [10].

## 6. Applications to space forms

In this section we focus our attention on surfaces in space forms. We shall obtain several results as a consequence of the previous study.

We start by giving a geometrical argument for obtaining height bounds for a large class of families of surfaces which satisfy a maximum principle. The proof is based on some ideas used in [10].

**Definition 6.** We say that a family $\mathcal{A}$ of oriented surfaces in $\mathbb{R}^3$ satisfies the Hopf maximum principle if the following properties are fulfilled:

1. $\mathcal{A}$ is invariant under isometries of $\mathbb{R}^3$. In other words, if $\Sigma \in \mathcal{A}$ and $\varphi$ is an isometry of $\mathbb{R}^3$, then $\varphi(\Sigma) \in \mathcal{A}$.
2. If $\Sigma \in \mathcal{A}$ and $\widetilde{\Sigma}$ is another surface contained in $\Sigma$, then $\widetilde{\Sigma} \in \mathcal{A}$.

3. There is an embedded compact surface without boundary in $\mathcal{A}$.
4. Any two surfaces in $\mathcal{A}$ satisfy the maximum principle (interior and boundary).

Note that a large amount of families of surfaces verify the Hopf maximum principle. Classical examples of this fact are the family of surfaces with constant mean curvature $H \neq 0$ and the family of surfaces with positive constant extrinsic curvature $K$. And, more generally, the family of special Weingarten surfaces in $\mathbb{R}^3$ satisfying a relation of the type $H = f(H^2 - K)$, where $f$ is a smooth function defined on an interval $\mathcal{J} \subseteq [0, \infty)$ with $0 \in \mathcal{J}$, such that $f(0) \neq 0$ and $4tf'(t)^2 < 1$ for all $t \in \mathcal{J}$ (see [23]).

We also point out that if a family of surfaces $\mathcal{A}$ satisfies the Hopf maximum principle, then there exists, up to isometries of $\mathbb{R}^3$, a unique embedded compact surface $\Sigma$ without boundary in $\mathcal{A}$. Such surface is, necessarily, a totally umbilical sphere.

To see this, it suffices to observe that the Alexandrov reflection principle works for surfaces in $\mathcal{A}$. Thus, for every plane $P \subseteq \mathbb{R}^3$ there exists a plane, parallel to $P$, which is a symmetry plane of $\Sigma$. Therefore, $\Sigma$ is a round sphere.

In addition, there cannot be two totally umbilical spheres $\Sigma_1, \Sigma_2$ in $\mathcal{A}$ which are non-isometric. Otherwise, up to isometries, we can suppose that one of them, let us say $\Sigma_1$, is contained in the bounded region determined by $\Sigma_2$. If we move $\Sigma_1$ until it first meets $\Sigma_2$ and at the contact point the normal vectors to $\Sigma_1$, $\Sigma_2$ coincide, from the maximum principle we can conclude that $\Sigma_1 = \Sigma_2$. If the normal vectors at that point do not coincide, we keep on moving $\Sigma_1$ in such a way that its center moves along a half-line, until it meets $\Sigma_2$ at the last contact point. At that contact point the normal vectors do necessarily coincide, which allows us, as before, to assert that $\Sigma_1 = \Sigma_2$.

Now, let us see that there exists a constant $c_{\mathcal{A}}$ such that for all compact embedded surfaces $\Sigma \in \mathcal{A}$ whose boundary is contained in a plane $P$, the maximum distance from a point $p \in \Sigma$ to $P$ is bounded by $c_{\mathcal{A}}$. This bound only depends on the radius of the unique totally umbilical sphere contained in the family $\mathcal{A}$.

Although we shall not provide optimal estimates here, the existence of such height estimates with respect to planes will allow us to get interesting consequences regarding several aspects of embedded surfaces in $\mathbb{R}^3$ (see [19–21,23]).

We shall start by studying graphs $\Sigma$ with boundary contained in a plane $P$ of $\mathbb{R}^3$. Up to an isometry, we can assume that $P$ is the $xy$-plane, and so

$$\Sigma = \bigl\{\bigl(x, y, u(x, y)\bigr) \in \mathbb{R}^3\colon\ (x, y) \in \Omega \subseteq \mathbb{R}^2\bigr\}.$$

**Theorem 2.** *Let $\mathcal{A}$ be a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf maximum principle, and let $\Sigma \in \mathcal{A}$ be a compact graph on a domain $\Omega$ in the $xy$-plane with $\partial\Sigma$ contained in this plane. Then for all $p \in \Sigma$, the distance in $\mathbb{R}^3$ from $p$ to the $xy$-plane is less than or equal to $4R_{\mathcal{A}}$. Here, $R_{\mathcal{A}}$ stands for the radius of the unique totally umbilical sphere in the family $\mathcal{A}$.*

**Proof.** Let $\Sigma \in \mathcal{A}$ be a graph on a domain $\Omega$ in the $xy$-plane and let $\Sigma_0$ be the unique totally umbilical sphere of $\mathcal{A}$. Let $P(t)$ be the foliation of $\mathbb{R}^3$ by horizontal planes, where $P(t)$ is the plane at height $t$.

**Claim 1.** *For every $t > 2R_{\mathcal{A}}$, the diameter of any open connected component bounded by $\Sigma(t) = P(t) \cap \Sigma$ is less than or equal to $2R_{\mathcal{A}}$.*

Indeed, let us suppose that this claim is not true. Then, for some connected component $C(t)$ of $\Sigma(t)$, there are points $p$, $q$ in the interior of the domain $\Omega(t)$ in $P(t)$ bounded by $C(t)$ such that $\operatorname{dist}(p,q) > 2R_{\mathcal{A}}$. Let $Q$ be the domain in $\mathbb{R}^3$ bounded by $\Sigma \cup \Omega$. Let $\beta$ be a curve in $\Omega(t)$ joining $p$ and $q$, and so that $\beta$ and $C(t)$ are disjoint.

Let $\Pi$ be the *rectangle* given by

$$\Pi = \{\alpha_s(r)\colon\ s \in \mathcal{I},\ r \in [0,t]\}$$

where $\mathcal{I}$ is the interval where $\beta$ is defined, and $\alpha_s$ is the geodesic with initial data $\alpha_s(0) = \beta(s)$ and $\alpha_s'(0) = -e_3$, $r$ being the arc length parameter along $\alpha_s$ and $e_3 = (0,0,1)$.

Since $\Sigma$ is a graph and $\beta$ is contained in the interior of the domain determined by $C(t)$, then $\Pi \subset Q$. Let $\widetilde{p} \in \Pi$ be a point whose distance to $\partial\Pi$ is greater than $R_{\mathcal{A}}$. Note that, according to our construction of $\Pi$, the point $\widetilde{p}$ necessarily exists.

Let $\eta(r)$ be a horizontal geodesic passing through $\widetilde{p}$ and such that every point in $\eta(r)$ is at a distance from $\partial\Pi$ greater than $R_{\mathcal{A}}$. Observe that such a geodesic can be chosen as the horizontal line in the plane $P(t_1)$ that contains the point $\widetilde{p}$ and is orthogonal to the segment joining $p$ and $q$. Let $\widetilde{q_1}$ be the first point where $\eta$ meets $Q$, and let $\widetilde{q_2}$ be the last one.

Now, let us consider at each point of $\eta(r)$ the sphere $\Sigma_0(r) \in \mathcal{A}$ centered at $\eta(r)$. Note that these spheres can be obtained from the rotational sphere $\Sigma_0$ by means of a translation of $\mathbb{R}^3$.

Then, there exists a first sphere in this family (coming from $\widetilde{q_1}$) which meets $\Sigma$. If the normal vectors of both surfaces coincide at this point, we conclude that both surfaces agree by the maximum principle. On the other hand, if the normal vectors are opposite, we argue as follows.

Let us consider the first sphere $\Sigma_0(r_0)$ in the family above (coming from $\widetilde{q_1}$) which meets $\Pi$ at an interior point of $\Pi$.

For every $r > r_0$ we consider the piece $\widetilde{\Sigma_0}(r)$ of the sphere $\Sigma_0(r)$ which has gone through $\Pi$. Since these spheres leave $Q$ at $\widetilde{q_2}$ and none of them meets $\partial\Pi$, there exists a first value $r_1$ such that $\widetilde{\Sigma_0}(r_1)$ first meets $\partial Q \cap \Sigma$ at a point $\widetilde{q_0}$. Thus, applying the maximum principle to $\Sigma_0(r_1)$ and $\Sigma$ at $\widetilde{q_0}$, we conclude that both surfaces agree, which is a contradiction.

Therefore we obtain that, for the height $t = 2R_{\mathcal{A}}$, the diameter of every open connected component bounded by $\Sigma(t) = P(t) \cap \Sigma$ is less than or equal to $2R_{\mathcal{A}}$.

To finish, we shall see that $P(t) \cap \Sigma$ is empty for $t > 4R_{\mathcal{A}}$. To do that, it suffices to prove the following assertion:

**Claim 2.** *Let $\Omega_1$ be a connected component bounded by $\Sigma(2R_{\mathcal{A}})$ in $P(2R_{\mathcal{A}})$. Then, the distance from any point in $\Sigma$ (which is a graph on $\Omega_1$) to the plane $P(2R_{\mathcal{A}})$ is less than or equal to the diameter of $\Omega_1$.*

Let $\sigma$ be a support line of $\partial\Omega_1$ in $P(2R_{\mathcal{A}})$ with exterior unit normal vector $v$, and let us take $\eta(r)$ a geodesic such that $\eta(0) \in \sigma$ and $\eta'(0) = \frac{1}{\sqrt{2}}(v + e_3)$.

Now, let us consider for every $r$ the plane $\Pi(r)$ in $\mathbb{R}^3$ passing through $\eta(r)$ which is orthogonal to $\eta'(r) = \eta'(0)$. Such planes intersect every horizontal plane in a line parallel to $\sigma$, being $\pi/4$ the angle between them.

If Claim 2 was not true, there would exist a point $p \in \Sigma$ over $\Omega_1$ whose height over the plane $P(2R_{\mathcal{A}})$ would be greater than the diameter of $\Omega_1$.

Let $\Sigma_1$ be the compact piece of $\Sigma$ which is a graph on $\Omega_1$. Observe that, for $r$ big enough, $\Pi(r)$ does not meet $\Sigma_1$. In addition, for $r = 0$ the plane $\Pi(0)$ contains the line $\sigma$, and the reflection of $p$ with respect to $\Pi(0)$ is a point whose vertical projection on $P(2R_{\mathcal{A}})$ is not in $\Omega_1$. Therefore, using the Alexandrov reflection principle (see for instance [13]) for the planes $\Pi(r)$ with $r$ coming from infinity, there exists a first value $r_0 > 0$ such that: (a) the reflection of the piece of $\Sigma_1$ which is over $\Pi(r)$ meets first $\Sigma_1$ at an interior point, or (b) both surfaces are tangent at a point in the boundary. But any of these is a contradiction, by the maximum principle.

This finishes the proof. $\square$

As a consequence of this result, we are able to bound the maximum height attained by an embedded compact surface whose boundary is contained in a plane.

**Corollary 3.** *Let $\mathcal{A}$ be a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf maximum principle. Then every embedded compact surface $\Sigma \in \mathcal{A}$ whose boundary is contained in a plane $P$ verifies that for every $p \in \Sigma$ the distance in $\mathbb{R}^3$ from $p$ to the plane $P$ is less than or equal to $8R_{\mathcal{A}}$. Here, $R_{\mathcal{A}}$ denotes the radius of the unique totally umbilical sphere contained in $\mathcal{A}$.*

This result follows from Theorem 2 as a standard consequence of the Alexandrov reflection principle for planes parallel to $P$.

**Remark 1.** The techniques used in Theorem 2 and Corollary 3 are valid not only in $\mathbb{R}^3$, but also more generally for hypersurfaces in $\mathbb{R}^n$. Even more, they can easily be adapted to study hypersurfaces in $\mathbb{H}^n$.

The existence of a maximum principle and height estimates with respect to planes for a family of surfaces $\mathcal{A}$ allows us to extend the theory developed by Korevaar, Kusner, Meeks and Solomon [19–21] for constant mean curvature surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ to our family $\mathcal{A}$. On the other hand, in [23] Rosenberg and Sa Earp showed that those techniques are also suitable to study some families of surfaces satisfying a relation of the type $H = f(H^2 - K)$. Actually, they do not use that the surfaces satisfy $H = f(H^2 - K)$, but only that they satisfy the Hopf maximum principle and that there exist height estimates for them. Thus, following [23] we get

**Theorem 3** *(Cylindrical bounds). Let $\mathcal{A}$ be a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf maximum principle and let $\Sigma \in \mathcal{A}$ be an annulus (i.e. $\Sigma$ is homeomorphic to a punctured closed disk of $\mathbb{R}^2$). If $\Sigma$ is properly embedded, then it is contained in a half-cylinder of $\mathbb{R}^3$.*

A unit vector $v \in \mathbb{S}^2$ is said to be an axial vector for $\Sigma \subseteq \mathbb{R}^3$ if there exists a sequence of points $p_n \in \Sigma$ such that $|p_n| \to \infty$ and $p_n/|p_n| \to v$. In particular, the theorem above asserts that for any properly embedded annulus there exists a unique axial vector. In addition, this vector is the generator of the rulings of the cylinder.

Finally, following [23] for properly embedded complete surfaces, we have

**Theorem 4.** *Let $\mathcal{A}$ be a family of surfaces in $\mathbb{R}^3$ satisfying the Hopf maximum principle. If $\Sigma \in \mathcal{A}$ is a properly embedded surface with finite topology in $\mathbb{R}^3$, then every end of $\Sigma$ is cylindrically bounded. Moreover, if $a_1, \ldots, a_k$ are the $k$ axial vectors corresponding to the ends, then these vectors cannot be contained in an open hemisphere of $\mathbb{S}^2$. In particular:*

- *$k = 1$ is impossible.*
- *If $k = 2$, then $\Sigma$ is contained in a cylinder and is a rotational surface with respect to a line parallel to the axis of the cylinder.*
- *If $k = 3$, then $\Sigma$ is contained in a slab.*

**Definition 7.** Let $(I, II)$ be a Codazzi pair on a surface $\Sigma$. We shall say that the pair is special Weingarten of elliptic type if its mean and extrinsic curvatures $H$ and $K$ satisfy that $H = f(H^2 - K)$, where $f$ is a smooth function defined on $[0, a)$, $0 < a \leqslant \infty$, such that

$$4tf'(t)^2 < 1$$

for all $t \in [0, a)$.

It was proved by Rosenberg and Sa Earp [23] (see also [6]) that the set of Weingarten surfaces of elliptic type in $\mathbb{R}^3$ and $\mathbb{H}^3$ with $f(0) \neq 0$ is a family satisfying the Hopf maximum principle. Thus, the above theorems are true for this kind of surfaces. Actually these results were also proved in [23], although under the additional hypothesis $f'(t)(1 - 2f(t)f'(t)) \geqslant 0$.

The special Weingarten surfaces in $\mathbb{R}^3$ and $\mathbb{H}^3$ satisfying $H = f(H^2 - K)$ have been widely studied. In particular, an exhaustive study of the rotational surfaces was developed by Sa Earp and Toubiana [24–27].

An open problem of this context, posed in [26], is the classification of the surfaces satisfying $H = f(H^2 - K)$ whose extrinsic curvature does not change sign. More specifically, the problem is if such surfaces are totally umbilical spheres, cylinders or surfaces of minimal type (i.e. with $f(0) = 0$). Observe that this fact is known for surfaces with constant mean curvature. In fact, a minimal surface has non-positive extrinsic curvature at every point, and a complete surface with non-zero constant mean curvature and whose extrinsic curvature does not change sign, must be a sphere or a cylinder [12,17].

Next, and as a consequence of the study developed for Codazzi pairs, we study the problem above for the general case of special Weingarten surfaces of elliptic type.

**Theorem 5.** *Let $\Sigma$ be a special Weingarten surface of elliptic type in $\mathbb{R}^3$ satisfying $H = f(H^2 - K)$. Let us suppose that its extrinsic curvature does not change sign.*

1. *If $\Sigma$ is complete and $K \geqslant 0$ at every point, then $\Sigma$ is a totally umbilical sphere, a plane or a right circular cylinder.*
2. *If $\Sigma$ is properly embedded and $K \leqslant 0$ at every point, then $\Sigma$ is either a right circular cylinder or a surface of minimal type (i.e. $f(0) = 0$).*

In order to prove this theorem, we shall first establish the following general lemma for Codazzi pairs.

**Lemma 6.** *Let $(I, II)$ be a special Weingarten pair of elliptic type on a surface $\Sigma$, with mean and extrinsic curvatures $H$ and $K$, respectively. If $H^2 - K \neq 0$ on $\Sigma$, then the new metric*

$$g_0 = \sqrt{H^2 - K}\,A$$

*is a flat metric on $\Sigma$. Here, $A$ is the metric given by* (12) *for the function $\varphi$ defined in* (13).

*Moreover, if $I$ is complete and $H^2 - K \geqslant c_0 > 0$ then the metric $g_0$ is complete. In particular, $\Sigma$ with the conformal structure given by $A$ (or by $g_0$) is conformally equivalent to the complex plane, the once punctured complex plane or a torus.*

**Proof.** From Corollary 2 we get that $2|Q| = tA$, where $t = \sqrt{H^2 - K}$ and $Q$ is a holomorphic quadratic form for $A$. Thus, since $H^2 - K > 0$, $g_0 = 2|Q|$ is a well-defined flat metric on $\Sigma$.

Let us see that $g_0$ is complete if $I$ is complete. In such a case $g_0$ would be a complete flat metric and, so, the universal Riemannian covering of $\Sigma$ for the metric $g_0$ would be the Euclidean plane. Hence, $\Sigma$ would be conformally equivalent to the complex plane, to the once punctured complex plane or to a torus.

Observe that, from (14), we get that

$$I \leqslant 2\cosh\varphi(t)A. \tag{15}$$

On the other hand, since $(I, II)$ is a special Weingarten pair of elliptic type it follows that $4s^2 f'(s^2)^2 < 1$ and so, from (13),

$$s^2\varphi'(s)^2 < 1, \quad \text{or equivalently} \quad -\frac{1}{s} < \varphi'(s) < \frac{1}{s}.$$

Hence, by integrating between a fixed point $s_0 > 0$ and $s$ one gets that there exists a constant $c_1 > 0$ such that $|\varphi(s)| \leqslant |\log s| + c_1$. Therefore, since

$$\lim_{s\to\infty} \frac{\cosh\log(s)}{s} = \frac{1}{2}$$

and $t \geqslant \sqrt{c_0}$, we deduce the existence of a constant $c_2 > 0$ such that $\cosh\varphi(t) \leqslant c_2 t$.

Finally, from (15) it follows that

$$I \leqslant 2c_2 tA = 2c_2 g_0,$$

that is, $g_0$ is complete. □

**Proof of Theorem 5.** Firstly, let us suppose that $\Sigma$ is a complete surface in $\mathbb{R}^3$ with $K \geqslant 0$.

If $K$ vanishes identically, then it is easy to conclude that $\Sigma$ is either a plane or a right circular cylinder (see [24]).

If there exists a point where the extrinsic curvature is positive, then $\Sigma$ is properly embedded and it is homeomorphic either to a sphere or to a plane [28]. In addition, we have $f(0) \neq 0$ (see [24]).

In the first case, $\Sigma$ must be a totally umbilical round sphere. This can be deduced from the existence of the holomorphic quadratic form given in Section 4, or from the classical results [8], [11] or [13].

The second case is not possible, as follows from Theorem 4 applied to our family of special Weingarten surfaces with $H = f(H^2 - K)$.

Now, let us suppose that $\Sigma$ is properly embedded and $K \leqslant 0$. Then we have that

$$0 \geqslant K = H^2 - (H^2 - K) = f(H^2 - K)^2 - (H^2 - K).$$

Hence, if $f(0) \neq 0$, since the function $f(s)^2 - s$ is continuous for $s \geqslant 0$ and takes a positive value at $s = 0$, then there exists $s_0 > 0$ such that $f(s)^2 - s > 0$ for $s \in [0, s_0]$. Consequently $H^2 - K \geqslant s_0 > 0$ on $\Sigma$ since $K \leqslant 0$.

From Lemma 6, $\Sigma$ is homeomorphic to the plane, to the once punctured plane or to a torus. Using once again Theorem 4, $\Sigma$ cannot be homeomorphic to a plane. In addition, every compact surface in $\mathbb{R}^3$ must have a point with positive extrinsic curvature, and so $\Sigma$ cannot be homeomorphic to a torus. With all of this, $\Sigma$ must be homeomorphic to the once punctured plane. So, from Theorem 4, $\Sigma$ must be a rotational surface and must be contained in a cylinder $C$ of $\mathbb{R}^3$.

To finish, let us see that $\Sigma$ is a right circular cylinder. In fact, up to an isometry of $\mathbb{R}^3$, we can suppose that $\Sigma$ is a rotational surface with respect to the $z$-axis. Let us denote by

$$\alpha = \Sigma \cap \{(x, y, z) \in \mathbb{R}^3 \colon\ x > 0,\ y = 0\}$$

a generatrix curve of $\Sigma$. It is clear that $\alpha$ is a line of curvature of $\Sigma$ and its signed-curvature on the plane $y = 0$ changes sign if and only if $K$ changes sign.

Since $K \leqslant 0$, the sign of the curvature of $\alpha$ on the plane $y = 0$ does not change, and so $\alpha$ is a convex curve. But, since $\alpha$ is contained in the strip determined by the $z$-axis and the parallel line $C \cap \{(x, y, z) \in \mathbb{R}^3 \colon\ x > 0,\ y = 0\}$, we conclude that $\alpha$ must be a parallel line to the $z$-axis, as we wanted to prove. $\square$

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