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Another step towards proving a conjecture by Plummer and Toft[☆]

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ABSTRACT

A cyclic colouring of a graph G embedded in a surface is a vertex colouring of G in which any two distinct vertices sharing a face receive distinct colours. The cyclic chromatic number $\chi_c(G)$ of G is the smallest number of colours in a cyclic colouring of G. Plummer and Toft in 1987 [M.D. Plummer, B. Toft, Cyclic coloration of 3-polytopes, J. Graph Theory 11 (1987) 507–515] conjectured that $\chi_c(G) \le \Delta^* + 2$ for any 3-connected plane graph G with maximum face degree Δ^*. It is known that the conjecture holds true for $\Delta^* \le 4$ and $\Delta^* \ge 24$. The validity of the conjecture is proved in the paper for $\Delta^* \ge 18$.

1. Introduction

In this article we are dealing with plane graphs, that is embeddings of planar graphs in a plane (or, equivalently, in a sphere). Various types of vertex colourings of plane graphs were intensively studied by many graph theorists. The famous Four Colour Theorem states that in a proper vertex colouring (where adjacent vertices receive distinct colours) of a plane graph four colours are sufficient.

In [8] Ore and Plummer came up with a strengthening of a proper vertex colouring of a plane graph by requiring distinct colours for vertices sharing a common face (a cyclic colouring). As usual, a minimum number of colours was searched for. Evidently, any such colouring must use at least as many colours as the maximum number of vertices incident to a face of the involved graph. Therefore, the minimum number of colours depends on the structure of the graph and can be arbitrarily large. However, for plane triangulations proper and cyclic colourings coincide, and so a cyclic colouring with at most four colours can be found.

Let us now look at the problem in a more general setting. Consider a cell-embedding $G = (V, E, F)$ of a 2-connected graph in a 2-manifold. The *degree* $\deg(x)$ of $x \in V \cup F$ is the number of edges incident to x. A vertex of degree k is a *k-vertex*, a face of degree k is a k-face. By $V(x)$ we denote the set of all vertices incident to $x \in E \cup F$; similarly, $F(y)$ is the set of all faces incident to $y \in V \cup E$. If $e \in E$, $F(e) = \{f_1, f_2\}$ and $\deg(f_1) \le \deg(f_2)$, then the pair $(\deg(f_1), \deg(f_2))$ is called the *type* of e. A (d_1, d_2)*-neighbour* of a vertex x is a vertex y such that the edge xy is of type (d_1, d_2). Paths and cycles in G will be understood as vertex sequences in which any two vertices placed on neighbouring positions are adjacent in G. A cycle in G is *facial* if its vertex set is equal to $V(f)$ for some $f \in F$. Though graphs we are dealing with are nonoriented, sometimes it will be useful to equip certain edges with one of two possible orientations.

A vertex x_1 is *cyclically adjacent* to a vertex $x_2 \ne x_1$ if there is a face f with $x_1, x_2 \in V(f)$. The *cyclic neighbourhood* $N_c(x)$ of a vertex x is the set of all vertices that are cyclically adjacent to x and the *closed cyclic neighbourhood* of x is $\overline{N}_c(x) = N_c(x) \cup \{x\}$. (The usual neighbourhood of x is denoted by $N(x)$.) The *cyclic degree* of x is $\mathrm{cd}(x) = |N_c(x)|$. A *cyclic colouring* of G is a mapping $\varphi : V \to C$ in which $\varphi(x_1) \ne \varphi(x_2)$ whenever x_1 is cyclically adjacent to x_2 (elements of C are *colours* of φ). The cyclic chromatic number $\chi_c(G)$ of the graph G is the minimum number of colours in a cyclic colouring of G.

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For $p, q \in \mathbb{Z}$ let $[p, q] = \{z \in \mathbb{Z} : p \le z \le q\}$ and $[p, \infty) = \{z \in \mathbb{Z} : p \le z\}$. The *concatenation* of finite sequences $A = (a_1, \dots, a_m)$ and $B = (b_1, \dots, b_n)$ is the sequence $AB = (a_1, \dots, a_m, b_1, \dots, b_n)$. Because of the obvious associativity of concatenation we can use the symbol $\prod_{i=1}^{k} A_i$ for the concatenation of $k \in [0, \infty)$ finite sequences in the order given by the sequence $(A_1, \dots, A_k)$. If $A_i = A$ for all $i \in [1, k]$, then $\prod_{i=1}^{k} A_i$ is replaced by A^k, where $A^0 = (\,)$ is the empty sequence.

Let G be an embedding of a 2-connected graph and let v be its vertex of degree n. Consider a sequence $(f_1, \dots, f_n)$ of faces incident to v in a cyclic order around v (there are altogether $2n$ such sequences) and the sequence $D = (d_1, \dots, d_n)$ in which $d_i = \deg(f_i)$ for $i \in [1, n]$. The sequence D is called the *type* of the vertex v provided it is the lexicographical minimum of the set of all such sequences corresponding to v, that is, of the set

$$\bigcup_{i=1}^{n} (\{(d_i, \dots, d_{i+n-1})\} \cup \{(d_i, \dots, d_{i-n+1})\}),$$

where indices are taken modulo n in the interval $[1, n]$. It is easy to see that $\mathrm{cd}(v) = \sum_{i=1}^{n} (d_i - 2)$. The multiset $\mathrm{dm}(v) = \{d_1, \dots, d_n\}$ is the *degree multiset* of the vertex v. A *contraction* of an edge $xy \in E(G)$ consists in a continuous identification of the vertices x and y forming a new vertex $x \leftrightarrow y$ and the removal of the created loop together with all possibly created multiedges; if G/xy is the result of such a contraction, then, clearly, $\Delta^*(G/xy) \le \Delta^*(G)$. An edge xy of a 3-connected plane graph G is *contractible* if G/xy is again 3-connected.

If the graph G is 2-connected, any face f of G is incident to $\deg(f)$ vertices. In such a case $\chi_c(G)$ is naturally lower bounded by $\Delta^*(G)$, the *maximum face degree* of G. Sanders and Zhao [10] proved that $\chi_c(G) \le \lceil \frac{5}{3} \Delta^*(G) \rceil$ for any 2-connected plane graph G. On the other hand, for any $d \in [4, \infty)$ there is a 2-connected plane graph $\tilde{G}_d$ satisfying $\Delta^*(\tilde{G}_d) = d$ and $\chi_c(\tilde{G}_d) = \lfloor \frac{3}{2} d \rfloor$. It is conjectured that $\chi_c(G) \le \lfloor \frac{3}{2} \Delta^*(G) \rfloor$ for *any* 2-connected plane graph G.

However, our interest is concentrated on 3-connected plane graphs. By a classical result of Whitney [11] all plane embeddings of a 3-connected planar graph are essentially the same. This means that $\chi_c(G_1) = \chi_c(G_2)$ if G_1, G_2 are plane embeddings of a fixed 3-connected planar graph G; thus, we can speak simply about the cyclic chromatic number of G. On the other hand, when analysing $\chi_c(G)$ for a 3-connected planar graph G, any edge of G can be chosen to be incident or not to be incident to the unbounded face of an embedding of G in the plane.

Plummer and Toft proposed the following conjecture in [9].

Conjecture 1. *If G is a 3-connected plane graph, then $\chi_c(G) \le \Delta^*(G) + 2$.*

Note that they were able to show a weaker inequality $\chi_c(G) \le \Delta^*(G) + 9$. Let $\mathrm{PTC}(d)$ denote Conjecture 1 restricted to graphs with $\Delta^*(G) = d$. Because of the Four Colour Theorem we know that for a triangulation G we have $\chi_c(G) \le 4 = \Delta^*(G) + 1$. PTC(4) is known to be true by the work of Borodin [2]. Horňák and Jendrol' [6] proved $\mathrm{PTC}(d)$ for any $d \ge 24$. The bound was improved to 22 by Morita [7], but to the best of our knowledge, the proof was never published. Enomoto et al. [4] obtained for $\Delta^*(G) \ge 60$ even a stronger result, namely that $\chi_c(G) \le \Delta^*(G) + 1$. The example of the (graph of) d-sided prism with maximum face degree d and cyclic chromatic number $d + 1$ shows that the bound is the best possible. The best known general result (with no restriction on $\Delta^*(G)$) is the inequality $\chi_c(G) \le \Delta^*(G) + 5$ of Enomoto and Horňák [3].

Conjecture 1 is still open. This means that we do not know any G with $\chi_c(G) - \Delta^*(G) \ge 3$. On the other hand, all G's with $\chi_c(G) - \Delta^*(G) = 2$ we are aware of satisfy $\Delta^*(G) = 4$. Therefore, the conjecture could even be strengthened:

Conjecture 2. *If G is a 3-connected plane graph G with $\Delta^*(G) \ne 4$, then $\chi_c(G) \le \Delta^*(G) + 1$.*

The main goal of our paper is to show that $\mathrm{PTC}(d)$ is true for any $d \in [18, \infty)$. That is, we are going to prove

Theorem 1. *If G is a 3-connected plane graph with $\Delta^*(G) \ge 18$, then $\chi_c(G) \le \Delta^*(G) + 2$.*

2. Strategy of the proof

Clearly, Theorem 1 is a direct consequence of a more general result:

Theorem 2. *If $d \in [18, \infty)$ and G is a 3-connected plane graph with $\Delta^*(G) \le d$, then $\chi_c(G) \le d + 2$.*

Theorem 2 will be proved in Section 4 by contradiction with the help of the Discharging Method that showed its power by serving as a main tool for the proof of the Four Colour Theorem. Let us now describe the basic idea of the method.

If there is a counterexample to Theorem 2, then there is certainly also a counterexample that is minimal with respect primarily to the number of vertices and secondarily to the number of edges. Let $d \in [5, \infty)$ and $k \in [2, 4]$. A 3-connected plane graph G is said to be (d, k)-*minimal* if $\Delta^*(G) \le d$ and $\chi_c(G) > d + k$, but $\Delta^*(H) \le d$ implies $\chi_c(H) \le d + k$ for any 3-connected plane graph H such that the pair $(|V(H)|, |E(H)|)$ is lexicographically smaller than the pair $(|V(G)|, |E(G)|)$. Thus, the above mentioned counterexample to Theorem 2 is a $(d, 2)$-minimal graph for some $d \in [18, \infty)$.

We shall see in Section 3 (Lemma 3) that the structure of a $(d, 2)$-minimal graph $G = (V, E, F)$ is quite restricted. If $d \ge 18$, the restriction is so strong that the existence of G is incompatible with Euler's Theorem $|V| - |E| + |F| = 2$. From this theorem it is easy to derive that $\sum_{v \in V} c_0(v) = 2$ for the mapping $c_0 : V \to \mathbb{Q}$ (called the *initial* charge) with

$$c_0(v) = 1 - \frac{\deg(v)}{2} + \sum_{f \in F(v)} \frac{1}{\deg(f)}.$$

Putting $\Sigma(c_0, W) = \sum_{v \in W} c_0(v)$ for $W \subseteq V$ we have $\Sigma(c_0, V) = 2$. In Section 4 we shall find consecutively in four phases charge mappings $c_i : V \to \mathbb{Q}$, $i = 1, 2, 3, 4$, such that $\Sigma(c_i, V) = 2$, which means that passing from c_{i-1} to c_i is simply a redistribution of charges of vertices that is governed by redistribution rules. The restriction on the structure of G yielded by Lemma 3 enables us to prove that $c_4(v) \leq 0$ for any $v \in V$, which represents a contradiction with $\Sigma(c_4, V) = 2$.

3. Auxiliary results

In the proof of Theorem 2 we shall need a special information on the structure of 3-connected plane graphs contained in Lemma 1 proved by Halin [5] and in Lemma 2 that follows by the results of Ando et al. [1].

Lemma 1. *Any 3-vertex of a 3-connected plane graph G with $|V(G)| \geq 5$ is incident to a contractible edge.* ■

Lemma 2. *If a vertex of degree at least four of a 3-connected plane graph G with $|V(G)| \geq 5$ is not incident to a contractible edge, then it is adjacent to three 3-vertices.* ■

The next lemma shows that a $(d, 2)$-minimal graph cannot contain some configurations.

Lemma 3. *Let $d \in [6, \infty)$ and let G be a $(d, 2)$-minimal graph. Then G does not contain any of the following configurations:*

1. *a 3-vertex x with $\mathrm{cd}(x) \leq d + 1$;*
2. *a vertex x with $\deg(x) \geq 4$ and $\mathrm{cd}(x) \leq d + 1$ that is incident to a contractible edge;*
3. *a vertex x with $\deg(x) \geq 4$ and $\mathrm{cd}(x) \leq d + 1$ that is adjacent to a 3-vertex y with $\mathrm{cd}(y) \leq d + 2$;*
4. *a triangle t incident to exactly one 3-vertex such that the face adjacent to t along the edge joining vertices of degree at least four is of degree at most $d - 1$;*
5. *a separating 3-cycle;*
6. *an edge of type $(3, d_2)$ with $d_2 \in [3, 4]$;*
7. *the configuration $\mathcal{C}_i$, $i \in [1, 7]$, that is depicted in one of Figs. 1–4, where encircled numbers represent degrees of corresponding vertices, vertices without degree specification are of an arbitrary degree and dashed lines are parts of facial cycles.*

Proof. 1–4. The statements have already been proved in [6, (Lemma 3.1(e), 3.3(i), 3.3(ii) and 3.4)]. For the rest of the proof suppose that G contains a configuration $\mathcal{C}$ described in Lemmas 3.5, 3.6 or 3.7.

5. Let $\mathcal{C}$ be a separating 3-cycle (x_1, x_2, x_3, x_1) and let G_1 and G_2 be components of the graph $G - \{x_1, x_2, x_3\}$. It is easy to see that the subgraph H_i of G induced by $V(G_i) \cup \{x_1, x_2, x_3\}$ is a 3-connected plane graph with $\Delta^*(H_i) \leq \Delta^*(G) \leq d$ and $|V(H_i)| < |V(G)|$, hence there is a cyclic colouring $\varphi_i : V(H_i) \to C$, $i = 1, 2$, where $|C| = d + 2$. Without loss of generality we may suppose that $\varphi_1(x_i) = \varphi_2(x_i)$, $i = 1, 2, 3$. Then $\psi : V(G) \to C$ determined by $\psi(x) = \varphi_i(x) \overset{\text{df.}}{\Leftrightarrow} x \in V(H_i)$, $i = 1, 2$, is a cyclic colouring of G in contradiction with $\chi_c(G) > d + 2$.

6. Now let G contain a triangle xy_1y_2 adjacent to a quadrangle $y_1y_2z_2z_1$. Without loss of generality we may suppose that neither of the two faces incident to y_1y_2 is unbounded. By Lemma 3.1 we have $\deg(y_i) \geq 4$, $i = 1, 2$, and consequently, by Lemma 3.4, $\deg(x) \geq 4$. Consider the graph $G' = G - y_1y_2$ with $|V(G')| = |V(G)|$ and $|E(G')| = |E(G)| - 1$. If G' is 3-connected, then it has a cyclic colouring using at most $d + 2$ colours which is also a cyclic colouring of G, a contradiction.

Therefore, G' has to be 2-connected. Let $\{v_1, v_2\}$ be a cutset of G'. Clearly, $\{v_1, v_2\} \cap \{y_1, y_2\} = \emptyset$, so there is a component $C(y_i)$ of the graph $G'' = G' - \{v_1, v_2\}$ containing the vertex y_i, $i = 1, 2$. From 3-connectedness of G it follows that any vertex of G'' belongs either to $C(y_1)$ or to $C(y_2)$, hence $C(y_1) \neq C(y_2)$, $x \in \{v_1, v_2\}$ and $\{v_1, v_2\} \subseteq \{x, z_1, z_2\}$ (otherwise there is a path joining y_1 to y_2 in G''). Thus we may suppose without loss of generality that $v_1 = x$ and $v_2 = z_j$ for some $j \in [1, 2]$. Then both x and z_j are incident to the unbounded face f of G. Because of Lemma 3.5 the vertices x and z_j are not adjacent in G, otherwise (x, y_j, z_j, x) would be a separating 3-cycle of G. Therefore, the facial cycle of the unbounded face of G is of the form $(x)P^1(z_j)P^2(x)$, where both paths P^1 and P^2 are nonempty. For $i = 1, 2$ consider the cycle $C^i = (x)P^i(z_j, y_j, x)$, the plane subgraph G^i of G induced by all vertices lying in the closed disc bounded by the closed Jordan curve corresponding to C^i, and join vertices x and z_j of G^i by an arc lying in the unbounded face of G^i. It is easy to see that we obtain a 3-connected plane graph H^i with $\Delta^*(H^i) \leq \Delta^*(G) \leq d$ and $|V(H^i)| < |V(G)|$, hence there is a cyclic colouring $\varphi^i : V(H^i) \to C$; if f^i is the unbounded face of H^i, then $V(f^1) \cup V(f^2) = V(f)$ has at most d vertices, and so we may suppose without loss of generality that $\varphi^1(v) = \varphi^2(v)$ for any $v \in \{x, y_j, z_j\}$ (note that xy_jz_j is a 3-face of both H^1 and H^2) and $\varphi^1(V(f^1) - \{x, z_j\}) \cap \varphi^2(V(f^2) - \{x, z_j\}) = \emptyset$. As in Lemma 3.5, the colouring $\psi : V(G) \to C$ with $\psi(x) = \varphi_i(x) \overset{\text{df.}}{\Leftrightarrow} x \in V(H^i)$, $i = 1, 2$, yields a contradiction.

If G contains a triangle xy_1y_2 adjacent to a triangle y_1y_2z, we proceed similarly as above having in mind that the cutset of the 2-connected graph G' is $\{x, z\}$.

7. If $\mathcal{C} = \mathcal{C}_i$, $i \in \{1, 3, 5, 6, 7\}$, then the configuration $\mathcal{C}$ contains a 3-vertex x_1 incident to a contractible edge u_ix_1; the oriented edge (u_i, x_1) is indicated by an arrow. The graph $G' = G/u_ix_1$ is a 3-connected plane graph satisfying $\Delta^*(G') \leq \Delta^*(G) \leq d$ and $|V(G')| = |V(G)| - 1$, hence there is a cyclic colouring $\varphi : V(G') \to C$. This colouring will be used

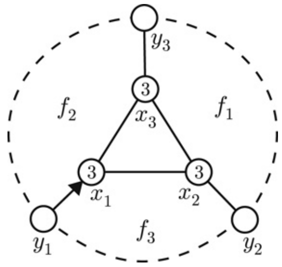

Fig. 1. Configuration $\mathcal{C}_1$ with $\mathrm{cd}(x_1) \le d + 2$.

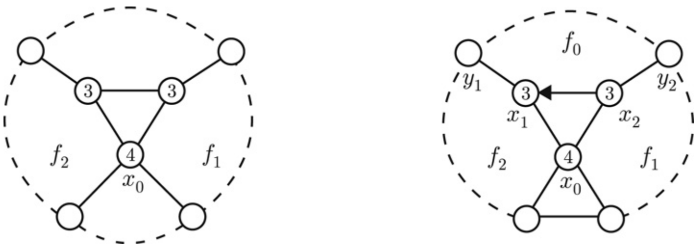

Fig. 2. Configurations $\mathcal{C}_2$ (left) with $\mathrm{cd}(x_0) \le d + 1$ and $\mathcal{C}_3$ (right) with $\mathrm{cd}(x_1) \le d + 2$.

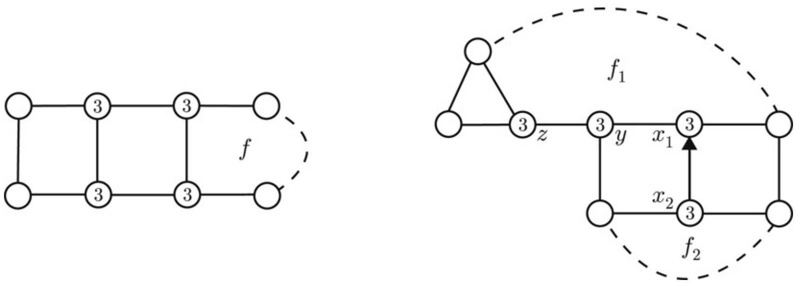

Fig. 3. Configurations $\mathcal{C}_4$ (left) with $\deg(f) \in [4, 5]$ and $\mathcal{C}_5$ (right).

to find a cyclic colouring $\psi : V(G) \to C$ in order to obtain a contradiction with $\chi_c(G) > d + 2$. We put $\psi(u) = \varphi(u)$ for any $u \in V(G) - \{u_i, x_1\}$ and $\psi(u_i) = \varphi(u_i \leftrightarrow x_1)$ (if not stated explicitly otherwise) so that we have (in general) to determine only $\psi(x_1)$.

$i = 1$: First note that Lemma 3.1 yields $\mathrm{cd}(x_1) = d + 2$. If there is a colour that appears twice on vertices of $N_c(x_1)$ (under φ), then we see that at least one colour is available as $\psi(x_1)$. Henceforth suppose that $|\varphi(N_c(x_1))| = d + 2$. With $W = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ and $C_j = \varphi(V(f_j) - W), j = 1, 2, 3$, we have $C_2 \cap C_3 = \emptyset$. If there is $j \in [2, 3]$ such that $C_j - C_1 \neq \emptyset$, then we take $\psi(x_j) \in C_j - C_1$ and define $\psi(x_1) = \varphi(x_j)$. To conclude this case notice that $C_2 - C_1$ and $C_3 - C_1$ cannot be both empty, since then $C_j \subseteq C_1, j = 2, 3$, and $\deg(f_1) = |C_1| + 4 \ge |C_2| + |C_3| + 4 = d + 1$, a contradiction.

$i = 2$: Since, by Lemma 3.6, $\deg(f_j) \ge 5$, the configuration $\mathcal{C}_2$ is not present in G by Lemma 3.2 of [6].

$i = 3$: As for $i = 1$ it is sufficient to analyse the case in which $|\varphi(N_c(x_1))| = d + 2$. With $W = \{x_0, x_1, x_2, y_1, y_2\}$ and $C_j = \varphi(V(f_j) - W), j = 0, 1, 2$, we obtain $C_0 \cap C_2 = \emptyset$. If $C_2 - C_1 \neq \emptyset$, then we are done by taking $\psi(x_2) \in C_2 - C_1$ and $\psi(x_1) = \varphi(x_2)$. On the other hand, $C_2 - C_1 = \emptyset$ implies $C_1 \subseteq C_2$, and so defining $\psi(x_1) = \varphi(x_0)$ leaves at least one colour available for $\psi(x_0)$.

$i = 4$: For the proof see Lemma 3.1(c) and 3.1(d) of [6].

$i = 5$: In this case $\varphi(x_2 \leftrightarrow x_1)$ can be used as either $\psi(x_1)$ or $\psi(x_2)$. By Lemma 3.1 we have $\deg(f_1) = \deg(f_2) = d$, and so we may suppose (similarly as for $i = 1$ or $i = 3$) that $|\varphi(N_c(x_1))| = d + 2$ and $|\varphi(N_c(x_2) - \{x_1\})| = d + 1$. Since $N_c(z) \subseteq \bar{N}_c(y)$, this allows us to define $\psi(x_1) = \varphi(x_1 \leftrightarrow x_2)$, $\psi(x_2) = \varphi(y)$, $\psi(y) = \varphi(z)$ and $\psi(z) = \varphi(y)$.

$i = 6, 7$: By Lemmas 3.1, 3.7.1 and 3.7.3 (for $i = 7$) we have $\deg(f_1) = \deg(f_2) = \deg(f) = d$ and $\mathrm{cd}(v) = d + 3$ for any $v \in \{x_1, x_2, z_1, z_2\}$. If there is a colour (of C) not present in $\varphi(\bar{N}_c(x_2) - \{x_1\}) = \varphi(N_c(x_1))$, then we use it as $\psi(x_1)$. Henceforth we suppose that the vertex x_2 is *saturated* — all colours of C appear on vertices of its closed cyclic neighbourhood; as x_1 is not coloured under φ, on vertices of the cyclic neighbourhood of x_2 one colour appears twice and d colours appear once. If $\varphi(z_j) \notin \varphi(V(f))$ and $c \in C - \varphi(N_c(z_j) - \{x_1\})$, then we are done (that is, we obtain a contradiction) by putting $\psi(z_j) = c$, $\psi(x_j) = \varphi(z_j)$ and $\psi(x_{3-j}) = \varphi(x_2 \leftrightarrow x_1)$. Therefore, we assume that $\varphi(z_j) \notin \varphi(V(f))$ implies that the vertex x_j is saturated, $j = 1, 2$. There is $j \in [1, 2]$ such that the x_2-*duplicated* colour, that is, one that appears twice on vertices of $N_c(x_2)$, is either $\varphi(t_j)$ or $\varphi(z_j)$. If $\varphi(t_j)$ is x_2-duplicated, then obviously $\varphi(z_j) \notin \varphi(V(f))$, so z_j is saturated, at most one of $\varphi(t_{3-j})$ and $\varphi(z_{3-j})$ is

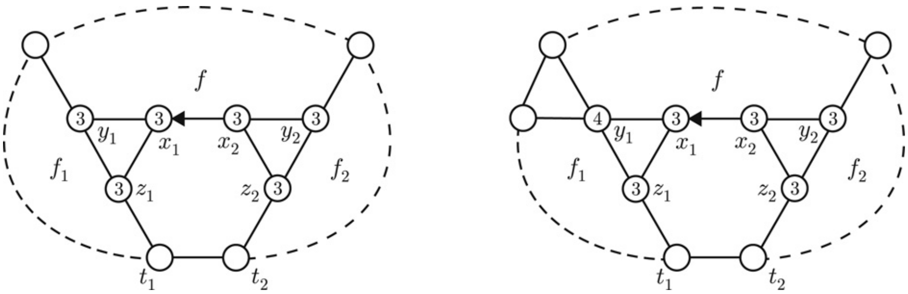

Fig. 4. Configurations $\mathcal{C}_6$ (left) and $\mathcal{C}_7$ (right).

z_j-duplicated and $\{\varphi(t_{3-j}), \varphi(z_{3-j})\} - \varphi(V(f_j)) \neq \emptyset$. If, say, $\varphi(t_{3-j}) \notin \varphi(V(f_j))$, then, having in mind that $\varphi(t_{3-j}) \notin \varphi(V(f))$, we can take $\psi(y_j) = \varphi(t_{3-j})$ and $\psi(x_1) = \varphi(y_j)$. Now let $\varphi(z_j)$ be x_2-duplicated; as a consequence, z_{3-j} is saturated. If one of $\varphi(t_{3-j}), \varphi(z_{3-j})$ is out of $\varphi(V(f_j))$, then we use it as $\psi(y_j)$ and put $\psi(x_1) = \varphi(y_j)$. On the other hand, provided $\{\varphi(t_{3-j}), \varphi(z_{3-j})\} \subseteq \varphi(V(f_j))$, there is a colour $c \in C - \varphi(\bar{N}_c(z_j) - \{x_1\})$, which allows us to define $\psi(z_j) = c$ together with either $\psi(z_{3-j}) = \varphi(z_j)$ and $\psi(x_1) = \varphi(z_{3-j})$ (if $\varphi(t_j)$ is z_{3-j}-duplicated) or $\psi(y_{3-j}) = \varphi(t_j)$ and $\psi(x_1) = \varphi(y_{3-j})$ (otherwise). ■

Note that the configurations of Lemma 3, except for $\mathcal{C}_6$ and $\mathcal{C}_7$, do not appear even in (5, 2)-minimal graphs.

From the definition of the type of a vertex v in Section 1 we see that the type of v determines the initial charge (see Section 2) of v. Namely, if v is of type $(d_1, \dots, d_n)$, then

$$c_0(v) = \gamma(d_1, \dots, d_n) = 1 - \frac{n}{2} + \sum_{i=1}^{n} \frac{1}{d_i}.$$

Clearly, if π is a permutation of the set $[1, n]$, then $\gamma(d_{\pi(1)}, \dots, d_{\pi(n)}) = \gamma(d_1, \dots, d_n)$. Let the *weight* of a sequence $D = (d_1, \dots, d_n) \in \mathbb{Z}^n$ be defined by $\mathrm{wt}(D) = \sum_{i=1}^{n} d_i$. For $n \in [2, \infty)$, $q \in [0, n-2]$, $(d_1, \dots, d_{n-1}) \in [1, \infty)^{n-1}$ and $w \in [\sum_{i=1}^{n-1} d_i + 1, \infty)$ let $S_q(d_1, \dots, d_{n-1}; w)$ be the set of all sequences $D = (d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \in \mathbb{Z}^n$ satisfying $d'_i \geq d_i$ for any $i \in [q+1, n-1]$ and $\mathrm{wt}(D) \geq w$. An analogue of the following statement has been proved as Lemma 4 in [6] (with a different definition of γ).

Lemma 4. *The maximum of* $\gamma(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n)$ *over all sequences* $(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \in S_q(d_1, \dots, d_{n-1}; w)$ *is equal to* $\gamma(d_1, \dots, d_{n-1}, w - \sum_{i=1}^{n-1} d_i)$.

Proof. Pick a sequence $(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \in S_q(d_1, \dots, d_{n-1}; w)$. Decrease d'_i to d_i and increase d'_n by $d'_i - d_i$ successively for all $i \in [q+1, n-1]$. If $a_1, a_2, a_3, a_4 \in [1, \infty)$, $a_1 + a_2 = a_3 + a_4$ and $a_1 < \min(a_3, a_4)$, then $\frac{1}{a_3} + \frac{1}{a_4} < \frac{1}{a_1} + \frac{1}{a_2}$. Moreover, with $d''_n = d'_n + \sum_{i=q+1}^{n-1}(d'_i - d_i)$ we have $\sum_{i=1}^{n-1} d_i + d''_n = \mathrm{wt}(d_1, \dots, d_n, d''_n) = \mathrm{wt}(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \geq w$, hence $(d_1, \dots, d_{n-1}, d''_n) \in S_q(d_1, \dots, d_{n-1}; w)$ and $\gamma(d_1, \dots, d_q, d'_{q+1}, \dots, d'_n) \leq \gamma(d_1, \dots, d_{n-1}, d''_n) \leq \gamma(d_1, \dots, d_{n-1}, w - \sum_{i=1}^{n-1} d_i)$. Here equalities apply if and only if $d'_i = d_i$ for any $i \in [q+1, n-1]$ and $d'_n = d''_n = w - \sum_{i=1}^{n-1} d_i$. ■

4. Proof of Theorem 2

As already mentioned at the beginning of Section 2, for the proof by contradiction we suppose that $G = (V, E, F)$ is a $(d, 2)$-minimal graph with $d \in [18, \infty)$. A set $W \subseteq V$ is *positive* if $\Sigma(c_0, W) > 0$, otherwise it is nonpositive; a negative set and a nonnegative set are defined similarly. If $W = \{w\}$ or $W = V(f)$, $f \in F$, then we shall speak simply about a positive (nonpositive, negative, nonnegative) vertex w or face f, respectively. A triangle $t \in F$ is an *i-triangle* if the number of 3-vertices in $V(t)$ is i. For a vertex $v \in V$ let $N_{4+}(v)$ denote the set of all neighbours of v of degree at least four and put $n_{4+}(v) = |N_{4+}(v)|$. Now we are going to prove a series of claims concerning vertices of V and faces of F (which is implicitly assumed in those claims).

Claim 1. 1. *If faces* f_1 *and* f_2 *are adjacent to each other, then* $\deg(f_1) + \deg(f_2) \geq 8$.
2. *If a vertex is of type* (d_1, d_2, d_3)*, then* $d_3 \geq d + 8 - d_1 - d_2$.
3. *If a vertex is positive, then it is of degree* 3.
4. *If a vertex of type* (d_1, d_2, d_3) *is positive, then either* $d_1 = 3$ *and* $d_2 \in [5, 11]$ *or* $d_1 = 4$ *and* $d_2 \in [4, 5]$.
5. *If a vertex of type* $(3, d_2, d_3)$ *is nonpositive, then* $d_2 \geq 7$.

Proof. 1. The inequality follows from Lemma 3.6.

For the rest of the proof consider an n-vertex v of type $D = (d_1, \dots, d_n)$ and put $d_{n+i} = d_i$ for $i \in [1, n]$. Notice that $d_1 \leq d_2$ and that $n = 3$ implies $d_2 \leq d_3$.

Table 1
Positive upper bounds $u(d_1, d_2)$.

d_1	3	3	3	3	3	3	3	4	4
d_2	5	6	7	8	9	10	11	4	5
$u(d_1, d_2)$	$\frac{4}{45}$	$\frac{1}{17}$	$\frac{13}{336}$	$\frac{1}{40}$	$\frac{1}{63}$	$\frac{2}{195}$	$\frac{1}{132}$	$\frac{1}{18}$	$\frac{3}{340}$

2. If $\deg(v) = 3$, then $\mathrm{cd}(v) = d_1 + d_2 + d_3 - 6$. To obtain the desired inequality use Lemma 3.1.

3. Suppose that $n \geq 4$. By Claim 1.1 we have $d_i + d_{i+1} \geq 8$ and $\frac{1}{d_i} + \frac{1}{d_{i+1}} \leq \max\{\frac{1}{3} + \frac{1}{5}, \frac{1}{4} + \frac{1}{4}\} = \frac{8}{15}$ for any $i \in [1, 2n-1]$, hence $\sum_{i=1}^{n} \frac{1}{d_i} = \frac{1}{2} \sum_{i=1}^{n} (\frac{1}{d_{2i-1}} + \frac{1}{d_{2i}}) \leq \frac{4n}{15}$ and $c_0(v) = 1 - \frac{n}{2} + \sum_{i=1}^{n} \frac{1}{d_i} \leq 1 - \frac{7n}{30}$. If $n \geq 5$, then $c_0(v) \leq -\frac{1}{6}$. It remains to analyse the case $n = 4$. If $d_1 \geq 4$, then $c_0(v) \leq -1 + 4 \cdot \frac{1}{4} = 0$. If $d_3 \geq 4$, then $c_0(v) \leq -1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{4} + \frac{1}{5} = -\frac{1}{60}$. Finally, suppose that v is of type $(3, d_2, 3, d_4)$. If $d_2 \geq 6$, then $c_0(v) = -\frac{1}{3} + \frac{1}{d_2} + \frac{1}{d_4} \leq -\frac{1}{3} + 2 \cdot \frac{1}{6} = 0$. If $d_2 = 5$ and $d_2 \geq 8$, then $c_0(v) \leq -\frac{1}{3} + \frac{1}{5} + \frac{1}{8} < 0$. So, let $d_2 = 5$ and $d_4 \in [5, 7]$. If v has at least three neighbours of degree three, then, because of $\mathrm{cd}(v) \leq 10 \leq d + 1$, we obtain a contradiction with (the fact that G does not contain) $\mathcal{C}_2$. On the other hand, if v has at least two neighbours of degree at least four, then by Lemma 2 the vertex v is incident to a contractible edge. Since $\mathrm{cd}(v) \leq d + 1$, this contradicts Lemma 3.2.

4. From Claim 1.2 we know that $\mathrm{wt}(D) \geq d + 8$ so that $D \in S_3(d_1, d_2; d + 8)$. If $d_1 \geq 5$, then, by Lemma 4, $c_0(v) \leq -\frac{1}{2} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{d-2} \leq -\frac{1}{10} + \frac{1}{16} < 0$. If $d_1 = 4$ and $d_2 \geq 6$, then, again by Lemma 4, $c_0(v) \leq -\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{d-2} \leq -\frac{1}{12} + \frac{1}{16} < 0$. If $d_1 = 3$, then $d_2 \geq 5$ (Claim 1.1) and with $d_3 \geq d_2 \geq 12$ we have $c_0(v) \leq -\frac{1}{6} + \frac{1}{12} + \frac{1}{12} = 0$.

5. If $d_1 = 3$ and $d_2 \leq 6$, then $c_0(v) = -\frac{1}{6} + \frac{1}{d_2} + \frac{1}{d_3} \geq \frac{1}{d_3} > 0$. ■

By Claim 1.2 and Lemma 4, provided v is a vertex of type (d_1, d_2, d_3), we have $c_0(v) \leq \gamma(d_1, d_2, d + 8 - d_1 - d_2) \leq \gamma(d_1, d_2, 26 - d_1 - d_2) =: u(d_1, d_2)$. The positive upper bounds $u(d_1, d_2)$ are presented in Table 1.

A triangle is of *type* (d_1, d_2, d_3) if it is adjacent to three distinct faces f_1, f_2, f_3 with $\deg(f_1) = d_1 \leq \deg(f_2) = d_2 \leq \deg(f_3) = d_3$.

Claim 2. *If a 3-triangle t of type (d_1, d_2, d_3) is positive, then $d_1 \in [6, 7]$, $d_2 \geq d+6-d_1$ and $\Sigma(c_0, V(t)) \leq -\frac{1}{2} + \frac{2}{d_1} + \frac{4}{d+6-d_1} =: \beta(d_1, d)$.*

Proof. From Claim 1.1 and $\mathcal{C}_1$ it follows that $d_1 \geq 6$. Put $d_4 = d_1$. If $d_1 \geq 12$, then $\Sigma(c_0, V(t)) = \sum_{i=1}^{3} \gamma(3, d_i, d_{i+1}) = -\frac{1}{2} + 2 \sum_{i=1}^{3} \frac{1}{d_i} \leq -\frac{1}{2} + 2 \cdot \frac{3}{12} = 0$. Let $x \in V(t)$ be a vertex of type $(3, d_1, d_2)$. From $\mathcal{C}_1$ we obtain $d + 3 \leq \mathrm{cd}(x) = d_1 + d_2 - 3$, $d_3 \geq d_2 \geq d + 6 - d_1$, and so $\Sigma(c_0, V(t)) \leq -\frac{1}{2} + 2(\frac{1}{d_1} + \frac{2}{d+6-d_1}) \leq -\frac{1}{2} + \frac{2}{d_1} + \frac{4}{24-d_1}$. With $d_1 \in [8, 11]$ we have $\Sigma(c_0, V(t)) \leq -\frac{1}{2} + \frac{2}{8} + \frac{4}{16} = 0$, hence $d_1 \in [6, 7]$. ■

Let us define *absorbing* vertices as follows: Any vertex of degree at least four is absorbing. A 3-vertex is absorbing if it is either of type $(5, d_2, d_3)$ with $d_2 \geq 11$ and $d_3 \geq d - 1$ or of type $(7, d_2, d_3)$ with $d_2 \geq 10$.

Claim 3. *If a 5-face f is incident to a vertex of type $(4, 5, d_3)$, then f is incident to an absorbing vertex.*

Proof. Let $C = (x_1, x_2, x_3, x_4, x_5, x_1)$ be a facial cycle of f and let f_i be the face adjacent to f along the edge $x_i x_{i+1}$ (with indices taken modulo 5). If $\deg(x_i) \geq 4$ for some $i \in [1, 5]$, then x_i is absorbing. If $\deg(x_i) = 3$ for any $i \in [1, 5]$, then we may suppose without loss of generality that $\deg(f_3) = 4$. By Claim 1.2 then $\deg(f_i) \geq d - 1$ for $i = 2, 4$. By the same Claim we have $\max\{\deg(f_1), \deg(f_5)\} \geq 11$, and so at least one of the vertices x_2, x_5 is absorbing. ■

Claim 4. *If a 7-face f is adjacent to a 3-triangle, then f is incident to an absorbing vertex.*

Proof. Let $C = (x_1, x_2, \ldots, x_7, x_1)$ be a facial cycle of f and let f_i be the face adjacent to f along the edge $x_i x_{i+1}$ (with indices taken modulo 7). If $\deg(x_i) \geq 4$ for some $i \in [1, 7]$, then x_i is absorbing. Henceforth assume that $\deg(x_i) = 3$ for any $i \in [1, 7]$. Since 3-triangles adjacent to f cover an even number of vertices of f, there is a subpath P of C of an odd order $k \in \{1, 3, 5\}$, without loss of generality $P = \prod_{i=1}^{k}(x_i)$, such that none of x_i with $i \in [1, k]$ is incident to a 3-triangle, but x_i is incident to a 3-triangle for any $i \in \{k + 1\} \cup \{7\}$. By $\mathcal{C}_1$ then $\min\{\deg(f_k), \deg(f_7)\} \geq d - 1$. If $k = 1$, then the vertex x_1 is absorbing. If $k \in \{3, 5\}$ and $\max\{\deg(f_1), \deg(f_{k-1})\} \geq 10$, then at least one of the vertices x_1, x_k is absorbing; note that, by Claim 1.2, the inequality is certainly true if $k = 3$. Finally, if $k = 5$ and $\max\{\deg(f_1), \deg(f_4)\} \leq 9$, then, again by Claim 1.2, $\min\{\deg(f_2), \deg(f_3)\} \geq 10$, and hence the vertex x_3 is absorbing. ■

A *transition edge* of a vertex x of type $(4, 5, d_3)$ is an oriented edge (v, w) whose endvertex is an absorbing vertex of the 5-face f incident to x that is closest to x in one of the two possible orientations of the cycle bounding f. Similarly, a *transition edge* of a 3-triangle t adjacent to a 7-face f is an oriented edge (v, w) whose endvertex is an absorbing vertex of f that is closest to (a vertex of) t in one of the two possible orientations of the cycle bounding f. Finally, a *transition edge* of a 3-triangle t adjacent to a 6-face f is an oriented edge (v, w) with $v \in V(t)$ and $w \in V(f) - V(t)$. From Claims 1.1, 2–4 it follows that

Table 2
Positive upper bounds $\bar{u}(d'_1, d'_2)$.

d'_1	3	3	3	3	3	3	3	3	3	3	3
d'_2	5	6	7	8	9	10	11	$\in [12, d-3]$	$d-2$	$d-1$	d
$\bar{u}(d'_1, d'_2)$	$\frac{41}{408}$	$\frac{1}{12}$	$\frac{20}{357}$	$\frac{1}{40}$	$\frac{1}{63}$	$\frac{2}{195}$	$\frac{1}{132}$	$\frac{1}{40}$	$\frac{13}{336}$	$\frac{1}{17}$	$\frac{4}{45}$

d'_1	4	4	4	5	6	7	7
d'_2	4	5	d	$d-1, d$	d	$d-2$	$d-1, d$
$\bar{u}(d'_1, d'_2)$	$\frac{1}{18}$	$\frac{3}{170}$	$\frac{1}{18}$	$\frac{3}{170}$	$\frac{1}{36}$	$\frac{5}{238}$	$\frac{15}{476}$

any vertex of type $(4, 5, d_3)$ and any positive 3-triangle has exactly two transition edges. Moreover, the initial vertex of any transition edge is a 3-vertex.

Let us now present redistribution rules leading from c_0 to c_4. The first "co-ordinate" i of a rule RR $i.j$ means that RR $i.j$ is used when passing from c_{i-1} to c_i.

RR 1.1 If (v, w) is a transition edge of a vertex x of type $(4, 5, d_3)$, then x sends to w the amount $\frac{1}{2}c_0(x)$ through (v, w).

RR 1.2 If (v, w) is a transition edge of a positive 3-triangle t, then t sends to w the amount $\frac{1}{2}\Sigma(c_0, V(t))$ through (v, w) and $c_1(x) = 0$ for any $x \in V(t)$.

RR 1.3 If (v, w) is a transition edge involved in RR 1.1 or RR 1.2 and $c_0(v) < 0$, then v sends to w the amount $c_0(v)$ through (v, w).

RR 1.4 If t is a nonpositive 3-triangle, then $c_1(x) = \frac{1}{3}\Sigma(c_0, V(t))$ for any $x \in V(t)$.

RR 2.1 If v is a vertex of type $(4, d_2, d)$ with $c_1(v) < 0$ and $\tilde{N}(v) = \{w \in N(v) : c_1(w) > 0\} = \{w_i : i \in [1, \tilde{n}(v)]\} \neq \emptyset$, then v sends to w_i the amount $\frac{c_1(v)}{\tilde{n}(v)}$ for any $i \in [1, \tilde{n}(v)]$.

RR 3.1 A vertex v of type $(3, d_2, d_3)$ with $c_2(v) > 0$, that is incident to a 1-triangle, sends to its $(3, d_3)$-neighbour w the amount $c_2(v)$ through (v, w). (The rule is correct, since $c_2(v) > 0$ implies $c_0(v) > 0$, and so, by Claims 1.2 and 1.4, $d_3 > d_2$.)

RR 3.2 If t is a 2-triangle with $V(t) = \{v_1, v_2, w\}$, where v_1, v_2 are 3-vertices, then v_i sends to w the amount $c_2(v_i)$ through (v_i, w), $i = 1, 2$.

RR 3.3 If v is a vertex of type $(4, 4, d)$ satisfying $c_2(v) > 0$ and $n_{4+}(v) = 0$ and $n_{4+}(w) \geq 1$ for the $(4, 4)$-neighbour w of v, then v sends to w the amount $c_2(v)$.

RR 4.1 If v is a 3-vertex with $c_3(v) > 0$ and $N_{4+}(v) = \{w_i : i \in [1, n_{4+}(v)]\} \neq \emptyset$, then v sends to w_i the amount $\frac{c_3(v)}{n_{4+}(v)}$ through (v, w_i) for any $i \in [1, n_{4+}(v)]$.

Recall that our aim is to show that $c_4(w) \leq 0$ for any $w \in V$. The case $\deg(w) = 3$ will be treated separately at the end of our analysis. If $\deg(w) \geq 4$ and $v \in N(w)$, then let $a(v, w)$ be the total amount received by w through the oriented edge (v, w) (according to one of RR 1.1, 1.2, 1.3, 3.1, 3.2 and 4.1). If $\deg(v) \geq 4$, then $a(v, w) = 0$. If $\deg(v) = 3$, then $a(v, w)$ depends among other things on the type of the edge vw. Let $\bar{u}(d'_1, d'_2)$ be a nonnegative upper bound for $a(v, w)$ provided vw is of type (d'_1, d'_2). If $\bar{u}(d'_1, d'_2)$ is not mentioned at all, then it is considered to be 0. We shall assume that $\mathrm{dm}(v) = \{d'_1, d'_2, d'_3\}$.

First suppose that $d'_1 = 3$. If $d'_2 = 5$, then v is of type $(3, 5, d)$ (Claim 1.2), and so, because of RR 1.1 and RR 3.2, we have $a(v, w) \leq \gamma(3, 5, d) + \frac{1}{2}\gamma(4, 5, d) + \gamma(4, 5, d-1) = -\frac{1}{24} + \frac{1}{d-1} + \frac{3}{2d} \leq \frac{41}{408}$. Let $d'_2 = 6$. If $c_2(v) \neq c_0(v)$, then it is because of RR 1.2; in such a case, by $\mathfrak{C}_1$, $d'_3 = d$, and so, by Claim 2, $a(v, w) = c_2(v) \leq \gamma(3, 6, d) + \frac{1}{2}\beta(6, d) = \frac{3}{d} - \frac{1}{12} \leq \frac{1}{12}$. If $c_2(v) = c_0(v)$, then Claim 1.2 yields $d'_3 \geq d-1$ and $a(v, w) = c_0(v) = \frac{1}{d'_3} \leq \frac{1}{17}$. Thus, we can take $\bar{u}(3, 6) = \frac{1}{12}$. Similarly, we can define $\bar{u}(3, 7) = \gamma(3, 7, 17) + \beta(7, 18)$. If $d'_2 \in [8, d]$, then $c_2(v) = c_0(v)$, $\mathrm{cd}(v) = d'_2 + d'_3 - 3 \geq d+2$ and $d'_3 \geq d+5-d'_2$. Therefore, because of RR 3.1 or RR 3.2, $a(v, w) \leq \gamma(3, d'_2, 23-d'_2)$. Moreover, $\gamma(3, d'_2, 23-d'_2) \leq \gamma(3, 8, 15) =: \bar{u}(3, d'_2)$ for any $d'_2 \in [12, d-3]$; for $d'_2 \in [8, 11] \cup [d-2, d]$ we put $\bar{u}(3, d'_2) = \gamma(3, d'_2, 23-d'_2)$.

Now consider the case $d'_1 = 4$. If $d'_2 = 4$, then RR 4.1 yields $a(v, w) \leq c_0(v) \leq \gamma(4, 4, 18) =: \bar{u}(4, 4)$. If $d'_2 = 5$, then, by RR 1.1, $a(v, w) \leq 2\gamma(4, 5, 17) =: \bar{u}(4, 5)$. If $d'_2 = 6$ and $\deg(v) = 3$, then, by RR 1.2 and Claim 2, $a(v, w) \leq \gamma(4, 6, d) + \frac{1}{2}\beta(6, d) = \frac{3}{d} - \frac{1}{6} \leq 0$ and we can take $\bar{u}(4, 6) = 0$. If $d'_2 = 7$ and $\deg(v) = 3$, then, by RR 1.2 with Claim 2 and by RR 1.3 with Claim 1.2, $a(v, w) \leq \beta(7, 18) + \gamma(4, 7, 17) < 0$; therefore, we take again $\bar{u}(4, 7) = 0$. If $(d'_1, d'_2) = (4, d)$, then, using $\mathfrak{C}_4$, $\mathfrak{C}_5$, RR 2.1 and RR 3.3 we can obtain $a(v, w) \leq c_0(v) \leq \gamma(4, 4, 18) = \bar{u}(4, d)$.

With $d'_1 \in [5, 7]$ the following bounds are easily derived: $\bar{u}(5, d'_2) = 2\gamma(4, 5, 17)$ for $d'_2 \in [d-1, d]$, $\bar{u}(6, d) = \frac{1}{2}\beta(6, 18)$, $\bar{u}(7, d-2) = \beta(7, 18)$, and $\bar{u}(7, d'_2) = \frac{3}{2}\beta(7, 18)$ for $d'_2 \in [d-1, d]$. The (positive) upper bounds $\bar{u}(d'_1, d'_2)$ are summarised in Table 2; for our analysis it is helpful to have them ordered in a decreasing sequence $(\frac{41}{408}, \frac{4}{45}, \frac{1}{12}, \frac{1}{17}, \frac{20}{357}, \frac{1}{18}, \frac{13}{336}, \frac{15}{476}, \frac{1}{36}, \frac{1}{40}, \frac{5}{238}, \frac{3}{170}, \frac{1}{63}, \frac{2}{195}, \frac{1}{132})$. Finally, for $d'_1 > d'_2$ we put $\bar{u}(d'_1, d'_2) = \bar{u}(d'_2, d'_1)$.

Now consider an n-vertex w of type $D = (d_1, \ldots, d_n)$ and let $(v_1, \ldots, v_n)$ be a sequence of neighbours of w in a cyclic order around w such that the edge $v_i w$ is incident to faces f_i of degree d_i and f_{i+1} of degree d_{i+1} (if $i \in [n+1, \infty)$, then the index i in v_i, f_i or d_i is taken modulo n so as to belong to $[1, n]$). Then $c_0(w) = 1 - \frac{n}{2} + \sum_{i=1}^{n} \frac{1}{d_i} = \sum_{i=1}^{n} p_i^n(w)$, where $p_i^n(w) = \frac{1}{n} - \frac{1}{2} + \frac{1}{2d_i} + \frac{1}{2d_{i+1}}$ is the ith *partial* charge of the vertex w (corresponding to the edge $v_i w$). If $n \geq 4$, then we have $c_4(w) = c_0(w) + \sum_{i=1}^{n} a(v_i, w) = \sum_{i=1}^{n}(p_i^n(w) + a(v_i, w)) \leq \sum_{i=1}^{n}(p_i^n(w) + \bar{u}(d_i, d_{i+1}))$. To bound $p_i^n(w)$

we use the following inequality yielded by Claim 1.1: $\frac{1}{2d_i}+\frac{1}{2d_{i+1}} \le \max\{\frac{1}{6}+\frac{1}{10}, \frac{1}{8}+\frac{1}{8}\} = \frac{4}{15}$ for any $i \in [1, n]$. By $F_k = |\{i \in [1, n] : d_i = k\}|$ we denote the *frequency* of k in D; we put $F_{k+} = \sum_{l=k}^{d} F_l$.

(1) If $n \ge 8$, then using Table 2 we see that $p_i^n(w) + \bar{u}(d_i, d_{i+1}) \le \frac{1}{8} - \frac{1}{2} + \frac{4}{15} + \frac{41}{408} < 0$ for any $i \in [1, n]$, and so $c_4(w) < 0$.

(2) $n \in [5, 7]$.

(21) If $\mathrm{cd}(w) \le d+1$, then, by Claim 1.1, $d_i \le d-5$ for any $i \in [1, n]$. Further, by Lemma 3.3, $\deg(v_i) = 3$ implies $\mathrm{cd}(v_i) \ge d+3$, and so from $d_i + d_{i+1} = 8$ it follows that $a(v_i, w) = 0$ and $\frac{1}{2d_i} + \frac{1}{2d_{i+1}} + a(v_i, w) \le \frac{1}{6} + \frac{1}{10} = \frac{4}{15}$. Using Table 2 it is easy to check that $d_i + d_{i+1} \ge 9$ yields $\frac{1}{2d_i} + \frac{1}{2d_{i+1}} + a(v_i, w) \le \frac{1}{6} + \frac{1}{12} + \frac{1}{12} = \frac{1}{3}$; moreover, if $\{d_i, d_{i+1}\} \ne \{3, 6\}$, then $\frac{1}{2d_i} + \frac{1}{2d_{i+1}} + a(v_i, w) \le \frac{1}{6} + \frac{1}{14} + \frac{20}{357} = \frac{5}{17}$.

(211) If $n \in [6, 7]$, then $p_i^n(w) + a(v_i, w) \le \frac{1}{n} - \frac{1}{2} + \max\{\frac{4}{15}, \frac{1}{3}\} \le 0$ for any $i \in [1, n]$ and $c_4(w) \le 0$.

(212) If $n = 5$, then, since $\frac{1}{5} - \frac{1}{2} + \max\{\frac{4}{15}, \frac{5}{17}\} < 0$, $p_i^5(w) + a(v_i, w)$ can be positive only if $\{d_i, d_{i+1}\} = \{3, 6\}$. Let $k = |\{i \in [1, 5] : \{d_i, d_{i+1}\} = \{3, 6\}\}|$.

(2121) If $k = 0$, then $c_4(w) < 0$ as a sum of five negative summands.

(2122) If $k \ge 1$, then, by Claim 1.1, $F_3 \in [1, 2]$. If $\deg(v_i) = 3$, $v_i w$ is of type $(3, 6)$ and v_i is not involved in RR 1.2, then $a(v_i, w) \le \gamma(3, 6, d) \le \frac{1}{18}$; notice that the number of i's such that $\deg(v_i) = 3$, $v_i w$ is of type $(3, 6)$ and v_i is involved in RR 1.2 is at most F_6.

(21221) If $F_3 = 1$, then, by Claim 1.1 and Table 2, $c_0(w) + \sum_{i=1}^{5} a(v_i, w) \le (-\frac{3}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + 2\cdot\frac{1}{4}) + 2\cdot\frac{1}{12} + 3\cdot\frac{3}{170} < 0$.

(21222) If $F_3 = 2$, then, by Claim 1.1, $F_4 = 0$. In such a case $a(v_i, w) = 0$ for (the unique) $i \in [1, 5]$ satisfying $\min\{d_i, d_{i+1}\} \ge 5$.

(212221) If $k \ge 4$, then w is of type $(3, 6, 3, 6, 6)$ and $c_0(w) + \sum_{i=1}^{5} a(v_i, w) \le -\frac{1}{3} + (3\cdot\frac{1}{12} + \frac{1}{18}) < 0$.

(212222) If $k = 3$, then $F_6 = 2$, $c_0(w) \le \gamma(3, 5, 6, 3, 6) = -\frac{3}{10}$, $\sum_{i=1}^{5} a(v_i, w) \le 2\cdot\frac{1}{12} + \frac{1}{18} + \frac{20}{357} < \frac{3}{10}$ and $c_4(w) < 0$.

(212223) $k = 2$.

(2122231) If $F_6 = 1$, then $c_0(w) \le \gamma(3, 5, 5, 3, 6) = -\frac{4}{15}$, $\sum_{i=1}^{5} a(v_i, w) \le \frac{1}{12} + \frac{1}{18} + 2\cdot\frac{20}{357} < \frac{4}{15}$ and $c_4(w) < 0$.

(2122232) If $F_6 = 2$, then $c_0(w) \le \gamma(3, 5, 3, 6, 6) = -\frac{3}{10}$, $\sum_{i=1}^{5} a(v_i, w) \le 2\cdot\frac{1}{12} + 2\cdot\frac{20}{357} < \frac{3}{10}$ and $c_4(w) < 0$.

(212224) If $k = 1$, then $c_0(w) \le \gamma(3, 5, 3, 5, 6) = -\frac{4}{15}$, $\sum_{i=1}^{5} a(v_i, w) \le \frac{1}{12} + 3\cdot\frac{20}{357} < \frac{4}{15}$ and $c_4(w) < 0$.

(22) $\mathrm{cd}(w) \ge d+2$

(221) If $n = 7$, then, by Claim 1.1, $F_{5+} \ge F_3$, $F_3 \le 3$, and so, by Lemma 4, $c_0(w) \le \gamma((3)^{F_3}(5)^{F_3}(4)^{6-2F_3}(d-8)) = -1 + \frac{F_3}{30} + \frac{1}{d-8} \le -\frac{4}{5}$. On the other hand, $\sum_{i=1}^{7} a(v_i, w) \le 7\cdot\frac{41}{408} < \frac{4}{5}$ and $c_4(w) < 0$.

(222) $n = 6$

(2221) If $F_3 \le 2$, then using Claim 1.1 and the assumption $\mathrm{cd}(w) \ge d+2$ we see that $F_{5+} \ge F_3 + 1$, and so, by Lemma 4, $c_0(w) \le \gamma((3)^{F_3}(5)^{F_3}(4)^{5-2F_3}(d-6)) = -\frac{3}{4} + \frac{F_3}{30} + \frac{1}{d-6} \le -\frac{2}{3} + \frac{F_3}{30}$. On the other hand, Table 2 yields $\sum_{i=1}^{6} a(v_i, w) \le 2F_3\cdot\frac{41}{408} + (6 - 2F_3)\cdot\frac{1}{18}$. Therefore, $c_4(w) \le \frac{377F_3}{3060} - \frac{1}{3} \le \frac{377}{1530} - \frac{1}{3} < 0$.

(2222) If $F_3 = 3$, then, by Claim 1.1, w is of type $(3, d_2, 3, d_4, 3, d_6)$ and, by Lemma 4, $c_0(w) \le \gamma(3, 5, 3, 5, 3, d-5) = -\frac{3}{5} + \frac{1}{d-5} \le -\frac{3}{5} + \frac{1}{13} = -\frac{34}{65}$. So, it is sufficient to show that $\sum_{i=1}^{6} a(v_i, w) \le \frac{34}{65}$.

(22221) If there is $i \in [1, 6]$ with $\deg(v_i) \ge 4$, then $\sum_{i=1}^{6} a(v_i, w) \le 5\cdot\frac{41}{408} < \frac{34}{65}$.

(22222) If $\deg(v_i) = 3$ for any $i \in [1, 6]$, then consider the expression $c_4(w) = \sum_{i=1}^{6} q_i$, where $q_i = \frac{1}{6} - \frac{1}{2} + \frac{1}{6} + \frac{1}{2\max\{d_i, d_{i+1}\}} + a(v_i, w) \le -\frac{1}{6} + \frac{1}{2\max\{d_i, d_{i+1}\}} + \bar{u}(3, \max\{d_i, d_{i+1}\})$ and $\max\{d_i, d_{i+1}\} \in [5, d]$. Using Table 2 it is easy to check that three maximal values of $f(s) = -\frac{1}{6} + \frac{1}{2s} + \bar{u}(3, s)$ for $s \in [5, d]$ are $f(5) = \frac{23}{680}$, $f(6) = 0$ and $f(7) = -\frac{2}{51}$. Notice that $c_4(w) = \sum_{i=1}^{3}(q_{2i-1} + q_{2i}) \le 2\sum_{i=1}^{3} f(d_{2i})$.

(222221) If $d_2 \ge 6$, then, as $\min\{d_4, d_6\} \ge d_2$, we obtain $c_4(w) \le 0$.

(222222) $d_2 = 5$.

(2222221) If $\min\{d_4, d_6\} \ge 7$, then $c_4(w) \le 2\cdot(\frac{23}{680} - 2\cdot\frac{2}{51}) < 0$.

(2222222) If there is $j \in \{4, 6\}$ with $d_j \in [5, 6]$, then $d_{10-j} \ge d - d_j$. Let d' be the degree of the face adjacent to both f_j and f_{10-j}. By Claim 1.2 we know that $d' \ge d + 5 - d_j$. Therefore, by RR 3.2, the summand $a(v_k, w)$ corresponding to the vertex v_k with $\mathrm{dm}(v_k) = \{3, d_{10-j}, d'\}$ is equal to $\gamma(3, d_{10-j}, d') = -\frac{1}{6} + \frac{1}{d_{10-j}} + \frac{1}{d'} \le -\frac{1}{6} + \frac{1}{d-6} + \frac{1}{d-1} \le -\frac{1}{6} + \frac{1}{12} + \frac{1}{17} < 0$ and $\sum_{i=1}^{6} a(v_i, w) < 5\cdot\frac{41}{408} < \frac{34}{65}$.

(223) $n = 5$.

(2231) If $F_3 = 0$, then, due to Lemma 4, $c_0(w) \le \gamma((4)^4(d-4)) \le -\frac{3}{7}$, and so $c_4(w) \le -\frac{3}{7} + 5\cdot\frac{1}{18} < 0$.

(2232) If $F_3 = 1$, then $c_4(w) \le \gamma(3, 5, 4, 4, d-4) = -\frac{7}{15} + \frac{1}{d-4} \le -\frac{83}{210}$, $\sum_{i=1}^{5} a(v_i, w) \le 2\cdot\frac{41}{408} + 3\cdot\frac{1}{18} < \frac{83}{210}$ and $c_4(w) < 0$.

(2233) If $F_3 = 2$, then, by Claim 1.1, $F_4 = 0$. By Lemma 4 we have $c_0(w) \le \gamma(3, 5, 3, 5, d-4) = -\frac{13}{30} + \frac{1}{d-4} \le -\frac{38}{105}$, and so it is sufficient to prove that $\sum_{i=1}^{5} a(v_i, w) \le \frac{38}{105}$.

(22331) If there is $i \in [1, 5]$ such that v_i is incident to a triangle and $\deg(v_i) \ge 4$, then $\sum_{i=1}^{5} a(v_i, w) \le 3\cdot\frac{41}{408} + \frac{15}{476} < \frac{38}{105}$.

Table 3
Upper bounds $\bar{\bar{u}}(d_1', d_2')$.

d_1'	3	3	3	3	3	3	4	4	4
d_2'	5	6	7	8	$\in [9, d-2]$	$d-1, d$	4	5	$\in [6, d-5]$
$\bar{\bar{u}}(d_1', d_2')$	$\frac{239}{2040}$	$\frac{1}{12}$	$\frac{3}{68}$	$\frac{1}{240}$	$-\frac{1}{84}$	$\frac{1}{30}$	$\frac{1}{18}$	$-\frac{1}{136}$	$-\frac{1}{24}$

d_1'	4	4	5	5	5
d_2'	$\in [d-4, d-1]$	d	$\in [5, d-5]$	$d-4, d-3$	$\in [d-2, d]$
$\bar{\bar{u}}(d_1', d_2')$	$-\frac{5}{56}$	$-\frac{1}{24}$	$-\frac{1}{20}$	$-\frac{4}{35}$	$-\frac{7}{68}$

d_1'	6	6	6	7	7	$\in [8, d]$
d_2'	$\in [6, d-6]$	$\in [d-5, d-1]$	d	$\in [7, d-7]$	$\in [d-6, d]$	$\in [d_1', d]$
$\bar{\bar{u}}(d_1', d_2')$	$-\frac{1}{12}$	$-\frac{5}{39}$	$-\frac{1}{9}$	$-\frac{3}{28}$	$-\frac{2}{17}$	$-\frac{1}{8}$

(22332) Now suppose that all neighbours of w incident to a triangle are of degree three. Let f_j be the face adjacent to two triangles.

(223321) If $d_j \in [5, 7]$, then there is $k \in [1, 5]$ such that $d_k \geq 9$. The face $\tilde{f}$ adjacent to both f_j and f_k is of degree $d' \geq d-2$ (Claim 1.2), hence for the vertex v_l incident to f_k and $\tilde{f}$ we have $a(v_l, w) = -\frac{1}{6} + \frac{1}{d_k} + \frac{1}{d'} \leq \frac{1}{144}$ and, by Table 2, $\sum_{i=1}^{5} a(v_i, w) \leq 3 \cdot \frac{41}{408} + \frac{1}{144} + \frac{15}{476} < \frac{38}{105}$.

(223322) If $d_j \in [8, d-3]$, then $\sum_{i=1}^{5} a(v_i, w) \leq 2 \cdot \frac{41}{408} + 2 \cdot \frac{1}{40} + \frac{15}{476} < \frac{38}{105}$.

(223323) If $d_j \in [d-2, d]$, then notice that from Table 2 it follows that if $\min\{d_i, d_{i+1}\} \geq 5$, then $p_i^5(w) + \bar{u}(d_i, d_{i+1}) < 0$. Therefore, it suffices to show that if $d_l = 3$, then $\sum_{i=l-1}^{l}(p_i^5(w) + a(v_i, w)) \leq 0$. Let d' be the degree of the face adjacent to f_{l-1}, f_l and f_{l+1}. Claim 1.2 then yields $d' \geq \max\{d+5-d_{l-1}, d+5-d_{l+1}\}$, and so, by RR 3.2, $\sum_{i=l-1}^{l}(p_i^5(w) + a(v_i, w)) = -\frac{3}{5} + \frac{3}{2d_{l-1}} + \frac{3}{2d_{l+1}} + \frac{2}{d'}$. If $m \in \{-1, 1\}$, then $\frac{3}{2d_{i+m}} + \frac{2}{d'} \leq \frac{3}{2d_{i+m}} + \frac{2}{23-d_{i+m}} \leq \frac{3}{36} + \frac{2}{5} = \frac{29}{60}$, and so, as $j \in \{m-1, m+1\}$, we have $\sum_{i=m-1}^{m}(p_i^5(w) + a(v_i, w)) \leq -\frac{3}{5} + \frac{3}{32} + \frac{29}{60} < 0$.

(3) $n = 4$.

(31) If $\mathrm{cd}(w) \leq d+1$, then by Lemma 3.2 the vertex w is not incident to a contractible edge, hence, by Lemma 2, w has at least three neighbours of degree three. Since $d_i < d$ for any $i \in [1, 4]$, using Lemma 3.4 and $\mathcal{C}_2$ we see that $d_1 \geq 4$. As in (21), $d_i = d_{i+1} = 4$ implies $a(v_i, w) = 0$ and $p_i^4(w) + a(v_i, w) = 0$. Moreover, with the help of Table 2 it is easy to check that $p_i^4(w) + \bar{u}(d_i, d_{i+1}) \leq 0$ whenever $d_i + d_{i+1} \geq 9$ (and $\min\{d_i, d_{i+1}\} \geq 4$); as a consequence, $c_4(w) \leq 0$.

(32) If $\mathrm{cd}(w) \geq d+2$, then put $q_i = p_i^4(w) + a(v_i, w)$ for $i \in [1, \infty)$.

(321) If $F_3 = 2$, then, by Claim 1.1, w is of type $(3, d_2, 3, d_4)$, where $d_2 + d_4 \geq d+4$. Since $c_4(w) = (q_2 + q_3) + (q_4 + q_5)$, it is sufficient to show that $q_i + q_{i+1} \leq 0$ for any $i \in \{2, 4\}$. So, in what follows we assume that $i \in \{2, 4\}$.

(3211) If $\min\{\deg(v_i), \deg(v_{i+1})\} \geq 4$, then $q_i + q_{i+1} = -\frac{1}{6} + \frac{1}{2d_2} + \frac{1}{2d_4} \leq -\frac{1}{6} + \frac{1}{10} + \frac{1}{34} < 0$.

(3212) If there is $j \in [i, i+1]$ such that $\deg(v_j) = 3$ and $\deg(v_{2i+1-j}) \geq 4$, then, by Lemma 3.4, $d_4 = d$ and $q_i + q_{i+1} = -\frac{1}{6} + \frac{1}{2d_2} + \frac{1}{2d} + a(v_j, w) \leq -\frac{1}{6} + \frac{1}{10} + \frac{1}{36} + a(v_j, w) = -\frac{7}{180} + a(v_j, w)$.

(32121) If $a(v_j, w) \leq 0$, then $q_i + q_{i+1} < 0$.

(32122) If $a(v_j, w) > 0$, then, by RR 3.1, v_j is of type $(3, d', d_2)$ (where d_2 appears either without loss of generality, namely if w is of type $(3, d, 3, d)$, or due to Lemma 3.4). By Claim 1.4 we obtain $d' \in [5, 11]$, and so, by Claim 1.2, $d_2 \geq d+5-d' \geq d-6$. Therefore, $q_i + q_{i+1} \leq -\frac{1}{6} + \frac{1}{2(d-6)} + \frac{1}{2d} + \frac{4}{45} \leq -\frac{1}{6} + \frac{1}{24} + \frac{1}{36} + \frac{4}{45} < 0$.

(3213) If $\deg(v_i) = \deg(v_{i+1}) = 3$, then, by $\mathcal{C}_3$, $\min\{\mathrm{cd}(v_i), \mathrm{cd}(v_{i+1})\} \geq d+3$. Therefore, Claim 1.2 yields $\min\{d_2, d_4\} \geq 6$. Let d' be the degree of the face adjacent to the triangle $v_i w v_{i+1}$ along the edge $v_i v_{i+1}$. Then $d_2 + d' - 3 = \min\{\mathrm{cd}(v_i), \mathrm{cd}(v_{i+1})\} \geq d+3$, hence $d' \geq d+6-d_2$.

(32131) If $d_2 \leq 8$, then $q_i \leq -\frac{1}{12} + \frac{1}{2d_2} + \bar{u}(3, d_2)$ and $q_{i+1} = -\frac{1}{4} + \frac{3}{2d_4} + \frac{1}{d'} \leq -\frac{1}{4} + \frac{3}{2(d+4-d_2)} + \frac{1}{d+6-d_2}$.

(321311) If $d_2 = 6$, then $q_i + q_{i+1} \leq \frac{1}{12} - \frac{1}{4} + \frac{3}{32} + \frac{1}{18} < 0$.

(321312) If $d_2 \in [7, 8]$ then $q_i + q_{i+1} \leq -\frac{1}{12} + \frac{1}{14} + \frac{20}{357} - \frac{1}{4} + \frac{3}{28} + \frac{1}{16} < 0$.

(32132) If $d_2 \in [9, 14]$, then $d' \geq 10$ and $q_i + q_{i+1} = -\frac{1}{2} + \frac{3}{2d_2} + \frac{3}{2d_4} + \frac{2}{d'} \leq -\frac{1}{2} + \frac{3}{18} + \frac{3}{26} + \frac{2}{10} < 0$.

(32133) If $d_2 \in [15, d-2]$, then $q_i + q_{i+1} \leq -\frac{1}{2} + 2 \cdot \frac{3}{30} + \frac{2}{8} < 0$.

(32134) If $d_2 = d-1$, then $q_i + q_{i+1} \leq -\frac{1}{2} + 2 \cdot \frac{3}{34} + \frac{2}{7} < 0$.

(32135) If $d_2 = d$, then $q_i + q_{i+1} \leq -\frac{1}{2} + 2 \cdot \frac{3}{36} + \frac{2}{6} = 0$.

(322) If $F_3 = 1$, then consider the inequalities $q_i \leq -\frac{1}{4} + \frac{1}{2d_i} + \frac{1}{2d_{i+1}} + \bar{u}(d_i, d_{i+1}) \leq \bar{\bar{u}}(d_i, d_{i+1})$, where $i \in [1, \infty)$, $\bar{\bar{u}}(d_1', d_2')$ with $d_1' \leq d_2'$ is an upper bound for $-\frac{1}{4} + \frac{1}{2d_1'} + \frac{1}{2d_2'} + \bar{u}(d_1', d_2')$ presented in Table 3 (that is created using Table 2) and, provided $d_1' > d_2'$, $\bar{\bar{u}}(d_1', d_2') = \bar{\bar{u}}(d_2', d_1')$. Since $d_1 = 3$, by Claim 1.2 we have $d_4 \geq d_2 \geq 5$; as $d_3 \geq 4$, from Table 3 we see that $q_i < 0$, $i = 2, 3$.

(3221) If $q_i \leq 0$, $i = 1, 4$, then $c_4(w) = \sum_{i=1}^{4} q_i < 0$.

(3222) $\max\{q_1, q_4\} > 0$.

(32221) If $q_j + q_{j+2} \leq 0$ for $j = 1, 4$, then $c_4(w) = (q_1 + q_3) + (q_4 + q_6) \leq 0$.

(32222) Let $i \in \{1, 4\}$ be such that $q_i + q_{i+2} \geq q_{5-i} + q_{7-i}$ and $q_i + q_{i+2} > 0$ (so that $q_{i+2} < 0$ implies $q_i > 0$).

(322221) If $a(v_i, w) = 0$, then $q_i = -\frac{1}{12} + \frac{1}{2\max\{d_i, d_{i+1}\}}$, and so $\max\{d_i, d_{i+1}\} = 5$ and $q_i = \frac{1}{60}$ (for otherwise $q_i \leq 0$). Then, however, $d_{i+2} + d_{i+3} = \mathrm{cd}(w) \geq d + 2$ and $\min\{d_{i+2}, d_{i+3}\} \geq 4$, so that Table 3 yields $q_{i+2} \leq -\frac{3}{32}$ and $q_i + q_{i+2} < 0$, a contradiction.

(322222) If $a(v_i, w) \neq 0$, then $\deg(v_i) = 3$ and $\mathrm{dm}(v_i) = \{3, s, d'\}$, where $s = \max\{d_i, d_{i+1}\}$.

(3222221) If v_i is incident to a 1-triangle, then $s > d'$ (we are using RR 3.1), and so, by Claim 1.2, $s \geq 12$; then, by Table 3, $s \geq d - 1$ and $q_i \leq \frac{1}{30}$. Moreover, $a(v_{5-i}, w) = 0$ and, by Lemma 3.4, the edge $v_{5-i}w$ is of type $(3, d)$ so that $q_{5-i} = -\frac{1}{12} + \frac{1}{2d} \leq -\frac{1}{12} + \frac{1}{36} = -\frac{1}{18}$ and $\sum_{j=1}^{4} q_j < q_1 + q_4 \leq \frac{1}{30} - \frac{1}{18} < 0$.

(3222222) Now suppose that v_i is incident to a 2-triangle (which means that $\deg(v_1) = \deg(v_4) = 3$). From Table 3 it follows that $s \in [5, 8] \cup [d - 1, d]$. We have $s + d_{i+2} + d_{i+3} - 5 = \mathrm{cd}(w) \geq d + 2$, hence $d_{i+2} + d_{i+3} \geq d + 7 - s$.

(32222221) If $s = 5$, then $d' = d$ (by Claim 1.2) and either $\min\{d_{i+2}, d_{i+3}\} \in [4, 5]$ or $\{d_{i+2}, d_{i+3}\} = \{6, d\}$, since otherwise $q_{i+2} \leq -\frac{2}{17}$ and $q_i + q_{i+2} \leq \frac{239}{2040} - \frac{2}{17} < 0$. Thus, w is of one of types $(3, 5, 4, d_4)$, $(3, 5, 5, d_4)$, $(3, 5, 6, d)$, $(3, 5, d, 6)$ and $(3, 5, d_3, 5)$; in the first four cases we have immediately $i = 1$ and in the last case we may suppose without loss of generality that $i = 1$.

(322222211) If $d_3 = 4$, then $d_4 \geq d - 2$, $q_3 \leq \bar{\bar{u}}(4, d_4)$ and $q_4 = -\frac{1}{4} + \frac{1}{d} + \frac{3}{2d_4} \leq -\frac{7}{36} + \frac{3}{2d_4}$. Since $\bar{\bar{u}}(4, d_4) + \frac{3}{2d_4} \leq \max\{-\frac{5}{56} + \frac{3}{32}, -\frac{1}{24} + \frac{3}{36}\} = \frac{1}{24}$, we obtain $c_4(w) \leq \frac{239}{2040} - \frac{1}{136} - \frac{7}{36} + \frac{1}{24} < 0$.

(322222212) If w is of type $(3, 5, 5, d_4)$, then $d_4 \geq d - 3$, $a(v_4, w) = -\frac{1}{6} + \frac{1}{d_4} + \frac{1}{d} \leq -\frac{1}{6} + \frac{1}{15} + \frac{1}{18} = -\frac{2}{45}$, $q_4 \leq -\frac{1}{12} + \frac{1}{30} - \frac{2}{45} = -\frac{17}{180}$ and $c_4(w) \leq \frac{239}{2040} - \frac{1}{20} - \frac{7}{68} - \frac{17}{180} < 0$.

(322222213) If w is of type $(3, 5, d_3, 5)$, then $d_3 \geq d-3$ and $c_0(w) \leq \gamma(3, 5, d-3, 5) = -\frac{4}{15} + \frac{1}{d-3} \leq -\frac{4}{15} + \frac{1}{15} = -\frac{1}{5}$. It is easy to see that if a face f_j with $j \in \{2, 4\}$ is incident to a vertex of type $(4, 5, \hat{d})$, then the number of such vertices is at most two and besides w there is at least one other absorbing vertex incident to f_j. Therefore, the total amount received by w due to RR 1.1 is bounded from above by $2\gamma(4, 5, 17)$, $\sum_{j=1}^{4} a(v_j, w) \leq 2\gamma(3, 5, 18) + 2\gamma(4, 5, 17) = \frac{299}{1530}$ and $c_4(w) \leq -\frac{1}{5} + \frac{299}{1530} < 0$.

(322222214) If $\{d_3, d_4\} = \{6, d\}$, then $c_0(w) = \gamma(3, 5, 6, d) = -\frac{3}{10} + \frac{1}{d} \leq -\frac{3}{10} + \frac{1}{18} = -\frac{11}{45}$, $\sum_{j=1}^{4} a(v_j, w) \leq \frac{41}{408} + \frac{1}{36} + \max\{\frac{3}{170} + \frac{1}{12}, 0 + \frac{4}{45}\} < \frac{11}{45}$, and so $c_4(w) < 0$.

(32222222) If $s \in [6, 8]$, then $q_i \leq \bar{\bar{u}}(3, s)$ and $q_{i+2} \leq \max\{\bar{\bar{u}}(d'_1, d'_2) : d'_1 \geq 4, d'_1 + d'_2 \geq d + 7 - s\}$. From Table 3 it follows that $i = 1$, $d_3 = 4$ and $d_4 = d$ (for otherwise $q_i + q_{i+2} < 0$, a contradiction). Claim 1.2 yields $d' \geq d + 5 - s$, hence $q_4 = -\frac{1}{12} + \frac{1}{2d} + (-\frac{1}{6} + \frac{1}{d} + \frac{1}{d'}) \leq -\frac{1}{4} + \frac{3}{36} + \frac{1}{15} = -\frac{1}{10}$ and, by Table 3, $\sum_{j=1}^{4} q_j \leq \frac{1}{12} - \frac{1}{24} - \frac{1}{24} - \frac{1}{10} < 0$.

(32222223) If $s \in [d - 1, d]$, then $\{d_{i+2}, d_{i+3}\} = [4, 5]$, for otherwise $q_i + q_{i+2} \leq \frac{1}{30} - \frac{1}{24} < 0$. By Claim 1.1 then w is of type $(3, 5, 4, d_4)$, hence $i = 4$ and $d' = d$ (by Claim 1.2). Therefore, $q_4 = -\frac{1}{4} + \frac{1}{d} + \frac{3}{2d_4} \leq -\frac{1}{4} + \frac{1}{18} + \frac{3}{34} < 0$, a contradiction.

(323) $F_3 = 0$.

(3231) If $q_i \leq 0$ or $q_i + q_{i+2} \leq 0$ for every $i \in [1, 4]$, then $c_4(w) \leq 0$.

(3232) Let $i \in [1, 4]$ be such that $q_i > 0$ and $q_i + q_{i+2} > 0$. From Table 3 it follows that $d_i = d_{i+1} = 4$ and $q_i \leq \frac{1}{18}$. Since $d_{i+2} + d_{i+3} = \mathrm{cd}(w) \geq d + 2$, Table 3 yields also $\{d_{i+2}, d_{i+3}\} = \{4, d\}$. Thus, w is of type $(4, 4, 4, d)$, we may suppose without loss of generality that $i = 1$ and $c_0(w) = \gamma(4, 4, 4, d) = -\frac{1}{4} + \frac{1}{d} \leq -\frac{7}{36}$.

(32321) If $\max\{\deg(v_j) : j \in [1, 4]\} \geq 4$, then $c_4(w) \leq -\frac{7}{36} + 3 \cdot \frac{1}{18} < 0$.

(32322) If $\deg(v_j) = 3$ for any $j \in [1, 4]$, then consider the quadrangle $v_1 w v_2 x$.

(323221) If $\deg(x) = 3$, then x is of type $(4, d, d)$ and, by RR 2.1, $c_2(v_1) = \gamma(4, 4, d) + \frac{1}{2}\gamma(4, d, d) = -\frac{1}{8} + \frac{2}{d} \leq -\frac{1}{72}$, hence $q_1 = a(v_1, w) = 0$, which contradicts $q_i > 0$.

(323222) If $\deg(x) \geq 4$, then, by RR 4.1, $q_1 = a(v_1, w) \leq \frac{1}{2}c_3(v_1) \leq \frac{1}{2}\gamma(4, 4, d) = \frac{1}{2d} \leq \frac{1}{36}$ and $q_1 + q_3 \leq \frac{1}{36} - \frac{1}{24} < 0$ in contradiction with $q_i + q_{i+2} > 0$.

(4) $n = 3$.

(41) If $d_1 = 3$, then w belongs to an i-triangle t, $i \in [1, 3]$.

(411) $i = 1$.

(4111) If $c_0(w) \leq 0$, then $d_2 \geq 9$ (Claim 1.5), hence $c_4(w) = c_0(w) \leq 0$.

(4112) If $c_0(w) > 0$, then $c_2(w) \geq c_0(w) > 0$, and so, by RR 3.1, $c_4(w) = 0$.

(412) If $i = 2$, then applying RR 3.2 yields $c_4(w) = 0$.

(413) $i = 3$.

(4131) If t is positive, then, by RR 1.2 and $\mathcal{C}_6$, we have $c_4(w) = 0$.

(4132) If t is nonpositive, then, by RR 1.4, $c_4(w) = \frac{1}{3}\Sigma(c_0, V(t)) \leq 0$.

(42) $d_1 = 4$.

(421) $d_2 = 4$.

(4211) If $c_3(w) \leq 0$, then $c_4(w) = c_3(w) \leq 0$.

(4212) If $c_3(w) > 0$, then necessarily also $c_2(w) > 0$.

(42121) If $n_{4+}(w) \geq 1$, then, by RR 4.1, $c_4(w) = 0$.

(42122) $n_{4+}(w) = 0$.

(421221) If $n_{4+}(v_1) \geq 1$, then, by RR 3.3, $c_4(w) = 0$.

(421222) If $n_{4+}(v_1) = 0$, then, by $\mathcal{C}_4$, for any $i \in [2, 3]$ the type $(4, d_i', d)$ of the vertex v_i is such that $d_i' \geq 6$. Therefore, by $\mathcal{C}_5$ and RR 2.1, $c_3(w) = \gamma(4, 4, d) + \gamma(4, d_2', d) + \gamma(4, d_3', d) = -\frac{1}{2} + \frac{3}{d} + \frac{1}{d_2'} + \frac{1}{d_3'} \leq -\frac{1}{2} + \frac{3}{18} + 2 \cdot \frac{1}{6} = 0$, a contradiction.

(422) If $d_2 = 5$, then, by RR 1.1, $c_4(w) = 0$.

(423) If $d_2 \geq 6$, then $c_0(w) \leq 0$ (Claim 1.4).

(4231) If w has not received any amount, then $c_0(w) \leq c_4(w) \leq 0$.

(4232) If w has received an amount, then $d_2 = 6$ and the rule RR 1.2 has been applied; then, by Claim 2, $c_1(w) \leq \gamma(4, 6, d) + \frac{1}{2}\beta(6, d) = -\frac{1}{6} + \frac{3}{d} \leq 0$, and so $c_1(w) \leq c_4(w) \leq 0$.

(43) If $d_1 \geq 5$, then, by Claim 1.4, $c_0(w) \leq 0$.

(431) If w has not received any amount, then $c_0(w) \leq c_4(w) \leq 0$.

(432) If w has received an amount, then either $d_1 = 5$ and RR 1.1 has been applied or $[6, 7] \cap \mathrm{dm}(w) \neq \emptyset$ and RR 1.2 has been applied.

(4321) If $d_1 = 5$, then $d_2 \geq 11$, $d_3 \geq d - 1$ and $c_4(w) \leq \gamma(5, 11, d - 1) + 4\gamma(4, 5, d - 1) \leq -\frac{9}{22} + \frac{5}{17} < 0$.

(4322) If $6 \in \mathrm{dm}(w)$, then $\mathrm{dm}(w) = \{6, s, d\}$ with $s \in [5, d]$ and $c_4(w) \leq \gamma(6, 5, d) + \frac{1}{2}\beta(6, d) = -\frac{13}{60} + \frac{3}{d} \leq -\frac{13}{60} + \frac{3}{18} < 0$.

(4323) If $7 \in \mathrm{dm}(w)$, then $d_1 = 7$, $d_2 \geq 10$ and $c_4(w) \leq \gamma(7, 10, 10) + 3\beta(7, d) \leq -\frac{4}{5} + \frac{12}{17} < 0$.

Since $c_4(w) \leq 0$ for any $w \in V$, the proof is complete. ■

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