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Formulas for the number of $(n-2)$-gaps of binary objects in arbitrary dimension

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ABSTRACT

In this paper we define the notion of a gap in an arbitrary digital binary object S in a digital space of arbitrary dimension. Then we obtain an explicit formula for the number of gaps in S of maximal dimension, derive combinatorial relations for digital curves, and discuss possible applications to image analysis of digital surfaces (in particular planes) and curves.

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1. Introduction

Informally speaking, gap[2] is a location in a digital object (that is any finite set of pixels/voxels in two or three dimensions) through which a "discrete path" can penetrate. Theoretical studies of objects of this sort are related to combinatorial geometry and topology, but are also of interest in several other disciplines, such as digital geometry, combinatorial image analysis, and theory of computer graphics. A classical result of combinatorial topology is the famous Descartes–Euler formula $v - e + f = 2$ that relates the number of vertices (v), edges (e), and facets (f) of a polytope. Other related well-known concepts are the Euler–Poincaré characteristic of an object and its Betti numbers. For various applications of these to image analysis and digital geometry, see Chapters 4 and 6 of [25]. Gaps are considered in rendering pixelized/voxelized scenes, which is done by casting digital rays from the image to the scene [18,23]. This is particularly interesting when dealing with digital curves or surfaces. Assume, for example, that an unknown closed continuous surface Γ has been digitized, e.g., by a tomography scanner, and a digital surface S is obtained. Information about the gaps of S is used when it is traced through digital rays (e.g., for visualization or illumination purposes), since the penetration of a ray through the surface causes a false hole in it. Knowledge about the type of gaps of S may predetermine the usage of an appropriate type of digital rays for tracing the surface in order to avoid wrong conclusions about the topology of the original continuous surface Γ. Then, for the purposes of surface reconstruction, one will be able to faithfully model the geometry of the original three-dimensional set. This is of importance for three-dimensional imaging in medicine (e.g., organ and tumor measurements in CT images, beating

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2 A brief discussion concerning the use of the term "gap" versus "tunnel" is provided in a later section.

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heart, or lung simulations), bioinformatics (e.g., protein binding simulations), robotics (e.g., motion planning), or engineering (e.g., finite element stress simulations). In fact, a small hole in a heart surface created by imperfections of the synthetic representation, while possibly insignificant (or simply unnoticeable) for visualization, renders the synthetic surface useless for blood flow simulation. Also note that digital picture gapfreeness appears to be equivalent to the notion of well-composedness of a set of pixels/voxels proposed by Latecki, Eckhardt, and Rosenfeld [28]. This last paper demonstrates the advantages of using well-composed (gapfree) sets in image analysis. Therefore, it is important to have sound mathematical definitions and results about the gaps in a digital object, that can facilitate achieving correctness of key topological properties of synthetic surfaces.

It is also useful to have an estimation for the number of gaps (if any) in a digital object, possibly as a function of other object parameters. Such a kind of information may help us to better understand the topological structure of a binary picture and is of potential interest in property-based image analysis. Of special interest are the gaps of maximal dimension (to be defined later) since they can be penetrated by a digital ray of any connectivity. Moreover, estimations of the number of such gaps may be useful for evaluating the performance of some polyhedra decomposition algorithms (see comments in Section 6). Other arguments are provided in Section 4.1.

Studies about gaps and tunnels in digital lines, planes, and polyhedral surfaces are available, e.g., in [1–3,5–8,11,12]. A recent work [15] provides the formula $g = v - 2(p + c - h) + b$, where g is the number of gaps, v the number of vertices, p the number of pixels, h the number of holes, c the number of connected components, and b the number of 2×2 grid squares in a two-dimensional digital picture. For another similar result we refer to [16].

The notion of a gap has been used in higher dimensions, too [4]. However, rigorous definitions and results that apply to an arbitrary binary object were missing. Approaches to estimating the number of gaps have been, overall, unclear.

Formula for the number of gaps of dimension $n - 2$ was recently obtained in [17]. Here we extend the results of this last work and exhibit relations to other studies, such as image analysis of surfaces and theory of digital planes.

In the next section we recall some basic notions and notations of digital geometry and topology. In Section 3 we introduce the concepts of tandems, gaps, and brims that are used to obtain the main results. In Section 4.2 we obtain an explicit formula for the number of $(n - 2)$-gaps of a binary object in a digital space of arbitrary dimension. We also demonstrate why knowledge about $(n - 2)$-gaps is important in the case of digital surfaces and planes and obtain some results about gaps in digital planes. In Section 5 we obtain combinatorial relations for digital curves. We conclude with some remarks in Section 6.

2. Basic notions

In this section we introduce some basic notions to be used in what follows. We conform to the terminology used in [25] (see also [26]).

All considerations take place in the *grid cell model* that consists of the grid cells of $\mathbb{Z}^n$, together with the related topology. In that model we represent n-cells as hyper-cubes (sometimes called *hyper-voxels*, or *voxels*, for short). Their edges and vertices are 1-*cells* and 0-*cells*, respectively. For every $i = 0, 1, \ldots, n$, the set of all cells of dimension i (or i-cells) is denoted by $\mathbb{C}_n^{(i)}$. Further, we define the space $\mathbb{C}_n = \bigcup_{k=0}^{n} \mathbb{C}_n^{(i)}$.

We say that two n-cells e, e' are k-adjacent for $0 \leq k \leq n - 1$ if they share a k-cell. Two n-cells are *strictly* k-adjacent if they are k-adjacent but not $(k + 1)$-adjacent. The n-cells that are k-adjacent to a given n-cell c constitute its *k-neighborhood* and are called *k-neighbors* of c.

One can consider the grid cell model as an *incidence structure*, i.e., as a triple $(\mathbb{C}_n, I, \dim)$, where I is an *incidence relation* on $\mathbb{C}_n$ that is reflexive and symmetric and dim is a function defined on $\mathbb{C}_n$ and into the set $\{0, 1, \ldots, n\}$. For example, a 2-cell c is incident with the 1-cells corresponding to its sides as well as with the 0-cells corresponding to its vertices. These 1- and 0-cells are also incident to c. The grid cell model can also be considered as an *abstract cell complex* $(\mathbb{C}_n, <, \dim)$ (see [27]), where $<$ is a *bounding relation*, that is antisymmetric, irreflexive, and transitive, and such that for every $e, e' \in \mathbb{C}_n$, $e < e'$ if and only if eIe' and $\dim(e) < \dim(e')$. Relation $<$ is a partial order on $\mathbb{C}_n$. The corresponding order topology $\tau(<)$ is called the *grid cell topology*.[3] In the rest of the paper, we will assume that the abstract cell complex $(\mathbb{C}_n, <, \dim)$ is equipped with the topology $\tau(<)$. Then, for any subset A of $\mathbb{C}_n$, its *boundary* $\partial(A)$ is defined as the set of all points x of $\mathbb{C}_n$ such that every open neighborhood of x meets A and $\mathbb{C}_n \setminus A$, while its *interior* $\mathrm{int}(A)$ is the set of all points x of $\mathbb{C}_n$ such that there exists some open neighborhood of x contained in A. The points of $\mathrm{int}(A)$ are the *internal points* of A. Given a digital object S, note that its closure $\bar{S}$ is naturally a subcomplex of $\mathbb{C}_n$. In what follows, we will denote by S_k the set of k-cells of $\bar{S}$, i.e. $S_k = \bar{S} \cap \mathbb{C}_n^{(k)}$. In particular, we have $S_n = \bar{S} \cap \mathbb{C}_n^{(n)} = S$.

A digital object $S \subset \mathbb{C}_n$ is a finite set of n-cells. A *k-path* $(0 \leq k \leq n - 1)$ in S is a sequence of voxels from S such that every two consecutive voxels on the path are k-adjacent. Two voxels of a digital object S are *k-connected* (in S) iff there is a k-path in S between them. A subset G of S is *k-connected* iff there is a k-path connecting any two voxels of G. The maximal (by inclusion) k-connected subsets of a digital object S are called *k-components* of S. Components are nonempty, and distinct k-components are disjoint. For a given subset M of a digital object D, if $D \setminus M$ is not m-connected then the set M is said to be *m-separating* in D.

[3] In that topology the open sets are precisely the sets $U \subseteq \mathbb{C}_n$, such that, for every $u \in U$ and every $v \in \mathbb{C}_n$ with $u < v$, we have $v \in U$.

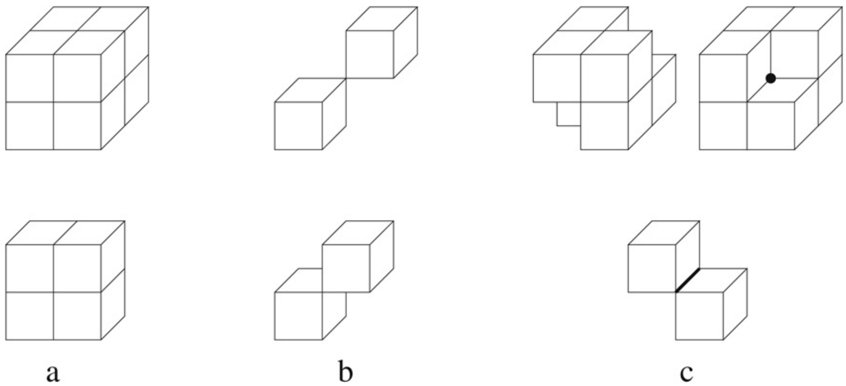

Fig. 1. Illustration to some notions in three dimensions. (a) *Top*: 2^3-block; *Bottom*: $2^2 1^1$-block. (b) *Top*: 0-tandem; *Bottom*: 1-tandem. (c) *Top*: configuration exposing a 0-gap (in two different orientations); *Bottom*: configuration exposing a 1-gap.

Let M be an m-separating digital object in D such that $D \setminus M$ has exactly two m-components. An *m-simple cell* (or *m-simple point*) of M (with respect to D) is an n-cell c such that $M \setminus \{c\}$ is still m-separating in D. An m-separating digital object in D is *m-minimal* (or *m-irreducible*) if it does not contain any m-simple cell (with respect to D).

Let $N_\alpha(c)$ be the *unit α-ball* (also called the *α-neighborhood* of c) with center c, consisting of all α-neighbors of c. Furthermore, let $A_\alpha(c) = N_\alpha(c) \setminus \{c\}$ be the *α-adjacency set* of c. Mylopoulos and Pavlidis [29] proposed the following recursive definition of dimension of a (finite or infinite) set of n-cells S with respect to an adjacency relation A_α.[4] A nonempty set $D \subseteq \mathbb{C}_n^{(n)}$ is called *totally α-disconnected* iff $A_\alpha(x) \cap D = \emptyset$ for any $x \in D$ (i.e., there is no pair of cells $c, c' \in D$ such that $c \neq c'$ and $\{c, c'\}$ is α-connected). $D \subseteq \mathbb{C}_n^{(n)}$ is called *linearly α-connected* whenever $|A_\alpha(x) \cap D| \leq 2$ for all $x \in D$ and $|A_\alpha(x) \cap D| > 0$ for at least one $x \in D$. Further, let $B_\alpha(c)$ be the union of $N_\alpha(c)$ with all n-cells c' for which there exist $c_1, c_2 \in N_\alpha(c)$ such that a shortest α-path from c_1 to c_2 not passing through c passes through c'. (For example, $B_1(c) = B_0(c) = N_0(c)$ for $n = 2$, and $B_2(c) = B_1(c) = N_1(c)$, $B_0(c) = N_0(c)$ for $n = 3$.) Denote $B^*_\alpha(c) = B_\alpha(c) \setminus \{c\}$. Now let D be a digital object in $\mathbb{C}_n^{(n)}$ and A_α an adjacency relation on $\mathbb{C}_n^{(n)}$. Then the (discrete) *dimension* $\dim_\alpha(D)$ of D is defined as follows:

(1) $\dim_\alpha(D) = -1$ iff $D = \emptyset$;
(2) $\dim_\alpha(D) = 0$ iff D is a nonempty, totally α-disconnected set;
(3) $\dim_\alpha(D) = 1$ if D is linearly α-connected;
(4) $\dim_\alpha(D) = \max_{c \in S} \dim_\alpha(B^*_\alpha(c) \cap D) + 1$ otherwise.

3. Tandems, gaps, and brims

In this section we introduce the notions of tandem, gap, and brim of arbitrary dimension. These notions, that were first introduced in [17], will be instrumental in obtaining our main results.

A $\underbrace{2 \times \cdots \times 2}_{k} \times \underbrace{1 \times \cdots \times 1}_{n-k}$ grid parallelepiped in $\mathbb{C}_n$ will be called $2^k 1^{n-k}$*-block* ($0 \leq k \leq n$). In particular, any voxel is a 1^n-block. See Fig. 1a for illustrations.

Now we are able to give the following definition.[5]

Definition 1. A pair $t_k = (v_1, v_2)$ of two strictly k-adjacent voxels v_1 and v_2, for $0 \leq k \leq n-1$, is called a *k-tandem*. Then the complement of t_k w.r.t. a $2^{n-k}1^k$-block, for $0 \leq k \leq n-2$, determines a *k-gap* of S.

A k-gap ($0 \leq k \leq n-3$) that is not a $(k+1)$-gap is called a *proper k-gap*.

Remark 2. Technically, the complement of an $(n-1)$-tandem to a $2^1 1^{n-1}$-block can be considered as defining an $(n-1)$-gap. In classic combinatorial topology these are called "tunnels", or sometimes also "holes" (see [25]). The fact is that digital geometers have not reached an agreement so far on the terminology concerning "gaps" and "tunnels". To some authors, k-gaps ($0 \leq k \leq n-1$) are better called k-tunnels. To others, using the word "tunnel" (that had a specific topological meaning

[4] Formally, A_α can be considered as a relation over S defined as follows: $A_\alpha \subseteq S \times S$, where for $c_1, c_2 \in S$, $(c_1, c_2) \in A_\alpha$ if and only if $c_1 \in A_\alpha(c_2)$ (then $c_2 \in A_\alpha(c_1)$ also).

[5] Usually, gaps are defined through separability as follows: Let a digital object M be m-separating but not $(m-1)$-separating in a digital object D. Then M is said to have *k-gaps* for any $k < m$. A digital object without m-gaps is called *m-gapfree*. See Fig. 5. Our technical definition of a gap seems to better fit the considerations that follow.

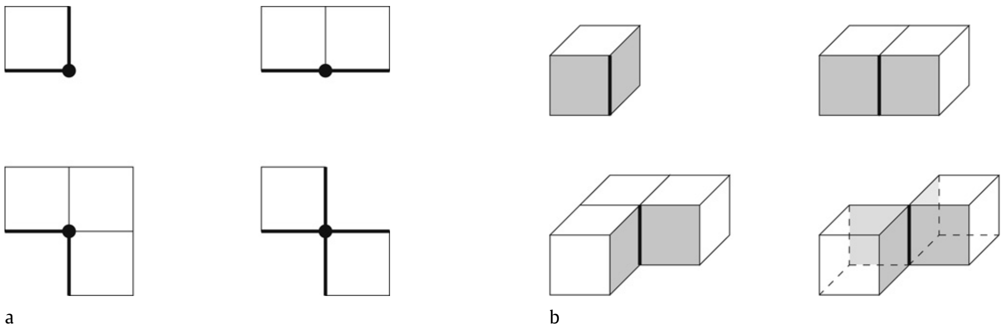

Fig. 2. (a) Possible 1-brims in two dimensions. (b) Possible 2-brims in three dimensions.

for a long time) for what they call "gap" is unacceptable. More specifically, to these last researchers mixing gaps (that have dimension $0 \le k \le n-2$) with tunnels (that are $(n-1)$-dimensional) is not a good idea because of the above-mentioned historical reasons. Agreement on this issue is definitely desirable. Here we do not take side since it seems irrelevant to the purpose of the present paper. Anyway, we have decided to use the word "k-gap" instead of "k-tunnel" for $0 \le k \le n-2$ and to use the word "tunnel" instead of "$(n-1)$-gap".

There are $n-1$ types of gaps: $0, 1, 2, \dots,$ and $(n-2)$-gaps. Given a digital object S, the number of its tandems and gaps will be denoted by $b_0, b_1, \dots, b_{n-1}$ and $g_0, g_1, \dots, g_{n-2}$, respectively. Fig. 1b and c illustrates tandems and gaps in dimension three.

In what follows we will also use the following technical notion.

Definition 3. Let $c \in \partial(S_{k-1})$ for some k $(1 \le k \le n)$ and let $b_k(c)$ be the set of elements of $\partial(S_k)$ incident to it. Then the pair $br_k(c) = (c, b_k(c))$ is called a *k-brim* of S. We will say that $br_k(c)$ is *hinged* on c.

Basically, k-brims of a digital object delineate its "k-dimensional" boundary. A set of voxels in a digital object will be called *configuration*. Fig. 2 displays possible configurations of pixels/voxels that expose 1-brims in $\mathbb{C}_2$ and 2-brims in $\mathbb{C}_3$ (note that there is one-to-one correspondence between both). We remark that there are 19 distinct configurations of voxels that expose 1-brims in $\mathbb{C}_3$.

4. $(n-2)$-gaps in binary objects

4.1. Why $(n-2)$-gaps are important?

In this section we provide some evidence for the importance of $(n-2)$-gaps in binary objects. We also propose a generalization of the notion of gap, which will be further discussed in Section 4.3 in the context of digital planes.

4.1.1. Digital surfaces

Because of their relevance to practical problems, digital surfaces have been widely studied over the years (see [14] and the bibliography therein). A recent paper [14] provides the first definition of a digital surface involving the notion of dimension in digital spaces, with the explicit goal to make the notions of a digital surface/curve compatible with corresponding classical definitions in continuous topology. On this basis, classification of digital surfaces was obtained with respect to the type of gaps they possess. In order to make the paper self-contained and to be able to make meaningful comments and conclusions in view of the title of Section 4.1, we briefly recall the definition from [14][6] (further details and results are available therein).

The first step is defining a one-dimensional digital manifold, i.e., a simple digital curve. The latter admits various equivalent definitions [13], one of which is the following. A *simple digital α-curve* is a set $\gamma = \{c_1, c_2, \dots, c_l\}$ of voxels that satisfy the following two axioms: (A1) c_i is α-adjacent to c_j iff $i = j \pm 1$(modulo l), and (B1) γ is one-dimensional with respect to α-adjacency. Then a digital manifold is recursively defined as follows:

(i) M is a one-dimensional $(n-1)$-manifold in $\mathbb{C}_n^{(n)}$ if it is an $(n-1)$-curve in $\mathbb{C}_n^{(n)}$. For $2 \le k \le n-1$, M is a k-dimensional $(n-1)$-manifold in $\mathbb{C}_n^{(n)}$ if: (1) M is $(n-1)$-connected (or, equivalently, M consists of a single $(n-1)$-component); (2) for any $x \in M$, the set $A_0(x) \cap M$ is a $(k-1)$-dimensional $(n-1)$-manifold in $\mathbb{C}_n^{(n)}$.

6 The definition applies to digital analogs of hole-free hypersurfaces, i.e., ones without tunnels ($(n-1)$-gaps). In fact, in the framework of the approach of [14], a hypersurface with tunnels can be an $(n-2)$-dimensional set of n-cells (see Fig. 3 (left)), which would not be in conformity with the continuous case.

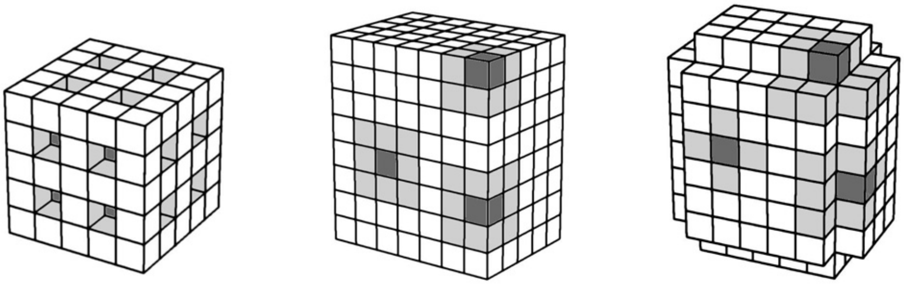

Fig. 3. *Left*: this digital object has tunnels and is one-dimensional with respect to 2-adjacency. *Middle* (*Right*): A 1-surface (2-surface) on which three sample voxels are emphasized (in dark gray), together with the 1-curves (2-curves) adjacent to them (in light gray).

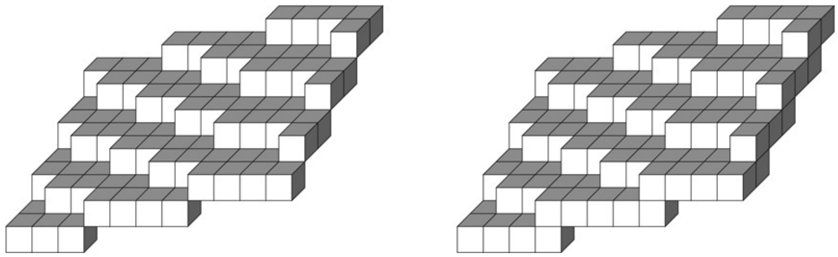

Fig. 4. Portion of a naive arithmetic plane (*left*) and a standard arithmetic plane (*right*).

(ii) M is a one-dimensional α-manifold ($0 \leq \alpha \leq n-2$) in $\mathbb{C}_n^{(n)}$ if M is an α-curve in $\mathbb{C}_n^{(n)}$. M is a k-dimensional α-manifold ($0 \leq k \leq n-1$, $0 \leq \alpha \leq n-2$) in $\mathbb{C}_n^{(n)}$ if: (1) M is α-connected (or, equivalently, M consists of a single α-component); (2) for any $x \in M$ the set $A_0(x) \cap M$ is a $(k-1)$-dimensional α-manifold in $\mathbb{C}_n^{(n)}$ but is not a $(k-1)$-dimensional $(\alpha+1)$-manifold in $\mathbb{C}_n^{(n)}$. (Such an α-manifold is also called *proper*.)

In the particular case when M is an $(n-1)$-dimensional α-manifold in $\mathbb{C}_n^{(n)}$ for $\alpha = n-2$ or $n-1$, we say that M is a *(closed) digital α-hypersurface*. See Fig. 3 (middle and right) for illustrations of 1- and 2-surfaces. It has been shown in [14] that digital hypersurfaces admit a classification in which $(n-2)$-gaps play a special role:

There are two and only two basic types of α-hypersurfaces: one for $\alpha = n-1$ and one for $\alpha = n-2$:

For $\alpha = n-2$, a hypersurface S has $(n-2)$-gaps which appear on the $(n-2)$-manifolds that build it and, possibly, between adjacent pairs[7] of such $(n-2)$-manifolds.

For $\alpha = n-1$, the hypersurface S is gapfree.

Moreover, an $(n-2)$-surface is $(n-1)$-gapfree and $(n-1)$-minimal, while an $(n-1)$-surface is 0-gapfree and 0-minimal.

Thus, knowing the type of a given digital surface, one can have correct expectations about the result of tracing the surface by digital rays of a certain type. If the surface type is unknown, then information about the surface gaps is needed. If in particular it is known that $g_{n-2} = 0$, the surface can be traced by $(n-2)$-connected digital rays with no danger for loss of information [18,23].

4.1.2. Gaps and cracks

For the sake of clarity, we restrict ourselves to the practically important case of digital surfaces in three dimensions. Let S be an $(n-2)$-surface satisfying the definition of Section 4.1.1 and g an $(n-2)$-gap of S (i.e., a 1-gap). Geometrically, g can be regarded as a segment on the boundary of S through which the gap occurs. We will call that segment the *support* of gap g and denote it by $s(g)$. See Fig. 1c (bottom).

Now let g' and g'' be two gaps of S and $s(g')$ and $s(g'')$ their respective supports. If the segments $s(g')$ and $s(g'')$ are incident, then their union represents a "larger gap" that passes through $s(g') \cup s(g'')$. Obviously, starting from any particular gap, an agglomeration process can be performed until a maximal (non-extendable) gap is obtained. We will call such a gap a *crack*. The union of all cracks forms the *gap skeleton* for a digital surface. (See Figs. 3 (right), and 4 (left)).

Clearly, the above notions extend to arbitrary digital objects. It is not hard to realize that if S is a closed digital 1-surface in $\mathbb{C}_3$ (in the sense of the definition above), the gap skeleton can be represented by a planar graph. The gap skeleton and its graph may provide useful information about the structure of the surface, in particular about its 2-connected components. For example, the digital surface in Fig. 3 (right) has only one crack (whose edges form a parallelepiped). That crack constitutes the gap skeleton of the surface. It partitions the surface into six 2-connected components. Cracks in digital planes will be

[7] Actually, two such manifolds, called "adjacent", may have both adjacent and common n-cells.

discussed in Section 4.3. Note that the structure of a set of 0-gaps in a digital surface is as that of a set of isolated points where the 0-gaps are located (remember Fig. 1c, top), i.e., in this case the notions of crack and gap skeleton do not admit a reasonable counterpart.

4.1.3. Polyhedron decomposition

Knowledge of the number of gaps of maximal dimension can also be applied to the well-known polyhedron decomposition problem [21,22], that is to partition a given non-convex polyhedron into as small as possible number of convex polytopes. Specifically, let P be the rectilinear polyhedron defined as a union of a set of voxels of $\mathbb{C}_3$. The fact is that all bounds on the number of convex polytopes obtained by decomposition algorithms, as well as the computational complexity of these algorithms, are in terms of the number r of "notches" of P, that are locations causing non-convexity.[8] It is not hard to see that the number of gaps in the discrete surface constituted by the boundary voxels of P is an upper bound for r. A more careful study of this point is seen as a further task.

4.2. Formula for the number of $(n-2)$-gaps

For a given digital object $S \subset \mathbb{C}_n$, let $s_i = |S_i|$, $0 \le k \le n$. In this section we prove the following theorem.

Theorem 4. *For a given digital object $S \subset \mathbb{C}_n$,*

$$g_{n-2} = -2n(n-1)s_n + 2(n-1)s_{n-1} - s_{n-2} + b, \tag{1}$$

where b is the number of $2^2 1^{n-2}$-blocks of S.

Proof. For any $c \in S_{k-1}$, $1 \le k \le n-1$, we define

$$I_k(c) = \{c' \in S_k : c \text{ is incident with } c'\}.$$

We also define

$$\begin{aligned} \mathrm{int}(S_{k-1}) &= \{c \in S_{k-1} : c \in \mathrm{int}(S)\} \\ \partial(S_{k-1}) &= \{c \in S_{k-1} : c \in \partial(S)\} \\ \partial(S_k) &= \{c \in S_k : c \in \partial(S)\}. \end{aligned}$$

It is easy to see that a $(k-1)$-cell belongs to $\mathrm{int}(S_{k-1})$ iff it is incident with $2^{n-(k-1)}$ n-cells of S. Otherwise, it belongs to the boundary of S.

For $c \in S_{n-1}$ we can consider $I_n(c) = \{c' \in S_n : c \text{ is incident with } c'\}$. The possible values for $|I_n(c)|$ are 1 and 2. More precisely, we have

$$\begin{aligned} \mathrm{int}(S_{n-1}) &= \{c \in S_{n-1} : I_{n-1}(c) = 2\}, \\ \partial(S_{n-1}) &= \{c \in S_{n-1} : I_{n-1}(c) = 1\}, \end{aligned}$$

and so

$$S_{n-1} = \mathrm{int}(S_{n-1}) \cup \partial(S_{n-1}).$$

Let us denote $s^{\mathrm{int}}_{n-1} = |\mathrm{int}(S_{n-1})|$ and $s^{\partial}_{n-1} = |\partial(S_{n-1})|$. Then $s_{n-1} = s^{\mathrm{int}}_{n-1} + s^{\partial}_{n-1}$. Since every n-cell of S is incident with $2n$ $(n-1)$-cells from S_{n-1}, we obtain

$$2n|S| = s^{\partial}_{n-1} + 2s^{\mathrm{int}}_{n-1}.$$

From here we get

$$s^{\mathrm{int}}_{n-1} = ns_n - \frac{s^{\partial}_{n-1}}{2}.$$

Next we consider incidence relations between elements of $\partial(S_{n-1})$ and S_{n-2}. For any $c \in S_{n-2}$ we consider the brim hinged on c:

$$br_{n-1}(c) = \{c' \in \partial(S_{n-1}) : c \text{ is incident with } c'\}.$$

The possible values for $|br_{n-1}(c)|$ are 0, 2, and 4. This partitions S_{n-2} as follows:

$$S_{n-2} = S^0_{n-2} \cup S^2_{n-2} \cup S^4_{n-2}, \tag{2}$$

[8] Notch (or reflex edge) is an edge of a polyhedron where the inner dihedral angle subtended by two incident facets is greater than 180 degrees.

where $S^i_{n-2} = \{c \in S_{n-2} : |br_{n-1}(c)| = i\}$, for $i = 0, 2, 4$. If we denote $\bar{s}^i_{n-2} = |S^i_{n-2}|$, $i = 0, 2, 4$, we get $s_{n-2} = \bar{s}^0_{n-2} + \bar{s}^2_{n-2} + \bar{s}^4_{n-2}$. From here, we obtain $\bar{s}^2_{n-2} = s_{n-2} - \bar{s}^0_{n-2} - \bar{s}^4_{n-2}$.

Every cell $x \in S^\partial_{n-1}$ is incident with $2(n-1)$ cells $y \in S_{n-2}$. Then it follows that

$$\begin{aligned} 2(n-1)s^\partial_{n-1} &= 4\bar{s}^4_{n-2} + 2\bar{s}^2_{n-2} = 4\bar{s}^4_{n-2} + 2(s_{n-2} - \bar{s}^0_{n-2} - \bar{s}^4_{n-2}) \\ &= 2\bar{s}^4_{n-2} + 2s_{n-2} - 2\bar{s}^0_{n-2}, \end{aligned}$$

from where we obtain

$$s^\partial_{n-1} = \frac{\bar{s}^4_{n-2} + s_{n-2} - \bar{s}^0_{n-2}}{n-1}.$$

Then

$$s_{n-1} = s^{\mathrm{int}}_{n-1} + s^\partial_{n-1} = ns_n - \frac{s^\partial_{n-1}}{2} + s^\partial_{n-1} = ns_n + \frac{s^\partial_{n-1}}{2},$$

i.e.,

$$s_{n-1} = ns_n + \frac{\bar{s}^4_{n-2} + s_{n-2} - \bar{s}^0_{n-2}}{2(n-1)}. \tag{3}$$

Thus

$$2(n-1)s_{n-1} = 2n(n-1)s_n + \bar{s}^4_{n-2} + s_{n-2} - \bar{s}^0_{n-2},$$

and

$$\bar{s}^4_{n-2} = -2n(n-1)s_n + 2(n-1)s_{n-1} - s_{n-2} + \bar{s}^0_{n-2}.$$

We also have the following fact:

For any $n \geq 2$, the sets of $(n-2)$-gaps and $(n-2)$-tandems are determined by the same configurations.

Then it is enough to observe that $\bar{s}^4_{n-2} = g_{n-2}$ is the number of $(n-2)$-gaps (that are also $(n-2)$-tandems) and $\bar{s}^0_{n-2} = b$ the number of $2^2 1^{n-2}$-blocks of S, and we obtain the result stated. $\square$

The above theorem implies different combinatorial relations, e.g., in lower dimensions. In particular, we can easily obtain the formula from [15] recalled in the Introduction (its original proof by induction was pretty long). To this end, first observe that for $n = 2$ the only gaps in S are the 0-gaps. For this case equality (3) has the form $s_1 = 2s_2 + \frac{1}{2}(g_0 + s_0 - b)$, where b is the number of (2×2)-blocks in S. Now, by the Euler–Poincaré characteristic we have $s_0 - s_1 + s_2 = \beta_0 - \beta_1 + \beta_2$, where $\beta_0, \beta_1, \beta_2$ are the Betti numbers [25]. From here we get $s_2 - (2s_2 + \frac{1}{2}(s_0 - b + g_0)) + s_0 = \beta_2 - \beta_1 + \beta_0$. Since S is homotopic to a one-dimensional CW-complex, we have $\beta_2 = 0$. Moreover, β_0 is the number of connected components of S, while β_1 is the number of its holes. From here we immediately obtain the formula of [15].

4.3. Gaps and cracks in digital planes

Important examples of digital hypersurfaces are the *digital hyperplanes* (digital planes, in three dimensions). These are well studied from various points of view. In particular, digital hyperplanes admit an analytical description (see [4,12,30]):

A set $P(b, a_1, a_2, \ldots, a_n, \omega) = \{x \in \mathbb{Z}^n \mid 0 \leq b + \sum_{i=1}^n a_i x_i < \omega\}$ is a *digital hyperplane*[9] with coefficients $a_1, a_2, \ldots, a_n$, b and *thickness* ω. A digital hyperplane with thickness $\omega = |a|_{\max} = \max\{|a_1|, |a_2|, \ldots, |a_n|\}$ is called *naive*, and one with thickness $\omega = \sum_{i=1}^n |a_i|$ is called *standard*. For $n = 2$ and 3 one obtains a definition of a digital line and digital plane, respectively. See Fig. 4 for illustration of naive and standard planes and Fig. 5 (left), for illustrations of disconnected, naive, and standard lines. The standard lines and planes are the most widely used digital lines and planes.

Conditions under which a digital line/plane may have gaps are well known. The main result due to Andres et al. is the following theorem.

Theorem 5 *([4, Proposition 9]). Let $P = P(b, a_1, a_2, \ldots, a_n, \omega) = \{x \in Z^n \mid 0 \leq b + \sum_{i=1}^n a_i x_i < \omega\}$ be a discrete hyperplane, where $b \geq 0$, $a_i \geq 0$ for all i, and $a_i \leq a_{i+1}$ for $1 \leq i \leq n-1$. Then, if $\omega < a_n$, the hyperplane has tunnels; for $0 < k < n$, if $\sum_{i=k+1}^n a_i \leq \omega < \sum_{i=k}^n a_i$, the hyperplane has $(k-1)$-gaps and is k-separating; if $\omega \geq \sum_{i=1}^n a_i$, the hyperplane is gapfree.*

[9] Also called an "arithmetic" hyperplane.

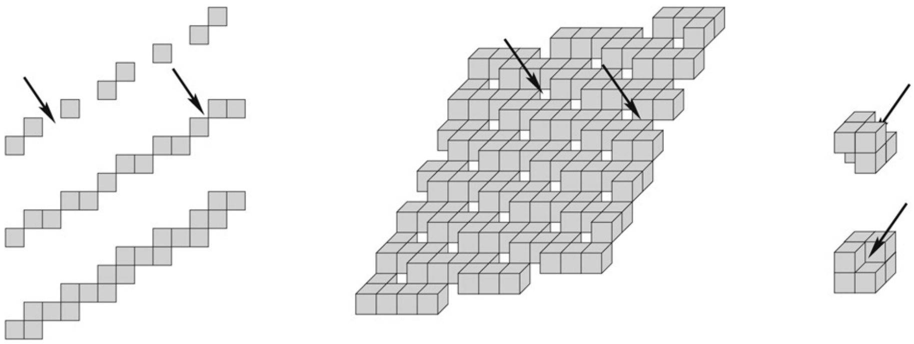

Fig. 5. *Left*: from top to bottom: portions of arithmetic lines defined by $0 \le 3x_1 - 5x_2 < 3$ (disconnected line), $0 \le 3x_1 - 5x_2 < 5$ (naive line), and $0 \le 3x_1 - 5x_2 < 8$ (standard line). The first one has tunnels (and also 0-gaps; a tunnel is pointed out by an arrow), the second one has 0-gaps (one of them pointed out by an arrow) but no tunnels, and the third one is gapfree. *Middle*: portion of an arithmetic plane defined by $0 \le 2x_1 + 5x_2 + 9x_3 < 7$. It has tunnels (and also 1- and 0-gaps). A tunnel and a 1-gap are pointed out by arrows. *Right*: configuration of voxels (in two different orientations) that features a 0-gap (pointed out by an arrow).

For the particular case $n = 3$, a digital plane with $\omega < a_3$ may have tunnels; a naive plane with $\omega = a_3$ has no tunnels; a plane with $a_3 \le \omega < a_2+a_3$ may have 1-gaps; a plane with $\omega = a_2+a_3$ has no 1-gaps; a plane with $a_2+a_3 \le \omega < a_1+a_2+a_3$ may have 0-gaps; a standard plane with $\omega = a_1 + a_2 + a_3$ is gapfree, and so is any plane with $\omega > a_1 + a_2 + a_3$.

Note that the two basic types of planes – naive and standard – cannot have proper 0-gaps. A naive plane may have only 1-gaps while a standard one is always gapfree. This fact demonstrates the importance of $(n-2)$-gaps in the geometry of digital planes once more.

In view of the discussion of the preceding section, Theorem 5 implies that any naive digital hyperplane is an $(n-2)$-hypersurface and a standard digital plane is an $(n-1)$-hypersurface. This follows from the fact that a digital plane with $\omega \ge \sum_{i=1}^{n} |a_i|$ is gapfree and 0-minimal while one with $\omega = |a|_{\max}$ is $(n-1)$-gapfree and $(n-1)$-minimal.

Theorem 5 provides an analytic characterization of digital planes with respect to their gaps, while Theorem 6 below provides a combinatorial characterization. Recall that an (p, q)-*cube* of a digital plane P at a point $(i, j) \in \mathbb{Z}^2$ is the set $\{(x, y, z) \in P : i \le x \le i + p - 1 \text{ and } j \le y \le j + q - 1\}$. Then we can formulate the following statement. (For the sake of simplicity it is done for $n = 3$.)

Theorem 6. *Let $P = P(b, a_1, a_2, a_3, \omega)$ be a digital plane whose coefficients satisfy $b > 0$, $0 < a_1 \le a_2 \le a_3$. Then the following three assertions are equivalent:*

(i) *P has 1-gaps;*
(ii) *$\omega < a_2 + a_3$;*
(iii) *for a sufficiently large (p, q)-cube C of P, $g_{n-2}(C) > 0$.*

Alternatively, the following are equivalent:

(i) *P has no 1-gaps;*
(ii) *$\omega \ge a_2 + a_3$;*
(iii) *for any (p, q)-cube C of P, $g_{n-2}(C) = 0$.*

One can also apply the formula of Theorem 4 to a (p, q)-cube C of a digital plane. It is well known that all digital planes $P(b, a_1, a_2, a_3, |a_3|)$, for $b = \pm1, \pm2, \ldots$ are equivalent up to a translation [9]. So, w.l.o.g., we may assume that $b = 0$. We also suppose that $0 < a_1 \le a_2 \le a_3$, and that the corresponding Euclidean plane $\bar{P} : a_1x_1 + a_2x_2 + a_3x_3 = 0$ makes with the coordinate plane Ox_1x_2 an angle θ with $0 \le \theta \le \arctan\sqrt{2}$ (see Fig. 6). It is known that these assumptions do not restrict the generality (see, e.g., [4,6,19]). The last one guarantees that each pixel of the integer grid in Ox_1x_2 is a projection of exactly one voxel from $P(0, a_1, a_2, a_3, |a_3|)$. Since such a digital plane is uniquely determined by its three coefficients, we can simplify its notation to $P(a_1, a_2, a_3)$.

Also recall that if $a_3 < a_1 + a_2$, then $P(a_1, a_2, a_3)$ has *jumps*, that are configurations as the one in Fig. 6 (right). If $a_3 \ge a_1 + a_2$, jumps do not occur [8].

Finally, recall that a digital plane can be represented by its *level lines*, defined by the following assertion (see [10,12,19, 20]).

Fact 7. *Let a naive plane $P(a_1, a_2, a_3) : 0 \le a_1x_1 + a_2x_2 + a_3x_3 < a_3$ be given.*

(a) *For a fixed value $x_3 = x_3^0 \in \mathbb{Z}$, that determines a level of $P(a_1, a_2, a_3)$, the projection $P(a_1, a_2, a_3)_{x_3=x_3^0} : 0 \le a_1x_1 + a_2x_2 + a_3x_3^0 < a_3$ of $P(a_1, a_2, a_3)$ on Ox_1x_2 is a digital line $L_{x_3^0} = L(a_1, a_2, -a_3x_3^0)$, called* level line *for $x_3 = x_3^0$. If $a_3 = a_1 + a_2$, then L is standard, and if $a_3 > a_1 + a_2$, then L is thicker than standard. If $a_3 < a_1 + a_2$, then L is thicker than naive and thinner than standard.*

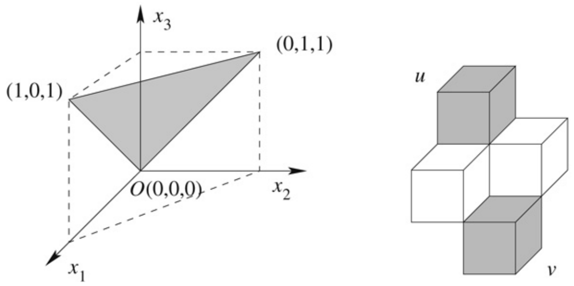

Fig. 6. *Left*: a plane forming an angle arctan $\sqrt{2}$ with the plane Ox_1x_2. *Right*: possible configuration of voxels in a discrete plane satisfying Condition (2). The voxels u and v form a jump.

a

0	0	−1	−1	−2	−2	−2	−3	−3	−3	−4	−4	−5	−5
0	0	0	−1	−1	−2	−2	−2	−3	−3	−3	−4	−4	−5
1	0	0	0	−1	−1	−1	−2	−2	−3	−3	−3	−4	−4
1	1	0	0	0	−1	−1	−1	−2	−2	−3	−3	−3	−4
2	1	1	1	0	0	−1	−1	−1	−2	−2	−2	−3	−3
2	2	1	1	1	0	0	−1	−1	−1	−2	−2	−2	−3
3	2	2	1	1	1	0	0	0	−1	−1	−2	−2	−2
3	3	2	2	1	1	1	0	0	0	−1	−1	−2	−2
3	3	3	2	2	2	1	1	0	0	0	−1	−1	−1
4	3	3	3	2	2	2	1	1	0	0	0	−1	−1

b

−1	−2	−2	−2	−3	−3	−3	−4	−4	−5	−5	−5	−6	−6
−1	−1	−1	−2	−2	−3	−3	−3	−4	−4	−4	−5	−5	−6
0	0	−1	−1	−2	−2	−2	−3	−3	−3	−4	−4	−5	−5
0	0	0	−1	−1	−1	−2	−2	−3	−3	−3	−4	−4	−4
1	1	0	0	0	−1	−1	−2	−2	−2	−3	−3	−3	−4
2	1	1	0	0	0	−1	−1	−1	−2	−2	−3	−3	−3
2	2	1	1	1	0	0	0	−1	−1	−2	−2	−2	−3
3	2	2	2	1	1	0	0	0	−1	−1	−1	−2	−2
3	3	3	2	2	1	1	1	0	0	0	−1	−1	−2
4	3	3	3	2	2	2	1	1	0	0	0	−1	−1

Fig. 7. (a) Level line code for $P(6, 7, 16)$ restricted to a (10, 14)-cube. All level lines have identical shape since $\gcd(6, 7) = 1$. There are 9 cracks in the cube. (b) Level line code for $P(6, 9, 16)$ restricted to a (10, 14)-cube. Distinct level lines may be different in shape since $\gcd(6, 9) = 3 \neq 1$ (see [10]). There are 10 cracks in the cube.

−1	−2	−2	−3	−3	−3	−4	−4	−5	−5	−6	−6	−7	−7
−1	−1	−1	−2	−2	−3	−3	−4	−4	−5	−5	−6	−6	−6
0	0	−1	−1	−2	−2	−3	−3	−4	−4	−4	−5	−5	−6
1	0	0	−1	−1	−2	−2	−2	−3	−3	−4	−4	−5	−5
1	1	0	0	0	−1	−1	−2	−2	−3	−3	−4	−4	−5
2	2	1	1	0	0	−1	−1	−2	−2	−3	−3	−3	−4
3	2	2	1	1	0	0	−1	−1	−1	−2	−2	−3	−3
3	3	2	2	1	1	1	0	0	−1	−1	−2	−2	−3
4	3	3	3	2	2	1	1	0	0	−1	−1	−2	−2
5	4	4	3	3	2	2	1	1	0	0	0	−1	−1

Fig. 8. Level line code for $P(5, 7, 11)$ restricted to a (10, 14)-cube. Connected subset (in gray) of the level code corresponds to a disconnected set of voxels of $P(5, 7, 11)$, since $a_1 + a_2 = 5 + 7 = 12 > a_3 = 11$. There are 12 pseudo-cracks in the cube.

(b) *As x_3 runs over the integers, the lines $L_0, L_{\pm 1}, L_{\pm 2}, \ldots$ form a partition Π of the discrete plane Ox_1x_2. This partition defines an equivalence relation, as any equivalence class of that relation corresponds to a discrete line obtained for a certain particular value of x_3.*

All level lines form the *level line code* of $P(a_1, a_2, a_3)$. See for illustration Figs. 7 and 8, as well as [10] for more details about level lines code.

Fact 7 leads us to the following observations. If $a_3 \geq a_1 + a_2$, then $P(a_1, a_2, a_3)$ has infinitely many ("horizontal") cracks that partition it into 2-connected components. These are the levels of $P(a_1, a_2, a_3)$ that correspond to the digital lines in its level lines code. If $a_3 < a_1 + a_2$, then $P(a_1, a_2, a_3)$ has a single crack (that is the gap skeleton of $P(a_1, a_2, a_3)$). It is composed of horizontal "pseudo-cracks"[10] (determining the levels of $P(a_1, a_2, a_3)$ that correspond to the digital lines in its level lines code). Two pseudo-cracks corresponding to consecutive levels k and $k + 1$ are periodically connected by vertical gaps in the locations of the jumps of $P(a_1, a_2, a_3)$ (see Fig. 8).

A (p, q)-cube C of $P(a_1, a_2, a_3)$ has a finite number c of cracks (if $a_3 \geq a_1 + a_2$)/pseudo-cracks (if $a_3 < a_1 + a_2$) that partition it into l levels. Here $l = c + 1$, where c is the number of cracks/pseudo-cracks in C.

10 We say "pseudo" since these are not maximal (non-extendable).

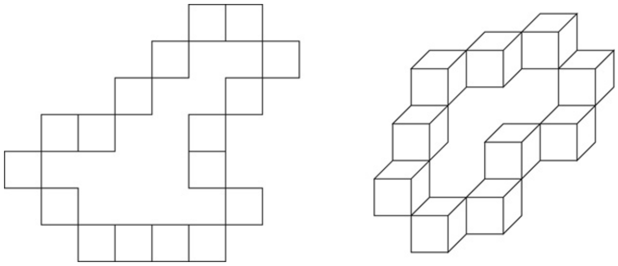

Fig. 9. *Left*: simple closed 0-curves in C_2; *Right*: a curve in C_3. It can be viewed both as a non-proper 0-curve and as a proper 1-curve.

After this preparation, assume that C is a (p, q)-cube of a naive digital plane $P(a_1, a_2, a_3)$. Then formula (1) can be written as

$$g_1 = -12s_3 + 4s_2 - s_1 + b.$$

We are able to obtain formulas that relate the number of gaps with the number of blocks and cracks/pseudo-cracks in C.

It is easy to see that the following relations hold:

$$s_3 = pq,$$
$$s_2 = 4pq + p + q + g_1,$$
$$s_1 = 5pq + 3p + 3q + 1 + g_1 + c.$$

Then, after substitution we obtain

$$g_1 = -pq + p + q + 3g_1 - c + b - 1$$

and

$$g_1 = \frac{1}{2}(pq - p - q + c - b + 1).$$

More specifically, by Fact 7 we obtain the following theorem.

Theorem 8. *Let C be a (p, q)-cube of a digital plane $P(a_1, a_2, a_3)$. Then the following hold:*

- *if $a_3 \geq a_1 + a_2$, then $b \neq 0$, and so $g_1 = \frac{1}{2}(pq - p - q + c - b + 1)$;*
- *if $a_3 < a_1 + a_2$, then $b = 0$, and so $g_1 = \frac{1}{2}(pq - p - q + c + 1)$,*

where b is the number of $2^2 1$-blocks and c the number of cracks/pseudo-cracks in C.

5. Relations for digital curves

In the definition of digital surfaces of Section 4.1 we used a definition of a simple closed curve (Fig. 9). Recall that by the classical Urysohn–Menger definition, a curve $\gamma \subset \mathbb{R}^2$ is known to be a one-dimensional continuum. One can define a *digital continuum* to be any nonempty, finite, and α-connected set of cells in a digital space (where α is the adopted adjacency relation). Then Urysohn–Menger's definition can be adapted to the case of digital curves as follows:

Definition 9. A *(general) digital curve* $\gamma \subset \mathbb{C}_2$ (with respect to a certain adjacency relation) is a one-dimensional digital continuum.

See Fig. 10 for the illustration of general digital curves in $\mathbb{C}_2$. The above definition straightforwardly generalizes for digital curves in a space of arbitrary dimension. General curves allow modeling of complex structures (see, e.g., [24] and Section 5(B) of [14]). For further developments and various results see [13,25] and the bibliography therein. For example, we have the following:

Fact 10 *([14]). Let $D \subseteq \mathbb{C}_n^{(n)}$ be a nonempty, α-connected set.*

(a) *If $0 \leq \alpha \leq n - 2$, then D is one-dimensional with respect to adjacency A_α iff it does not contain as a proper subset an elementary grid triangle consisting of three cells such that any two of them are α-adjacent.*
(b) *If $\alpha = (n - 1)$, then D is one-dimensional with respect to adjacency A_α iff it does not contain as a proper subset an elementary grid square that consists of four cells c_1, c_2, c_3, c_4 with coordinates $c_1 = (x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n)$, $c_2 = (x_1, \ldots, x_i + 1, \ldots, x_j, \ldots, x_n)$, $c_3 = (x_1, \ldots, x_i, \ldots, x_j + 1, \ldots, x_n)$, $c_4 = (x_1, \ldots, x_i + 1, \ldots, x_j + 1, \ldots, x_n)$, for some indices i, j, $1 \leq i, j \leq n$.*

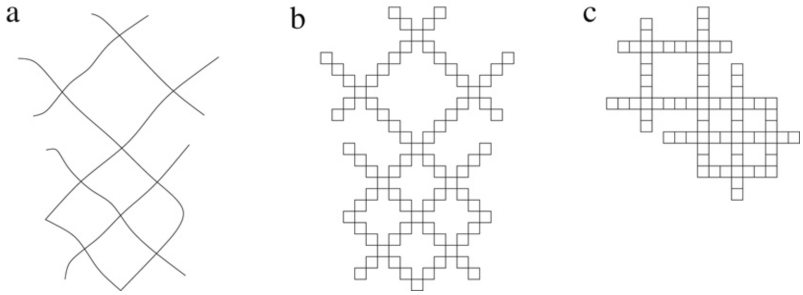

Fig. 10. *Left*: curve in R^2. *Middle*: 0-curve in C_2. *Right*: 1-curve in C_2.

We have the following theorem.

Theorem 11. *Let $\gamma \subset \mathbb{C}_n$ be any (general) digital α-curve $(0 \le \alpha \le n-1)$ according to Definition 9. Then:*

$$g_{n-2} = -2n(n-1)s_n + 2(n-1)s_{n-1} - s_{n-2}.$$

Proof. Clearly, if an α-curve γ contains an elementary grid square, then for any α $(0 \le \alpha \le n-2)$ it also contains an elementary grid triangle that is a part of the grid square. Since γ is one-dimensional, then, by Fact 10, for any α $(0 \le \alpha \le n-1)$ it does not contain an elementary grid square, that is a $2^2 1^{n-2}$-block. Then in (1) the parameter b will equal 0, i.e., we have the stated formula. □

For the special case of a simple closed curve we have the following combinatorial relation.

Theorem 12. *Let $\gamma \subset \mathbb{C}_n$ be a simple closed digital 0-curve. Let $b_0, \ldots, b_{n-1}$ be the number of its k-tandems, for $0 \le k \le n-1$. Then we have the relation*

$$s_k = 2^{n-k}\binom{n}{k} s_n - \sum_{i=0}^{n-k-1} 2^i \binom{k+i}{k} b_{i+k}. \tag{4}$$

Proof. Let c be a k-cell for $k \neq n$. We say that c is a *totally boundary cell* (tb) if c is incident with exactly one n-cell. If c is non-totally boundary (ntb), then c belongs to the closure of the shared face of a tandem t_j in dimension $j \ge k$; we then say that *c is involved with t_j*.

Since γ is a 0-curve, every k-cell is incident with at most two n-cells and, thus, every non-totally boundary cell is involved with exactly one tandem. Now the number of k-cells involved with a j-dimensional tandem t_j is easily seen to be $2^{j-k}\binom{j}{k}$. Therefore the number of non-totally boundary cells s_k^{ntb} is:

$$s_k^{\text{ntb}} = b_k + 2\binom{k+1}{k} b_{k+1} + \cdots + 2^{n-1-k}\binom{n-1}{k} b_{n-1}, \tag{5}$$

whereas the number of totally boundary k-cells is given by $s_k^{\text{tb}} = s_k - s_k^{\text{ntb}}$. Since every n-cell is incident with $2^{n-k}\binom{n}{k}$ k-cells, we have:

$$2^{n-k}\binom{n}{k} s_n = 1 \cdot s_k^{\text{tb}} + 2 \cdot s_k^{\text{ntb}} = s_k + s_k^{\text{ntb}}. \tag{6}$$

The assertion now follows straightforwardly from Eqs. (5) and (6). □

Remark 13. Note that $(n-2)$-gaps are the only gaps a digital curve γ may have. Also note that if γ is a digital $(n-2)$-curve,[11] then the number of $(n-2)$-gaps of γ matches the number of "linear segments" into which γ can be decomposed.

Remark 14. Since γ is a closed curve, its Euler–Poincaré characteristic $\chi(\gamma)$ is zero. We then have:

$$0 = \chi(\gamma) = \sum_{k=0}^{n} (-1)^k s_k.$$

Using the expression for the s_i found in Eq. (4), we recover, after elementary manipulations, the not-surprising relation:

$$s_n = b_0 + b_1 + \cdots + b_{n-1}.$$

[11] That is, any two consecutive voxels of γ are $(n-2)$-adjacent.

6. Concluding remarks

In this paper we provided a rigorous definition of gaps in a digital picture and derived a formula for the number of $(n-2)$-gaps, as well as certain combinatorial relations for digital curves. Our approach could be applied to obtain relations for k-curves with $k \neq 0$, as well as for an arbitrary digital object. Note however that the case of $(n-2)$-gaps admitted a comparatively compact solution because of its specific properties. For example, in the proof of Theorem 4 we took advantage of the fact that the cardinality of the set of $(n-1)$-brims for a cell c may have only values 0, 2, and 4, while the case of $(n-2)$-brims has a much more complex description even for $n = 3$. Because of such technical reasons a possible formula for the number of gaps of lower dimensions will have complexity that would make it to be of little use. Therefore, a possible future task is seen in seeking other approaches that would allow obtaining more compact characterizations of lower-dimensional gaps in binary objects.

The theoretical results described in the present paper are accompanied by an experimental computer program. Given a digital picture S represented by the coordinates of its voxels, our program takes as an input the list of the voxel coordinates and outputs the number of the 0-, 1-, and 2-facets of S, as well as of its 0- and 1-gaps. Alternatively, the number of 1-gaps can be calculated by formula (1) as soon as the involved parameters are found at the first pass. The program allows us to visualize the digital picture S and interactively rotate it along the Ox-, Oy-, and Oz-axes so that the object can be seen from different viewpoints. The algorithm is linear and the computation is immediate even for very large datasets.

A more challenging task is seen in designing an efficient algorithm for identifying cracks and the gap skeleton of a digital surface.

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