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On sampling theory and basic Sturm–Liouville systems

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Abstract

We investigate the sampling theory associated with basic Sturm–Liouville eigenvalue problems. We derive two sampling theorems for integral transforms whose kernels are basic functions and the integral is of Jackson's type. The kernel in the first theorem is a solution of a basic difference equation and in the second one it is expressed in terms of basic Green's function of the basic Sturm–Liouville systems. Examples involving basic sine and cosine transforms are given.

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1. Introduction

Consider the Sturm–Liouville problem

$$\ell(u) := -u'' + p(x)u = \lambda u, \quad -\infty < a \leqslant x \leqslant b < \infty, \quad \lambda \in \mathbb{C}, \tag{1.1}$$

$$U_1(u) := a_1 u(a) + a_2 u'(a) = 0, \quad |a_1| + |a_2| \neq 0, \tag{1.2}$$

$$U_2(u) := b_1 u(b) + b_2 u'(b) = 0, \quad |b_1| + |b_2| \neq 0, \tag{1.3}$$

where $p(\cdot)$ is a real-valued continuous function on $[a, b]$ and $a_i, b_i \in \mathbb{R}$, $i = 1, 2$. From the theory of Sturm–Liouville problems, see e.g. [24,23], problem (1.1)–(1.3) has a denumerable set of real and simple eigenvalues, $\{\mu_k\}_{k=0}^{\infty}$ say, and the corresponding set of eigenfunctions $\{\phi_k(\cdot)\}_{k=0}^{\infty}$ is a complete orthogonal set of $L^2(a, b)$. If $u(\cdot, \lambda)$ and $v(\cdot, \lambda)$ are the solutions of (1.1) which satisfy (1.2) and (1.3), respectively, then Green's function of problem (1.1)–(1.3) is given

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by, cf. e.g. [23],

$$K(x,\xi,\lambda)=\frac{1}{W(u,v)}\begin{cases}u(\xi,\lambda)v(x,\lambda), & \xi\leqslant x,\\ u(x,\lambda)v(\xi,\lambda), & x\leqslant \xi,\end{cases} \tag{1.4}$$

where $W(u,v)$ is the Wronskian of u, v, which is a function of λ only.

The idea of connecting the sampling theory of signal analysis together with Sturm–Liouville problems goes back to Weiss [25] and Kramer [21]. In [6,8,15,26,27] it is shown that Sturm-Liouville type integral transforms can be sampled via Lagrange-type interpolation series. Thus, for instance, the integral transform

$$f(\lambda)=\int_a^b g(x)u(x,\lambda)\,\mathrm{d}x,\quad g(\cdot)\in L^2(a,b), \tag{1.5}$$

has the sampling representation

$$f(\lambda)=\sum_{k=0}^{\infty} f(\mu_k)\frac{W(u,v)(\lambda)}{(\lambda-\mu_k)W'(u,v)(\mu_k)},\quad \lambda\in\mathbb{C}, \tag{1.6}$$

where series (1.6) converges uniformly on compact subsets of $\mathbb{C}$ and absolutely on $\mathbb{C}$.

The derivation of sampling (interpolation) representations of integral transforms whose kernels are expressed in terms of Green's function is first investigated by Haddad et al. [17]. The theory is developed and extended in [9,10,13] to more general situations. Thus for a fixed $\xi_0\in[a,b]$, the integral transform

$$h(\lambda)=\int_a^b \omega(\lambda)K(x,\xi_0,\lambda)g(x)\,\mathrm{d}x,\quad g(\cdot)\in L^2(a,b), \tag{1.7}$$

can be reconstructed via the sampling representation

$$h(\lambda)=\sum_{k=0}^{\infty} h(\mu_k)\frac{\omega(\lambda)}{(\lambda-\mu_k)\omega'(\mu_k)},\quad \lambda\in\mathbb{C}, \tag{1.8}$$

where $\omega(\lambda)$ is the well defined canonical product

$$\omega(\lambda)=\begin{cases}\lambda\prod_{k=1}^{\infty}\left(1-\dfrac{\lambda}{\mu_k}\right) & \text{if } \mu_0=0 \text{ is an eigenvalue},\\ \prod_{k=0}^{\infty}\left(1-\dfrac{\lambda}{\mu_k}\right) & \text{if zero is not an eigenvalue}.\end{cases} \tag{1.9}$$

Also series (1.8) converges uniformly on compact subsets of $\mathbb{C}$.

Our purpose in this paper is to derive basic analogs of the results outlined above. Thus the derivative in (1.1)–(1.3) will be replaced by the q-derivative, see definitions in the next section and the Lebesgue integration will be replaced by the q-integral of Jackson [19]. In the next section, we state the q-notations and results which will be needed for the derivation of the sampling theorems. In Section 3, we introduce the basic Sturm–Liouville systems studied in [11,12] and derive two associated sampling theorems. The first simulates (1.5)–(1.6) and the second is a basic analog of (1.7)–(1.8). In the last section we give three examples exhibiting the obtained results. It is worthy to mention here that the theory of q-sampling theory is a new subject in sampling theory. To the best of our knowledge the papers [1,7,18] have opened research in that field. One should note that the treatments in the present paper and that of [1,7] differ from that of [18], although both ways are q-type sampling analysis. While our present work and that of [1,7] are established using the q-Jackson difference operator that of [18] uses the Askey–Wilson q-difference operator.

2. Basic definitions

In this section we introduce some of the needed q-notations and results. Throughout this paper q is a positive number with $0<q<1$. For $\mu\in\mathbb{R}$, a set $A\subseteq\mathbb{R}$ is called a μ-geometric set if $\mu x\in A$ for all $x\in A$. If $A\subseteq\mathbb{R}$ is a μ-geometric set, then it contains all geometric sequences $\{x\mu^n\}_{n=0}^{\infty}$, $x \in A$. Let f be a function, real or complex valued, defined on a q-geometric set A. The q-difference operator is defined by

$$D_q f(x) := \frac{f(x) - f(qx)}{x - qx}, \quad x \in A/\{0\}. \tag{2.1}$$

If $0 \in A$, the q-derivative at zero is defined by

$$D_q f(0) := \lim_{n\to\infty} \frac{f(xq^n) - f(0)}{xq^n}, \quad x \in A, \tag{2.2}$$

if the limit exists and does not depend on x. As a converse of the q-difference operator, Jackson's q-integration, cf. [19], is given by

$$\int_0^x f(t)\,\mathrm{d}_q t := x(1-q)\sum_{n=0}^{\infty} q^n f(xq^n), \quad x \in A, \tag{2.3}$$

provided that the series converges, and

$$\int_a^b f(t)\,\mathrm{d}_q t := \int_0^b f(t)\,\mathrm{d}_q t - \int_0^a f(t)\,\mathrm{d}_q t, \quad a, b \in A. \tag{2.4}$$

We also have the following q-analog of the fundamental theorem of calculus, cf. [3]

$$D_{q,x}\int_0^x f(t)\,\mathrm{d}_q t = f(x), \qquad \int_0^x D_q f(t)\,\mathrm{d}_q t = f(x) - f(0), \quad x \in A. \tag{2.5}$$

The second identity in (2.5) holds if $f(\cdot)$ is q-regular at zero, i.e.

$$\lim_{n\to\infty} f(xq^n) = f(0), \quad x \in A. \tag{2.6}$$

The third type of q-Bessel functions is defined for $v > -1$ by

$$J_v(x;q) := \frac{(q^{2v+2};q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}}{(q^{2v+2};q^2)_n (q^2;q^2)_n} x^{2n+v}, \tag{2.7}$$

where $(a;q)_n$, $a \in \mathbb{C}$, is the q-shifted factorial defined by

$$(a;q)_n := \begin{cases} 1, & n = 0, \\ \prod_{i=0}^{n-1}(1-aq^i), & n = 1, 2, \ldots. \end{cases} \tag{2.8}$$

The limit of $(a;q)_n$ as n tends to infinity exists and will be denoted by $(a;q)_\infty$. The basic trigonometric functions $\cos(x;q)$ and $\sin(x;q)$ are defined on $\mathbb{C}$ by

$$\cos(x;q) := \sum_{n=0}^{\infty}(-1)^n \frac{q^{n^2}(x(1-q))^{2n}}{(q;q)_{2n}} \tag{2.9}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(xq^{-1/2}(1-q))^{1/2} J_{-1/2}(x(1-q)/\sqrt{q};q^2), \tag{2.10}$$

$$\sin(x;q) := \sum_{n=0}^{\infty}(-1)^n \frac{q^{n(n+1)}(x(1-q))^{2n+1}}{(q;q)_{2n+1}} \tag{2.11}$$

$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}(x(1-q))^{1/2} J_{1/2}(x(1-q);q^2), \tag{2.12}$$

and they are q-analogs of the cosine and sine functions, see [5,16]. It is proved in [20] that $J_\nu(\cdot, q^2)$ has a denumerable set of real and simple zeros only. Moreover, cf. [2], if $w_m^{(\nu)}, m \geqslant 1$, denote the positive zeros of $J_\nu(\cdot, q^2)$ and $q^{2\nu+2} < (1-q^2)^2$, then

$$w_m^{(\nu)} = q^{-m+\varepsilon_m^{(\nu)}}, \quad \sum_{m=1}^{\infty} \varepsilon_m^{(\nu)} < \infty, \quad 0 \leqslant \varepsilon_m^{(\nu)} < 1. \tag{2.13}$$

It is also proved in [2] that for any $q \in (0, 1)$, the zeros $\{w_m^{(\nu)}\}_{m=0}^{\infty}$ of $J_\nu(z; q^2)$ have the form (2.13) for sufficiently large m. From (2.10), (2.12) and (2.13) $\sin(\cdot; q)$ and $\cos(\cdot; q)$ have only real and simple zeros $\{0, \pm x_m\}_{m=1}^{\infty}$ and $\{\pm y_m\}_{m=1}^{\infty}$, respectively, where $x_m, y_m > 0$, $m \geqslant 1$ and

$$x_m = (1-q)^{-1} q^{-m+\varepsilon_m^{(-1/2)}} \quad \text{if } q < (1-q^2)^2, \tag{2.14}$$

$$y_m = (1-q)^{-1} q^{-m+1/2+\varepsilon_m^{(1/2)}} \quad \text{if } q^3 < (1-q^2)^2. \tag{2.15}$$

Moreover, for any $q \in (0, 1)$, (2.14) and (2.15) hold for sufficiently large m. Let $L_q^2(0, a)$ be the space of all complex valued functions defined on $[0, a]$ such that

$$\|f\| := \left(\int_0^a |f(x)|^2 \, d_q x \right)^{1/2} < \infty. \tag{2.16}$$

The space $L_q^2(0, a)$ is a separable Hilbert space with the inner product

$$\langle f, g \rangle := \int_0^a f(x)\overline{g(x)} \, d_q x, \quad f, g \in L_q^2(0, a), \tag{2.17}$$

and the complete orthonormal basis

$$\varphi_n(x) = \begin{cases} \dfrac{1}{\sqrt{x(1-q)}} & x = aq^n, \\ 0 & \text{otherwise}, \end{cases} \tag{2.18}$$

$n = 0, 1, 2, \ldots$, cf. [7]. The q-type of Gronwall's inequality introduced in [4] is given in the following lemma.

Lemma A. *Let f and g be real-valued, non negative and bounded on* $\mathbb{R}^+$. *If*

$$f(x) \leqslant c + \int_0^x f(qt) g(qt) \, d_q t, \tag{2.19}$$

where c is a non negative constant, then

$$f(x) \leqslant c \exp\left(\int_0^x g(qt) \, d_q t \right). \tag{2.20}$$

3. The sampling theorems

Consider the basic Sturm–Liouville problem

$$\ell_q(y) := \frac{-1}{q} D_{q^{-1}} D_q y(x) + v(x) y(x) = \lambda y(x), \quad 0 \leqslant x \leqslant a < \infty, \tag{3.1}$$

$$V_1(y) := \cos \alpha y(0) + \sin \alpha D_{q^{-1}} y(0) = 0, \tag{3.2}$$

$$V_2(y) := \cos \beta y(a) + \sin \beta D_{q^{-1}} y(a) = 0, \tag{3.3}$$

where $v(\cdot)$ is a real-valued function defined on $[0, a]$ and is continuous at zero and $\alpha, \beta \in (0, \pi]$. Problem (3.1)–(3.3) enjoys the following properties, cf. [11,12]:

(i) Problem (3.1)–(3.3) is self-adjoint in $L_q^2(0, a)$.

(ii) Eq. (3.1) has a fundamental set of solutions $\{\phi_i(\cdot,\lambda)\}_{i=1}^2$, $x \in [0,a]$, determined by the initial conditions

$$D_{q^{-1}}^{j-1}\phi_i(0,\lambda)=\delta_{ij}, \quad i,j=1,2, \tag{3.4}$$

where δ_{ij} is the Kronecker's delta. Moreover, $\phi_i(x,\lambda)$ are well defined continuous functions on $[0,a]$ for all $\lambda \in \mathbb{C}$ and for all $x \in [0,a]$, $\phi_i(x,\lambda)$ are entire in λ.

(iii) Problem (3.1)–(3.3) has a denumerable set of real eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ with no finite limit points and all eigenvalues are simple from both geometric and algebraic points of view. The eigenvalues are exactly the simple zeros of the entire function

$$\Delta(\lambda) := \cos\beta\theta_1(a,\lambda)+\sin\beta D_{q^{-1}}\theta_1(a,\lambda), \tag{3.5}$$

where $\theta_i(x,\lambda)$, $i=1,2$, are the solutions defined by

$$\theta_1(x,\lambda) := V_1(\phi_2)\phi_1(x,\lambda)-V_1(\phi_1)\phi_2(x,\lambda),$$

$$\theta_2(x,\lambda) := V_2(\phi_2)\phi_1(x,\lambda)-V_2(\phi_1)\phi_2(x,\lambda). \tag{3.6}$$

(iv) Green's function associated with problem (3.1)–(3.3) has the form

$$G(x,t,\lambda)=\frac{-1}{\Delta(\lambda)}\begin{cases}\theta_2(x,\lambda)\theta_1(t,\lambda), & 0\leqslant t\leqslant x,\\ \theta_1(x,\lambda)\theta_2(t,\lambda), & x<t\leqslant a,\end{cases} \tag{3.7}$$

where λ is not an eigenvalue.

(v) If $\{\phi_n(\cdot)\}_{n=0}^{\infty}$ denotes a set of eigenfunctions corresponding to $\{\lambda_n\}_{n=0}^{\infty}$, then $\{\phi_n(\cdot)\}_{n=0}^{\infty}$ is a complete orthogonal set of $L_q^2(0,a)$. Moreover since $v(\cdot)$ is real-valued, we can assume that $\phi_n(\cdot)$ are also real-valued.

(vi) For $y,z \in C_q^2(0,a)$, the following Lagrange's identity holds

$$\langle \ell_q(y), z\rangle = [y,\bar{z}]_0^a + \langle y, \ell_q(z)\rangle, \tag{3.8}$$

where

$$[y,z](x) := y(x)D_{q^{-1}}z(x)-z(x)D_{q^{-1}}y(x). \tag{3.9}$$

(vii) $\Delta(\lambda)$ has an infinite number of positive zeros and it may have a finite number of negative zeros. The positive zeros $\{\lambda_n\}$ are given asymptotically as $n\to\infty$ by

$$\sqrt{\lambda_n}=\begin{cases}\dfrac{q^{-n+1/2}}{a(1-q)}(1+\mathrm{O}(q^n)), & \sin\alpha\neq 0,\\[2ex] \dfrac{q^{-n+1}}{a(1-q)}(1+\mathrm{O}(q^n)), & \sin\alpha=0.\end{cases} \tag{3.10}$$

The following lemma is important for the proof of the uniform convergence of the obtained sampling series.

Lemma 3.1. *The functions* $\theta_1(x,\lambda)$, $\theta_2(x,\lambda)$, *defined above are uniformly bounded on the subsets of the form* $[0,a]\times\Omega$, *where* $\Omega\subset\mathbb{C}$ *is compact.*

Proof. Let $\Omega\subset\mathbb{C}$ be compact. We prove the lemma for $\theta_1(x,\lambda)$ only since the proof for $\theta_2(x,\lambda)$ is similar. From the properties of the basic sine and cosine functions, there is a constant $C_0>0$ which is independent of $(x,\lambda)\subset[0,a]\times\Omega$ such that

$$|\cos(\sqrt{\lambda}x;q)|, \quad \left|\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}\right|\leqslant C_0, \quad (x,\lambda)\in[0,a]\times\Omega. \tag{3.11}$$

Using a q-version of the variation of constant methods, we can see easily that

$$\theta_1(x,\lambda) = \sin\alpha\cos(\sqrt{\lambda}x;q) + \cos\alpha\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}} - \frac{q}{\sqrt{\lambda}}\int_0^x \{\sin(\sqrt{\lambda}x;q)\cos(\sqrt{\lambda}qt;q) - \cos(\sqrt{\lambda}x;q)\sin(\sqrt{\lambda}qt;q)\}v(qt)\theta_1(qt,\lambda)\,\mathrm{d}_q t. \tag{3.12}$$

Hence from (3.11), we have

$$|\theta_1(x,\lambda)| \leqslant 2C_0 + \int_0^x 2C_0^2|v(qt)||\theta_1(qt,\lambda)|\,\mathrm{d}_q t, \quad x\in[0,a], \quad \lambda\in\Omega. \tag{3.13}$$

Applying the q-type Gronwall's inequality we obtain for every $\lambda \in \Omega$

$$|\theta_1(x,\lambda)| \leqslant 2C_0^2 \exp\left(2C_0^2\int_0^x |v(qt)|\,\mathrm{d}_q t\right). \tag{3.14}$$

If $M := 2C_0^2 \sup_{0\leqslant x\leqslant a}\int_0^x |v(t)|\,\mathrm{d}t$, then

$$|\theta_1(x,\lambda)| \leqslant 2C_0^2 e^{aM}, \quad x\in[0,a]. \tag{3.15}$$

Hence $\theta_1(x,\lambda)$ is uniformly bounded on $[0,a]\times\Omega$. □

The property of $\theta_1(\cdot,\lambda)$ proved in the previous lemma suffices to prove the uniform convergence of the obtained sampling series below. It is a basic analog of the classical result of Coddington and Levinson [14, p. 225]. Now we are ready to state and prove the sampling results of this paper.

Theorem 3.2. *Let $g(\cdot)\in L_q^2(0,a)$ and $f(\lambda)$ be the q-type transform*

$$f(\lambda) = \int_0^a g(x)\theta_1(x,\lambda)\,\mathrm{d}_q x, \quad \lambda\in\mathbb{C}. \tag{3.16}$$

Then $f(\lambda)$ is an entire function which admits the sampling form

$$f(\lambda) = \sum_{k=0}^{\infty} f(\lambda_k)\frac{\Delta(\lambda)}{(\lambda-\lambda_k)\Delta'(\lambda_k)}, \tag{3.17}$$

where $\Delta(\lambda)$ is defined in (3.5) *above. Series* (3.17) *converges absolutely on $\mathbb{C}$ and uniformly on compact subsets of $\mathbb{C}$.*

Proof. The set of eigenfunctions $\{\theta_1(\cdot,\lambda_k)\}_{k=0}^{\infty}$ is a complete orthogonal set of $L_q^2(0,a)$. Thus, the application of Parserval's identity to (3.16) yields

$$f(\lambda) = \sum_{k=0}^{\infty}\frac{\widehat{\bar{g}}(k)\widehat{\theta}_1(k,\lambda)}{\|\theta_1(\cdot,\lambda_k)\|^2}, \tag{3.18}$$

where $\widehat{\bar{g}}(k)$, $\widehat{\theta}_1(k,\lambda)$ are the Fourier coefficients

$$\widehat{\bar{g}}(k) = \int_0^a \bar{g}(x)\bar{\theta}_1(x,\lambda_k)\,\mathrm{d}_q x, \quad \widehat{\theta}_1(k,\lambda) = \int_0^a \theta_1(x,\lambda)\bar{\theta}_1(x,\lambda_k)\,\mathrm{d}_q x. \tag{3.19}$$

Thus

$$f(\lambda) = \sum_{k=0}^{\infty} f(\lambda_k)\frac{\widehat{\theta}_1(k,\lambda)}{\|\theta_1(\cdot,\lambda_k)\|^2}, \quad \lambda\in\mathbb{C}. \tag{3.20}$$

Let $\lambda \in \mathbb{C}$, $\lambda \neq \lambda_k$ and $k \in \mathbb{N}$ be fixed. Applying Lagrange's identity (3.8), with $y(x) = \theta_1(x, \lambda)$ and $z(x) = \bar{\theta}_1(x, \lambda_k)$ we obtain

$$(\lambda - \lambda_k) \int_0^a \theta_1(x, \lambda)\bar{\theta}_1(x, \lambda_k)\, \mathrm{d}_q x = [\theta_1(x, \lambda), \bar{\theta}_1(x, \lambda_k)]_0^a. \tag{3.21}$$

From the definition of $\theta_1(\cdot, \lambda)$, we have

$$[\theta_1(x, \lambda), \bar{\theta}_1(x, \lambda_k)](0) = \theta_1(0, \lambda) D_{q^{-1}} \bar{\theta}_1(0, \lambda_k) - \bar{\theta}_1(0, \lambda_k) D_{q^{-1}} \theta_1(0, \lambda) = 0. \tag{3.22}$$

Substituting in (3.21) we obtain

$$(\lambda - \lambda_k) \int_0^a \theta_1(x, \lambda)\bar{\theta}_1(x, \lambda_k)\, \mathrm{d}_q x = \theta_1(a, \lambda) D_{q^{-1}} \bar{\theta}_1(a, \lambda_k) - \bar{\theta}_1(a, \lambda_k) D_{q^{-1}} \theta_1(a, \lambda). \tag{3.23}$$

Assume that $\sin \beta \neq 0$. Since $\theta_1(\cdot, \lambda_k)$ is an eigenfunction, then it satisfies (3.3). Hence

$$D_{q^{-1}} \bar{\theta}_1(a, \lambda_k) = -\cot \beta \bar{\theta}_1(a, \lambda_k). \tag{3.24}$$

Again, substituting from (3.24) in (3.23), we obtain

$$\begin{aligned}(\lambda - \lambda_k) \int_0^a \theta_1(x, \lambda)\bar{\theta}_1(x, \lambda_k)\, \mathrm{d}_q x &= -\bar{\theta}_1(a, \lambda_k)\{\cot \beta \theta_1(a, \lambda) + D_{q^{-1}} \theta_1(a, \lambda)\} \\ &= -\sin \beta \bar{\theta}_1(a, \lambda_k) \Delta(\lambda). \end{aligned} \tag{3.25}$$

In this case $\theta_1(a, \lambda_k) \neq 0$ since otherwise we get

$$\int_0^a |\theta_1(x, \lambda_k)|^2\, \mathrm{d}_q x = 0,$$

contradicting the fact that $\theta_1(\cdot, \lambda_k)$ is an eigenfunction. Hence

$$\int_0^a \theta_1(x, \lambda)\bar{\theta}_1(x, \lambda_k)\, \mathrm{d}_q x = -\sin \beta \bar{\theta}_1(a, \lambda_k) \frac{\Delta(\lambda)}{\lambda - \lambda_k}, \tag{3.26}$$

and from the simplicity of the zeros of $\Delta(\lambda)$, cf. [12, pp. 3786–3787]

$$\int_0^a |\theta_1(x, \lambda_k)|^2\, \mathrm{d}_q x = -\sin \beta \bar{\theta}_1(a, \lambda_k) \Delta'(\lambda_k). \tag{3.27}$$

Combining (3.26), (3.27) and (3.20) proves the interpolation representation (3.17) when λ is not an eigenvalue and $\sin \beta \neq 0$. The other cases are similar and the convergence is pointwise in $\mathbb{C}$. Now we prove that series (3.17) converges absolutely on $\mathbb{C}$. Indeed, let $\lambda \in \mathbb{C}$ be fixed. By Cauchy–Schwarz inequality,

$$\begin{aligned}\sum_{k=0}^{\infty} \left| f(\lambda_k) \frac{\Delta(\lambda)}{(\lambda - \lambda_k)\Delta'(\lambda_k)} \right| &= \sum_{k=0}^{\infty} \left| \frac{\widehat{g}(k)\widehat{\theta}_1(k, \lambda)}{\|\theta_1(\cdot, \lambda_k)\|^2} \right| \\ &\leqslant \left(\sum_{k=0}^{\infty} \left| \frac{\widehat{g}(k)}{\|\theta_1(\cdot, \lambda_k)\|} \right|^2 \right)^{1/2} \left(\sum_{k=0}^{\infty} \left| \frac{\widehat{\theta}_1(k, \lambda)}{\|\theta_1(\cdot, \lambda_k)\|} \right|^2 \right)^{1/2} \\ &< \infty, \end{aligned} \tag{3.28}$$

since $g(\cdot), \theta_1(\cdot, \lambda) \in L_q^2(0, \infty)$. As for uniform convergence on compact subsets of $\mathbb{C}$, let $\Omega_M := \{\lambda \in \mathbb{C} : |\lambda| \leqslant M\}$, M is a fixed positive number. Let $N > 0$ and $\sigma_N(\lambda)$, $\lambda \in \Omega_M$, be the function

$$\sigma_N(\lambda) := \left| f(\lambda) - \sum_{k=0}^{N-1} f(\lambda_k) \frac{\Delta(\lambda)}{(\lambda - \lambda_k)\Delta'(\lambda_k)} \right|. \tag{3.29}$$

The use of Cauchy–Schwarz inequality implies

$$\sigma_N(\lambda) = \left|\sum_{k=N}^{\infty} \frac{\widehat{\bar{g}}(k)\widehat{\theta}_1(k,\lambda)}{\|\theta_1(\cdot,\lambda_k)\|^2}\right| \leqslant \left(\sum_{k=N}^{\infty} \frac{|\widehat{g}(k)|^2}{\|\theta_1(\cdot,\lambda_k)\|^2}\right)^{1/2} \|\theta_1(\cdot,\lambda)\|, \quad \lambda \in \Omega_M. \tag{3.30}$$

From Lemma 3.1 above there exists a positive constant C_Ω which is independent of λ such that $\|\theta_1(\cdot,\lambda)\| \leqslant C_\Omega, \lambda \in \Omega_M$. Thus

$$\sigma_N(\lambda) \leqslant C_\Omega \left(\sum_{k=N}^{\infty} \frac{|\widehat{g}(k)|^2}{\|\theta_1(\cdot,\lambda_k)\|^2}\right)^{1/2} \longrightarrow 0 \tag{3.31}$$

as $N \to \infty$ without depending on λ, proving the uniform convergence of (3.17) on compact subsets of $\mathbb{C}$. Hence $f(\lambda)$ is entire and the proof is complete. □

In the following we state and prove another sampling theorem for the q-integral transform whose kernel is Green's function of problem (3.1)–(3.2) defined in (3.7) above. Since Green's function may have simple poles at the eigenvalues we define the function $\Phi(\cdot,\lambda)$ to be

$$\Phi(x,\lambda) = \Delta(\lambda) G(x, t_0, \lambda), \tag{3.32}$$

where t_0 is a fixed point in $\{0, aq^m, m \in \mathbb{N}\}$. Obviously $\Phi(x,\lambda)$ is entire in λ for every $x \in [0,a]$. Moreover, as in Lemma 3.1 above $\Phi(x,\lambda)$ is uniformly bounded on any set of the form $[0,a] \times \Omega$, $\Omega \subset \mathbb{C}$ is compact. The second sampling theorem of this paper is the following

Theorem 3.3. *Any function f defined by means of the basic integral transform*

$$f(\lambda) = \int_0^a g(x)\Phi(x,\lambda)\,\mathrm{d}_q x, \quad g(\cdot) \in L_q^2(0,a), \tag{3.33}$$

can be reconstructed via the sampling form

$$f(\lambda) = \sum_{k=0}^{\infty} f(\lambda_k) \frac{\Delta(\lambda)}{(\lambda-\lambda_k)\Delta'(\lambda_k)}. \tag{3.34}$$

Series (3.34) *converges absolutely on* $\mathbb{C}$ *and uniformly on compact subsets of* $\mathbb{C}$.

Proof. We notice that for all $k \in \mathbb{N}$ there exists a constant γ_k such that

$$\theta_1(x,\lambda_k) \equiv \gamma_k \theta_2(x,\lambda_k).$$

Hence

$$\Phi(x,\lambda_k) = -\gamma_k \theta_1(t_0,\lambda_k)\theta_1(x,\lambda_k). \tag{3.35}$$

Thus, the set $\{\Phi(\cdot,\lambda_k)\}_{k=0}^{\infty}$ is a complete orthogonal set of $L_q^2(0,a)$. The application of Parserval's identity to (3.16) yields

$$f(\lambda) = \sum_{k=0}^{\infty} \frac{\langle \Phi, \Phi_k\rangle \overline{\langle \bar{g}, \Phi_k\rangle}}{\|\Phi_k\|^2}. \tag{3.36}$$

But

$$\overline{\langle \bar{g}, \Phi_k\rangle} = \int_0^a g(x)\Phi(x,\lambda_k)\,\mathrm{d}_q x = f(\lambda_k). \tag{3.37}$$

Hence

$$f(\lambda)=\sum_{k=0}^{\infty} f(\lambda_k)\frac{\langle \Phi, \Phi_k\rangle}{\|\Phi(\cdot,\lambda_k)\|^2}, \quad \lambda\in\mathbb{C}. \tag{3.38}$$

Let $\lambda\in\mathbb{C}$, $\lambda\neq\lambda_k$ and $k\in\mathbb{N}$ be fixed. Applying Lagrange's identity (3.8), with $y(x)=\Phi(x,\lambda)$ and $z(x)=\Phi(x,\lambda_k)$ we obtain

$$\begin{aligned}
&(\lambda-\lambda_k)\int_0^a \Phi(x,\lambda)\bar{\Phi}(x,\lambda_k)\,\mathrm{d}_q x\\
&=(\lambda-\lambda_k)\int_0^{t_0} \Phi(x,\lambda)\bar{\Phi}(x,\lambda_k)\,\mathrm{d}_q x+(\lambda-\lambda_k)\int_{t_0}^{a} \Phi(x,\lambda)\bar{\Phi}(x,\lambda_k)\,\mathrm{d}_q x\\
&=-(\lambda-\lambda_k)\left(\bar{\theta}_2(t_0,\lambda_k)\theta_2(t_0,\lambda)\int_0^{t_0}\theta_1(x,\lambda)\bar{\theta}_1(x,\lambda_k)\,\mathrm{d}_q x\right.\\
&\quad\left.+\ \theta_1(t_0,\lambda)\bar{\theta}_1(t_0,\lambda_k)\int_{t_0}^{a}\theta_2(x,\lambda)\bar{\theta}_2(x,\lambda_k)\,\mathrm{d}_q x\right)\\
&=-\bar{\theta}_1(t_0,\lambda_k)\left(\gamma_k\theta_1(t_0,\lambda)[\theta_1(x,\lambda),\bar{\theta}_1(x,\lambda_k)]_0^{t_0}+\bar{\theta}_1(t_0,\lambda_k)[\theta_2(x,\lambda),\bar{\theta}_2(x,\lambda_k)]_{t_0}^{a}\right).
\end{aligned}$$

From the definition of $\theta_1(x,\lambda)$ we have

$$[\theta_1(x,\lambda),\bar{\theta}_1(x,\lambda_k)](0)=[\theta_2(x,\lambda),\bar{\theta}_2(x,\lambda_k)](a)=0.$$

Consequently,

$$\begin{aligned}
\int_0^a \Phi(x,\lambda)\bar{\Phi}(x,\lambda_k)\,\mathrm{d}_q x&=\frac{-1}{\lambda-\lambda_k}\theta_2(t_0,\lambda)\bar{\theta}_2(t_0,\lambda_k)[\theta_1(x,\lambda),\bar{\theta}_1(x,\lambda_k)](t_0)\\
&\quad+\frac{1}{\lambda-\lambda_k}\theta_1(t_0,\lambda)\bar{\theta}_1(t_0,\lambda_k)[\theta_2(x,\lambda),\bar{\theta}_2(x,\lambda_k)](t_0)\\
&=\frac{-1}{\lambda-\lambda_k}(\Delta(\lambda_k)\theta_1(t_0,\lambda)\theta_2(t_0,\lambda)-\Delta(\lambda)\bar{\theta}_1(t_0,\lambda_k)\bar{\theta}_2(t_0,\lambda_k))\\
&=\frac{\Delta(\lambda)}{\lambda-\lambda_k}\bar{\theta}_1(t_0,\lambda_k)\bar{\theta}_2(t_0,\lambda_k).
\end{aligned}\tag{3.39}$$

Hence

$$\begin{aligned}
\int_0^a \Phi(x,\lambda_k)\bar{\Phi}(x,\lambda_k)\,\mathrm{d}_q x&=\lim_{\lambda\to\lambda_k}\int_0^a \Phi(x,\lambda_k)\bar{\Phi}(x,\lambda_k)\,\mathrm{d}_q x\\
&=\bar{\theta}_1(t_0,\lambda_k)\bar{\theta}_2(t_0,\lambda_k)\Delta'(\lambda_k).
\end{aligned}\tag{3.40}$$

Assume that $\theta_i(t_0,\lambda_k)\neq 0$, $i=1,2$, $k\in\mathbb{N}$. Combining (3.39), (3.40), proves the interpolation representation (3.34) when λ is not an eigenvalue. The other cases are similar and the convergence is pointwise in $\mathbb{C}$. The proof of the absolute and uniform convergence of (3.34) on compact subsets of $\mathbb{C}$ is similar to the one in the proof of Theorem 3.2, so it is omitted. The proof when $\theta_i(t_0,k)=0$ for some k is similar to the above proof. The only change is that $f(\lambda_k)=0$. $\square$

Remark. From the asymptotics of the eigenvalues, (3.10), the canonical product

$$\omega(\lambda)=\begin{cases}\lambda\prod_{k=1}^{\infty}\left(1-\dfrac{\lambda}{\lambda_k}\right) & \text{if } \lambda_0=0 \text{ is an eigenvalue},\\[2ex] \prod_{k=0}^{\infty}\left(1-\dfrac{\lambda}{\mu_k}\right) & \text{if zero is not an eigenvalue}.\end{cases}\tag{3.41}$$

is an entire function of order zero cf. [22]. So we can define $\Phi(x,\lambda)$ to be

$$\Phi_0(x,\lambda)=\omega(\lambda)G(x,t_0,\lambda), \tag{3.42}$$

where $t_0\in\{0, aq^m, m\in\mathbb{N}\}$ is fixed. The previous sampling theorem holds if we replace $\Phi(x,\lambda)$ in (3.33) by $\Phi_0(x,\lambda)$. The only change will be the appearance of $\omega(\lambda)$ in (3.34) instead of $\Delta(\lambda)$.

4. Examples

This section includes three examples that exhibit the sampling theorems of the previous section. The first example is a sampling reconstruction of basic sine transforms at the zeros of the q-sine function and the second is a reconstruction of basic cosine transforms at the zeros of the basic sine function. The last example is a sampling theorem for a q-transform where the kernel is a combination of q-trigonometric functions.

Example 4.1. Consider the q-Sturm–Liouville boundary value problem

$$-\frac{1}{q}D_{q^{-1}}D_q y(x)=\lambda y(x),\quad 0\leqslant x\leqslant 1, \tag{4.1}$$

$$V_1(y)=y(0)=0,\quad V_2(y)=y(1)=0. \tag{4.2}$$

A fundamental set of solutions of (4.1) is

$$\phi_1(x,\lambda)=\cos(\sqrt{\lambda}x;q),\quad \phi_2(x,\lambda)=\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}. \tag{4.3}$$

Obviously $\phi_2(x,\lambda)$ satisfies the first boundary condition of (4.2). Hence the eigenvalues of problem (4.1)–(4.2) are the solutions of the equation

$$\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}=0. \tag{4.4}$$

Thus, the eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are the zeros of $\sin(\sqrt{\lambda};q)$. From (2.15),

$$\lambda_n=(1-q)^{-2}q^{-2n+2\varepsilon_n^{(-1/2)}},\quad n=1,2,\ldots, \tag{4.5}$$

for sufficiently large n and the corresponding set of eigenfunctions $\{\sin(\sqrt{\lambda_n}x;q)/\sqrt{\lambda_n}\}_{n=1}^{\infty}$ is an orthogonal basis of $L_q^2(0,1)$. We can see easily that zero is not an eigenvalue. Indeed, when $\lambda=0$, a fundamental set of solutions of (4.1) is $1, x$ and we cannot find any linear combinations of these solutions which satisfy (4.2) except for the trivial one. In the previous notations

$$\theta_1(x,\lambda)=\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}, \tag{4.6}$$

and

$$\theta_2(x,\lambda)=\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}\cos(\sqrt{\lambda}x;q)+\cos(\sqrt{\lambda};q)\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}. \tag{4.7}$$

So, if λ is not an eigenvalue, Green's function of problem (4.1)–(4.2) is given by

$$G(x,t,\lambda)=\frac{\sin(\sqrt{\lambda}t;q)}{\sin(\sqrt{\lambda};q)}\left(\cos(\sqrt{\lambda}x;q)\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}-\cos(\sqrt{\lambda};q)\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}\right), \tag{4.8}$$

for $0\leqslant t\leqslant x$, and

$$G(x,t,\lambda)=\frac{\sin(\sqrt{\lambda}x;q)}{\sin(\sqrt{\lambda};q)}\left(\cos(\sqrt{\lambda}t;q)\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}-\cos(\sqrt{\lambda};q)\frac{\sin(\sqrt{\lambda}t;q)}{\sqrt{\lambda}}\right), \tag{4.9}$$

for $x \leqslant t \leqslant 1$. Hence for a fixed $t_0 \in \{0, q^m, m \in \mathbb{N}\}$ the function $\Phi(x,\lambda)$ will be

$$\Phi(x,\lambda) = \sin(\sqrt{\lambda}t_0; q)(\cos(\sqrt{\lambda}x; q)\sin(\sqrt{\lambda}; q) - \cos(\sqrt{\lambda}; q)\sin(\sqrt{\lambda}x; q)), \tag{4.10}$$

$0 \leqslant t_0 \leqslant x$, and

$$\Phi(x,\lambda) = \sin(\sqrt{\lambda}x; q)(\cos(\sqrt{\lambda}t_0; q)\sin(\sqrt{\lambda}; q) - \cos(\sqrt{\lambda}; q)\sin(\sqrt{\lambda}t_0; q)), \tag{4.11}$$

$x \leqslant t_0 \leqslant a$. Applying Theorem 3.2 and Theorem 3.3, the q-transforms

$$f_1(\lambda) = \int_0^1 g(x)\frac{\sin(\sqrt{\lambda}x; q)}{\sqrt{\lambda}}\,\mathrm{d}_q x, \tag{4.12}$$

and

$$f_2(\lambda) = \int_0^1 g(x)\Phi(x,\lambda)\,\mathrm{d}_q x, \quad g(\cdot) \in L_q^2(0,1), \tag{4.13}$$

have the sampling formulae

$$f_i(\lambda) = \sum_{k=0}^{\infty} f_i(\lambda_k)\frac{\sin(\sqrt{\lambda}; q)}{\sqrt{\lambda}(\lambda - \lambda_k)\Delta'(\lambda_k)}, \quad i = 1, 2. \tag{4.14}$$

Example 4.2. Consider Eq. (4.1) together with the q-Neumann boundary conditions

$$V_1(y) = D_{q^{-1}}y(0) = 0, \quad V_2(y) = D_{q^{-1}}y(1) = 0. \tag{4.15}$$

In this case $\theta_1(x,\lambda) = \cos(\sqrt{\lambda}x; q)$, and

$$\theta_2(x,\lambda) = \cos(\sqrt{\lambda}q^{-1/2}; q)\cos(\sqrt{\lambda}x; q) + \sqrt{q}\sin(\sqrt{\lambda}q^{-1/2}; q)\sin(\sqrt{\lambda}x; q).$$

Since $\Delta(\lambda) = \sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2}; q)$, then for sufficiently large n, the non zero eigenvalues are given by

$$\lambda_n = q^{-2n+1+2\varepsilon_n^{(-1/2)}}, \quad n = 1, 2, \ldots. \tag{4.16}$$

Here $\lambda_0 = 0$ is an eigenvalue whose eigenfunction is 1. Therefore, $\{1, \cos(\sqrt{\lambda_n}x; q)\}_{n=1}^{\infty}$ is an orthogonal basis of $L_q^2(0,1)$. If λ is not an eigenvalue, Green's function $G(x,t,\lambda)$ is defined for $0 \leqslant t \leqslant x$ by

$$G(x,t,\lambda) = -\frac{\cos(\sqrt{\lambda}t; q)}{\sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2}; q)}(\cos(\sqrt{\lambda}q^{-1/2}; q)\cos(\sqrt{\lambda}x; q) + \sqrt{q}\sin(\sqrt{\lambda}q^{-1/2}; q)\sin(\sqrt{\lambda}x; q))$$

and for $x \leqslant t \leqslant 1$

$$G(x,t,\lambda) = -\frac{\cos(\sqrt{\lambda}x; q)}{\sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2}; q)}(\cos(\sqrt{\lambda}q^{-1/2}; q)\cos(\sqrt{\lambda}t; q) + \sqrt{q}\sin(\sqrt{\lambda}q^{-1/2}; q)\sin(\sqrt{\lambda}t; q)).$$

Hence for a fixed $t_0 \in \{0, q^m, m \in \mathbb{N}\}$, the function $\Phi(x,\lambda)$ is given for $0 \leqslant t_0 \leqslant x$ by

$$\Phi(x,\lambda) = -\cos(\sqrt{\lambda}t_0; q)(\cos(\sqrt{\lambda}q^{-1/2}; q)\cos(\sqrt{\lambda}x; q) + \sqrt{q}\sin(\sqrt{\lambda}q^{-1/2}; q)\sin(\sqrt{\lambda}x; q)) \tag{4.17}$$

and for $x \leqslant t_0 \leqslant 1$ by

$$\Phi(x,\lambda) = -\cos(\sqrt{\lambda}x; q)(\cos(\sqrt{\lambda}q^{-1/2}; q)\cos(\sqrt{\lambda}t_0; q) + \sqrt{q}\sin(\sqrt{\lambda}q^{-1/2}; q)\sin(\sqrt{\lambda}t_0; q)). \tag{4.18}$$

Applying Theorems 3.2 and Theorem 3.3 above to the q-transforms

$$f_1(\lambda)=\int_0^1 g(x)\cos(\sqrt{\lambda}x;q)\,\mathrm{d}_q x, \tag{4.19}$$

and

$$f_2(\lambda)=\int_0^1 g(x)\Phi(x,\lambda)\,\mathrm{d}_q x, \quad g(\cdot)\in L_q^2(0,1), \tag{4.20}$$

we obtain

$$f_i(\lambda)=\sum_{k=0}^{\infty} f_i(\lambda_k)\frac{\sqrt{q\lambda}\sin(\sqrt{\lambda}q^{-1/2};q)}{(\lambda-\lambda_k)\Delta'(\lambda_k)}, \quad i=1,2. \tag{4.21}$$

Example 4.3. Consider Eq. (4.1) together with the following boundary conditions

$$V_1(y)=y(0)+D_{q^{-1}}y(0)=0, \quad V_2(y)=y(1)=0. \tag{4.22}$$

For this problem $\theta_1(x,\lambda)=\cos(\sqrt{\lambda}x;q)-\sin(\sqrt{\lambda}x;q)/\sqrt{\lambda}$, and

$$\theta_2(x,\lambda)=\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}\cos(\sqrt{\lambda}x;q)-\cos(\sqrt{\lambda};q)\sin(\sqrt{\lambda}x;q).$$

Since $\Delta(\lambda)=\sin(\sqrt{\lambda};q)/\sqrt{\lambda}-\cos(\sqrt{\lambda};q)$, then the eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ of this boundary value problems are the solutions of the equation

$$\frac{\sin(\sqrt{\lambda};q)}{\sqrt{\lambda}}=\cos(\sqrt{\lambda};q). \tag{4.23}$$

Here $\lambda_0=0$ is also an eigenvalue whose eigenfunction is 1. If λ is not an eigenvalue, Green's function $G(x,t,\lambda)$ is defined for $0\leqslant t\leqslant x$ by

$$G(x,t,\lambda)=\frac{-(\sin(\sqrt{\lambda};q)\cos(\sqrt{\lambda}x;q)-\cos(\sqrt{\lambda};q)\sin(\sqrt{\lambda}x;q))(\cos(\sqrt{\lambda}t;q)-\sin(\sqrt{\lambda}t;q)/\sqrt{\lambda})}{\sin(\sqrt{\lambda};q)-\sqrt{\lambda}\cos(\sqrt{\lambda};q)}$$

and for $x\leqslant t\leqslant 1$

$$G(x,t,\lambda)=\frac{-(\sin(\sqrt{\lambda};q)\cos(\sqrt{\lambda}t;q)-\cos(\sqrt{\lambda};q)\sin(\sqrt{\lambda}t;q))(\cos(\sqrt{\lambda}x;q)-\sin(\sqrt{\lambda}x;q)/\sqrt{\lambda})}{\sin(\sqrt{\lambda};q)-\sqrt{\lambda}\cos(\sqrt{\lambda};q)}.$$

Hence for fixed $t_0\in\{0,q^m,m\in\mathbb{N}\}$, the function $\Phi(x,\lambda)$ is given for $0\leqslant t_0\leqslant x$ by

$$\Phi(x,\lambda)=-(\sin(\sqrt{\lambda};q)\cos(\sqrt{\lambda}x;q)-\cos(\sqrt{\lambda};q)\sin(\sqrt{\lambda};q))\left(\cos(\sqrt{\lambda}t_0;q)-\frac{\sin(\sqrt{\lambda}t_0;q)}{\sqrt{\lambda}}\right) \tag{4.24}$$

and for $x\leqslant t_0\leqslant 1$,

$$\Phi(x,\lambda)=-(\sin(\sqrt{\lambda};q)\cos(\sqrt{\lambda}t_0;q)-\cos(\sqrt{\lambda};q)\sin(\sqrt{\lambda}t_0;q))\left(\cos(\sqrt{\lambda}x;q)-\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}\right) \tag{4.25}$$

Applying Theorems 3.2 and 3.3 above to the q-transforms

$$f_1(\lambda)=\int_0^1 g(x)\left(\cos(\sqrt{\lambda}x;q)-\frac{\sin(\sqrt{\lambda}x;q)}{\sqrt{\lambda}}\right)\mathrm{d}_q x, \tag{4.26}$$

and

$$f_2(\lambda)=\int_0^1 g(x)\Phi(x,\lambda)\,\mathrm{d}_q x, \quad g(\cdot)\in L_q^2(0,1), \tag{4.27}$$

we obtain the sampling representation

$$f_i(\lambda)=\sum_{k=0}^{\infty} f_i(\lambda_k)\frac{\sin(\sqrt{\lambda};q)-\sqrt{\lambda}\cos(\sqrt{\lambda};q)}{\sqrt{\lambda}(\lambda-\lambda_k)\Delta'(\lambda_k)}, \quad i=1,2. \tag{4.28}$$

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