

# Combinatorial Structures in Loops

## I. Elements of the Decomposition Theory

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Difference sets have been extensively studied in groups, principally in Abelian groups. Here we extend the notion of a difference set to loops. This entails considering the class of  $\langle v, k \rangle$  systems and the special subclasses of  $\langle v, k, \lambda \rangle$  principal block partial designs (PBPDs) and  $\langle v, k, \lambda \rangle$  designs. By means of a certain permutation matrix decomposition of the incidence matrices of a system and its complement, we can isomorphically identify an abstract  $\langle v, k \rangle$  system with a corresponding system in a loop. Special properties of this decomposition correspond to special algebraic properties of the loop. Here we investigate the situation when some or all of the elements of the loop are right inverse. We identify certain classes of  $\langle v, k, \lambda \rangle$  designs, including skew-Hadamard designs and finite projective planes, with designs and difference sets in right inverse property loops and prove a universal existence theorem for  $\langle v, k, \lambda \rangle$  PBPDs and corresponding difference sets in such loops.

### 1. INTRODUCTION

Difference sets first arose in finite cyclic groups and can be traced back at least as far as Kirkman [10]. They have been extensively studied since the important work of Singer [15]. For a rather complete survey of this

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area see [2]. The notion of a difference set in a general finite group was introduced by Bruck [4] and has since been studied extensively for finite Abelian groups. For a partial survey of this area see [12]. Difference sets need not, however, be confined to groups, but may be defined and studied in loops generally. In a finite loop, a difference set is a type of block "design" in which the elements of the loop are the elements of the "design" and the difference set and all its left translates are the blocks of the "design." Because of this connection, we define, both abstractly and in finite loops, certain block systems which are of direct interest. The strongest such systems are the familiar  $\langle v, k, \lambda \rangle$  designs or symmetric balanced incomplete block designs. We show that every abstract block system can be identified with a block system in a loop in exactly the same way that a  $\langle v, k, \lambda \rangle$  design with a sharply transitive collineation group can be identified with a block design in the group (equivalent to a group difference set) [4, §2]. This involves a major use of the well-known König Theorem, which states that an incidence matrix of such a system can always be decomposed into a sum of permutation matrices. Now, special forms of this König decomposition can be related to special algebraic properties of the loop. The special form we consider here is related to the right inverse property (RIP) in the loop. In an RIP loop, a special type of block system, called a principal block partial design (PBPD), is equivalent to a difference set. Finite projective planes, skew-Hadamard designs, and  $\langle v, k, \lambda \rangle$  designs having a polarity for which either all or none of the elements and blocks are absolute are interesting classes of block designs which can be identified with block designs, and hence with difference sets, in RIP loops. Finally, for every set of integers  $v, k, \lambda$  satisfying  $0 < \lambda < k < v - 1$  and  $(v - 1)\lambda = k(k - 1)$ , we construct a PBPD which can be identified with a PBPD, and hence with a difference set, in an RIP loop.

## 2. PRELIMINARIES

Let  $\mathbb{L} = \{0 (= I_0), I_1, \dots, I_{v-1}\}$  be an additive loop of order  $v$ . The *right negative* of an element  $I \in \mathbb{L}$  is the element  $(-I)_R$  which satisfies the equation  $I + (-I)_R = 0$ , and the *left negative* is defined similarly. Let  $D = \{d_1, d_2, \dots, d_k\}$  be a  $k$ -subset of  $\mathbb{L}$ . If every element  $I \neq 0$  in  $\mathbb{L}$  appears exactly  $\lambda$  times in the set of all right differences  $\{d_i + (-d_j)_R\}$  where  $0 < \lambda < k < v - 1$ , then we call the combination  $(\mathbb{L}, D)_R$  a  $\langle v, k, \lambda \rangle$  *right loop difference set*. Left loop difference sets are defined similarly. The condition  $0 < \lambda < k < v - 1$  is imposed in order to avoid the trivial

situations in which  $|D| = 0, 1, v - 1$ , or  $v$ . By an easy counting argument, we see that  $v, k$ , and  $\lambda$  satisfy

$$(2.1) \quad (v - 1)\lambda = k(k - 1).$$

In distinction to the situation for right and left group difference sets, a right loop difference set need not be a left loop difference set. Again, consider the loop  $\mathbb{L}$  of order  $v$  and the  $k$ -subset  $D$  of  $\mathbb{L}$ . We form the left translates of  $D$ ,  $D = l_0 + D, l_1 + D, \dots, l_{v-1} + D$ . This combination, denoted by  $[\mathbb{L}, D]_R$ , is called a  $\langle v, k \rangle$  right loop system. If, in addition,  $|D \cap (l_j + D)| = \lambda$  for all  $l_j \neq 0$  in  $\mathbb{L}$ , where  $0 < \lambda < k < v - 1$ , then  $[\mathbb{L}, D]_R$  is called a  $\langle v, k, \lambda \rangle$  right loop principal block partial design (PBPD) where  $D$  is the principal block. If, further,  $|(l_i + D) \cap (l_j + D)| = \lambda$  for all  $l_i, l_j$  in  $\mathbb{L}, l_i \neq l_j$ , then  $[\mathbb{L}, D]_R$  is called a  $\langle v, k, \lambda \rangle$  right loop design. Left loop systems, PBPDs, and designs are defined similarly. The parameter values for a  $\langle v, k, \lambda \rangle$  right loop PBPD also satisfy (2.1). In distinction to the situation for right and left group PBPDs (which are the same as group designs), a right loop PBPD or design need not be, respectively, a left loop PBPD or design. Note that, if  $\hat{\mathbb{L}}$  is the addition table for a loop  $\mathbb{L}$  involved with one of these left loop structures, then  $(\hat{\mathbb{L}})^T = \hat{\mathbb{L}}_*$  is the addition table for a loop  $\mathbb{L}_*$  involved with the corresponding right loop structure. So, henceforth, we shall consider only the right loop structures.

Now, let  $S = \{x_0, x_1, \dots, x_{v-1}\}$  be a set of  $v$  elements and  $\mathcal{S} = \{X_0, X_1, \dots, X_{v-1}\}$  be a selection of  $v$  not necessarily distinct  $k$ -subsets of  $S$  such that each  $x_i$  in  $S$  appears in exactly  $k$  of the subsets in  $\mathcal{S}$ . This combination  $[S, \mathcal{S}]$  is called a  $\langle v, k \rangle$  system, also known as a square tactical configuration. Note that a  $\langle v, k \rangle$  right loop system is a  $\langle v, k \rangle$  system. If, in addition,  $|X_0 \cap X_j| = \lambda$  for all  $j, 1 \leq j \leq v - 1$ , where  $0 < \lambda < k < v - 1$ , then  $[S, \mathcal{S}]$  is called a  $\langle v, k, \lambda \rangle$  principal block partial design (PBPD) where  $X_0$  is the principal block. If, further,  $|X_i \cap X_j| = \lambda$  for all  $i, j, i \neq j, 0 \leq i, j \leq v - 1$ , then  $[S, \mathcal{S}]$  is known as a  $\langle v, k, \lambda \rangle$  design. Without loss of generality, we shall henceforth assume that the elements and sets of a  $\langle v, k \rangle$  system  $[S, \mathcal{S}]$  are labeled so that either  $x_i \notin X_i$  for all  $i$ , or  $x_i \in X_i$  for all  $i, 0 \leq i \leq v - 1$ . It is again easy to verify that the parameter values for a  $\langle v, k, \lambda \rangle$  PBPD satisfy (2.1). A  $\langle v, k \rangle$  system  $[S, \mathcal{S}]$  is characterized by its incidence matrix  $A = [a_{ij}]$  of order  $v$  with rows and columns indexed, in order,  $0, 1, \dots, v - 1$ , where we set

$$(2.2) \quad a_{ij} = \begin{cases} 1, & x_j \in X_i, \\ 0, & x_j \notin X_i. \end{cases}$$

By the definition of  $[S, \mathcal{S}]$ , every row and column sum of  $A$  is  $k$ , and by the labeling assumption in  $[S, \mathcal{S}]$ , either  $\text{tr}(A) = 0$  or  $\text{tr}(A) = v$ . We

denote the set of all 0,1 matrices of order  $v$  with every row and column sum equal to  $k$  and with trace equal to either 0 or  $v$  by  $\mathfrak{A}_v^*(k)$ . If  $[S, \mathcal{S}]$  is a  $\langle v, k \rangle$  system, then the *complement* of  $[S, \mathcal{S}]$  is the  $\langle v, v - k \rangle$  system  $[S, \mathcal{S}_c]$  where  $\mathcal{S}_c = \{S - X_0, S - X_1, \dots, S - X_{v-1}\}$ . Let  $J$  be the matrix of order  $v$  all of whose entries are equal to 1. If  $A$  is the incidence matrix for  $[S, \mathcal{S}]$ , then  $A_c = J - A$  is the incidence matrix for  $[S, \mathcal{S}_c]$ .

### 3. MAIN RESULTS

By the König Theorem [11, p. 239, B],  $J$  can be decomposed into a sum of  $v$  permutation matrices of order  $v$  in the following way:

$$(3.1) \quad J = \sum_{j=0}^{v-1} P_j, \quad P_0 \equiv I,$$

where the rows and columns of a matrix of order  $v$  are labeled in order,  $0, 1, \dots, v - 1$ , and where  $P_j$  denotes a permutation matrix which has its entry 1 of row 0 in column  $j$ . We call (3.1) a *König decomposition* of  $J$ . From this König decomposition of  $J$  we form an addition table  $\hat{\mathbb{L}}$  of a loop  $\mathbb{L} = \{0(=I_0), I_1, \dots, I_{v-1}\}$  as follows: We label the rows and columns of  $\hat{\mathbb{L}}$ , in order,  $0, 1, \dots, v - 1$ . Then if  $P_j$  has its entry 1 of row  $i$  in column  $m$ , the entry in row  $i$  and column  $j$  of  $\hat{\mathbb{L}}$  is  $I_m$ ,  $i, j, m = 0, 1, \dots, v - 1$ . Since  $P_j$  is a permutation matrix, each element of  $\mathbb{L}$  appears exactly once in column  $j$  of  $\hat{\mathbb{L}}$  and  $I_j$  appears in row 0 of this column,  $j = 0, 1, \dots, v - 1$ . Since each  $P_j$  has exactly one entry 1 in row  $i$  and no two such permutation matrices have their entries 1 of row  $i$  in the same column, each element of  $\mathbb{L}$  appears exactly once in row  $i$  of  $\hat{\mathbb{L}}$ . Furthermore, since column 0 of  $\hat{\mathbb{L}}$  is constructed from  $P_0 = I$ , the element of  $\mathbb{L}$  in row  $i$  of column 0 of  $\hat{\mathbb{L}}$  is  $I_i$ ,  $i = 0, 1, \dots, v - 1$ . Thus,  $\hat{\mathbb{L}}$  is the addition table for a loop  $\mathbb{L}$  of order  $v$  in which the permutation matrices of the König decomposition (3.1) determine the entries in the corresponding columns of  $\hat{\mathbb{L}}$ . Conversely, given an addition table  $\hat{\mathbb{L}}$  for a loop  $\mathbb{L}$ , we can reverse the above procedure and obtain a König decomposition for  $J$  as in (3.1), where the entries in the columns of  $\hat{\mathbb{L}}$  determine the particular permutation matrices in the decomposition. Since each column  $j$  of  $\hat{\mathbb{L}}$  is determined by its lead element  $I_j$ , we have established a one-to-one correspondence between the elements of  $\mathbb{L}$  and the permutation matrices in a König decomposition of  $J$ , called the *right König correspondence*.

Now, let  $[S, \mathcal{S}]$ ,  $S = \{x_0, x_1, \dots, x_{v-1}\}$ ,  $\mathcal{S} = \{X_0, X_1, \dots, X_{v-1}\}$ , be a  $\langle v, k \rangle$  system, let  $A \in \mathfrak{A}_v^*(k)$  be the incidence matrix for  $[S, \mathcal{S}]$ , and let  $A_c \in \mathfrak{A}_v^*(v - k)$  be the incidence matrix for  $[S, \mathcal{S}_c]$ . Then, by the König

Theorem,  $A$  and  $A_c$  can be decomposed into sums of permutation matrices of order  $v$

$$(3.2) \quad A = \delta P_{r_0} + \sum_{t=1}^{k-\delta} P_{r_t}, \quad P_{r_0} \equiv I, \quad \delta = \begin{cases} 0, & \text{tr}(A) = 0, \\ 1, & \text{tr}(A) = 1, \end{cases}$$

and

$$(3.3) \quad A_c = (1 - \delta)P_{s_0} + \sum_{u=1}^{v-k-1+\delta} P_{s_u}, \quad P_{s_0} \equiv I,$$

called *König decompositions* of  $A$  and  $A_c$ , respectively. These, in turn, determine the König decomposition of  $J$

$$(3.4) \quad J = P_0 + \sum_{t=1}^{k-\delta} P_{r_t} + \sum_{u=1}^{v-k-1+\delta} P_{s_u}, \quad P_0 = I,$$

which, in turn, determines a loop  $\mathbb{L} = \{0(=l_0), l_1, \dots, l_{v-1}\}$  under the right König correspondence

$$(3.5) \quad \rho: \begin{cases} P_0 = I \leftrightarrow 0, \\ P_{r_t} \leftrightarrow l_{r_t}, & t = 1, \dots, k - \delta, \\ P_{s_u} \leftrightarrow l_{s_u}, & u = 1, \dots, v - k - 1 + \delta. \end{cases}$$

In  $\mathbb{L}$ , let  $D = l_0 + D = \{l_{r_t} \mid t = 1 - \delta, 2 - \delta, \dots, k - \delta\}$  and form the left translates  $l_q + D$ ,  $q = 1, 2, \dots, v - 1$ . Then  $[\mathbb{L}, D]_R$  is a  $\langle v, k \rangle$  right loop system. We set up the one-to-one correspondence  $\varphi : S \leftrightarrow \mathbb{L}$  and  $\mathcal{S} \leftrightarrow \{l_i + D \mid l_i \in \mathbb{L}\}$  given by

$$(3.6) \quad \varphi: \begin{cases} x_j \leftrightarrow l_j \\ X_i \leftrightarrow l_i + D \end{cases}; i, j = 0, 1, \dots, v - 1.$$

Now  $x_j \in X_i$  if and only if  $l_i + l_{r_t} = l_j$  for some  $l_{r_t} \in D$  if and only if  $l_j \in l_i + D$ , for all  $i, j = 0, 1, \dots, v - 1$ , i.e.,  $\varphi$  preserves incidence. Hence  $[S, \mathcal{S}]$  and  $[\mathbb{L}, D]_R$  are isomorphic  $\langle v, k \rangle$  systems and by means of  $\varphi$ ,  $[S, \mathcal{S}]$  can be identified with  $[\mathbb{L}, D]_R$ . A similar identification holds between  $[S, \mathcal{S}_c]$  and  $[\mathbb{L}, \mathbb{L} - D]_R$ . We thus have the following result.

**THEOREM 3.1.** *A  $\langle v, k \rangle$  system  $[S, \mathcal{S}]$  can be identified with a  $\langle v, k \rangle$  right loop system  $[\mathbb{L}, D]_R$  according to (3.6). Furthermore, under this identification,*

- (i)  $[S, \mathcal{S}]$  is a  $\langle v, k, \lambda \rangle$  PBPD with principal block  $X_0$  if and only if  $[\mathbb{L}, D]_R$  is a  $\langle v, k, \lambda \rangle$  right loop PBPD with principal block  $D$ , and
- (ii)  $[S, \mathcal{S}]$  is a  $\langle v, k, \lambda \rangle$  design if and only if  $[\mathbb{L}, D]_R$  is a  $\langle v, k, \lambda \rangle$  right loop design.

It should be noted that, although the  $\langle v, k \rangle$  right loop system identified

with a  $\langle v, k \rangle$  system is unique combinatorially, the particular loop involved in the identification is not unique algebraically. If different König decompositions of  $A$  and  $A_0$  are employed, different loops may be obtained. The question of which loops are obtainable for a given  $\langle v, k \rangle$  system  $[S, \mathcal{S}]$  is worth investigating; we do not, however, pursue it here.

Now let  $\mathbb{L}$  be a loop and  $M$  be a subset of  $\mathbb{L}$ . We say that an element  $a \in \mathbb{L}$  is *right inversive* if, for each  $x \in \mathbb{L}$ ,  $(x + a) + (-a)_R = x$ , and that  $M$  is *right inversive* if every element of  $M$  is right inversive. If  $\mathbb{L}$  is right inversive then  $\mathbb{L}$  is said to have the *right inverse property* (RIP) and is called an *RIP loop*. Suppose that  $M$  is a right inversive subset of  $\mathbb{L}$ . Then, for  $a \in M$ ,  $((-a)_R + a) + (-a)_R = (-a)_R$  or  $(-a)_R + a = 0$ , whence  $(-a)_R = -a$ , that is, every element  $a \in M$  has a unique two-sided negative and so does every element of  $-M \equiv \{-a \mid a \in M\}$ . Now let  $-a \in -M$ . Then for each  $x \in \mathbb{L}$  there exists a  $y \in \mathbb{L}$  such that  $(x + (-a)) + a = y$ . Since  $M$  is right inversive, this becomes  $x + (-a) = y + (-a)$  or  $x = y$ , whence, since  $x \in \mathbb{L}$  and  $-a \in -M$  were arbitrary,  $-M$  is also right inversive. Thus, if  $M \subseteq \mathbb{L}$  is right inversive, the elements of  $M$  have unique two-sided negatives and  $-M$  is also right inversive. We now relate two of the combinatorial loop structures we have considered:

**THEOREM 3.2.** *Let  $\mathbb{L}$  be a loop of order  $v$  and  $D$  be a right inversive  $k$ -subset of  $\mathbb{L}$ . Then  $(\mathbb{L}, D)_R$  is a  $\langle v, k, \lambda \rangle$  right loop difference set if and only if  $[\mathbb{L}, D]_R$  is a  $\langle v, k, \lambda \rangle$  right loop PBPD with principal block  $D$ .*

*Proof.* Suppose  $(\mathbb{L}, D)_R$  is a  $\langle v, k, \lambda \rangle$  right loop difference set. Since the elements of  $D$  have unique two-sided negatives, for each  $l \neq 0$  in  $\mathbb{L}$  there are exactly  $\lambda$  ordered pairs of elements  $d_i, d_j \in D$  such that  $d_i - d_j = l$ . Since  $-D$  is right inversive, this equation becomes  $d_i = l + d_j$ , so that for all  $l \neq 0$  in  $\mathbb{L}$  we have  $|D \cap (l + D)| = \lambda$ . Then, since  $0 < \lambda < k < v - 1$ ,  $[\mathbb{L}, D]_R$  is a  $\langle v, k, \lambda \rangle$  right loop PBPD with principal block  $D$ . Since  $D$  is right inversive, this argument is reversible, which yields the converse.

In the last section of this paper we shall construct a  $\langle v, k, \lambda \rangle$  right loop PBPD, and hence a right loop difference set, in an RIP loop for every set of integers  $v, k, \lambda$  satisfying  $0 < \lambda < k < v - 1$  and  $(v - 1)\lambda = k(k - 1)$ . Since these PBPDs are in general *not*  $\langle v, k, \lambda \rangle$  designs, this will show that, in distinction to the situation for right group difference sets, a  $\langle v, k, \lambda \rangle$  right loop difference set, even in an RIP loop, need not be a  $\langle v, k, \lambda \rangle$  design.

We now investigate some of the relationships between the algebraic structure of a  $\langle v, k \rangle$  right loop system and the structure of its incidence

matrix. If  $B = [b_{ij}]$  and  $C = [c_{ij}]$  are two 0,1 matrices of size  $m \times n$ , we write  $B \cap C \equiv [e_{ij}]$  for the matrix of size  $m \times n$  where  $e_{ij} = \min\{b_{ij}, c_{ij}\}$  for all  $i$  and  $j$ . If  $B \cap C = B$ , we say that  $C$  contains  $B$  or that  $B$  is contained in  $C$ . Now let  $A \in \mathfrak{A}_v^*(k)$ ,  $0 \leq k \leq v$ . Then  $A$  has a König decomposition as given in (3.2). Suppose that for every  $P_{r_t}$  in this decomposition either  $P_{r_t}^T \cap A = 0$  or  $P_{r_t}^T \cap A = P_{r_u}$  for some  $P_{r_u}$  in the decomposition. We call such a decomposition a *type RI König decomposition* of  $A$ . If  $A = 0$  or  $A = I$  then  $A$  is considered to vacuously have a type RI König decomposition. (Note: A type RI König decomposition is called a *special König decomposition* in [9].) When  $P_{r_t}^T \cap A = 0$ ,  $P_{r_t} = [p_{ij}]$  has the properties that  $\text{tr}(P_{r_t}) = 0$  and for no  $i, j, i \neq j$ , does  $p_{ij} = p_{ji} = 1$ . Such a permutation matrix is called *skew*. When  $P_{r_t}^T \cap A = P_{r_u} \neq I$  for some  $P_{r_u}$  in the decomposition, then  $P_{r_u} = P_{r_t}^T$  and either  $P_{r_t}$  is skew or else  $P_{r_t}^T = P_{r_t}$  and  $\text{tr}(P_{r_t}) = 0$ . In the latter case  $P_{r_t}$  is called a *0-symmetric permutation matrix*. Thus, a type RI König decomposition of  $A$  has the form

$$(3.7) \quad A = \delta P_0 + \sum_{\alpha} P_{r_{\alpha}} + \sum_{\beta} (P_{r_{\beta}} + P_{r_{\beta}}^T) + \sum_{\gamma} P_{r_{\gamma}}, \quad P_0 = I,$$

where

$$(3.8) \quad \delta = \begin{cases} 0, & \text{tr}(A) = 0, \\ 1, & \text{tr}(A) = v, \end{cases}$$

and where the  $P_{r_{\alpha}}$ 's are the skew permutation matrices whose transposes do not occur in the decomposition, the  $P_{r_{\beta}}$ 's are the skew permutation matrices whose transposes do occur in the decomposition, and the  $P_{r_{\gamma}}$ 's are the 0-symmetric permutation matrices in the decomposition. The question of when a given  $A \in \mathfrak{A}_v^*(k)$  has a type RI König decomposition has been investigated in [9] and more or less satisfactorily settled; we shall use these results as needed. We now derive an important relationship:

**THEOREM 3.3.** *A  $\langle v, k \rangle$  right loop system  $[\mathbb{L}, D]_R$  where  $D$  is right invertive has an incidence matrix  $A \in \mathfrak{A}_v^*(k)$  which has a type RI König decomposition. Conversely, a matrix  $A \in \mathfrak{A}_v^*(k)$  which has a type RI König decomposition is the incidence matrix of a  $\langle v, k \rangle$  right loop system  $[\mathbb{L}, D]_R$  where  $D$  is right invertive.*

*Proof.* Let  $[\mathbb{L}, D]_R$  be a  $\langle v, k \rangle$  right loop system where  $D$  is right invertive, and let  $A \in \mathfrak{A}_v^*(k)$  be its incidence matrix. Under the right König correspondence  $\rho$  between  $\mathbb{L}$  and the König decomposition of  $J$ , each element  $l \in \mathbb{L}$  corresponds to a permutation matrix which we specially denote by  $P_l$ , and for  $D = \{d_1, \dots, d_k\}$  we have  $A = P_{d_1} + \dots + P_{d_k}$ . Since  $D$  is right invertive, the elements of  $D$  have unique two-sided

negatives and  $-D$  is right inversive. Consider  $d \in D$  and  $-d \in -D$  where  $\rho : d \leftrightarrow P_d$  and  $-d \leftrightarrow P_{-d}$ ,  $P_d = [p_{ij}]$  and  $P_{-d} = [q_{ij}]$ . Now,  $p_{rs} = 1$  if and only if  $l_r + d = l_s$  if and only if  $l_s + (-d) = l_r$  if and only if  $q_{sr} = 1$  for all  $r, s = 0, 1, \dots, v - 1$ ; hence,  $P_{-d} = P_d^T$ . If  $-d = d$ , then  $P_d = P_d^T$  and either  $d = 0$  and  $P_d = I$  or  $P_d$  is a 0-symmetric permutation matrix and  $P_d^T \cap A = P_d$ . If  $-d \neq d$ , then  $P_d^T \cap A = P_{-d}$  or  $0$  according as  $-d \in D$  or  $-d \notin D$ . Thus  $A$  has a type RI König decomposition. Now suppose that  $A \in \mathfrak{A}_v^*(k)$  has a type RI König decomposition as given in (3.7) and (3.8). Since each  $P_{r_\alpha}^T$  is contained in  $A_c$ , we can construct a König decomposition of  $A_c$  in which all of the  $P_{r_\alpha}^T$ 's appear:

$$(3.9) \quad A_c = (1 - \delta)P_0 + \sum_{\alpha} P_{r_\alpha}^T + \sum_{\epsilon} P_{s_\epsilon}.$$

The sum of these König decompositions of  $A$  and  $A_c$  is the König decomposition of  $J$ :

$$(3.10) \quad J = P_0 + \sum_{\alpha} (P_{r_\alpha} + P_{r_\alpha}^T) + \sum_{\beta} (P_{r_\beta} + P_{r_\beta}^T) + \sum_{\gamma} P_{r_\gamma} + \sum_{\epsilon} P_{s_\epsilon},$$

$$P_0 = I.$$

Under the right König correspondence  $\rho$  we obtain from (3.10) a loop  $\mathbb{L} = \{0(=l_0), l_1, \dots, l_{v-1}\}$  where  $\rho : P_i \leftrightarrow l_i$ ,  $i = 0, 1, \dots, v - 1$ . Let  $D = \{d_1, \dots, d_k\}$  be the elements in  $\mathbb{L}$  corresponding to the permutation matrices in (3.7), which we denote here by  $P_{d_1}, \dots, P_{d_k}$ , respectively. Then  $[\mathbb{L}, D]_R$  is a  $\langle v, k \rangle$  right loop system. Let  $d \in D$  where  $\rho : P_d \leftrightarrow d$ . If  $P_d = P_0 = I$ , then  $d = 0$  is trivially right inversive. If  $P_d = P_{r_\gamma} = P_{r_\gamma}^T$ , then  $d = l_{r_\gamma}$ , and we have for each  $l_r \in \mathbb{L}$  that  $l_r + d = l_s$  and  $l_s + d = l_r$  for some  $l_s \in \mathbb{L}$ , whence  $(l_r + d) + d = l_r$ . Hence  $d$  is right inversive and  $-d = d$ . If  $P_d = P_{r_\beta}$  and  $P_{d_*} = P_{r_\beta}^T$ ,  $d_* \in D$ , then  $d = l_{r_\beta}$ , and we have for each  $l_r \in \mathbb{L}$  that  $l_r + d = l_s$  and  $l_s + d_* = l_r$  for some  $l_s \in \mathbb{L}$ , whence  $(l_r + d) + d_* = l_r$ . Hence  $d$  is right inversive and  $d_* = -d$ . Finally, if  $P_d = P_{r_\alpha}$  and  $P_{d_*} = P_{r_\alpha}^T$ ,  $d_* \notin D$ , we again have by this last argument that  $d$  is right inversive and  $d_* = -d$ . Thus  $D$  is right inversive.

Taking  $k = v$  in this theorem we obtain the following result:

**COROLLARY 3.4.** *Let a loop  $\mathbb{L}$  and a König decomposition of  $J$  be related by the right König correspondence. Then  $\mathbb{L}$  is an RIP loop if and only if the König decomposition is of type RI.*

Our results have application to some interesting classes of  $\langle v, k \rangle$  systems. Let  $[S, \mathcal{S}]$ ,  $S = \{x_0, x_1, \dots, x_{v-1}\}$ ,  $\mathcal{S} = \{X_0, X_1, \dots, X_{v-1}\}$  be a  $\langle v, k \rangle$  system. We call  $[S, \mathcal{S}]$  skew (coskew) if  $x_i \notin X_i$  ( $x_i \in X_i$ ) for all  $i = 0, \dots, v - 1$ , and  $x_i \in X_j$  if and only if  $x_j \notin X_i$  for all  $i, j = 0, 1, \dots, v - 1$ ,  $i \neq j$ . Here  $v = 2k + 1$  if  $[S, \mathcal{S}]$  is skew and  $v = 2k - 1$  if



$[S, \mathcal{S}]$  is coskew. The incidence matrix for a skew  $\langle v, k \rangle$  system satisfies  $A + A^T = J - I$  and can be interpreted as a tournament matrix for a round-robin tournament in which all the contestants are tied. We call  $[S, \mathcal{S}]$  0-symmetric (cosymmetric) if  $x_i \notin X_i (x_i \in X_i)$  for all  $i = 0, 1, \dots, v - 1$ , and  $x_i \in X_j$  if and only if  $x_j \in X_i$  for all  $i, j = 0, 1, \dots, v - 1, i \neq j$ . The incidence matrix  $A$  for a 0-symmetric or cosymmetric  $\langle v, k \rangle$  system satisfies  $A^T = A$  and is called 0-symmetric and cosymmetric, respectively. A particularly interesting class of skew  $\langle v, k \rangle$  systems are the skew-Hadamard designs. These are the skew  $\langle v, k \rangle$  systems which are  $\langle v, k, \lambda \rangle$  designs. For further discussion of these designs and the special subclass of these designs which are Abelian group difference sets, the reader is referred to [7], [8], and [16]. The class of designs complementary to the skew-Hadamard designs are called coskew-Hadamard. An 0-symmetric (cosymmetric)  $\langle v, k \rangle$  system is one that has a polarity for which none (all) of the elements and blocks are absolute. An interesting class of 0-symmetric and cosymmetric  $\langle v, k \rangle$  systems are those which are  $\langle v, k, \lambda \rangle$  designs. A special subclass of these designs are the Abelian group difference sets having the inverse multiplier [6]. The 0-symmetric  $\langle v, k, \lambda \rangle$  designs are equivalent to  $\langle v, k, \lambda \rangle$  graphs. For further discussion of  $\langle v, k \rangle$  systems and  $\langle v, k, \lambda \rangle$  designs having a polarity and of  $\langle v, k, \lambda \rangle$  graphs, the reader is referred to [1], [3], [5], and [13]. Finally, a finite projective plane of order  $n$  is a  $\langle v, k, \lambda \rangle$  design in which  $v = n^2 + n + 1, k = n + 1, \text{ and } \lambda = 1, n \geq 2$  an integer. For a survey of finite projective planes, the reader is referred to [5]. We can now obtain several results for these special classes of systems and designs:

**THEOREM 3.5.** *A skew (coskew)  $\langle v, k \rangle$  system  $[S, \mathcal{S}]$  can be identified with a  $\langle v, k \rangle$  right loop system  $[\mathbb{L}, D]_R$  in an RIP loop  $\mathbb{L}$  where  $0 \notin D$  ( $0 \in D$ ) and where  $l \in D$  if and only if  $-l \notin D$  for all  $l \neq 0$  in  $\mathbb{L}$ ; and conversely.*

*Proof.* Let  $[S, \mathcal{S}]$  be a skew  $\langle v, k \rangle$  system. Then the incidence matrix  $A$  of  $[S, \mathcal{S}]$  satisfies  $A \cap A^T = 0$ , whence any König decomposition of  $A$  is of type RI,

$$(3.11) \quad A = \sum_{\alpha} P_{r_{\alpha}},$$

where the  $P_{r_{\alpha}}$ 's are all skew permutation matrices. Corresponding to (3.11) is the type RI König decomposition of  $A_c$  :

$$(3.12) \quad A_c = P_0 + \sum_{\alpha} P_{r_{\alpha}}^T, \quad P_0 = I.$$

The sum of these two König decompositions is a type RI König decomposition of  $J$ . Under the right König correspondence  $\rho$  this König decomposition of  $J$  corresponds, by Corollary 3.4, to an RIP loop  $\mathbb{L}$  and the König decomposition of  $A$  in (3.11) corresponds to a  $k$ -subset  $D \subseteq \mathbb{L}$ . Thus  $A$  is the incidence matrix of the  $\langle v, k \rangle$  right loop system  $[\mathbb{L}, D]_R$ . Since  $r_\alpha \neq 0$  in (3.11) we have  $0 \notin D$ , and, by the proof of Theorem 3.3, if  $\rho : l_r \leftrightarrow P_r$  then  $\rho : l_r \leftrightarrow P_r^T$  for all  $l_r \in \mathbb{L}$ . Hence, by (3.11),  $l \in D$  if and only if  $-l \notin D$  for all  $l \neq 0$  in  $\mathbb{L}$ . Now suppose that  $[\mathbb{L}, D]_R$  is a  $\langle v, k \rangle$  right loop system in an RIP loop  $\mathbb{L}$  where  $0 \notin D$  and where  $l \in D$  if and only if  $-l \notin D$  for all  $l \neq 0$  in  $\mathbb{L}$ . Then, by Theorem 3.3, the incidence matrix  $A$  of  $[\mathbb{L}, D]_R$  and  $A_c$  have type RI König decompositions whose sum is the corresponding type RI König decomposition of  $J$ . Since  $0 \notin D$ ,  $P_0 = I$  must be in the König decomposition of  $A_c$ . By the proof of Theorem 3.3, if  $\rho : l_r \leftrightarrow P_r$  then  $\rho : -l_r \leftrightarrow P_r^T$  for all  $l_r \in \mathbb{L}$ . Hence,  $P_r$  is contained in the König decomposition of  $A$  if and only if  $P_r^T$  is contained in the König decomposition of  $A_c$ , for all  $P_r \neq I$ . Since  $J = A + A_c = A + A^T + I$ ,  $A$  is the incidence matrix of a skew  $\langle v, k \rangle$  system  $[S, \mathcal{S}]$ . A similar proof obtains when  $[S, \mathcal{S}]$  is coskew and  $0 \in D$ . Here we merely need to put  $P_0 = I$  into the type RI König decomposition of  $A$ .

By Theorems 3.5 and 3.2 we then have the following result:

**COROLLARY 3.6.** *A skew (coskew) Hadamard design  $[S, \mathcal{S}]$  can be identified with a right loop design  $[\mathbb{L}, D]_R$  in an RIP loop  $\mathbb{L}$  where  $0 \notin D$  ( $0 \in D$ ) and where  $l \in D$  if and only if  $-l \notin D$  for all  $l \neq 0$  in  $\mathbb{L}$ ; and conversely. Furthermore,  $(\mathbb{L}, D)_R$  is a right loop difference set.*

**THEOREM 3.7.** *A 0-symmetric (cosymmetric)  $\langle v, k \rangle$  system  $[S, \mathcal{S}]$  whose incidence matrix has a type RI König decomposition can be identified with a  $\langle v, k \rangle$  right loop system  $[\mathbb{L}, D]_R$  where  $D$  is right inverse,  $0 \notin D$  ( $0 \in D$ ), and where  $l \in D$  if and only if  $-l \in D$  for all  $l \neq 0$  in  $\mathbb{L}$ ; and conversely.*

*Proof.* Let  $[S, \mathcal{S}]$  be a 0-symmetric  $\langle v, k \rangle$  system whose incidence matrix  $A$  has a type RI König decomposition. By Theorem 3.3,  $[S, \mathcal{S}]$  can be identified with a  $\langle v, k \rangle$  right loop system  $[\mathbb{L}, D]_R$  where  $D$  is right inverse. Since  $A$  is 0-symmetric, we have by the proof of Theorem 3.3 that if  $\rho : l_a \leftrightarrow P_a$  then  $\rho : -l_a \leftrightarrow P_a^T$  for all  $l_a \in D$ ; whence  $0 \notin D$ , and  $l \in D$  if and only if  $-l \in D$  for all  $l \neq 0$  in  $\mathbb{L}$ . Now suppose that  $[\mathbb{L}, D]_R$  is a  $\langle v, k \rangle$  right loop system where  $D$  is right inverse,  $0 \notin D$ , and where  $l \in D$  if and only if  $-l \in D$  for all  $l \neq 0$  in  $\mathbb{L}$ . Then, by Theorem 3.3, the incidence matrix  $A$  of  $[\mathbb{L}, D]_R$  has a type RI König decomposition. By the proof of Theorem 3.3, if  $\rho : l_a \leftrightarrow P_a$  then  $\rho : -l_a \leftrightarrow P_a^T$  for all  $l_a \in D$ . Thus,

since  $0 \notin D$ ,  $P_0 = I$  is not in the König decomposition of  $A$ , and since  $l_r \in D$  if and only if  $-l_r \in D$  for all  $l_r \neq 0$  in  $\mathbb{L}$ , we have  $P_r$  is in the König decomposition of  $A$  if and only if  $P_r^T$  is, for all  $r \neq 0$ . Hence  $A^T = A$  and  $\text{tr}(A) = 0$ , whence  $[S, \mathcal{S}]$  is 0-symmetric. A similar proof obtains when  $[S, \mathcal{S}]$  is cosymmetric and  $0 \in D$ . Here we include  $P_0 = I$  in the type RI König decomposition of  $A$ .

Note that in Theorem 3.7 we need to assume that the incidence matrix of  $[S, \mathcal{S}]$  has a type RI König decomposition. In [9] the authors have, in fact, constructed counterexamples of 0-symmetric and cosymmetric  $\langle v, k, \lambda \rangle$  PBPDs whose incidence matrices do not have type RI König decompositions. On the other hand, the following result was verified there:

LEMMA 3.8. *The matrix  $A \in \mathfrak{A}_v^*(k)$  has a type RI König decomposition when*

- (i)  *$A$  is 0-symmetric and  $k$  is even,*
- (ii)  *$A$  is cosymmetric and  $k$  is odd.*

The authors also showed there that, regardless of the value of  $k$ , an 0-symmetric or cosymmetric incidence matrix  $A$  for a  $\langle v, k, \lambda \rangle$  design always has a type RI König decomposition. Then  $A_o$  is a cosymmetric (0-symmetric) incidence matrix for a  $\langle v, v - k, v - 2k + \lambda \rangle$  design and hence also has a type RI König decomposition. Since the sum of these two König decompositions is a type RI König decomposition of  $J$ , we have by Theorems 3.1(ii), 3.2, and 3.7 and Corollary 3.4 the following result:

COROLLARY 3.9. *A 0-symmetric (cosymmetric)  $\langle v, k, \lambda \rangle$  design (i.e., one having a polarity for which none (all) of the elements and blocks are absolute) can be identified with a  $\langle v, k, \lambda \rangle$  right loop design  $[(\mathbb{L}, D)_R]$  in an RIP loop  $\mathbb{L}$  where  $0 \notin D$  ( $0 \in D$ ) and where  $l \in D$  if and only if  $-l \in D$  for all  $l \neq 0$  in  $\mathbb{L}$ ; and conversely. Furthermore,  $(\mathbb{L}, D)_R$  is a right loop difference set.*

We now consider the class of finite projective planes:

THEOREM 3.10. *A finite projective plane of order  $n$  can be identified with an  $\langle n^2 + n + 1, n + 1, 1 \rangle$  right loop design  $[(\mathbb{L}, D)_R]$  in an RIP loop  $\mathbb{L}$  where  $0 \in D$ . Furthermore,  $(\mathbb{L}, D)_R$  is a right loop difference set.*

*Proof.* Let  $[S, \mathcal{S}]$ ,  $S = \{x_0, x_1, \dots, x_{n^2+n}\}$ ,  $\mathcal{S} = \{X_0, X_1, \dots, X_{n^2+n}\}$ , be a finite projective plane of order  $n$ . Since  $[S, \mathcal{S}]$  has a system of distinct representatives, we may assume that the elements and blocks are labelled so that the incidence matrix  $A = [a_{ij}]$  of  $[S, \mathcal{S}]$  has  $\text{tr}(A) = n^2 + n + 1$ .

Let

$$(3.13) \quad A = P_0 + \sum_t P_{\tau_t}, \quad P_0 = I,$$

be any König decomposition of  $A$ . If  $a_{rs} = 1$  for  $r \neq s$ , then  $a_{sr} = 0$ , else with  $a_{rr} = a_{ss} = 1$  we would have  $|X_r \cap X_s| \geq 2$ , a contradiction. Hence  $P_{\tau_t}^T \cap A = 0$  for every  $r_t \neq 0$ , whence (3.13) is a type RI König decomposition. We construct a partial König decomposition of  $A_c$ ,

$$(3.14) \quad A_c = \sum_t P_{\tau_t}^T + A_c^*,$$

where the summation consists of the transposes of the non-identity permutation matrices in (3.13) and

$$(3.15) \quad A_c^* = J + I - (A + A^T)$$

is a 0-symmetric matrix in  $\mathfrak{A}_{n^2+n+1}^*(n^2 - n)$ . Now, since  $n^2 - n$  is always even,  $A_c^*$  has a type RI König decomposition by Lemma 3.8(i). The sum of this König decomposition of  $A_c^*$  and the type RI König decomposition  $\sum_t P_{\tau_t}^T$  yields, by (3.14), a type RI König decomposition of  $A_c$ . Then the sum of this König decomposition of  $A_c$  and the König decomposition (3.13) forms a type RI König decomposition of  $J$ . Hence, by Theorem 3.1(ii) and Corollary 3.4,  $[S, \mathcal{S}]$  is identified with an  $\langle n^2 + n + 1, n + 1, 1 \rangle$  right loop design  $[\mathbb{L}, D]_R$  in an RIP loop  $\mathbb{L}$ . Since  $\rho : 0 \leftrightarrow P_0 = I$  and  $P_0$  is contained in  $A$ , we have  $0 \in D$ . By Theorem 3.2,  $(\mathbb{L}, D)_R$  is a right loop difference set.

#### 4. CONSTRUCTIONS OF $\langle v, k, \lambda \rangle$ RIGHT LOOP PBPDs IN RIP LOOPS

We denote the set of all 0,1 matrices of size  $m \times n$  with every row sum equal to  $r$  and every column sum equal to  $s$  by  $\mathfrak{A}_{m,n}(r, s)$ . The *term rank* of a matrix  $A \in \mathfrak{A}_{m,n}(r, s)$ , denoted by  $\rho(A)$ , is the maximum number of entries equal to 1 in  $A$  such that no two of them occur in the same row or the same column. The following result follows readily from two theorems in Ryser [14, p. 63 and p. 56].

LEMMA 4.1. *Suppose  $m, n$  and  $r, s$  are integers such that  $0 < r \leq n$ ,  $0 < s \leq m$ , and  $mr = ns$ . Then  $\mathfrak{A}_{m,n}(r, s) \neq \emptyset$  and  $\rho(A) = \min\{m, n\}$  for every  $A \in \mathfrak{A}_{m,n}(r, s)$ .*

We are now ready to proceed with our constructions.

**THEOREM 4.2.** *For every set of integers  $v, k, \lambda$  satisfying  $(v - 1)\lambda = k(k - 1)$  and  $0 < \lambda < k < v - 1$ , there exists a  $\langle v, k, \lambda \rangle$  right loop PBPD  $[\mathbb{L}, D]_R$  in an RIP loop  $\mathbb{L}$ . Furthermore,  $(\mathbb{L}, D)_R$  is a right loop difference set.*

*Proof.* For each set of integers  $v, k, \lambda$  satisfying the conditions of the theorem we shall construct an incidence matrix  $A \in \mathfrak{A}_v^*(k)$  for a  $\langle v, k, \lambda \rangle$  PBPD such that  $A$  and  $A_c$  have type RI König decompositions whose sum is a type RI König decomposition of  $J$ . Then, by Theorem 3.1(i) and Corollary 3.4, we shall have constructed a  $\langle v, k, \lambda \rangle$  right loop PBPD  $[\mathbb{L}, D]_R$  in an RIP loop  $\mathbb{L}$ , and, by Theorem 3.2,  $(\mathbb{L}, D)_R$  is a right loop difference set. The constructions depend on the parities of  $v, k$ , and  $\lambda$ , and on the parities of the complementary values  $k_c, \lambda_c$  where  $k_c = v - k$  and  $\lambda_c = v - 2k + \lambda$ . Now, the  $k, \lambda$  set of values and the  $k_c, \lambda_c$  set of values pair off in exactly six possible ways according to parities:

- $v$  odd: (a)  $k$  even,  $\lambda$  even  $\Leftrightarrow k_c$  odd,  $\lambda_c$  odd,
- (b)  $k$  odd,  $\lambda$  even  $\Leftrightarrow k_c$  even,  $\lambda_c$  odd,
- (c)  $k$  even,  $\lambda$  odd  $\Leftrightarrow k_c$  odd,  $\lambda_c$  even,
- (d)  $k$  odd,  $\lambda$  odd  $\Leftrightarrow k_c$  even,  $\lambda_c$  even;
- $v$  even: (e)  $k$  even,  $\lambda$  even  $\Leftrightarrow k_c$  even,  $\lambda_c$  even,
- (f)  $k$  odd,  $\lambda$  even  $\Leftrightarrow k_c$  odd,  $\lambda_c$  even.

Note that when we have constructed a matrix  $A$  for case (a) we will have automatically constructed a matrix for case (d), and likewise for cases (b) and (c). This does not happen for cases (e) and (f); however, in these two cases a further simplification occurs by choosing  $k < v/2$  ( $k = v/2$  is impossible). For, when we have constructed a matrix  $A$  for case (e) with  $k < v/2$  we will have automatically constructed a matrix for case (e) with  $k > v/2$ , and likewise for case (f). Thus, our constructions essentially reduce to the four cases (a), (b), (e), and (f), with  $k < v/2$  in (e) and (f). A fifth case (a1) comes in because of the impossibility of the given construction in case (a) when  $\lambda_c = 1$ .

*Case (a):  $v$  odd,  $k$  even,  $\lambda$  even,  $\lambda_c > 1$ .* Let  $\mathbf{e} = (1, \dots, 1)$  of length  $k$  and  $\mathbf{0} = (0, \dots, 0)$  of length  $v - k - 1$ . Since  $\lambda \leq k - 2$  and  $\lambda$  is even, we can form a 0-symmetric circulant matrix  $F \in \mathfrak{A}_k^*(\lambda)$  not containing the 0-symmetric circulant permutation matrix of order  $k$ . Since  $0 < k - \lambda - 1 \leq v - k - 1$ ,  $0 < \lambda \leq k$ , and  $k(k - \lambda - 1) = (v - k - 1)\lambda$ , we have  $\mathfrak{A}_{k, v-k-1}(k - \lambda - 1, \lambda) \neq \emptyset$  by Lemma 4.1. Let  $E \in \mathfrak{A}_{k, v-k-1}(k - \lambda - 1, \lambda)$ . Since  $\lambda_c \geq 3$  or  $k - \lambda \leq v - k - 3$  and  $k - \lambda$  is even, we can form a 0-symmetric circulant matrix  $G \in \mathfrak{A}_{v-k-1}^*$

$(k - \lambda)$  not containing the 0-symmetric circulant permutation matrix of order  $v - k - 1$ . We then construct

$$A = \left[ \begin{array}{c|c|c} 0 & \mathbf{e} & \mathbf{0} \\ \hline \mathbf{e}^T & F & E \\ \hline \mathbf{0}^T & E^T & G \end{array} \right] \in \mathfrak{A}_v^*(k).$$

Since the inner product of the first row of  $A$  with any other row is  $\lambda$ ,  $A$  is the incidence matrix of a  $\langle v, k, \lambda \rangle$  PBPD. Since  $A$  is 0-symmetric and  $k$  is even, we have by Lemma 3.8(i) that  $A$  has a type RI König decomposition. Furthermore,  $A_c \in \mathfrak{A}_v^*(v - k)$  and the inner product of the first row of  $A_c$  with any other row is  $\lambda_c$ , whence  $A_c$  is the incidence matrix of a  $\langle v, k_c, \lambda_c \rangle$  PBPD, and since  $A_c$  is cosymmetric and  $v - k$  is odd, we have by Lemma 3.8(ii) that  $A_c$  has a type RI König decomposition. Since  $A$  and  $A_c$  are both symmetric, the sum of their König decompositions is a type RI König decomposition of  $J$ .

*Case (a1):*  $v$  odd,  $k$  even,  $\lambda$  even,  $\lambda_c = 1$ . Here, instead of  $A$ , we construct  $A_c = J - A$ . Note that the parameter values for  $A_c$  are  $v = n^2 + n + 1$ ,  $k_c = n + 1$  (odd), and  $\lambda_c = 1$ . Let  $\mathbf{e} = (1, \dots, 1)$  of length  $n$  and  $\mathbf{0} = (0, \dots, 0)$  of length  $n$ . Let  $I_n$  be the identity matrix of order  $n$ ,  $0_n$  the zero matrix of order  $n$ , and  $P = [p_{ij}]$  the circulant permutation matrix of order  $n$  determined by  $p_{01} = 1$ . We then construct

$$A_c = \left[ \begin{array}{c|c|c|c|c|c} 1 & \mathbf{e} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0}^T & I_n & P & \cdots & P & P \\ \hline \mathbf{0}^T & I_n & I_n & \cdots & P & P \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \mathbf{0}^T & I_n & I_n & \cdots & I_n & P \\ \hline \mathbf{e}^T & 0_n & I_n & \cdots & I_n & I_n \end{array} \right] \in \mathfrak{A}_{n^2+n+1}^*(n + 1).$$

Since the inner product of the first row of  $A_c$  with any other row is  $\lambda_c = 1$ ,  $A_c$  is the incidence matrix of a  $\langle v, k_c, \lambda_c \rangle$  PBPD. Since  $A_c^T \cap A_c = I$ , every König decomposition of  $A_c$  is of type RI. We now

proceed as in the proof of Theorem 3.10 with the roles of  $A$  and  $A_c$  interchanged, to obtain the desired type RI König decomposition of  $J$ . We note that the inner product of the first row of  $A$  with any other row is  $n^2 - n = \lambda$ , whence  $A$  is the incidence matrix of a  $\langle v, k, \lambda \rangle$  PBPD.

Case (b):  $v$  odd,  $k$  odd,  $\lambda$  even. Let  $\mathbf{e} = (1, \dots, 1)$  of length  $k - 1$  and  $\mathbf{0} = (0, \dots, 0)$  of length  $v - k$ . Since  $\lambda - 1 \leq k - 2$  and  $\lambda - 1$  is odd, we can form a cosymmetric circulant matrix  $F \in \mathfrak{A}_{k-1}^*(\lambda - 1)$  not containing the 0-symmetric circulant permutation matrix of order  $k - 1$ . Since  $0 < k - \lambda \leq v - k$ ,  $0 < \lambda \leq k - 1$ , and  $(k - 1)(k - \lambda) = (v - k)\lambda$ , we have  $\mathfrak{A}_{k-1, v-k}(k - \lambda, \lambda) \neq \emptyset$  by Lemma 4.1. Let  $E \in \mathfrak{A}_{k-1, v-k}(k - \lambda, \lambda)$ . Since  $k - \lambda \leq v - k - 1$  and  $k - \lambda$  is odd, we can form a cosymmetric circulant matrix  $G \in \mathfrak{A}_{v-k}^*(k - \lambda)$  not containing the 0-symmetric circulant permutation matrix of order  $v - k$ . We then construct

$$A = \left[ \begin{array}{c|c|c} 1 & \mathbf{e} & \mathbf{0} \\ \hline \mathbf{e}^T & F & E \\ \hline \mathbf{0}^T & E^T & G \end{array} \right] \in \mathfrak{A}_v^*(k).$$

By an argument similar to that in case (a), we verify that  $A$  and  $A_c$  are the incidence matrices of the desired PBPDs and have type RI König decompositions whose sum is the desired type RI König decomposition of  $J$ .

Case (e):  $v$  even,  $k$  even,  $\lambda$  even. We assume here that  $k < v/2$ . Let  $\mathbf{e} = (1, \dots, 1)$  of length  $k - 1$  and  $\mathbf{0} = (0, \dots, 0)$  of length  $v - k$ . Since  $\lambda - 1 \leq k - 1$  and  $\lambda - 1$  is odd, we can form a cosymmetric circulant matrix  $F \in \mathfrak{A}_{k-1}^*(\lambda - 1)$ . Since  $0 < k - \lambda \leq v - k$ ,  $0 < \lambda \leq k - 1$ , and  $(k - 1)(k - \lambda) = (v - k)\lambda$ , we have by Lemma 4.1 that  $\mathfrak{A}_{k-1, v-k}(k - \lambda, \lambda) \neq \emptyset$  and that  $\rho(E') = \min\{k - 1, v - k\} = k - 1$  for every  $E' \in \mathfrak{A}_{k-1, v-k}(k - \lambda, \lambda)$ . By permutation of columns of an  $E'$  we can obtain a matrix  $E \in \mathfrak{A}_{k-1, v-k}(k - \lambda, \lambda)$  of the form

$$E = \left[ \begin{array}{c|c} \text{---} & \text{---} \\ \textcircled{1} & \\ \textcircled{1} & \cdot \\ & \cdot \\ & \cdot \\ & \textcircled{1} \\ \hline \underbrace{\hspace{10em}}_{k-2} & \left. \vphantom{\begin{matrix} \textcircled{1} \\ \textcircled{1} \\ \cdot \\ \cdot \\ \cdot \\ \textcircled{1} \end{matrix}} \right\} k-2 \end{array} \right],$$

where the entries in  $E$  other than the circled 1's are unspecified. Since  $0 \leq k - \lambda - 2 \leq v - k - 4$  and  $k - \lambda - 2$  is even, we can form a cosymmetric matrix  $G \in \mathfrak{A}_{v-k}^*(k - \lambda)$  of the form

$$G = \left[ \begin{array}{ccc|ccc} 1 & 1 & & & & \\ 1 & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & 1 & \\ & & & 1 & 1 & \\ \hline & & & 1 & \textcircled{1} & \\ & & & \textcircled{1} & 1 & \\ & & & & 1 & \textcircled{1} \\ & & & & \textcircled{1} & 1 \\ & & & & & \ddots \\ & & & & & & 1 & \textcircled{1} \\ & & & & & & \textcircled{1} & 1 \end{array} \right],$$

$\underbrace{\hspace{15em}}_{k-2}$

where the unspecified entries in  $G$  are the corresponding entries from the sum of  $(k - \lambda - 2)/2$  pairs of circulant permutation matrices and their transposes, no circulant permutation matrix being 0-symmetric. We then construct

$$A = \left[ \begin{array}{c|c|c} 1 & \mathbf{e} & \mathbf{0} \\ \hline \mathbf{e}^T & F & E \\ \hline \mathbf{0}^T & E^T & G \end{array} \right] \in \mathfrak{A}_v^*(k).$$

Since the inner product of the first row of  $A$  with any other row is  $\lambda$ ,  $A$  is the incidence matrix of a  $\langle v, k, \lambda \rangle$  PBPD. Now, the first 1 in  $\mathbf{e}$ , the first 1 in  $\mathbf{e}^T$ , the circled 1's in  $E$ , the corresponding circled 1's in  $E^T$ , and the circled 1's in  $G$  together form a 0-symmetric permutation matrix  $Q$  contained in  $A$ . The matrix  $A - Q \in \mathfrak{A}_v^*(k - 1)$  is cosymmetric and  $k - 1$  is odd, hence by Lemma 3.8(ii)  $A - Q$  has a type RI König decomposition. Furthermore,  $A_c \in \mathfrak{A}_v^*(v - k)$  and the inner product of the first row of  $A_c$  with any other row is  $\lambda_c$ , whence  $A_c$  is the incidence matrix of a  $\langle v, k_c, \lambda_c \rangle$  PBPD, and, since  $A_c$  is 0-symmetric and  $v - k$  is even, we have by Lemma 3.8(i) that  $A_c$  has a type RI König decomposition. Since  $Q$ ,  $A - Q$ , and  $A_c$  are all symmetric matrices, the sum of  $Q$  and





Since the inner product of the first row of  $A$  with any other row is  $\lambda$ ,  $A$  is the incidence matrix of a  $\langle v, k, \lambda \rangle$  PBPD. Now, the first 1 in  $\mathbf{e}$ , the first 1 in  $\mathbf{e}^T$ , the circled 1's in  $F$ , and the circled 1's in  $G$  together form a 0-symmetric permutation matrix  $Q$  contained in  $A$ . The remainder of the verification is similar to that in case (e).

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